NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

A Two-Dimensional Systems Stability Analysis of Vehicle Platoons

A dissertation submitted for the degree of
Doctor of Philosophy by

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Maynooth, 5. October 2012

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Table of Contents

List of Figures

Abstract

The main contributions of this dissertation are in the field of stability analysis of linear and nonlinear two-dimensional systems. The study of stability of such systems is motivated by the "string stability" or "platooning" problem: In order to achieve tighter spacing between vehicles travelling one after the other in one direction, i. e. in a string or platoon, the driver is replaced by an automatic controller designed to keep a specified distance towards the preceding vehicle.

It is shown how such a vehicle platoon can be modelled as a two-dimensional system. Here, two-dimensional refers to the fact that the system depends on two independent variables such as time t and position within the string k. However, two-dimensional systems describing a vehicle string generically exhibit a singularity at the stability boundary. The existence of this singularity at the stability boundary prevents application of most stability criteria known in the literature, since this marginal case is almost always explicitly or implicitly excluded.

Bounded-input bounded-output stability of linear continuous-discrete two-dimensional systems is studied in the frequency domain paying particular attention to systems with nonessential singularities of the second kind at the stability boundary. A two-dimensional version of Parseval's Theorem and the corresponding induced operator norm are derived. The results are applied to a string of vehicles and sufficient conditions for string stability are deduced.

Sufficient conditions for different forms of stability of linear two-dimensional systems in the time domain are developed using a two-dimensional quadratic Lyapunov function and linear matrix inequalities. It is shown that systems permitting a two-dimensional Lyapunov function with a negative definite divergence are exponentially stable.

It is proven, however, that two-dimensional systems with a singularity at the stability boundary (such as two-dimensional descriptions of vehicle strings) cannot satisfy the required conditions for exponential stability as the divergence of the Lyapunov function can never be sign definite. If the divergence is only negative semidefinite, stability of the system can be guaranteed. Provided additional conditions on the Lyapunov function and the initial conditions are satisfied, asymptotic stability of systems whose Lyapunov functions have a negative semidefinite divergence can be shown.

Extending the results mentioned above, sufficient conditions for stability, exponential stability and asymptotic stability of nonlinear two-dimensional systems are deduced. Similar to the results on linear two-dimensional systems, exponential stability can be guaranteed if the divergence of the Lyapunov function is strictly negative. For systems with merely nonpositive divergence stability is also shown. Asymptotic stability of nonlinear two-dimensional systems can be proven if not only the initial conditions but also the Lyapunov function itself and the state space equations satisfy additional smoothness conditions. Instead of a quadratic Lyapunov function, a wider class of Lyapunov functions is allowed in the proofs of stability of nonlinear two-dimensional systems. The notion and theory of *(integral) input to state stability is used instead of linear matrix inequalities to derive the results.*

All proofs and results for the stability of linear and nonlinear two-dimensional systems in the time domain are given in a unified notation, studying systems with continuous and discrete independent variables simultaneously.

The theoretical results on linear two-dimensional systems are used to analyse the (string) stability of a linear unidirectional homogenous string with different time headways and communication range 1 *and* 2*. The stability results for nonlinear two-dimensional systems are applied to rigorously prove string stability of a nonlinear string with variable time headway.*

Preface

Thanks be to God for his indescribable gift!
2 Cor 9:15

Acknowledgements

Throughout the last four years I have encountered many challenges both in my academic and private live. Even though at the end of it only I will – hopefully – have gained the doctor's degree, I know that this achievement was only possible due to the loving and extensive support of many people I was fortunate to meet.

First and foremost I would like to acknowledge the fact that my academic achievements and development are a result of the superior supervision of Prof. Richard H. Middleton. His outstanding knowledge and his patience in discussing and explaining mathematical headaches set a high standard. I am also very grateful for his guidance and help beyond academic matters and his friendship and support throughout the last years.

During my time at the Hamilton Institute I had the great pleasure to work with Prof. Robert N. Shorten and Dr. Oliver Mason. I appreciate not only their truly lived out open-door-policy and their support in all sorts of tricky mathematical dilemmas but also Bob's enthusiastic and motivating attitude and Olli's dry humour.

When I think about the Hamilton Institute I also think of many great colleagues and friends I have met there during the years making it an exceptionally great place to work and study. Special thanks to Arieh Schlote and Andrés Peters for printing and submitting my thesis to the Examinations Office in Maynooth. After relocating it was a delight to get to know so many friendly and cheerful new colleagues here in Newcastle creating an enjoyable and partly witty atmosphere during lunch breaks and barbecues.

Work and office is not everything in life and I am most grateful for the loving and caring friends I have found both in the Maynooth Community Church and Hamilton Baptist Church in Newcastle. I would like to thank specially Kevin and Claire Hargaden as they play a huge role in the development of my faith. Their ongoing support and friendship have shaped me and helped me become the person I am now.

My friend Ruth Middleton also deserves a special thank you. Without her emotional and "hands on" support in many aspects our move to Australia and settling in would have been much harder.

Specially during my first two years as a PhD student I probably would have given up without the cheerful, yet thoughtful help of Nicola Mays. Her jolly, spot on and honest emails and seemingly endless Skype phone calls encouraged me many times to hang in there as a wife, a mother, a friend, a sister, a PhD student and many other things at the same time.

I am very thankful for the help and support of my family in Germany through the last four years. Their love and care for me is a vital part of my life. I cannot overestimate, however, my gratitude towards my lovely husband Florian and my daughter Sophie. Sophie is such a bundle of joy and delight but without Florian I would not be able to appreciate and enjoy it. His kind and forgiving love for me makes me realise what a great man I have got at my side.

Publications during doctorate

The following publications were prepared in the course of this doctorate:

S. Klinge and R. H. Middleton. String stability analysis of homogeneous linear unidirectionally connected systems with nonzero initial conditions. In *IET Irish Signals and Systems Conference (ISSC 2009)*, June 2009a. DOI : 10.1049/cp.2009.1694

S. Klinge and R. H. Middleton. Time Headway Requirements for String Stability of Homogeneous Linear Unidirectionally Connected Systems. In *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 1992–1997, December 2009b. DOI : 10.1109/CDC.2009.5399965

S. Knorn and R. H. Middleton. Two-Dimensional Frequency Domain Analysis of String Stability. In *Proceedings of the Australian Control Conference (AUCC)*, pages 298–303, November 2012a

S. Knorn and R. H. Middleton. Asymptotic Stability of Two-Dimensional Continuous Roesser Models with Singularities at the Stability Boundary. In *Proceedings of the 51st IEEE Conference on Decision & Control*, pages 7787–7792, December 2012b. DOI : 10.1109/CDC.2012.6426968

CHAPTER 1

Introduction: A Motivating Example

The platooning problem is used here as an introductory example to motivate the study of two-dimensional systems in general and continuous-discrete two-dimensional systems in particular.

One of the most remarkable changes of modern times is the increased mobility, both in a private but also commercial context. Due to significant advances in the automotive industry cars became affordable for large parts of society and are now an essential part in the daily lives of many. Increased mobility has brought many advantages but also led to an increase in pollution and, in many places, a capacity overload of the existing infrastructure, despite many efforts to build and expand roads.

In general different methods are known to model the traffic on roads or highways. In a "macroscopic" system description the traffic is characterised similar to a flow or fluid. Measurements such as the vehicle flow or the traffic density (number of vehicles passing a certain point or stretch of road per unit of time), or the average speed of vehicles are used to analyse the traffic flow. Another approach is to model the dynamics of individual vehicles and the influence of the surrounding motor vehicles' movements on the vehicles' behaviour. In contrast to the approach mentioned above this is sometimes referred to as a "microscopic" system description.

Some fifty years ago researchers started to investigate ways to allow higher traffic throughput without expanding the existing road network. One of the proposed solutions first seemed simple and intuitive: Instead of allowing drivers to navigate their vehicle freely and independently on the roads, cars and trucks were to be organised in a string or "platoon". Every vehicle should then follow its predecessor and keep a prescribed close distance to it, whereas the first vehicle is to follow an independent reference. To minimise reaction times and thus to allow tighter spacing the cars would not be driven by human drivers but automatic controllers.

As the platooning problem focuses on a string of individual vehicles trying to maintain a specified distance to surrounding cars (or a reference) usually microscopic models are applied to describe the dynamics of vehicle platoon. Here, we will focus on the longitudinal dynamics only, ignoring manoeuvres like overtaking or lane changing. We also assume throughout this work that the platoon maintains its structure at all times. Thus, we do not consider the possibility of cars joining or leaving the platoon.

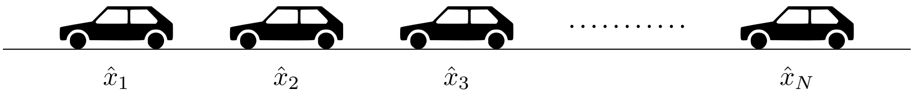


Figure 1.1: *Platoon / String of N vehicles*

Typically it is assumed that only the torque or the acceleration of a passenger car or heavy duty vehicle can be controlled directly — in contrast to the vehicle's velocity or position. Hence, we will use the acceleration as the input of our vehicle model

$$\dot{\hat{x}}_k(t) = \hat{v}_k(t), \tag{1.1}$$

$$\dot{\hat{v}}_k(t) = \hat{u}_k(t) - C_\mathrm{d}|\hat{v}_k(t)|\hat{v}_k(t), \tag{1.2}$$

where $\hat{x}_k(t)$ is the position of the kth vehicle in the string at time t, $\hat{v}_k(t)$ its velocity, $\hat{u}_k(t)$ the assigned acceleration (the output of the local controller and the input of the vehicle or plant model) and C_d the drag coefficient. Throughout, we will adhere to the convention that local variables are denoted with "ˆ". Neglecting time delays and linearising the model around the steady state velocity $v_0 > 0$, the Laplace transform of the simple plant model (1.1)-(1.2) yields

$$P(s) = \frac{1}{s^2 + 2C_\mathrm{d}v_0 s}. \tag{1.3}$$

In its simplest form every vehicle should keep a fixed distance $\hat{x}_\mathrm{d}$ to its predecessor and thus minimise the local error

$$\hat{e}_k(t) = \hat{x}_{k-1}(t) - \hat{x}_k(t) - \hat{x}_\mathrm{d}. \tag{1.4}$$

Now, deriving a stable controller (here a common PID controller) that minimises the location error between vehicles is, in most cases, a straight forward task. However, it became clear that this only covers one important aspect of the overall system dynamics: If a small disturbance is applied to the first vehicle in the string (or its reference), the local controller will aim to adjust the acceleration in order to minimise the local error. That will lead to a small deviation of the vehicles velocity and its relative position towards its follower. This transient then is a disturbance to the second vehicle in the string as its local controller also aims to maintain a specified distance to the leading vehicle. The disturbance response of the second car will act as a disturbance to the third vehicle's dynamics and so on. Thus the disturbance will travel down the string, i. e. from each car to its follower.

In some system settings — even though all local errors decrease to zero over time — the peek value of the local error grows without bound when the disturbance travels down the string. This effect became known as the "slinky effect" or "string instability", Chu (1974); Peppard (1974).

In the 1960s a similar problem was observed in supply chains, as discussed in Forrester (1961). It was thus called the "Forrester effect", or, later "bullwhip effect": Businesses attempt to forecast demand in order to plan inventory and other resources. A "safety stock" is maintained to buffer forecast errors. Each member of the supply chain, from end consumer to raw materials supplier, experiences greater fluctuation in demand and therefore needs a greater safety buffer. If the demand increases down-stream, members upstream will increase their orders. If the demand decreases, orders fall or even stop, and the inventory is not reduced. Hence, variations are amplified as one moves in the supply chain from the end customer to the raw material suppliers. A block diagram of a common form of a production control system is given in (Disney *et al.*, 2004, Fig. 1).

Similarly, imagine a chain of water reservoirs where the water flow from the kth pool is the inflow of pool $k+1$. If a controller with an integral component is used to regulate the water level independently of the inflow by adjusting the outflow, the result will be a chain of homogeneous systems with two integrators in the open loop. It is precisely this structural property that leads to the aforementioned slinky effect, Li *et al.* (2005).

Different approaches have been applied to analyse string stability. A common ansatz is to apply the Laplace transform with respect to time t and to study the dynamics of the kth vehicle in the frequency domain. Usually the aim is to minimise the infinity norm of the transfer function that describes how the Laplace transform of the local error $\hat{E}_k(s)$ depends on its predecessor's error $\hat{E}_{k-1}(s)$. Other attempts to study the platoon dynamics include the elimination of the index k (by applying the Z transform with respect to k) and the study of the poles of the resulting transfer function in z, as well as using graph theory and even partial differential equations. (For details and references please see Section 2.2.4.)

Note that in the platoon system description used so far the state variables depend on one independent continuous variable, the time t, and an index k. However, every local state variable, such as $\hat{e}_k(t)$, may also be described as a two-dimensional variable $\hat{e}(t,k)$. In this context, *two-dimensional* (2D) thus refers to the fact that the variable depends on two independent variables, e. g. the continuous time t and the discrete position or location k.

However, what appears to be only a simple change of notation, yields significant advantages. As we will see later different methods derived for two-dimensional systems in general can now be applied to study the platooning problem. For example the analysis of systems where vehicles know not only the position of their direct predecessor but a group of several predecessors can easily be applied in the two-dimensional setting by extending the state space accordingly.

Changing the system description from a one-dimensional into a two-dimensional system will also change the notation of "stability" and "string stability": Whether an indexed one-dimensional system with variables of the form $\hat{e}_k(t)$ is string stable is now equivalent to the two-dimensional system with variables of the form $\hat{e}(t,k)$ being stable. While stability of two-dimensional systems will be properly defined later, we simply note at this point that it requires all states to be bounded for all t and k. Consequently, if a string is stable in the two-dimensional sense, the bound for variables in the indexed one-dimensional description is independent of the position k. Thus the system is string stable.

The characterisation of vehicle strings as two-dimensional systems also involves some disadvantages and limitations. For example a heterogeneous string with different dynamics for each vehicle in the string cannot easily be modelled as a two-dimensional system. Also bidirectional systems where the local errors both towards the preceding and the following vehicle are used cannot be analysed without difficulty. In a bidirectional setting the last vehicle does not have a follower and thus no local error towards its follower exists. Hence, either an additional reference signal acting as the fictitious $N+1$st vehicle has to be introduced or the dynamics of the last vehicle depend on the error towards its predecessor only. In both cases the last element in the string (either the additional reference or the last vehicle) act differently compared to the other vehicles in the string. Hence, a string of N vehicles cannot be seen as a truncation of an infinite string. Since the two-dimensional model allows both directions t_1 and t_2 to grow without bound, it is not suitable to describe vehicle strings with bidirectional communication settings.

The main challenge when analysing vehicle platoons as two-dimensional systems, however, lies in an important structural singularity: Assume that the local state space variables of the kth vehicle (such as its position $\hat{x}(t,k)$, velocity $\hat{v}(t,k)$ and controller states) are summarised in the vector $\boldsymbol{x}_1(t,k) \in \mathbb{R}^{n_1}$ and the position of the preceding vehicle $\hat{x}(t,k-1)$, that is used as a reference for the kth vehicle, is set to be the scalar $\hat{x}(t,k-1) = x_2(t,k) \in \mathbb{R}$. The overall two-dimensional system can be described by

$$\begin{pmatrix} \dot{\boldsymbol{x}}_1(t,k) \\ \Delta x_2(t,k) \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_1(t,k) \\ x_2(t,k+1) - x_2(t,k) \end{pmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c} & -1 \end{bmatrix} \begin{pmatrix} \boldsymbol{x}_1(t,k) \\ x_2(t,k) \end{pmatrix}. \tag{1.5}$$

Assuming that the initial conditions $\boldsymbol{x}_1(0,k) = 0$ for all k and applying the Laplace transform with respect to the continuous time t yields

$$X_{2_\mathcal{L}}(s,k+1) = \underbrace{\left(\boldsymbol{c}\,(s\mathbf{I} - \boldsymbol{A})^{-1}\,\boldsymbol{b}\right)}_{=:\Gamma(s)} X_{2_\mathcal{L}}(s,k) \tag{1.6}$$

with $X_{2_\mathcal{L}}(s,k) = \mathcal{L}\{x_2(t,k)\}$. Thus,

$$X_{2_\mathcal{L}}(s,k) = \Gamma^k(s)\mathcal{L}\{x_2(t,0)\}. \tag{1.7}$$

Assume further that the desired distance $\hat{x}_\mathrm{d}$ is set to 0 for simplicity and a simple step response is chosen as a reference, $\mathcal{L}\{x_2(t,0)\} = 1/s$. Since this is the reference (position) signal for the first vehicle, the system has to be designed such that the position $\hat{x}(t,1)$ converges to 1 for $t \to \infty$. Consequently, this leads to $\lim_{t\to\infty} \hat{x}(t,k) = 1$ for all k. Thus, using the final value theorem, equation (1.7) becomes

$$\lim_{t\to\infty} x_2(t,k) = \lim_{s\to 0} sX_{2_\mathcal{L}}(s,k) = \lim_{s\to 0} s\Gamma^k(s)\frac{1}{s}. \tag{1.8}$$

Therefore, $\Gamma(s)$ has to be chosen such that $\Gamma(0) = 1$. However, applying the Laplace transform with respect to t and the Z transform with respect to k in (1.5) yields

$$\begin{pmatrix} \boldsymbol{X}_1(s,z) \\ X_2(s,z) \end{pmatrix} = \begin{bmatrix} s\mathbf{I} - \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{c} & z \end{bmatrix}^{-1} \begin{pmatrix} \mathcal{Z}\{\boldsymbol{x}_1(0,k)\} \\ z\mathcal{L}\{x_2(t,0)\} \end{pmatrix}. \tag{1.9}$$

For a formal definition of the Laplace-Z transform (a combination of the Laplace transform with respect to t and the Z transform with respect to k) and its properties see Section 3.2. Note that the determinant of the matrix above can be transformed using the Schur complement into

$$\begin{aligned}\det\left(\begin{bmatrix} s\mathbf{I}-\boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{c} & z \end{bmatrix}\right) &= \det\left(z - \boldsymbol{c}\left(s\mathbf{I}-\boldsymbol{A}\right)^{-1}\boldsymbol{b}\right) \\ &= \det\left(z-\Gamma(s)\right).\end{aligned} \tag{1.10}$$

Thus, the matrix is singular for $s = 0$ and $z = 1$. This is called a *singularity on the stability boundary* (SSB). Note that this is a structural property of a two-dimensional system describing a vehicle platoon and not a consequence of string instability. A discussion for the more general problem with a non scalar $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$, $n_2 > 1$ can be found in Section 3.5.1.

As we will discuss in more detail later in Chapter 2 almost all conditions for stability of two-dimensional systems known in the literature explicitly or implicitly exclude this case and the results therefore cannot be used to determine stability of two-dimensional systems with singularities on the stability boundary in general and vehicle platoons in particular.

Therefore, our aim is to study the stability of two-dimensional systems, explicitly *including* these marginally stable systems. Note that we primarily focus on the analysis of two-dimensional systems rather than engage in detailed discussions on how string stability can be achieved in a practical setting. However, the string stability problem will be used as an example throughout this work to illustrate our findings.

We will start by examining the existing literature on string stability, stability of linear and nonlinear two-dimensional systems and some results on input-to-state stability (ISS) used to discuss the stability of nonlinear two-dimensional systems later. In Chapter 3 we will then examine the stability of linear two-dimensional systems in the frequency domain after applying the combined Laplace-Z transform to obtain a two-dimensional variable in the frequency domain that depends on the two complex variables s and z. Strictly speaking, we will examine the poles of the resulting two-dimensional transfer function and thus study bounded-input bounded-output (BIBO) stability of linear two-dimensional systems. It will be revealed that although this approach simplifies the string stability discussion of linear two-dimensional systems, it requires a detailed and extensive examination of the singularity on the stability boundary. Therefore, Chapter 4 is dedicated to deriving conditions for stability, asymptotic stability and exponential stability of linear two-dimensional systems in the time domain using linear matrix inequalities (LMI). A unified approach will be used to study the stability of continuous-continuous, continuous-discrete and discrete-discrete two-dimensional systems simultaneously. To complete this work, we will extend the results obtained for linear systems and derive stability conditions for nonlinear two-dimensional systems based on the notion of input-to-state stability (ISS) in Chapter 5.

CHAPTER 2

Literature Review

The second chapter reviews related work reported in the literature and puts the thesis into the context of existing research. In particular, we discuss the areas of string stability, stability of linear and nonlinear two-dimensional systems and (integral) input-to-state stability.

Chapter contents

2.1 Introduction

As the string stability problem will be used to illustrate our main results on stability of two-dimensional systems, we will begin this chapter with reviewing the related literature on string stability and platooning. As the nature of this work is mainly theoretical we will focus on reviewing the main ideas and challenges of the platooning problem and known approaches to guarantee string stability. However, papers concerning the real world application of platooning will be mentioned briefly.

We will study the stability of linear two-dimensional systems in the frequency domain in Chapter 3 and in the time domain in Chapter 4. Even though our research was motivated by the string stability problem, a number of other applications can also be modelled as linear two-dimensional systems in general and linear discrete two-dimensional systems in particular. This led to significant advances and a wide range of publications on the stability theory for linear two-dimensional systems. The publications most closely related to our work are reviewed in Section 2.3. Since every linear two-dimensional system describing a vehicle platoon will exhibit a structural singularity at the stability boundary adding to the complexity of the stability theory of this field, special care is needed to distinguish between results that do and do not include systems with such singularities at the stability boundary.

In stark contrast to the wide study of linear two-dimensional systems, few publications discussing the stability of general nonlinear two-dimensional systems are available. They are discussed in Section 2.4. The stability results for nonlinear two-dimensional systems in Chapter 5 will aim to advance the stability theory of such nonlinear systems. As they are based on the notion of "input-to-state stability (ISS)" and "integral input-to-state stability (iISS)" related literature to this stability concept for nonlinear systems is reviewed in Section 2.5.

2.2 String Stability

2.2.1 General Ideas and Notation

Since the 1960s researchers pursued the idea of arranging a group of moving vehicles into formations called "platoons". The aim was and is to achieve tight spacing between the vehicles and therefore increase traffic throughput and safety while, at the same time, decreasing costs and fuel consumption. To do so, human drivers are replaced by automatic controllers. While the first vehicle in the string should follow a given trajectory or reference signal, every successive vehicle is required to maintain a specified distance to its predecessor.

Since the publication of Levine and Athans (1966) research in the area has advanced significantly. As many different variations have been discussed since then, a brief overview of the concepts and common notation will be helpful before presenting the most important contributions in the field.

Homogeneity and Heterogeneity This characterisation of vehicle platoons describes if the model of the individual vehicle in the string is the same for all vehicles (homogeneous), e. g. Chu (1974), or depends on the position in the string (heterogeneous), e. g. Lestas and Vinnicombe (2007). Heterogeneity can be a result of allowing a range of cars with varying vehicle dynamics in the same string as well as a desired consequence of designing a differing controller for each vehicle. However, as we will discus later, this effect might not be desirable from a string formation or coordination viewpoint and it is not clear how heterogeneity helps to solve the underlying problem of string instability.

Communication Range The communication range determines the amount of information available to the local controller of each vehicle. The system settings used most commonly are:

- Full access: All states of all vehicles in the string are available to every individual vehicle in the string, e. g. Melzer and Kuo (1971).

- Reference or steady state information: Some researchers implicitly or explicitly use the desired steady state velocity or position of the vehicle by using the relative velocity or position (to the desired or steady state value) as an input for the controller, e. g. Peppard (1974).
- Leader information: Instead of the reference information, some dynamic states of the leading / first vehicle of the string (such as its velocity v_l or acceleration a_l) are broadcast to every vehicle in the string and used to compute the local control signal, e. g. Sheikholeslam and Desoer (1990). Note that in some publications the term "lead vehicle" is used to denote to the direct predecessor. In order to avoid confusion we will use the term "lead" only to refer to the first vehicle of the string and "predecessor" for the direct neighbour in front of each vehicle.
- Local information: The relative position, velocity or acceleration error towards the members of a limited group of surrounding vehicles is used. In "unidirectional" settings only the relative information of *preceding* vehicles is used, see e. g. Darbha and Hedrick (1996), whereas in "bidirectional" strings state information of both preceding and subsequent vehicles is required, e. g. Barooah *et al.* (2007).

Often, however, combinations of these concepts are employed.

Linear and Nonlinear For both vehicles and controllers linear and nonlinear forms and descriptions have been presented in the literature. For an example of a fully linear system description see Eyre *et al.* (1998). A general nonlinear system description can be found in Sheikholeslam and Desoer (1992).

Spacing Policies Usually the local control objective is to maintain a constant or velocity-dependent inter-vehicle distance ("time headway approach"), e. g. Sheikholeslam and Desoer (1990) or Chien and Ioannou (1992), respectively.

2.2.2 The Platooning Problem and String Stability

One of the earliest discussions of the realisation of vehicle platoons can be found in Levine and Athans (1966) where the stability of a string of a finite number of vehicles is discussed. Simulations are presented for a platoon of three vehicles but the more general problem of "string instability" is not yet discussed. In Melzer and Kuo (1971) an infinite string of vehicles is studied. However, the authors use a centralised controller requiring the knowledge of the complete state space of all vehicles.

The study of "string stability" itself, i. e. the boundedness of the error states of the kth vehicle within the string independently of k, began with two publications in 1974: In Peppard (1974) the author notes that

> *For these systems, string stability, or the property of the vehicle string to attenuate disturbances as they propagate down the string, is an important performance criterion.*

This was probably the earliest informal definition of "string stability". Realising that string stability can always be achieved when the position within the string is known, Peppard (1974) designed a local PID controller using only the relative position error to the nearest two neighbours and the velocity error. The fact that for linear systems the magnitude of the transfer function from vehicle $k-1$ to k ($|X_k(j\omega)/X_{k-1}(j\omega)|$) needs to be less or equal to 1 for all ω in order to guarantee string stability is used. Shortly afterwards, stability of an infinite string of vehicles was discussed in Chu (1974), informally defining string stability as the requirement that the errors must be bounded for all k for any set of bounded initial conditions. The author than presents six different communication settings and discusses string stability and performance criteria for them. It should be noted, however, that most approaches discussed in these two papers require at least the knowledge of some reference information since the relative velocity towards the steady state or reference velocity is used.

The first formal definition of "l_p String Stability" was given in Darbha and Hedrick (1996) requiring an upper bound on the l_p norm of the states of the entire string for all t for a given bound on the l_p norm of the initial conditions. Two years later, Eyre *et al.* (1998) proposed three different definitions of string stability requiring the norm of the output error to be smaller than the norm of the input error.

It has been shown in Seiler *et al.* (2004) and Barooah and Hespanha (2005) that it is not possible to achieve string stability in a homogeneous string of strictly proper feedback systems when only using information of the states of the nearest neighbours, linear systems with two integrators in the open loop of each subsystem, and constant inter-vehicle spacing independently of the particular plant or controller model in place.

Different strategies have been proposed in the literature to guarantee string stability: In Chien and Ioannou (1992) string stability was guaranteed using a sufficiently large velocity dependent distance or "time headway" instead of a fixed inter-vehicle distance. The desired separation between the kth vehicle and its predecessor was now the product of a fixed time headway h and the speed $v_k(t)$, which thus grows linearly with the velocity. This approach was later extended in Yanakiev and Kanellakopoulos (1998) proposing a variable time headway that gives a nonlinear two-dimensional system. In Eyre *et al.* (1998) string stability and performance of systems without time headway, with fixed time headway and with variable time headway were compared and analysed.

When a constant spacing policy is required, however, string stability can be guaranteed using suitable information of the lead vehicles dynamics. Different communication settings and a discussion which states of the lead vehicle are necessary to guarantee string stability can be found in Darbha *et al.* (1994).

Another approach is to use a heterogeneous string structure. In Khatir and Davison (2004) a design for a local controller that depends on the position k was presented. Even though uniformly bounded local errors could be guaranteed, it should be noted that the control parameters do grow linearly with k and thus are unbounded for an infinite string. In Shaw and Hedrick (2007) it was shown that a heterogeneous string is string stable if the local transfer function $|G_k(j\omega)|$ is less or equal to 1 for all ω and k. Later in Middleton and Braslavsky (2010) an infimal average time headway was derived to permit heterogeneous string stability.

Although most researchers have worked on linear string models to analyse string stability there are also some results for nonlinear systems. In Sheikholeslam and Desoer (1992) the authors prove that strings of nonlinear systems using the lead velocity, the lead acceleration and local measurements are string stabile if the inputs vary sufficiently slow. In Darbha and Hedrick (1996) a global Lipschitz condition is used to guarantee string stability of nonlinear systems with sufficiently small Lipschitz constants or "weak coupling".

2.2.3 Application Papers

While most of the literature reviewed so far is of a predominantly theoretical nature it should be noted that a large body of work dealing with applications and practical problems also exists. Since the main focus of this thesis is to extend the theory of two-dimensional systems, we will only review a small sample of articles investigating real world road traffic, human driver models and difficulties in the implementation of platooning.

One of the first human driver models appeared in Chandler *et al.* (1958). Later contributions include Burnham *et al.* (1974) and Gipps (1981). In the review paper by Brackstone and McDonald (1999) it was shown, however, that the parameters of these models differ significantly, leading to the conclusion that an adequate driver model may not yet have been found.

Even though most papers deal with the theoretical aspects of string stability analysis, some researchers have used more detailed and accurate truck models, e. g. Chien and Ioannou (1992); Hedrick *et al.* (1993); Yanakiev and Kanellakopoulos (1998).

The effect of communication delays on string stability (which are typical real world phenomenon) was discussed for example in Liu *et al.* (2001) and Ploeg *et al.* (2011) while other problems relating to the real world implementation of platooning are discussed in Varaiya (1993); Hedrick *et al.* (1994); Tan *et al.* (1998); Zhang *et al.* (1999).

2.2.4 Analysis Methods

Different methods and mathematical concepts have been used to analyse string stability. Since most models considered are linear, the Laplace transform with respect to time t has been widely used to study the stability of a string in the frequency domain. Examples can be found in Sheikholeslam and Desoer (1990); Eyre *et al.* (1998); Stanković *et al.* (2000); Seiler *et al.* (2004); Barooah and Hespanha (2005). However, some researchers also employed the Z transform with respect to the discrete variable k, e. g. Melzer and Kuo

(1971); Chu (1974). Graph theory was used in Lestas and Vinnicombe (2006) to study the stability of a more general string. Barooah *et al.* (2009) approximated the string dynamics as a partial differential equation to study the string stability of strings with bidirectional communication settings.

As we will show later, there is yet another method to analyse whether a platoon of vehicles is string stable: The system can be modelled as a two-dimensional system, treating the index k as a discrete variable. Thus the indexed platoon system with state $\hat{x}_k(t)$ is transformed into a continuous-discrete two-dimensional system with state $\hat{x}(t, k)$. Since, as we shall see, this is a suitable reformulation of the problem, it is important to review some of the literature relating to the stability of two-dimensional systems.

2.3 Linear Two-Dimensional Systems

In this section we will review relevant literature on the stability of linear two-dimensional systems. Here, "two-dimensional" refers to the fact that functions and variables depend not only on one independent variable (such as time or space) but on two completely independent variables. Although, in general, three types of two-dimensional systems exist (i. e. discrete two-dimensional systems depending on two discrete variables; continuous two-dimensional systems depending on two continuous variables; and continuous-discrete two-dimensional systems depending on one continuous and one discrete variable), most research appears to focus on linear discrete two-dimensional systems due to a broad range of applications that can be modelled using this type of system.

Since linear two-dimensional systems have been studied in a very comprehensive way, there exists a large number of publications in the field. We will therefore focus only on stability of two-dimensional systems rather than stabilisability, controllability or controller design. Note that the most important results on the stability of two-dimensional state space models are also summarised in Bouagada and Van Dooren (2011).

2.3.1 Linear Discrete Two-Dimensional Systems

Two-Dimensional Models

Linear discrete two-dimensional systems can be modelled in different ways. Here, we will restrict the review to the most commonly used forms.

A model which is used mainly to study input-output stability is

$$Y(z_1, z_2) = \underbrace{\frac{\mathrm{num}(z_1, z_2)}{\mathrm{den}(z_1, z_2)}}_{G(z_1,z_2)} W(z_1, z_2) \tag{2.1}$$

where $Y(z_1, z_2)$ is the output, $W(z_1, z_2)$ is the input and $G(z_1, z_2)$ is the transfer function of the system in the frequency domain. Thus the variables depend on the two complex variables z_1 and z_2. The denominator $\mathrm{den}(z_1, z_2)$ is called "characteristic polynomial" of the system. This system description has been used for example in Shanks *et al.* (1972).

One of the best known explicit state space descriptions for two-dimensional discrete systems appeared in Roesser (1975):

$$\begin{pmatrix} \boldsymbol{x}_1(k+1,l) \\ \boldsymbol{x}_2(k,l+1) \end{pmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix}}_{\boldsymbol{A}} \begin{pmatrix} \boldsymbol{x}_1(k,l) \\ \boldsymbol{x}_2(k,l) \end{pmatrix} + \underbrace{\begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}}_{\boldsymbol{B}} \boldsymbol{u}(k,l) \tag{2.2}$$

$$\boldsymbol{y}(k,l) = \underbrace{\begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix}}_{\boldsymbol{C}} \begin{pmatrix} \boldsymbol{x}_1(k,l) \\ \boldsymbol{x}_2(k,l) \end{pmatrix} + \boldsymbol{D}\boldsymbol{u}(k,l) \tag{2.3}$$

where $\boldsymbol{x}_1(k,l) \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2(k,l) \in \mathbb{R}^{n_2}$ and the dimensions of $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are chosen appropriately.

Two well known models were introduced by Fornasini and Marchesini. They are typically referred to as Fornasini-Marchesini's first model (FM1), Fornasini and Marchesini (1976):

$$\boldsymbol{x}(k+l, l+l) = \boldsymbol{A}_0\boldsymbol{x}(k,l) + \boldsymbol{A}_1\boldsymbol{x}(k+1,l) + \boldsymbol{A}_2\boldsymbol{x}(k,l+1) + \boldsymbol{B}\boldsymbol{u}(k,l) \tag{2.4}$$

$$\boldsymbol{y}(k,l) = \boldsymbol{C}\boldsymbol{x}(k,l) \tag{2.5}$$

where $\boldsymbol{x}(k,l) \in \mathbb{R}^n$, as well as Fornasini-Marchesini's second model (FM2), Fornasini and Marchesini (1978):

$$\boldsymbol{x}(k+1, l+1) = \boldsymbol{A}_1\boldsymbol{x}(k,l+1) + \boldsymbol{A}_2\boldsymbol{x}(k+1,l) + \boldsymbol{B}_1\boldsymbol{u}(k,l+1) + \boldsymbol{B}_2\boldsymbol{u}(k+1,l) \tag{2.6}$$

$$\boldsymbol{y}(k,l) = \boldsymbol{C}\boldsymbol{x}(k,l) \tag{2.7}$$

with $\boldsymbol{x}(k,l) \in \mathbb{R}^n$. Note that the Roesser model and the Fornasini-Marchesini models can, however, be transformed into each other, Eising (1978). A corresponding input-output function also exists for each state space model if a suitable output equation exists. The transfer function of the two complex variables z_1 and z_2 can be deduced after applying the Z-Z Transform with respect to k and l, respectively.

Another (general) model appeared in Kurek (1985); a general model for singular two-dimensional systems can be found in Kaczorek (1988).

A special case of linear discrete two-dimensional systems are linear discrete repetitive processes. In these systems the second discrete variable specifies iterations. Since a new iteration can only start when the previous iteration has finished, the first variable (describing the time t) is usually assumed to be in the range between 0 and the "pass length" α. Also, the initial conditions for a new iteration may depend on the profile of the previous passes. Thus the stability analysis of theses systems inherits some important properties of general two-dimensional systems, but differs in some aspects. Even though repetitive processes are mostly described using a special state space description, this formulation can be transferred into the Roesser model, Galkowski *et al.* (1999).

Input-Output-Stability

One of the earliest discussions of stability of linear discrete two-dimensional systems was presented in Shanks *et al.* (1972). They claimed that the system given in (2.1) is bounded-input bounded-output (BIBO) stable if and only if the characteristic polynomial, $\mathrm{den}(z_1, z_2)$, has no zeros in the closed unit bi-disc $\bar{U}^2 = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1\}$. This led to different stability tests proposed in the literature, such as in Huang (1972) and Anderson and Jury (1973).

In Du *et al.* (1999) the authors extend theses results by showing different versions of the Bounded Real Lemma (BRL) for the discrete Fornasini-Marchesini second model. The noise attenuation (i. e. the upper bound of the norm of the output signal depending on the norm of the noise input and the initial / boundary conditions) can be obtained by solving a linear matrix inequality (LMI). A similar result (in combination with a suitable controller design guaranteeing a specified "H_∞ performance") for discrete Roesser models was presented in Du *et al.* (2001). (These and other related results on H_∞ control and filtering of discrete two-dimensional systems have been summarised in Du and Xie (2002).)

Stability of the Fornasini-Marchesini Models

In Fornasini and Marchesini (1978) the authors proved asymptotic stability for FM2. A straight diagonal separation set or "contour" and its norm $\|\mathcal{X}_r\| = \sup_{n\in\mathbb{Z}} |\boldsymbol{x}(r-n, n)|$ is defined. For $u = 0$ and $\|\mathcal{X}_0\| < \infty$ the system is asymptotically stable in the sense $\lim_{r\to\infty} \|\mathcal{X}_r\| = 0$ if and only if the characteristic polynomial is not zero for any (z_1, z_2) in $\bar{U}^2$.

An extension can be found in Fornasini and Marchesini (1980). Here the authors prove asymptotic stability using a more general contour using the linear matrix inequality constraint requiring a positive definite Hermitian solution $\boldsymbol{P}(\omega)$ for all real ω. An alternative proof of this condition was published in Cook (2000).

Shortly afterwards it was shown in Pandolfi (1984) that for exponentially decaying initial conditions the system is exponentially stable if and only if the characteristic polynomial is devoid of zeros in $\bar{U}^2$.

Based on the necessary and sufficient condition on the characteristic polynomial to be devoid of zeros in $\bar{U}^2$, a sufficient LMI based conditions for asymptotic stability was derived in Hinamoto (1993) providing the first LMI condition for FM2 with constant coefficients.

Necessary and sufficient conditions with constant coefficients for asymptotic stability using a "guardian map" were later presented in Ebihara *et al.* (2006).

Although the second model of Fornasini-Marchesini has attracted most attention, a necessary condition for asymptotic stability for Fornasini-Marchesini's first model appeared in Bose and Trautman (1992). In this analysis, the initial conditions are considered to satisfy $\boldsymbol{x}(k, 0) = 0$ for $k \geq K$ and $\boldsymbol{x}(0, l) = 0$ for $l \geq L$ for some $K, L < \infty$.

A different sufficient LMI condition for asymptotic stability was developed in Kar and Singh (2003). LMI based necessary and sufficient conditions for asymptotic stability can also be found in Zhou (2006).

Stability of the Roesser Model

Using the state space description by Roesser, it was claimed in Lodge and Fahmy (1981) that the characteristic polynomial $\mathrm{den}(z_1, z_2)$ fulfills Shank's stability criterion if and only if there exists a positive definite, symmetric matrix $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$, where $\oplus$ denotes the direct sum, i. e. $\boldsymbol{P}_1 \oplus \boldsymbol{P}_2 = \mathrm{diag}\{\boldsymbol{P}_1, \boldsymbol{P}_2\}$, $\boldsymbol{P}_1 \in \mathbb{R}^{n_1 \times n_1}$ and $\boldsymbol{P}_2 \in \mathbb{R}^{n_2 \times n_2}$, such that

$$\boldsymbol{A}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{A} - \boldsymbol{P} = \boldsymbol{Q} < 0. \tag{2.8}$$

An additional stability test based on these results appeared in Lu and Lee (1985). However, Anderson *et al.* later showed that, in general, the existence of such a $\boldsymbol{P}$ is sufficient but not necessary for stability, Anderson *et al.* (1986).

Necessary and sufficient conditions for asymptotic stability of positive two-dimensional systems described by the Roesser model were discussed in Kurek (2002).

Stability of Linear Discrete Repetitive Processes

In Galkowski *et al.* (1999) the authors study the stability and controllability of discrete repetitive processes modelled in a form similar to the Roesser model. The system is then transformed into a one-dimensional model and conditions for controllability for a certain kind of dynamic process initial conditions are derived. It was shown in Galkowski *et al.* (2002), that for linear discrete repetitive processes with known constant initial conditions and a finite pass length α, the same sufficient LMI condition for asymptotic stability as published in Lodge and Fahmy (1981) holds. These results can also be found in greater detail in Rogers *et al.* (2007). A version of the Bounded Real Lemma (BRL) for discrete repetitive processes as well as sufficient conditions for the design of suitable output feedback controller are introduced in Wu *et al.* (2007).

2.3.2 Linear Continuous Two-Dimensional Systems

For continuous two-dimensional systems similar results as for discrete two-dimensional systems have been presented in the literature. Stability conditions for the two-dimensional transfer function $G(s_1, s_2) = \mathrm{num}(s_1, s_2)/\mathrm{den}(s_1, s_2)$ appeared in Ansell (1964): The corresponding continuous two-dimensional system is BIBO stable if $G(s_1, s_2)$ is devoid of poles with nonnegative real parts of s_1 and s_2 (i. e. the characteristic polynomial is a "very strict Hurwitz polynomial", there are no poles in the region $\bar{S}^2 = \{(s_1, s_2) : \Re\{s_1\} \geq 0, \Re\{s_2\} \geq 0\}$). It was shown in Huang (1972) that this is the continuous equivalent to the condition presented in Shanks *et al.* (1972) for the discrete case. Necessary and sufficient conditions to guarantee a given polynomial in s_1 and s_2 is very strictly Hurwitz were published in Reddy and Rajan (1986). A necessary and sufficient algebraic condition for the characteristic polynomial of a continuous Roesser model to be very strictly Hurwitz can be found in Agathoklis *et al.* (1991). Stability margins of the characteristic polynomial are discussed in Mastorakis *et al.* (2000).

A different approach to study BIBO stability of continuous two-dimensional systems is based on the impulse response of the system, Jury and Bauer (1988).

As in the field of discrete systems, LMI-based stability conditions have also been developed for continuous two-dimensional systems. In contrast to discrete two-dimensional systems, researchers have focused on the continuous version of the Roesser model. Similar to the results in Lodge and Fahmy (1981) Piekarski claimed in Piekarski (1977) a necessary and sufficient stability condition: The system is stable (characteristic polynomial is very strict Hurwirtz) if and only if there exists a positive definite, symmetric matrix $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$ such that

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = \boldsymbol{Q} < 0. \tag{2.9}$$

However, it was shown, that this condition is only sufficient in Anderson *et al.* (1986). Algorithms to find such a $\boldsymbol{P}$ appeared in Xiao *et al.* (1997). Another necessary and sufficient LMI condition requiring the existence of a positive definite Hermitian solution $\boldsymbol{P}(\omega)$ for all real ω was given in Agathoklis *et al.* (1991).

Piekarski's LMI condition (2.9) was later found to be sufficient to guarantee asymptotic stability for systems with bounded initial conditions, Galkowski (2002).

The stability of uncertain continuous two-dimensional systems has been discussed in the time domain in Xu *et al.* (2005), and in the frequency domain in Fernando and Trinh (2007).

2.3.3 Linear Continuous-Discrete Two-Dimensional Systems

Alongside discrete repetitive processes some researchers also studied "differential" repetitive processes leading to the study of continuous-discrete two-dimensional systems.

Stability theory for continuous-discrete two-dimensional systems appears to be well developed. Different conditions for stability and asymptotic stability of differential repetitive processes with dynamic boundary conditions (depending on the pass profiles of the previous passes) are given in Owens and Rogers (1999). These results were extended to stability tests based on a one-dimensional Lyapunov function in Benton *et al.* (2002).

In Galkowski *et al.* (2003) the authors discuss stability along the pass (similar to asymptotic stability) for differential repetitive processes modelled in a form similar to the Roesser model. Here, the first independent variable is the continuous time t and the second variable is the discrete iteration k. They claim that such a system is stable along the pass if there exist two positive definite, symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that

$$\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{P}_1 \oplus \mathbf{0}) + (\boldsymbol{P}_1 \oplus \mathbf{0})\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}(\mathbf{0} \oplus \boldsymbol{P}_2)\boldsymbol{A} - (\mathbf{0} \oplus \boldsymbol{P}_2) = \boldsymbol{Q} < 0. \tag{2.10}$$

The proof in Galkowski *et al.* (2003) refers to Rogers and Owens (1992) for details. While this book covers extensive results in the area, a complete LMI based stability proof for the Roesser Model is not given. To the best of the author's knowledge, there is no published complete proof.

2.3.4 Nonessential Singularity of the Second Kind

Furthermore, an important special case is often excluded (either implicitly or explicitly) in the stability discussions mentioned above. This is the case when there exists a set of (z_1, z_2) (in the discrete-discrete case), (s_1, s_2) (in the continuous-continuous case), or (s, z) (in the continuous-discrete case) such that both the denominator and the numerator of the transfer function go to zero at the same time. In contrast to the case where the numerator is nonzero for (z_1, z_2), (s_1, s_2) or (s, z) (nonessential singularity of the first kind) these special points are often called nonessential singularities of the second kind (NSSK). Note that the state space matrix $\boldsymbol{A}$ of every system with an NSSK at a certain point of the bi-plane will also exhibit a singularity at the same point.

Most research avoids this singular case and assumes instead that the characteristic loci are strictly inside the stability bi-region. However, it cannot always be avoided due to structural properties of certain applications, or they are even desirable to obtain a system with special characteristics. One such example is the design of fan filters that inherently require an NSSK at the stability boundary, Bruton and Bartley (1989).

Another example are vehicle platoons. As discussed in Chapter 1 the characteristic polynomial of these two-dimensional systems include a singularity at $s = 0$ and $z = 1$, $\mathrm{den}(0, 1) = 0$. (A more detailed discussion on how a suitable transfer function is derived and the discussion of the corresponding numerator can be found in Section 3.4.)

It was shown in Goodman (1977) that some transfer functions with NSSK are BIBO stable, while some with NSSK at the same point in the bi-plane are BIBO unstable. A sufficient BIBO stability condition in the frequency domain for discrete two-dimensional systems with NSSK at the boundary of the bi-disc (i.e. $|z_1| = |z_2| = 1$) has been presented in Dautov (1981): The system is stable if $\mathrm{den}(z_1, z_2)$ has finitely many zeros at the stability border and can be continuously extended to the closed polydisc. Dautov (1981) conjectures this condition is also necessary. This was followed by a necessary condition that stability can only be achieved when the NSSK occur at the border (or outside) of the bi-disc in Reddy and Jury (1987).

Although the results in Goodman (1977); Dautov (1981); Reddy and Jury (1987) were obtained in the frequency domain, it should be noted that previous LMI-based results in the time domain also exclude systems where $\boldsymbol{A}$ has singularities on the stability boundary

(SSB) since a sign definite solution of the LMI is required. However, as we will show later in Lemma 4.1, such systems cannot achieve a sign definite solution to the required LMI. Hence, LMI based stability conditions presented in the literature so far cannot be employed to study the stability of two-dimensional systems including singularities at the stability boundary in general, and continuous-discrete two-dimensional systems describing a vehicle platoon in particular.

This is not surprising considering that the condition that the poles are inside the open stability region is *necessary and sufficient* for asymptotic stability. However, the definition used commonly for asymptotic stability requires the states to tend to zero in the presence of any set of bounded initial conditions. As discussed in Chapter 1 applying a step signal as a reference signal (i. e. bounded initial conditions in the two-dimensional sense) in a vehicle platoon leads to nonzero local states. Thus, the system is not asymptotically stable according to that definition. This is not due to poor design of the system but a necessary property of a functioning vehicle platoon.

Therefore, in order to analyse the stability of vehicle strings in a two-dimensional system setting, different definitions for stability have to be developed as well as conditions for (asymptotic) stability that explicitly include these marginally stable cases.

2.4 Nonlinear Two-Dimensional Systems

Compared to the large variety of results about the stability of linear two-dimensional systems only little work seems to be available concerning the stability of nonlinear two-dimensional systems. It is also worth mentioning that all results on stability of nonlinear two-dimensional systems known to the author exclusively study the stability of the *discrete time* version of such systems.

Most research appears to be focused on particular types of nonlinearities. A range of papers, for example, analyse overflow nonlinearities in general and saturations in particular. In Kar and Singh (2001) the authors propose a sufficient condition based on LMIs for global asymptotic stability of linear two-dimensional Roesser models with overflow nonlinearities. Similar LMI based sufficient conditions for global asymptotic stability of linear two-dimensional Roesser models with saturated derivatives appeared in Kar and Singh (2005) and were later extended in Singh (2007).

Sufficient stability conditions based on LMIs for systems with a more general set of nonlinearities in the sector $[0, g_k]$ were presented in Hinamoto (1993).

Stability of a general nonlinear discrete two-dimensional system of the form

$$\begin{pmatrix} \boldsymbol{x}_1(k+1,l) \\ \boldsymbol{x}_2(k,l+1) \end{pmatrix} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, k, l) \tag{2.11}$$

was first analysed in Kurek (1995). The main theorem guarantees different kinds of stability if a scalar, positive definite Lyapunov function $\phi(\boldsymbol{x}, k, l) = \phi_1(\boldsymbol{x}_1, k, l) + \phi_2(\boldsymbol{x}_2, k, l)$ with

$\phi'(\boldsymbol{x}, k, l) = \phi_1(\boldsymbol{x}, k+1, l) + \phi_2(\boldsymbol{x}, k, l+1)$ exists: The system is uniformly locally stable if $\phi' \leq \phi$, uniformly locally asymptotically stable if $\phi' < \phi$ (and additional conditions are satisfied), and uniformly globally asymptotically stable if there exists a real positive number $\alpha < 1$ such that $\phi' \leq \alpha\phi$.

In Zhu and Hu (2011) the general discrete two-dimensional Fornasini-Marchesini second model of the form $\boldsymbol{x}(k+1, l+1) = \boldsymbol{f}\left(\boldsymbol{x}(k+1, l) + \boldsymbol{x}(k, l+1)\right)$ was considered. Using Lyapunov arguments based on the theory of input to state stability (ISS), sufficient conditions for local and global asymptotic stability, in the presence of bounded, decaying initial conditions were derived. The authors also give sufficient LMI conditions for absolute stability of linear systems with nonlinear feedback functions in the sector $[0, K]$.

It should be noted that similar to stability results based on Lyapunov arguments for linear two-dimensional systems, all results of the Lyapunov type for nonlinear two-dimensional systems known to the author require the divergence or deviation of the Lyapunov function to be strictly negative. The only exception is the sufficient condition $\phi' - \phi \leq 0$ for uniform local stability in Kurek (1995). As noted in (Zhu and Hu, 2011, Remark 3) in order to show global asymptotic stability for nonpositive differences, the assumptions on the initial conditions need to be stronger than merely boundedness.

2.5 Input-to-State Stability

As we will use the notion of “input-to-state stability” (ISS) to prove stability of nonlinear two-dimensional systems in Chapter 5 we will briefly review the most relevant findings on ISS and its integral variant iISS.

The concept of input-to-state stability (ISS) was first proposed in Sontag (1989): It was shown that the system is “smoothly input-to-state stabilisable” if for a system (linear in control, described by $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})\boldsymbol{u}$) there exists a control law such that the system becomes ISS, i. e. there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for each measurable input $\boldsymbol{u}$ and each initial state $\boldsymbol{\xi}_0$ the solution $\boldsymbol{x}(t)$ exists and satisfies

$$|\boldsymbol{x}(t)| < \beta(|\boldsymbol{\xi}_0|, t) + \gamma(\|\boldsymbol{u}\|). \tag{2.12}$$

(Note that class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions will be defined in Section 5.2.) This result was than extended in Sontag (1990) to show that systems of the general form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ also become ISS when a stabilising control law of a more general form is applied. A wide range of different variations and analogies of ISS are summarised in Sontag and Wang (1996). These notions include zero (global) asymptotic stability, global and local stability, the (uniform) asymptotic gain and limit property and the existence of a “smooth ISS Lyapunov function” $V(\boldsymbol{x})$ of the form

$$\alpha_1(|\boldsymbol{x}|) \leq V(\boldsymbol{x}) \leq \alpha_2(|\boldsymbol{x}|) \tag{2.13}$$

$$\text{with } \dot{V}(\boldsymbol{x}) \leq -\alpha_3(|\boldsymbol{x}|) + \gamma(|\boldsymbol{u}|) \tag{2.14}$$

where $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mathcal{K}_\infty$.

Two integral variants of ISS were later proposed in Sontag (1998). One of the two (the "L_2 to L_2 property") was shown to be equivalent to ISS, whereas the second (the "L_2 to L_∞ property") became known as "integral input-to-state stability" (iISS): There exist functions $\alpha, \gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that the following estimate holds for all initial states $\boldsymbol{\xi}_0$ and measurable inputs $\boldsymbol{u}$

$$\alpha(|\boldsymbol{x}(t)|) < \beta(|\boldsymbol{\xi}_0|, t) + \int_0^t \gamma(|\boldsymbol{u}(s)|)\mathrm{d}s. \tag{2.15}$$

It was also shown that ISS implies iISS and that a system is iISS if there exist a positive-definite proper smooth function V, a constant $q > 0$ and functions $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all states $\boldsymbol{x}$ and inputs $\boldsymbol{u}$

$$\dot{V}(\boldsymbol{x}) \leq (\gamma_1(|\boldsymbol{u}|) - q)V(\boldsymbol{x}) + \gamma_2(|\boldsymbol{u}|). \tag{2.16}$$

An extension of this Lyapunov type stability argument appeared in Angeli *et al.* (2000): The system is iISS if there exists a "smooth iISS Lyapunov function" $V(\boldsymbol{x})$ of the form (2.13)-(2.14) where α_3 is now only required to be positive definite. An extensive summary of results on ISS and iISS can be found in Sontag (2008).

Although all results discussed above concern continuous time systems, a range of analogies for discrete time systems of the form $\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{u}(k))$ have been introduced in the literature. In Kazakos and Tsinias (1994) the authors extend the results on global stabilisation in Sontag (1989, 1990) to discrete time systems. A proof that a discrete version of the smooth ISS Lyapunov function exists if and only if the (discrete time) system is ISS appeared in Jiang and Wang (2001). A range of analogies of iISS for discrete time systems have been proposed in Angeli (1999), including the finding that the system is iISS if and only if there exists a discrete time analogue of the smooth iISS Lyapunov function.

2.6 Conclusion

We conclude that vehicle platoons and the related field of string stability have been studied in detail and different methods have been proposed to analyse the vehicle string dynamics and to ensure string stability. Besides other approaches to analyse the system, a vehicle platoon can be modelled as a two-dimensional system where the index or position k is treated as a second dimension.

However, this system description inherits a singularity at the stability boundary due to structural requirements of a vehicle platoon. As a consequence most published results on two-dimensional systems cannot be applied since this marginally case is frequently explicitly or implicitly excluded.

This work seeks to fill this gap in the existing theory and derive sufficient conditions for stability of two-dimensional systems with singularities at the stability boundary both in the frequency domain (in Chapter 3) and the time domain (in Chapter 4 and Chapter 5) using the example of platooning systems.

CHAPTER 3

BIBO Stability of Linear 2D Systems

This chapter aims to study bounded-input bounded-output stability of continuous-discrete two-dimensional systems in the presence of nonessential singularities of the second kind on the stability boundary. The string stability of a chain of linear, unidirectionally connected systems in the frequency domain using the Laplace-Z transform is studied as an illustrative example.

Chapter contents

3.1 Introduction

In this chapter we will study the bounded-input bounded-output (BIBO) stability of linear continuous-discrete two-dimensional systems. A combination of the Laplace transform (with respect to the continuous variable time t) and the Z transform (with respect to the discrete variable position k) will be used to transform the system into the frequency domain description. Several useful properties of the Laplace-Z transform, the proof for a version of Parseval's Theorem for continuous-discrete two-dimensional systems and other mathematical results are presented in Section 3.2. They will then be used in Section 3.3 to derive the L_2 induced operator norm in the frequency domain.

A linear homogeneous string of vehicles with unidirectional control will be modelled as a continuous-discrete two-dimensional system and will subsequently be studied in Section 3.4 using the L_2 induced operator norm. The same approach can also be used to study the string stability of a linear homogeneous string of vehicles with unidirectional control and communication range 2, Section 3.5. However, as it was revealed in Chapter 1 every two-dimensional system describing such a vehicle string will feature a structural nonessential singularity on the stability boundary and special attention is needed to guarantee that the induced operator norm is bounded at this point.

It should be noted that although the results in this chapter are given specifically for linear continuous-discrete two-dimensional systems, coinciding results for linear continuous two-dimensional and linear discrete two-dimensional systems follow directly from the findings of this chapter. In order to enhance readability, however, linear continuous-discrete two-dimensional systems will be studied in particular rather than the general linear two-dimensional model to be discussed in Chapter 4 and Chapter 5.

A short version of this chapter has been accepted for publications in Knorn and Middleton (2012a).

3.2 Mathematical Preliminaries

The Laplace transform $\boldsymbol{X}(s)$ exists for any piecewise continuous (on $0 \leq t < \infty$) function $\boldsymbol{x}(t)$ which is of exponential order as $t \to \infty$, that is: $\exists c, a < \infty$, such that $|\boldsymbol{x}(t)| \leq c\mathrm{e}^{at}$. In this case the Laplace transform $\boldsymbol{X}(s)$ in terms of the complex variable s is defined as

$$\boldsymbol{X}(s) = \mathcal{L}\left\{\boldsymbol{x}(t)\right\} := \int\limits_{0}^{\infty} \boldsymbol{x}(t)\mathrm{e}^{-st}\mathrm{d}t. \tag{3.1}$$

The inverse Laplace transform is given by

$$\boldsymbol{x}(t) = \mathcal{L}^{-1}\left\{\boldsymbol{X}(s)\right\} = \frac{1}{2\pi j}\int\limits_{\alpha-j\infty}^{\alpha+j\infty} \boldsymbol{X}(s)\mathrm{e}^{st}\mathrm{d}s \tag{3.2}$$

with $\alpha > a$ and j is the imaginary unit. If the region of convergence of $\boldsymbol{X}(s)$ includes the imaginary axis, (3.2) yields

$$\boldsymbol{x}(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \boldsymbol{X}(j\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega. \tag{3.3}$$

If the discrete signal $\boldsymbol{x}(k)$ for $k \in \mathbb{N}_0$ (with $\mathbb{N}_0$ being the set of natural numbers $\mathbb{N}$ and 0, i. e. all nonnegative integers) grows no faster than exponentially, that is: $\exists c, a < \infty$, such that $|\boldsymbol{x}(k)| \leq ca^k$, its unilateral Z transform $\boldsymbol{X}(z)$ in terms of the complex variable z exists:

$$\boldsymbol{X}(z) = \mathcal{Z}\left\{\boldsymbol{x}(k)\right\} := \sum_{k=0}^{\infty} \boldsymbol{x}(k)z^{-k}. \tag{3.4}$$

The inverse Z transform is

$$\boldsymbol{x}(k) = \mathcal{Z}^{-1}\left\{\boldsymbol{X}(z)\right\} = \frac{1}{2\pi j}\oint\limits_{\mathcal{C}_\alpha} \boldsymbol{X}(z)z^{k-1}\mathrm{d}z \tag{3.5}$$

where $\oint_{\mathcal{C}_\alpha}$ is the contour integral around a circle centred at the origin with radius $\alpha > a$. If the region of convergence of $\boldsymbol{X}(z)$ includes the unit circle, (3.5) can be transformed into

$$\boldsymbol{x}(k) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \boldsymbol{X}\left(\mathrm{e}^{j\theta}\right) \mathrm{e}^{j\theta k} \mathrm{d}\theta. \tag{3.6}$$

We assume that $\boldsymbol{x}(t,k)$ is a continuous-discrete two-dimensional signal, which does not grow faster than exponentially, i. e. $\exists c, a, b < \infty$ such that

$$|\boldsymbol{x}(t,k)| \leq c\mathrm{e}^{at} b^k. \tag{3.7}$$

Thus, the unilateral, combined Laplace-Z transform $\boldsymbol{X}(s,z)$ of $\boldsymbol{x}(t,k)$ is defined as

$$\boldsymbol{X}(s,z) = \mathcal{ZL}\left\{\boldsymbol{x}(t,k)\right\} := \sum_{k=0}^{\infty} \int\limits_{0}^{\infty} \boldsymbol{x}(t,k)\mathrm{e}^{-st}\mathrm{d}t z^{-k} \tag{3.8}$$

and its inverse

$$\begin{aligned} \boldsymbol{x}(t,k) &= \left\{\mathcal{ZL}\right\}^{-1} \left\{\boldsymbol{X}(s,z)\right\} = \mathcal{Z}^{-1}\left\{\ \mathcal{L}^{-1}\left\{\boldsymbol{X}(s,z)\right\}\right\} \\ &= \frac{1}{(2\pi j)^2} \oint\limits_{\mathcal{C}_\beta} \int\limits_{\alpha-j\infty}^{\alpha+j\infty} \boldsymbol{X}(s,z)\mathrm{e}^{st}\mathrm{d}s z^{k-1}\mathrm{d}z \end{aligned} \tag{3.9}$$

where $\alpha > a$ and $\mathcal{C}_\beta$ is the contour $|z| = \beta > b$. In the case where $\alpha = 0$ and $\beta = 1$ lie within the region of convergence, (3.9) yields

$$\boldsymbol{x}(t,k) = \frac{1}{(2\pi)^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\infty}^{\infty} \boldsymbol{X}\left(j\omega, \mathrm{e}^{j\theta}\right) \mathrm{e}^{j\omega t}\mathrm{d}\omega \mathrm{e}^{j\theta k}\mathrm{d}\theta. \tag{3.10}$$

To prove Parseval's Theorem later several properties of the Laplace-Z transformation are needed. (These properties are simple extensions of well known results on the Laplace and Z transform, which can be found in most textbooks, see e. g. Bellmann and Roth (1984); Mathews and Howell (2006); Debnath and Bhatta (2006).)

Permutability Assuming that both transforms and inverse transforms exist, we can write

$$\boldsymbol{X}(s,z) = \mathcal{ZL}\left\{\boldsymbol{x}(t,k)\right\} = \mathcal{Z}\left\{\ \mathcal{L}\left\{\boldsymbol{x}(t,k)\right\}\right\} = \mathcal{L}\left\{\ \mathcal{Z}\left\{\boldsymbol{x}(t,k)\right\}\right\} \tag{3.11}$$

and

$$\boldsymbol{x}(t,k) = \left\{\mathcal{ZL}\right\}^{-1}\left\{\boldsymbol{X}(s,z)\right\} = \mathcal{Z}^{-1}\left\{\ \mathcal{L}^{-1}\left\{\boldsymbol{X}(s,z)\right\}\right\} = \mathcal{L}^{-1}\left\{\ \mathcal{Z}^{-1}\left\{\boldsymbol{X}(s,z)\right\}\right\}. \tag{3.12}$$

Proof To prove the first part in (3.11) we will use the Interchange Theorem (for interchanging summation and integration based on the Uniform Convergence Theorem), which can be found in several text books (see e.g. LePage (1980); Priestley

(2003); Jeffrey (2005)). We need to show, that

$$\sum_{k=0}^{\infty} \int_{0}^{\infty} \left| \boldsymbol{x}(t,k) \mathrm{e}^{-st} z^{-k} \right| \mathrm{d}t < \infty. \tag{3.13}$$

Since we assume $\boldsymbol{x}(t,k)$ does not grow faster than exponentially, (3.7), we can bound the left hand side of (3.13) as

$$\sum_{k=0}^{\infty} \int_{0}^{\infty} |\boldsymbol{x}(t,k)| \, \mathrm{e}^{-\Re\{s\}t} |z|^{-k} \mathrm{d}t \leq \sum_{k=0}^{\infty} \int_{0}^{\infty} c \mathrm{e}^{(a-\Re\{s\})t} \left(\frac{b}{|z|} \right)^{-k} \mathrm{d}t \tag{3.14}$$

which will converge for all $\Re\{s\} > a$ and $|z| > b$. Note that this is called the Region of Convergence (ROC).

The property in (3.12) can be proven using the fact that the integration of a bounded function over a finite length contour is bounded. □

Integration If the Laplace-Z transform of $\boldsymbol{x}(t,k)$ is $\boldsymbol{X}(s,z)$ and the integral $\int_0^t \boldsymbol{x}(\tau,k)\mathrm{d}\tau$ exists, we can write

$$\mathcal{ZL}\left\{ \int_{0}^{t} \boldsymbol{x}(\tau,k)\mathrm{d}\tau \right\} = \frac{1}{s}\boldsymbol{X}(s,z). \tag{3.15}$$

Accumulation If the Laplace-Z transform of $\boldsymbol{x}(t,k)$ is $\boldsymbol{X}(s,z)$, then the Laplace-Z transform of the cumulative sum $\sum_{l=0}^{k} \boldsymbol{x}(t,l)$ can be written as

$$\mathcal{ZL}\left\{ \sum_{l=0}^{k} \boldsymbol{x}(t,l) \right\} = \frac{1}{1-z^{-1}}\boldsymbol{X}(s,z). \tag{3.16}$$

Final Value Theorem If the final values $\lim_{t\to\infty} \boldsymbol{x}(t,k)$ and $\lim_{k\to\infty} \boldsymbol{x}(t,k)$ exist, they can be expressed as

$$\lim_{t\to\infty} \boldsymbol{x}(t,k) = \lim_{s\to 0} s\boldsymbol{X}_{\mathcal{L}}(s,k) \text{ or} \tag{3.17}$$

$$\lim_{t\to\infty} \boldsymbol{X}_{\mathcal{Z}}(t,z) = \lim_{s\to 0} s\boldsymbol{X}(s,z) \tag{3.18}$$

and

$$\lim_{k\to\infty} \boldsymbol{x}(t,k) = \lim_{z\to 1} (1-z^{-1})\boldsymbol{X}_{\mathcal{Z}}(t,z) \text{ or} \tag{3.19}$$

$$\lim_{k\to\infty} \boldsymbol{X}_{\mathcal{L}}(s,k) = \lim_{z\to 1} (1-z^{-1})\boldsymbol{X}(s,z) \tag{3.20}$$

where $\boldsymbol{X}_{\mathcal{L}}(s,k) = \mathcal{L}\{\boldsymbol{x}(t,k)\}$, and $\boldsymbol{X}_{\mathcal{Z}}(t,z) = \mathcal{Z}\{\boldsymbol{x}(t,k)\}$. Consequently if the double limit $\lim_{t,k\to\infty} \boldsymbol{x}(t,k)$ exists

$$\lim_{(t,k)\to(\infty,\infty)} \boldsymbol{x}(t,k) = \lim_{(s,z)\to(0,1)} s(1-z^{-1})\boldsymbol{X}(s,z). \tag{3.21}$$

Multiplication If both Laplace-Z transforms of $\boldsymbol{x}_1(t,k)$ and $\boldsymbol{x}_2(t,k)$ exist ($\boldsymbol{X}_1(s,z)$ and $\boldsymbol{X}_2(s,z)$, respectively, and $|\boldsymbol{x}_i(t,k)| \leq c_i \mathrm{e}^{a_i t} b_i^k$ for $i \in \{1,2\}$), the Laplace-Z transform of $\boldsymbol{x}_1(t,k)\boldsymbol{x}_2(t,k)$ is a combined convolution in the frequency domain

$$\begin{aligned}&\mathcal{ZL}\left\{\boldsymbol{x}_1(t,k)\boldsymbol{x}_2(t,k)\right\}\\&= \frac{1}{(2\pi j)^2} \oint\limits_{\mathcal{C}_{\beta_1}} \int\limits_{\alpha_1 - j\infty}^{\alpha_1 + j\infty} \boldsymbol{X}_1(p,v)\boldsymbol{X}_2\left(s-p, \frac{z}{v}\right) v^{-1} \mathrm{d}p \mathrm{d}v\end{aligned} \tag{3.22}$$

where $\alpha_1 > a_1$ and $\beta_1 > b_1$. If the region of convergence of $\boldsymbol{X}_1$ includes the imaginary axis and the unit circle, i. e. $\alpha_1 = 0$ and $\beta_1 = 1$, respectively, (3.22) becomes

$$\begin{aligned}&\mathcal{ZL}\left\{\boldsymbol{x}_1(t,k)\boldsymbol{x}_2(t,k)\right\}\\&= \frac{1}{(2\pi)^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\infty}^{\infty} \boldsymbol{X}_1\left(j\omega', \mathrm{e}^{j\theta'}\right) \boldsymbol{X}_2\left(j(\omega-\omega'), \mathrm{e}^{j(\theta-\theta')}\right) \mathrm{d}\omega' \mathrm{d}\theta'.\end{aligned} \tag{3.23}$$

Lemma 3.1 (Parseval's Theorem for Continuous-Discrete Two-Dimensional Systems)

If there exist $a < 0$ and $b < 1$ such that $|\boldsymbol{x}(t,k)| \leq c\mathrm{e}^{at}b^k$, then the Laplace-Z transform $\boldsymbol{X}(s,z)$ exists and the L_2-norm of $\boldsymbol{x}(t,k)$ in the time domain is the same as the L_2-norm of $\boldsymbol{X}(s,z)$in the frequency domain:

$$\sum_{k=0}^{\infty} \int\limits_{0}^{\infty} \boldsymbol{x}^2(t,k)\mathrm{d}t = \|\boldsymbol{x}(\cdot,\cdot)\|_2^2 = \|\boldsymbol{X}\left(\cdot,\cdot\right)\|_2^2 = \frac{1}{(2\pi)^2} \int\limits_{-\pi}^{\pi} \int\limits_{-\infty}^{\infty} \boldsymbol{X}^2\left(j\omega, \mathrm{e}^{j\theta}\right) \mathrm{d}\omega \mathrm{d}\theta. \tag{3.24}$$

Proof First, we will define $\phi(t,k)$, $\psi(t,k)$, and $\xi(t,N)$ such that

$$\int\limits_0^t \boldsymbol{x}^2(\tau,k)\mathrm{d}\tau = \int\limits_0^t \phi(\tau,k)\mathrm{d}\tau = \psi(t,k) \quad \text{and} \quad \sum_{k=0}^{N} \psi(t,k) = \xi(t,N). \tag{3.25}$$

Since $\boldsymbol{x}(t,k) \in L_2\ [0,\infty) \times [0,\infty)$ and $\boldsymbol{x}(t,k) \in \mathbb{R}$

$$\sum_{k=0}^{N} \int\limits_0^t \boldsymbol{x}^2(t,k)\mathrm{d}t \leq \sum_{k=0}^{\infty} \int\limits_0^{\infty} \boldsymbol{x}^2(t,k)\mathrm{d}t = \sum_{k=0}^{\infty} \int\limits_0^{\infty} |\boldsymbol{x}(t,k)|_2^2 \mathrm{d}t < \infty. \tag{3.26}$$

Thus, the order of the summation, the integration and limits can be interchanged. We can write the norm of $\boldsymbol{x}(t,k)$ as

$$\begin{aligned}
\|\boldsymbol{x}(\cdot,\cdot)\|_2^2 &= \sum_{k=0}^{\infty}\int\limits_0^{\infty}\boldsymbol{x}^2(t,k)\mathrm{d}t \\
&= \sum_{k=0}^{\infty}\lim_{t\to\infty}\int\limits_0^{t}\boldsymbol{x}^2(\tau,k)\mathrm{d}\tau \\
&= \lim_{t\to\infty}\sum_{k=0}^{\infty}\psi(t,k) \\
&= \lim_{t\to\infty}\lim_{N\to\infty}\xi(t,N).
\end{aligned} \tag{3.27}$$

With the final value theorem for Laplace Transforms and for Z transforms the limit of $\xi(t,N)$ in (3.27) can be expressed as the limit in the frequency domain of the corresponding Laplace-Z transform $\Xi(s,z)$, (3.21),

$$\|\boldsymbol{x}(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1} s\left(1-z^{-1}\right)\Xi(s,z). \tag{3.28}$$

Because $\xi(t,N)$ is the accumulation of $\psi(t,k)$ we can apply (3.16), and (3.28) yields

$$\|\boldsymbol{x}(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1} s\Psi(s,z) \tag{3.29}$$

where $\Psi(s,z) = \mathcal{LZ}\{\psi(t,k)\}$. Since $\psi(t,k)$ is the integral of $\phi(t,k)$, $\mathcal{LZ}\{\phi(t,k)\} = \Phi(s,z) = s\Psi(s,z)$ and we can write according to (3.15)

$$\|\boldsymbol{x}(\cdot,\cdot)\|_2^2 = \lim_{s\to 0}\lim_{z\to 1}\Phi(s,z). \tag{3.30}$$

Furthermore, we know that a multiplication in the time domain corresponds to a convolution in the frequency domain, (3.22), and transform (3.30) into

$$\begin{aligned}
\|\boldsymbol{x}(\cdot,\cdot)\|_2^2 &= \lim_{s\to 0}\lim_{z\to 1}\frac{1}{(2\pi j)^2}\oint\limits_{\mathcal{C}_\beta}\int\limits_{\alpha-j\infty}^{\alpha+j\infty}\boldsymbol{X}(p,v)\boldsymbol{X}\left(s-p,\frac{z}{v}\right)v^{-1}\mathrm{d}p\mathrm{d}v \\
&= \frac{1}{(2\pi j)^2}\oint\limits_{\mathcal{C}_\beta}\int\limits_{\alpha-j\infty}^{\alpha+j\infty}\boldsymbol{X}(p,v)\boldsymbol{X}\left(-p,v^{-1}\right)v^{-1}\mathrm{d}p\mathrm{d}v.
\end{aligned} \tag{3.31}$$

Since we require that $|\boldsymbol{x}(t,k)| \leq c\mathrm{e}^{at}b^k$ with $a<0$ and $b<1$, the region of convergence of $\boldsymbol{X}^2(s,z)$ includes $\alpha=0$ and $|\beta|=1$. Thus (3.31) becomes

$$\begin{aligned}
\|\boldsymbol{x}(\cdot,\cdot)\|_2^2 &= \frac{1}{(2\pi)^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\infty}^{\infty}\boldsymbol{X}\left(j\omega,\mathrm{e}^{j\theta}\right)\boldsymbol{X}\left(-j\omega,\mathrm{e}^{-j\theta}\right)\mathrm{d}\omega\mathrm{d}\theta \\
&= \frac{1}{(2\pi)^2}\int\limits_{-\pi}^{\pi}\int\limits_{-\infty}^{\infty}\left|\boldsymbol{X}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta \\
&= \|\boldsymbol{X}(\cdot,\cdot)\|_2^2.
\end{aligned} \tag{3.32}$$

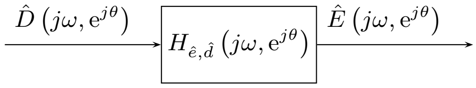


Figure 3.1: *Block diagram of a simple open loop system*

Thus, the Euclidean norm in the time domain $\|\boldsymbol{x}(\cdot,\cdot)\|_2$ is equivalent to the Euclidean norm in the frequency domain $\|\boldsymbol{x}(\cdot,\cdot)\|_2 = \|\boldsymbol{X}(\cdot,\cdot)\|_2$. □

3.3 Induced Operator Norm

We now want to find the induced 2-norm using the results for the norm in ω and θ introduced in (3.32). Consider the continuous-discrete two-dimensional system in the frequency domain in Figure 3.1 with $\hat{E}\left(j\omega, \mathrm{e}^{j\theta}\right) = H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right) \hat{D}\left(j\omega, \mathrm{e}^{j\theta}\right)$. The induced 2-norm of $H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)$ is the upper bound for the norm of $\hat{E}\left(j\omega, \mathrm{e}^{j\theta}\right)$ for all $\hat{D}\left(j\omega, \mathrm{e}^{j\theta}\right)$ with $\|\hat{D}(\cdot,\cdot)\|_2 = 1$.

Assume $H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)$ is continuous almost everywhere except at a finite number of points $\left(j\omega_{\mathrm{p}}, \mathrm{e}^{j\theta_{\mathrm{p}}}\right)$ of discontinuity at nonessential singularities of the second kind. In addition, we require that for each such point $(j\omega_{\mathrm{p}}, \mathrm{e}^{\theta_{\mathrm{p}}})$ there exists a neighbourhood around $(j\omega_{\mathrm{p}}, \mathrm{e}^{\theta_{\mathrm{p}}})$ such that for every possible curve $\theta = \theta_i(\omega)$ in this neighbourhood the limit superior of the function $g_i(\omega) = \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta_i(\omega)}\right)\right|$ exists, i. e. $\limsup_{\omega\to\omega_{\mathrm{p}}} g_i(\omega) = C_i$.

Then the induced operator norm of $H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)$ is

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\mathrm{i}_2} = \operatorname*{ess\,sup}_{\omega,\theta} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|. \tag{3.33}$$

Note that we include functions $H_{\hat{e},\hat{d}}$ that are discontinuous at a finite number of points. Thus we extend the sufficient stability condition given in Dautov (1981). Dautov (1981) proves that a system with a finite number of NSSK on the stability boundary is stable if the transfer function can be continuously extended. This implies that all curves in the neighbourhood around each point $\left(j\omega_{\mathrm{p}}, \mathrm{e}^{j\theta_{\mathrm{p}}}\right)$ of discontinuity leading towards $\left(j\omega_{\mathrm{p}}, \mathrm{e}^{j\theta_{\mathrm{p}}}\right)$ satisfy $\lim_{\omega\to\omega_{\mathrm{p}}} g_i(\omega) = C$. Thus, the limit as $\left(j\omega_{\mathrm{p}}, \mathrm{e}^{j\theta_{\mathrm{p}}}\right)$ is approached does not only exist but is equal for all possible curves. Then the transfer function can be continuously extended by setting $H_{\hat{e},\hat{d}}\left(j\omega_{\mathrm{p}}, \mathrm{e}^{j\theta_{\mathrm{p}}}\right) = C$.

The conjecture that this condition is sufficient and necessary (which was also used in Reddy and Jury (1987)), however, is disproved since the transfer function of a two-dimensional system describing a vehicle platoon is discontinuous around the NSSK at the origin ($\omega = 0$, $\theta = 0$) and cannot be continuously extended as there exist curves (such as spirals around the origin) such that the limit $\lim_{\omega\to\omega_{\mathrm{p}}} g_i(\omega)$ does not exists. Also, for curves that are chosen such that there exists a limit C_i, these limits might not be necessarily equivalent. However, the system can be designed such that the system is BIBO stable.

The induced norm of $H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is defined as

$$\begin{aligned}\left\|H_{\hat{e},\hat{d}}\left(\cdot,\cdot\right)\right\|_{\mathrm{i}_2}^2 &:= \sup_{\|\hat{D}\|_2^2=1}\left\|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right\|_2^2\\ &= \sup_{\|\hat{D}\|_2^2=1}\left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta\right). \end{aligned} \tag{3.34}$$

First, we will show that the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$ over all ω and θ is the upper bound of the induced operator norm: From (3.34) and using Hölder's inequality, we get

$$\begin{aligned}\left\|H_{\hat{e},\hat{d}}\left(\cdot,\cdot\right)\right\|_{\mathrm{i}_2}^2 &= \sup_{\|\hat{D}\|_2^2=1}\left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\left|\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta\right)\\ &\leq \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\sup_{\|\hat{D}\|_2^2=1}\left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta\right)\\ &= \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2. \end{aligned} \tag{3.35}$$

To show that the essential supremum also is a lower bound we will use the following Lemma:

Lemma 3.2

Given a two-dimensional operator $H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)$ which is continuous in $\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)$ the induced operator norm of $H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is always greater or equal to the magnitude of $H_{\hat{e},\hat{d}}\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)$:

$$\left\|H_{\hat{e},\hat{d}}\left(\cdot,\cdot\right)\right\|_{\mathrm{i}_2} \geq \left|H_{\hat{e},\hat{d}}\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)\right|. \tag{3.36}$$

Proof For $0<|\omega_0|<\infty$ and $0<\theta_0<2\pi$ we choose the disturbance signal

$$\hat{d}_\epsilon(t,k)=\alpha_{\omega_0}\mathrm{e}^{-\epsilon t}\cos\omega_0 t\cdot\alpha_{\theta_0}\mathrm{e}^{-\epsilon k}\cos\theta_0 k \tag{3.37}$$

with

$$\alpha_{\omega_0}^2=\frac{4\epsilon^2+4\omega_0^2}{2\epsilon+\omega_0^2/\epsilon} \tag{3.38}$$

and

$$\alpha_{\theta_0}^2=\frac{2}{\frac{1}{1-\mathrm{e}^{-2\epsilon}}+\frac{1-\mathrm{e}^{-2\epsilon}\cos 2\theta_0}{1-2\mathrm{e}^{-2\epsilon}\cos 2\theta_0+\mathrm{e}^{-4\epsilon}}} \tag{3.39}$$

to guarantee $\left\|\hat{d}_\epsilon(\cdot,\cdot)\right\|_2 = 1$. For details on how to choose α_{ω_0} and α_{θ_0} see Section 3.A.1. We will now use the following trick with $\mathcal{R}_\epsilon = \{(\omega,\theta) : |\omega \pm \omega_0| \leq \sqrt{\epsilon}, |\theta \pm \theta_0| \leq \sqrt{\epsilon}\}$:

$$\begin{aligned}
\left\|H_{\hat{e},\hat{d}}\left(\cdot,\cdot\right)\right\|_{\mathrm{i}_2}^2 &= \sup_{\|\hat{D}\|_2=1} \left\|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right\|_2^2 \\
&\geq \left\|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right\|_2^2 \\
&= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\left|\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta \\
&\geq \frac{1}{(2\pi)^2}\iint_{\omega,\theta\in\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\left|\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta \\
&\geq \frac{1}{(2\pi)^2}\left(\operatorname*{ess\,inf}_{\omega,\theta\in\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\right)\left(\iint_{\omega,\theta\in\mathcal{R}_\epsilon}\left|\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta\right).
\end{aligned} \tag{3.40}$$

To conclude the proof we will now take the limit for ϵ approaching 0. Given that $H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is continuous around ω_0 and θ_0 the limit of $\operatorname{ess\,inf}_{\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$ is

$$\lim_{\epsilon\to 0}\operatorname*{ess\,inf}_{\mathcal{R}_\epsilon}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2 = \left|H_{\hat{e},\hat{d}}\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)\right|^2. \tag{3.41}$$

To evaluate the integral of $\left|\hat{D}_\epsilon\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2$ over $\mathcal{R}_\epsilon$ requires some more work. We will use the Laplace-Z transform of $\hat{d}_\epsilon(t,k)$

$$\hat{D}_\epsilon(j\omega,\mathrm{e}^{j\theta}) = \alpha_{\omega_0}\frac{j\omega+\epsilon}{(j\omega+\epsilon)^2+\omega_0^2}\cdot\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}} \tag{3.42}$$

and show that in the limit $\epsilon \to 0$ the integral is equal to $(2\pi)^2$. For details refer to Section 3.A.2. Note that for $\omega_0 = 0$ or $\theta_0 = 0$ a simplified $d_\epsilon(t,k)$ can be chosen with

$$\hat{d}_{\epsilon_\omega}(t) = \alpha_{\omega_0}\mathrm{e}^{-\epsilon t} \qquad \text{or} \qquad \hat{d}_{\epsilon_\theta}(k) = \alpha_{\theta_0}\mathrm{e}^{-\epsilon k}, \tag{3.43}$$

respectively. The corresponding coefficients are $\alpha_{\omega_0}^2 = 2\epsilon$ and $\alpha_{\theta_0}^2 = 1-\mathrm{e}^{-2\epsilon}$, see Section 3.A.1. It can be shown in the same way that the integral of $|\hat{D}_\epsilon(j\omega,\mathrm{e}^{j\theta})|^2$ over $\mathcal{R}_\epsilon = \{(\omega,\theta) : |\omega| < \sqrt{\epsilon}, |\theta| < \sqrt{\epsilon}\}$ is $(2\pi)^2$, see Section 3.A.2.

Thus,

$$\left\|H_{\hat{e},\hat{d}}\left(\cdot,\cdot\right)\right\|_{\mathrm{i}_2} \geq \left|H_{\hat{e},\hat{d}}\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)\right|. \tag{3.44}$$

□

If the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$ exists it can be achieved in three different cases:

First we will assume that the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is achieved at $\bar{\omega}$ and $\bar{\theta}$ and $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is continuous in and around the supremum. In that case, we can set $(\omega_0, \theta_0) = (\bar{\omega}, \bar{\theta})$ in (3.36) and use Lemma 3.2.

However, it is also possible that the essential supremum is achieved at a point $(\omega_\mathrm{p}, \theta_\mathrm{p})$ of discontinuity of $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$. We will use the assumptions made at the beginning of this section. We require that for each such point $(j\omega_\mathrm{p}, \mathrm{e}^{\theta_\mathrm{p}})$ there exists a neighbourhood around $(j\omega_\mathrm{p}, \mathrm{e}^{\theta_\mathrm{p}})$ such that for every possible curve $\theta = \theta_i(\omega)$ in this neighbourhood the limit superior of the function $g_i(\omega) = \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta_i(\omega)}\right)\right|$ exists, i. e. $\limsup_{\omega\to\omega_\mathrm{p}} g_i(\omega) = C_i$. Given Lemma 3.2 above, for each $\epsilon_i > 0$ there exist a $\delta_i(\epsilon_i) > 0$ and a point $(j\omega_0, \mathrm{e}^{j\theta_i(\omega_0)})$ on g_i such that for all ω, θ in a circle with radius δ_i around $(\omega_0, \theta_i(\omega_0))$ (i. e. $|(\omega, \theta) - (\omega_0, \theta_i(\omega_0))| \leq \delta_i$) and $|(\omega_\mathrm{p}, \theta_\mathrm{p}) - (\omega_0, \theta_i(\theta_0))| = \epsilon_i$ we have

$$\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\|_{\mathrm{i}_2} \geq \operatorname*{ess\,inf}_{|(\omega,\theta)-(\omega_0,\theta_i(\omega_0))|\leq\epsilon_i} g_i(\omega). \tag{3.45}$$

Therefore, it must be true that $\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\|_{\mathrm{i}_2} \geq C_i$.

In the third case the supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ occurs as $\omega_0 \to \infty$ and θ_0

$$\operatorname*{ess\,sup}_{\omega,\theta} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right| = \lim_{\omega\to\infty} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta_0}\right)\right|. \tag{3.46}$$

For the time dependent part of $\hat{d}_\epsilon(t,k)$ we will choose $\hat{d}_N(t) = \sqrt{2N}\mathrm{e}^{-Nt}$. It can then be shown that for $N \to \infty$ the integral of $\left|\hat{D}_N(\omega)\right|^2$ over $\omega \in \left[-N^2, -\sqrt{N}\right] \cup \left[\sqrt{N}, N^2\right]$ is equal to 2π. (For details see Section 3.A.2.) At the same time we can use a similar argument as above to show that $\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\|_{\mathrm{i}_2} \geq \lim_{\omega\to\infty} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta_0}\right)\right|$.

Thus, it is always true that

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\mathrm{i}_2} \geq \operatorname*{ess\,sup}_{\omega,\theta} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|. \tag{3.47}$$

Together with (3.35) the induced L_2-norm of $H_{\hat{e},\hat{d}}(j\omega, \mathrm{e}^{j\theta})$ is

$$\left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\mathrm{i}_2} = \left\|H_{\hat{e},\hat{d}}(\cdot,\cdot)\right\|_{\infty} := \operatorname*{ess\,sup}_{\omega,\theta} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|. \tag{3.48}$$

3.4 Linear, Unidirectional Control

3.4.1 System Description

We wish to discuss the stability of a simple chain of vehicles where all but the first should keep a fixed distance $\hat{x}_\mathrm{d}$ to their predecessor. The first car follows a given trajectory $\hat{x}(t, 0)$. We will choose the same vehicle model with transfer function $P(s)$ and the same linear controller $C(s)$ for every subsystem, i. e. every car. The controller $C(s)$ is chosen such that the subsystem with $T(s) = C(s)P(s)/(1 + C(s)P(s))$ is stable.

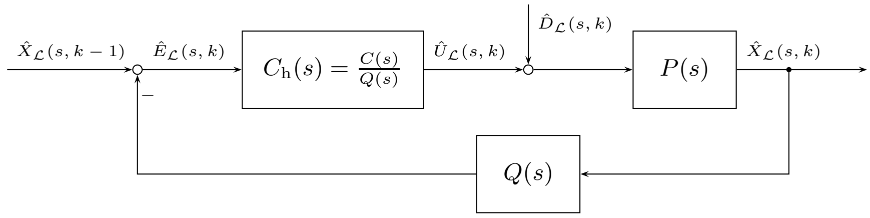


Figure 3.2: *Block diagram of the linear subsystem with time headway*

The open loop transfer function $L(s)$ has exactly two poles at the origin, $L(s) = P(s)C(s) = \frac{1}{s^2}\tilde{L}(s)$ with $\tilde{L}(0) \neq 0$. Feedback loops of this kind are also referred to as "type II servomechanisms". The position of the kth vehicle $\hat{x}(t,k)$ depends on the disturbance $\hat{d}(t,k)$ and the actuator signal of the kth controller $\hat{u}(t,k)$. The local control objective is to force the separation error $\hat{e}(t,k)$ to zero. Measurement noise is neglected for simplicity. Using the Laplace transform with respect to time t (denoted by $\cdot_{\mathcal{L}}$) the system with zero initial conditions is described by

$$\hat{X}_{\mathcal{L}}(s,k) = P(s)\left(\hat{U}_{\mathcal{L}}(s,k) + \hat{D}_{\mathcal{L}}(s,k)\right), \tag{3.49}$$

$$\hat{U}_{\mathcal{L}}(s,k) = C(s)\hat{E}_{\mathcal{L}}(s,k) \qquad \text{and} \tag{3.50}$$

$$\hat{E}_{\mathcal{L}}(s,k) = \hat{X}_{\mathcal{L}}(s,k-1) - \hat{X}_{\mathcal{L}}(s,k) - \frac{\hat{x}_{\mathrm{d}}}{s}. \tag{3.51}$$

It is known that for type II servomechanisms the absolute value of the complementary sensitivity function of a single subsystem, $T(s) = \frac{L(s)}{1+L(s)}$, is greater than 1 for a range of frequencies $\omega \in (\omega_-, \omega_+)$. The system therefore will be 'string unstable' for constant spacing ($\hat{x}_{\mathrm{d}} = \mathrm{const}$), Sheikholeslam and Desoer (1990); Seiler *et al.* (2004), which means that the peak over time of the error signal $\hat{e}(t,k)$ grows without bound as k increases.

Since when using a constant spacing policy the system is string unstable, a linear time headway h is incorporated in the feedback path. In addition to a fixed vehicle separation, a velocity $\hat{v}(t,k)$ dependent distance is required between the vehicles, $\hat{x}_{\mathrm{d}}(t,k) = \hat{x}_{\mathrm{d}_0} + h\hat{v}(t,k)$. Note that in order to preserve the closed loop poles of the time headway free system, an additional pole is inserted into the controller transfer function such that $C_{\mathrm{h}}(s) = \frac{C(s)}{Q(s)}$ with $Q(s) = hs + 1$. To simplify the following derivations and because we are interested in the disturbance to error behaviour we shall set $\hat{x}_{\mathrm{d}_0} = 0$ below. The new subsystem is shown in Figure 3.2.

After applying the Laplace transform with respect to t the error signal $\hat{E}_{\mathcal{L}}(s,k)$ yields

$$\begin{aligned}
\hat{E}_{\mathcal{L}}(s,k) =& \hat{X}_{\mathcal{L}}(s,k-1) - Q(s)\hat{X}_{\mathcal{L}}(s,k) \\
=& P(s)C_{\mathrm{h}}(s)\hat{E}_{\mathcal{L}}(s,k-1) + P(s)\hat{D}_{\mathcal{L}}(s,k-1) \\
& - Q(s)\left(P(s)C_{\mathrm{h}}(s)\hat{E}_{\mathcal{L}}(s,k) + P(s)\hat{D}_{\mathcal{L}}(s,k)\right) \\
=& \Gamma(s)\hat{E}_{\mathcal{L}}(s,k-1) + \Gamma(s)C_{\mathrm{h}}^{-1}(s)\left(\hat{D}_{\mathcal{L}}(s,k-1) - Q(s)\hat{D}_{\mathcal{L}}(s,k)\right)
\end{aligned} \tag{3.52}$$

with $\hat{E}_{\mathcal{L}}(s,k) = \mathcal{L}\{\hat{e}(t,k)\}$, $\hat{X}_{\mathcal{L}}(s,k) = \mathcal{L}\{\hat{x}(t,k)\}$, $\hat{D}_{\mathcal{L}}(s,k) = \mathcal{L}\left\{\hat{d}(t,k)\right\}$ and the single loop complementary sensitivity function

$$\Gamma(s) = \frac{P(s)C_{\mathrm{h}}(s)}{1+Q(s)P(s)C_{\mathrm{h}}(s)} = \frac{1}{Q(s)}\frac{P(s)C(s)}{1+P(s)C(s)} = \frac{T(s)}{Q(s)}. \tag{3.53}$$

Applying the Z transform with respect to k, (3.52) becomes

$$\begin{aligned}\hat{E}(s,z) =& \Gamma(s)z^{-1}E(s,z) + \Gamma(s)C_{\mathrm{h}}^{-1}(s)\left(z^{-1} - Q(s)\right)\hat{D}(s,z)\\ =& \underbrace{\frac{z^{-1}-Q(s)}{1-z^{-1}\Gamma(s)}\Gamma(s)C_{\mathrm{h}}^{-1}(s)}_{=H_{\hat{e},\hat{d}}(s,z)}\hat{D}(s,z)\end{aligned} \tag{3.54}$$

with $\hat{E}(s,z) = \mathcal{Z}\left\{\hat{E}_{\mathcal{L}}(s,k)\right\} = \mathcal{Z}\mathcal{L}\left\{\hat{e}(t,k)\right\}$ and $\hat{D}(s,z) = \mathcal{Z}\left\{\hat{D}_{\mathcal{L}}(s,k)\right\} = \mathcal{Z}\mathcal{L}\left\{\hat{d}(t,k)\right\}$.

3.4.2 Conditions for String Stability

To guarantee $\left\|\hat{E}(s,z)\right\|_2 < \infty$ for any $\hat{D}(s,z)$ satisfying $\left\|\hat{D}(s,z)\right\|_2 < \infty$, $\left|H_{\hat{e},\hat{d}}(s,z)\right|$ must be bounded for any $s = a + j\omega$ and $z = r\mathrm{e}^{j\theta}$ with $a \geq 0$ and $r \geq 1$.

This is always true if $H_{\hat{e},\hat{d}}(s,z)$ has no poles with $\{\Re\{s\} \geq 0\} \cap \{|z| \geq 1\}$. As discussing stability of the string only makes sense for strings with stable subsystems, $\Gamma(s)$ must not have any poles with $\Re\{s\} \geq 0$. Also a local controller with zeros with positive real parts has to be avoided to guarantee $\left|C_{\mathrm{h}}^{-1}(s)\right| < \infty$ for $\Re\{s\} \geq 0$.

Before discussing under which conditions the first part of $H_{\hat{e},\hat{d}}(s,z)$, i.e. the fraction $\left(z^{-1} - Q(s)\right) / \left(1 - z^{-1}\Gamma(s)\right)$, is devoid of poles with $\{\Re\{s\} > 0\} \cap \{|z| > 1\}$ we will focus on the region $\{s = j\omega\} \cap \left\{z = \mathrm{e}^{j\theta}\right\}$.

Note that $\hat{X}(s,k) = \Gamma(s)\hat{X}(s,k-1) + \Gamma(s)C_{\mathrm{h}}^{-1}(s)\hat{D}(s,k)$. Every vehicle should be able to follow its predecessor and the local error should be forced to 0 for $t \to \infty$. Therefore, the subsystem closed loop transfer function $\Gamma(s)$ is designed such that $\Gamma(0) = 1$. However, this implies that $H_{\hat{e},\hat{d}}(s,z)$ will always have a pole at $\{s = 0\} \cap \{z = 1\}$. Note that the numerator of $H_{\hat{e},\hat{d}}(s,z)$ is also 0 at the same point, i.e. $1 - Q(0) = 1 - 1 - h \cdot 0 = 0$. This is referred to as a nonessential singularity of the second kind (NSSK). (See also the discussion in Chapter 1.)

Therefore, we have to show that $\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$ is bounded:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right| \\
&\quad=\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{\mathrm{e}^{-j\theta}-Q(j\omega)}{1-\mathrm{e}^{-j\theta}\Gamma(j\omega)}\Gamma(j\omega)C_{\mathrm{h}}^{-1}(j\omega)\right| \\
&\quad=\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega)-\mathrm{e}^{j\theta}}{\mathrm{e}^{j\theta}-\Gamma(j\omega)}T(j\omega)C_{\mathrm{h}}^{-1}(j\omega)\right| \\
&\quad=\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega)-\Gamma(j\omega)}{\mathrm{e}^{j\theta}-\Gamma(j\omega)}-1\right|\left|T(0)C_{\mathrm{h}}^{-1}(0)\right| \\
&\quad\leq\left(\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega)-\Gamma(j\omega)}{\mathrm{e}^{j\theta}-\Gamma(j\omega)}\right|+1\right)\left|T(0)C_{\mathrm{h}}^{-1}(0)\right|.
\end{aligned} \tag{3.55}$$

Since $\left|T(0)C_{\mathrm{h}}^{-1}(0)\right|$ is bounded, we will focus on the first term on the right hand side of inequality (3.55):

$$\begin{aligned}
&\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega)-\Gamma(j\omega)}{\mathrm{e}^{j\theta}-\Gamma(j\omega)}\right| \\
&\qquad\leq\lim_{\epsilon\to 0}\sup_{\omega\in B_\epsilon(0)}\frac{\left|Q^{-1}(j\omega)-\Gamma(j\omega)\right|}{1-|\Gamma(j\omega)|} \\
&\qquad=\lim_{\epsilon\to 0}\sup_{\omega\in B_\epsilon(0)}\frac{\frac{1}{\sqrt{h^2\omega^2+1}}\left|1-\frac{\tilde{L}(j\omega)}{\tilde{L}(j\omega)-\omega^2}\right|}{1-\frac{1}{\sqrt{h^2\omega^2+1}}\left|\frac{\tilde{L}(j\omega)}{\tilde{L}(j\omega)-\omega^2}\right|} \\
&\qquad=\lim_{\epsilon\to 0}\sup_{\omega\in B_\epsilon(0)}\frac{\omega^2}{\sqrt{h^2\omega^2+1}\left|\tilde{L}(j\omega)-\omega^2\right|-\left|\tilde{L}(j\omega)\right|}.
\end{aligned} \tag{3.56}$$

Remember that $\tilde{L}(j\omega)$ is the open loop transfer function of a single subsystem without the two integrators. Around the origin we can therefore express $\tilde{L}(j\omega)$ as $\tilde{L}(j\omega)=a_0+a_2\omega^2+a_4\omega^4+\ldots+j(a_1\omega+a_3\omega^3+\ldots)$. Using that and L'Hôpital's Rule, (3.56) becomes

$$\begin{aligned}
&\lim_{\epsilon\to 0}\sup_{(\omega,\theta)\in B_\epsilon(0,0)}\left|\frac{Q^{-1}(j\omega)-\Gamma(j\omega)}{\mathrm{e}^{j\theta}-\Gamma(j\omega)}\right| \\
&\qquad\leq\limsup_{\omega\to 0}\left(\frac{1}{2}\frac{h^2}{\sqrt{h^2\omega^2+1}}\left|\tilde{L}(j\omega)-\omega^2\right|\right. \\
&\qquad\quad\left.+\sqrt{h^2\omega^2+1}\frac{\partial}{\partial\omega^2}\left|\tilde{L}(j\omega)-\omega^2\right|-\frac{\partial}{\partial\omega^2}\left|\tilde{L}(j\omega)\right|\right)^{-1} \\
&\qquad=\frac{1}{\frac{1}{2}h^2a_0-1}.
\end{aligned} \tag{3.57}$$

Thus, using a time headway greater than $\sqrt{2/a_0}=\sqrt{2/\tilde{L}(0)}$ will guarantee that $H_{\hat{e},\hat{d}}(s,z)$ is bounded at the NSSK at the origin.

To ensure $\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|<\infty$ for all $\omega\neq 0$ and θ, we must guarantee that $|\Gamma(j\omega)|<1$ for all $\omega\neq 0$. Otherwise, since we know that $\Gamma(s)$ is strictly proper, there must exist an

$\omega_0 \neq 0$ such that $|\Gamma(j\omega_0)| = 1$. To ensure $|\Gamma(j\omega)| < 1\ \forall \omega \neq 0$, the time headway h must be greater than the infimal time headway

$$h_0 := \sqrt{\sup_{\omega} \left(\frac{|T(j\omega)|^2 - 1}{\omega^2} \right)} \tag{3.58}$$

where $T(s) = \frac{L(s)}{1+L(s)}$ is the single loop complementary sensitivity function of the system with zero time headway and $L(s) = \frac{1}{s^2}\tilde{L}(s)$ is the corresponding open loop transfer function with exactly two integrators and $\tilde{L}(0) \neq 0$.

Since the maximum in (3.58) can be attained at $\omega = 0$ or at at least one $\omega_0 \neq 0$, we will distinguish between these two cases:

(a) The maximum in (3.58) is attained at $\omega = 0$ only. Using L'Hôpital's Rule and the fact that $\tilde{L}(0) = \bar{\tilde{L}}(0) = \left|\tilde{L}(0)\right|$ condition (3.58) becomes

$$h_0 = \lim_{\omega \to 0} \sqrt{\frac{\left|\frac{L(j\omega)}{1+L(j\omega)}\right|^2 - 1}{\omega^2}} = \sqrt{2/\left|\tilde{L}(0)\right|}. \tag{3.59}$$

Hence, choosing $h > \sqrt{2/\left|\tilde{L}(0)\right|}$ guarantees that $|\Gamma(j\omega)| \leq 1$ for all ω and $|\Gamma(j\omega)| = 1$ only at $\omega = 0$. Note that this is the same condition given above to guarantee that $|H_{\hat{e},\hat{d}}(s,z)|$ is bounded at the NSSK at the origin. In fact, this condition has a simple geometric interpretation. For $h = \sqrt{2/\left|\tilde{L}(0)\right|}$ the second derivative of $|\Gamma(j\omega)|$ at $\omega = 0$ is zero, $\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}|\Gamma(j\omega)|\Big|_{\omega=0} = 0$. Since $|\Gamma(j\omega)|$ is equal to 1 at the origin, it would be greater than 1 for some frequency $\omega' > 0$ if its second derivative at the origin is greater than zero.

(b) The maximum in (3.58) is attained at at least one $\omega_0 \neq 0$. In that case condition (3.58) becomes

$$h_0 = \frac{\sqrt{\left|\frac{L(j\omega_0)}{1+L(j\omega_0)}\right|^2 - 1}}{\omega_0}. \tag{3.60}$$

Choosing a time headway which is strictly greater than h_0, $h > h_0$, will guarantee $|\Gamma(j\omega)| \leq 1$ for all ω and $|\Gamma(j\omega)| = 1$ only at $\omega = 0$ in both cases.

Note that if the supremum of (3.58) is achieved at $\omega = 0$ and h_0 is chosen according to (3.59) the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is achieved at the origin. In that case approaching the origin along the curve $\theta = \theta(\omega) = -h\omega$ allows us to obtain the limit: $\lim_{\omega\to 0}\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{\theta(\omega)}\right)\right| = \frac{1/2h^2a_0}{1/2h^2a_0 - 1}|C_{\mathrm{h}}^{-1}(0)\Gamma(0)|$. This is in fact the supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ over ω and θ obtained at the origin (compare with (3.55) and (3.57)).

However, if h_0 is chosen according to (3.60) the point where the essential supremum of $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is reached is continuous.

Note that if the induced norm of $H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)$ is bounded, $\left\|H_{\hat{e},\hat{d}}\left(a+j\omega, r\mathrm{e}^{j\theta}\right)\right\|_{\infty}$ is also bounded for any $a \geq 0$ and $r \geq 1$. Given that $\Gamma(j\omega)$ and $|\Gamma(j\omega)| \leq 1$ and $|\Gamma(j\omega)| = 1$ if and only if $\omega = 0$, the Poisson Integral Formula yields

$$
\begin{aligned}
|\Gamma(a+j\omega_0)| =& \frac{1}{\pi}\int_{-\infty}^{\infty} |\Gamma(j\omega)| \frac{a}{a^2+(\omega_0-\omega)^2}\mathrm{d}\omega \\
\leq& \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2+\omega_0^2-2\omega_0\omega+\omega^2}\mathrm{d}\omega \\
=& \frac{1}{\pi}\frac{2a}{2a}\left[\arctan\frac{2\omega-\omega_0}{2a}\right]_{-\infty}^{\infty} \\
=& 1.
\end{aligned} \tag{3.61}
$$

Note that since $\Gamma(j\omega)$ is strictly proper, there exists a $\tilde{\omega}$ such that $|\Gamma(j\omega)| < \frac{1}{2}$ for all $\omega > \tilde{\omega}$. Therefore, the inequality above is strict and there are no poles of $H_{\hat{e},\hat{d}}(s,z)$ in $\{\Re\{s\} > 0\} \cap \{|z| \geq 1\}$. Furthermore, there are no poles of $H_{\hat{e},\hat{d}}(s,z)$ in $\{\Re(s) \geq 0\} \cap \{|z| > 1\}$ because

$$
\left|1 - z^{-1}\Gamma(s)\right| \geq 1 - \left|z^{-1}\right| |\Gamma(s)| \geq 1 - \left|z^{-1}\right| > 0 \qquad \text{for} \quad \Re\{s\} \geq 0, |z| > 1. \tag{3.62}
$$

The previous results can now be combined: Since $H_{\hat{e},\hat{d}}(s,z)$ is bounded for

$$
\{\{\Re\{s\} = 0\} \cap \{|z| = 1\}\} \cup \{\{\Re\{s\} > 0\} \cap \{|z| \geq 1\}\} \cup \{\{\Re\{s\} \geq 0\} \cap \{|z| > 1\}\}, \tag{3.63}
$$

it is bounded for $H_{\hat{e},\hat{d}}(s,z)$ in $\{\Re\{s\} \geq 0\} \cap \{|z| \geq 1\}$ provided that the time headway is greater than the infimal time headway, i. e. $h > h_0$.

3.4.3 Example and Simulations

To illustrate our results we will simulate a string of forty vehicles with the simplified, linearised second order model for each car

$$
P(s) = \frac{1}{s^2 + 2C_\mathrm{d}v_0 s} \tag{3.64}
$$

where the drag coefficient is $C_\mathrm{d} = 7 \cdot 10^{-4}$. A simple PID controller of the form

$$
C(s) = k_\mathrm{p} + \frac{k_\mathrm{i}}{s} + \frac{k_\mathrm{d}s}{Ts+1} \tag{3.65}
$$

with $k_\mathrm{p} = 1.66$, $k_\mathrm{i} = 0.17$, $k_\mathrm{d} = 4.1$ and $T = 1/30$ will be used. The infimal time headway which will guarantee string stability can be found examining the curve $\sqrt{|T(j\omega)|^2 - 1}/\omega$. According to equation (3.58) and Figure 3.3, we see that the infimal time headway is $h_0 \approx 1.18$.

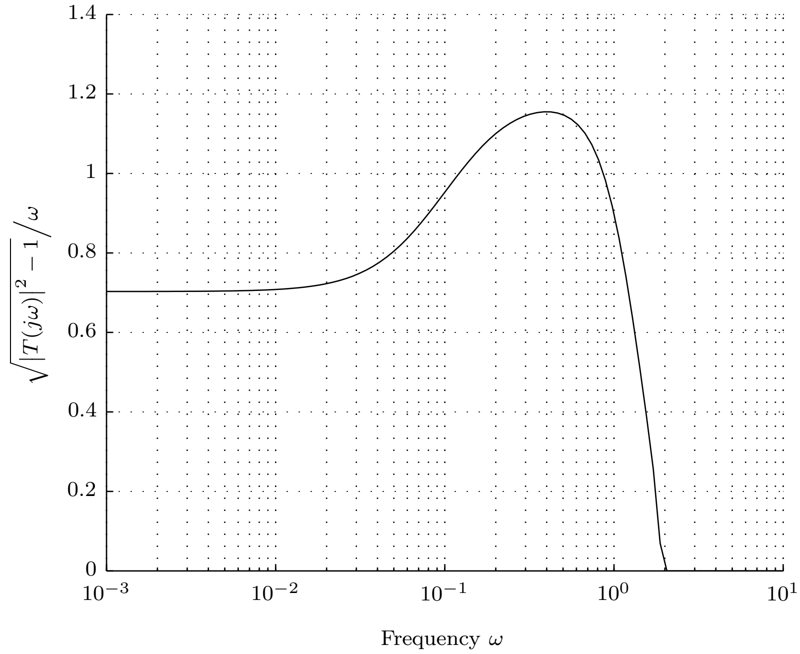


Figure 3.3: *Curve to determine the infimal time headway* h_0

Thus choosing constant spacing (zero time headway) or a small time headway will lead to an unstable string. However, if h is greater than h_0, the system will be string stable. The magnitude of $H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)$ for $h = 1.5$ is shown in Figure 3.4. The ridge leading towards the origin is displayed in Figure 3.5. Simulations for $h = 1$ and $h = 1.5$ are shown in Figure 3.6.

3.5 Linear, Unidirectional Control with Communication Range 2

We have shown how a linear, unidirectional string behaves depending on the time headway h. However, to obtain a string stable system often a fairly large time headway is required. That means that the distance between the vehicles at high speed can be very large.

This can be improved by a wider communication range, that is using information from a bigger set of vehicles. The easiest way to do so is to use the information of several predecessors instead of only the direct predecessor.

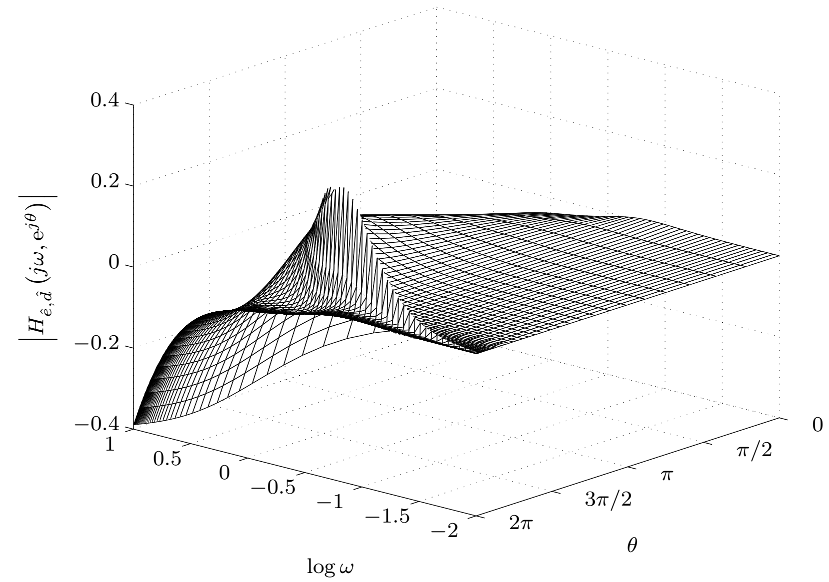


Figure 3.4: *Unidirectional string with* $h = 1.5$: $\left|H_{\hat{e},\hat{d}}\right|$

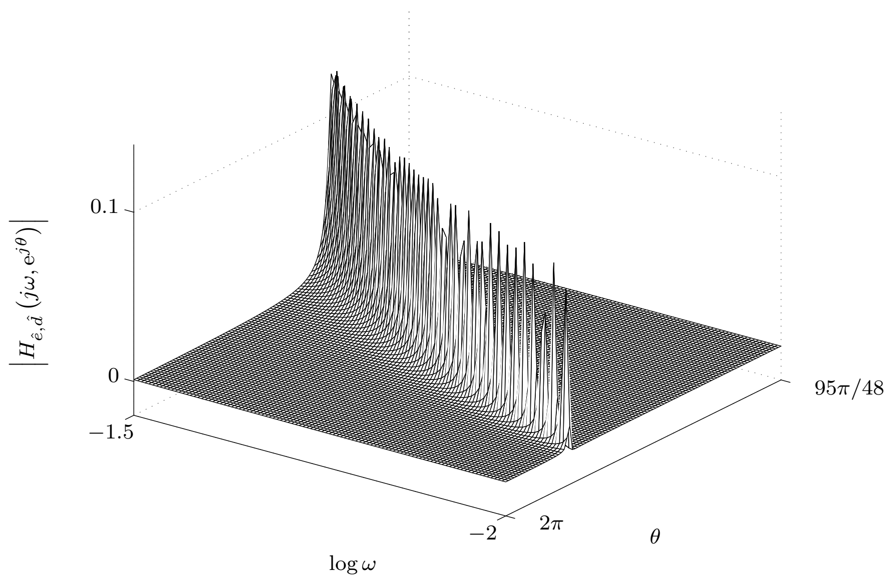
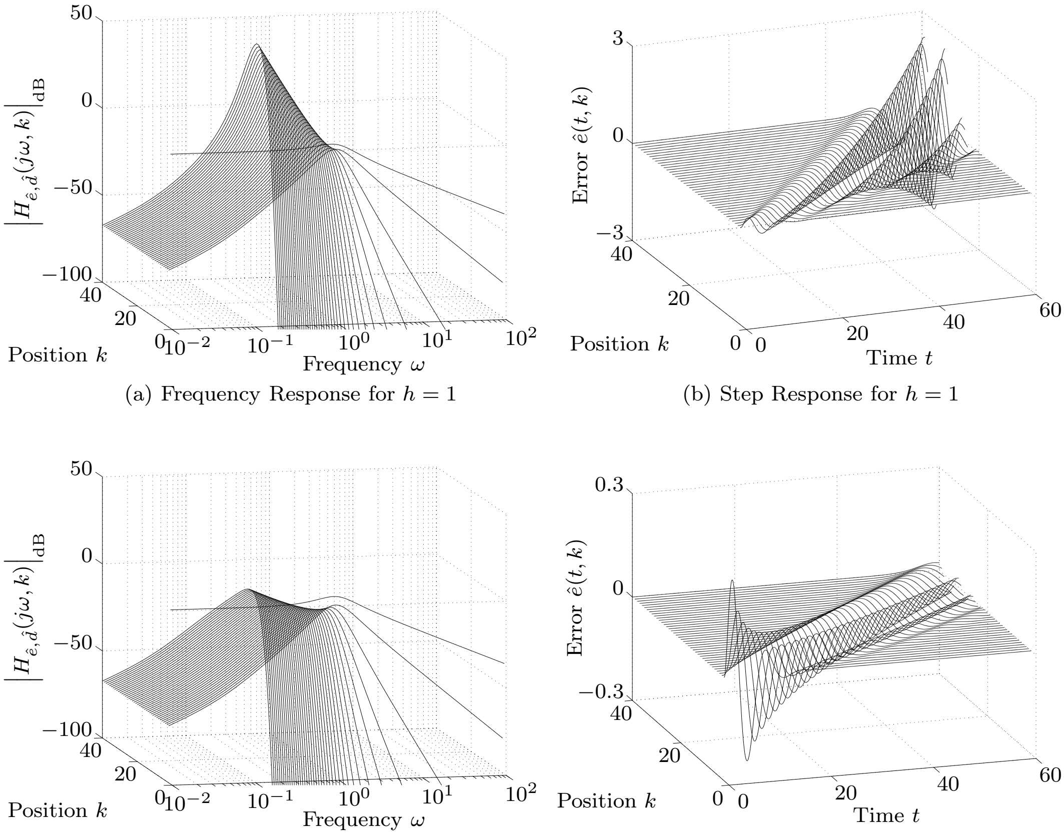


Figure 3.5: *Unidirectional string with* $h = 1.5$: $\left|H_{\hat{e},\hat{d}}\right|$ *around the origin*



(a) Frequency Response for $h = 1$

(b) Step Response for $h = 1$

(c) Frequency Response for $h = 1.5$

(d) Step Response for $h = 1.5$

Figure 3.6: *Unidirectional string with communication range* 1, $h = 1.5$

3.5.1 Singularity on the Stability Boundary

This implies that in the state space description the dimension of the variable $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$ is $n_2 > 1$ and equals at least the number of vehicles ahead whose information is used in the local error equation. Before considering a possible realisation with communication range $n_2 = 2$ we will show that similar to the case with $n_2 = 1$ discussed in Chapter 1 a singularity on the stability boundary is not only unavoidable but rather necessary.

Consider the state space description

$$\dot{\boldsymbol{x}}_1(t,k) = \boldsymbol{A}_{11}\boldsymbol{x}_1(t,k) + \boldsymbol{A}_{12}\boldsymbol{x}_2(t,k), \tag{3.66}$$

$$\Delta\boldsymbol{x}_2(t,k) = \boldsymbol{A}_{21}\boldsymbol{x}_1(t,k) + \boldsymbol{A}_{22}\boldsymbol{x}_2(t,k), \tag{3.67}$$

$$y(t,k) = \boldsymbol{c}_1\boldsymbol{x}_1(t,k) + \boldsymbol{c}_2\boldsymbol{x}_2(t,k). \tag{3.68}$$

Assume that the initial conditions $\boldsymbol{x}_{10}(k) = 0$ for all k. Since the vehicles need to be able to follow their predecessor and the first vehicle in the platoon a given trajectory, we require that for a vector of step responses as initial or boundary conditions $\boldsymbol{x}_{20}(t) = \boldsymbol{f}_2$ the output signal $y(t,k)$ tends to 1 for all k. Thus, $\boldsymbol{f}_2$ has to be suitably chosen such that there exists a $\boldsymbol{f}_1$ such that

$$0 = \boldsymbol{A}_{11}\boldsymbol{f}_1 + \boldsymbol{A}_{12}\boldsymbol{f}_2 \tag{3.69}$$

$$\boldsymbol{f}_2 = \boldsymbol{A}_{21}\boldsymbol{f}_1 + \boldsymbol{A}_{22}\boldsymbol{f}_2 \tag{3.70}$$

$$1 = \boldsymbol{c}_1\boldsymbol{f}_1 + \boldsymbol{c}_2\boldsymbol{f}_2 \tag{3.71}$$

Hence,

$$\forall k \lim_{t\to\infty} y(t,k) = 1. \tag{3.72}$$

Applying the Laplace transform with respect to t and the Z transform with respect to k this yields

$$\lim_{s\to 0} sY(s,z) = \frac{1}{1-z^{-1}}. \tag{3.73}$$

Applying the Laplace transform with respect to t and the Z transform with respect to k to (3.66)-(3.67) yields

$$\boldsymbol{X}_1(s,z) = (s\mathbf{I} - \boldsymbol{A}_{11})^{-1} \boldsymbol{A}_{12}\boldsymbol{X}_2(s,z), \tag{3.74}$$

$$\boldsymbol{X}_2(s,z) = (z\mathbf{I} - \mathbf{I} - \boldsymbol{A}_{22})^{-1} \left(\frac{z}{s}\boldsymbol{f}_2 + \boldsymbol{A}_{21}\boldsymbol{X}_1(s,z)\right). \tag{3.75}$$

Thus

$$\begin{aligned}
\boldsymbol{X}_2(s,z) &= (z\mathbf{I} - \mathbf{I} - \boldsymbol{A}_{22})^{-1} \left(\frac{z}{s}\boldsymbol{f}_2 + \boldsymbol{A}_{21}(s\mathbf{I} - \boldsymbol{A}_{11})^{-1} \boldsymbol{A}_{12}\boldsymbol{X}_2(s,z)\right) \\
&= \left(\mathbf{I} - (z\mathbf{I} - \mathbf{I} - \boldsymbol{A}_{22})^{-1} \boldsymbol{A}_{21}(s\mathbf{I} - \boldsymbol{A}_{11})^{-1} \boldsymbol{A}_{12}\right)^{-1} (z\mathbf{I} - \mathbf{I} - \boldsymbol{A}_{22})^{-1} \frac{z}{s}\boldsymbol{f}_2 \\
&= \left(z\mathbf{I} - \mathbf{I} - \boldsymbol{A}_{22} - \boldsymbol{A}_{21}(s\mathbf{I} - \boldsymbol{A}_{11})^{-1} \boldsymbol{A}_{12}\right)^{-1} \frac{z}{s}\boldsymbol{f}_2 \\
&= \left(\mathbf{I} - z^{-1} \underbrace{\left(\mathbf{I} + \boldsymbol{A}_{22} + \boldsymbol{A}_{21}(s\mathbf{I} - \boldsymbol{A}_{11})^{-1} \boldsymbol{A}_{12}\right)}_{=:\boldsymbol{\Gamma}(s)}\right)^{-1} \frac{1}{s}\boldsymbol{f}_2
\end{aligned} \tag{3.76}$$

with the transfer function $\boldsymbol{\Gamma}(s) \in \mathbb{C}^{n_2 \times n_2}$ describing how $\boldsymbol{X}_2(s, k+1)$ depends on $\boldsymbol{X}_2(s,k)$. Combining (3.76) and the Laplace-Z transform of (3.68), $Y(s,z)$ yields

$$\begin{aligned}
Y(s,z) &= \boldsymbol{c}_1\boldsymbol{X}_1(s,z) + \boldsymbol{c}_2\boldsymbol{X}_2(s,z) \\
&= \left(\boldsymbol{c}_1(s\mathbf{I} - \boldsymbol{A}_{11})^{-1} \boldsymbol{A}_{12} + \boldsymbol{c}_2\right) \boldsymbol{X}_2(s,z) \\
&= \left(\boldsymbol{c}_1(s\mathbf{I} - \boldsymbol{A}_{11})^{-1}\boldsymbol{A}_{12} + \boldsymbol{c}_2\right) \left(\mathbf{I} - z^{-1}\boldsymbol{\Gamma}(s)\right)^{-1} \frac{\boldsymbol{f}_2}{s}.
\end{aligned} \tag{3.77}$$

Thus, with (3.77), (3.73) yields

$$\begin{aligned}
\lim_{s\to 0} sY(s,z) &= \lim_{s\to 0} \left(\boldsymbol{c}_1(s\mathbf{I} - \boldsymbol{A}_{11})^{-1}\boldsymbol{A}_{12} + \boldsymbol{c}_2\right) \left(\mathbf{I} - z^{-1}\boldsymbol{\Gamma}(s)\right)^{-1} \boldsymbol{f}_2 \\
&= \left(-\boldsymbol{c}_1\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} + \boldsymbol{c}_2\right) \left(\mathbf{I} - z^{-1}\boldsymbol{\Gamma}(0)\right)^{-1} \boldsymbol{f}_2.
\end{aligned} \tag{3.78}$$

Since it is required in (3.73) that the right hand side of (3.78) is equal to $1/(1-z^{-1})$, the system description has to be chosen such that $\det(\mathbf{I}-z^{-1}\mathbf{\Gamma}(0))$ contains at least the factor $1-z^{-1}$. This, however, also produces a singularity at $s=0$ and $z=1$.

3.5.2 System Description

Although numerous different approaches are possible, we will focus on this version. Here $\hat{e}(t,k)$ depends not only on $\hat{x}(t,k)$, $\hat{x}(t,k-1)$, and $\hat{v}(t,k)$ but also on $\hat{x}(t,k-2)$:

$$\hat{e}(t,k) = (1-\alpha)\left(\hat{x}(t,k-1)-\hat{x}(t,k)\right)+\alpha\left(\hat{x}(t,k-2)-\hat{x}(t,k-1)\right)-h\hat{v}(t,k) \tag{3.79}$$

with $\alpha \in [0,1)$. The first vehicle in the platoon follows a given trajectory $\hat{x}(t,0)$. However, $\hat{x}(t,-1)$ does not exist. To ensure that the separation between the cars in steady state is equal, we will use the steady state separation $h\hat{v}(t,0)$ with $\hat{v}(t,0)=\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t,0)$ instead of $(\hat{x}(t,k-2)-\hat{x}(t,k-1))$ in the equation for $\hat{e}(t,1)$:

$$\hat{e}(t,1) = (1-\alpha)\left(\hat{x}(t,0)-\hat{x}(t,1)\right)+\alpha h\hat{v}(t,0)-h\hat{v}(t,1). \tag{3.80}$$

Thus, the separation between the cars in steady state is equal to the version with communication range 1 for the same time headway.

The system is set in a way that it is equivalent to the unidirectional case when the parameter α is 0. Using the equations for $\hat{X}_{\mathcal{L}}(s,k)$, (3.49), and $\hat{U}_{\mathcal{L}}(s,k)=C_{\mathrm{h},\alpha}(s)\hat{E}_{\mathcal{L}}(s,k)$ with $C_{\mathrm{h},\alpha}(s)=C(s)/(Q(s)-\alpha)$, and applying the Laplace-Z transform the disturbance to error transfer function $H_{\hat{e},\hat{d}}(s,z)$ is

$$H_{\hat{e},\hat{d}}(s,z) = \frac{(1-2\alpha)z^{-1}+\alpha z^{-2}-(Q(s)-\alpha)}{1-(1-2\alpha)\Gamma_\alpha(s)z^{-1}-\alpha\Gamma_\alpha(s)z^{-2}}\Gamma_\alpha(s)C_{\mathrm{h},\alpha}^{-1}(s) \tag{3.81}$$

with $\Gamma_\alpha(s)=T(s)/(Q(s)-\alpha)$.

3.5.3 Conditions for String Stability

Given the fact that $\Gamma_\alpha(s)$ and $C_{\mathrm{h},\alpha}^{-1}(s)$ do not have poles with real parts greater or equal to zero, string stability of the system will depend on the zeros of the denominator of (3.81).

The zeros of $p(s,z)=z^2-(1-2\alpha)\Gamma_\alpha(s)z-\alpha\Gamma_\alpha(s)$ for $\alpha<1/2$ are

$$z_{1,2}(s) = \frac{1-2\alpha}{2}\Gamma_\alpha(s) \pm \sqrt{\frac{(1-2\alpha)^2}{4}\Gamma_\alpha^2(s)+\alpha\Gamma_\alpha(s)}. \tag{3.82}$$

To guarantee string stability both poles have to lie within the closed unit circle around the origin in the z-plane for $s=j\omega$ for all ω. That means that we have to guarantee $|z_{1,2}(j\omega)|<1$ for all ω. We will now write $\Gamma_\alpha(j\omega)$ as $T(j\omega)/(Q(j\omega)-\alpha)$ where

$T(j\omega) = r(\omega)\exp(j\phi(\omega))$. The magnitude of pole $z_1(j\omega)$ can be bounded by

$$\begin{aligned} |z_1(j\omega)| \leq & \frac{1-2\alpha}{2}\frac{r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}} \\ & + \left|\sqrt{\frac{(1-2\alpha)^2}{4}\frac{r^2(\omega)\mathrm{e}^{2j\left(\phi-\arctan\frac{h\omega}{1-\alpha}\right)}}{h^2\omega^2+(1-\alpha)^2}+\frac{\alpha r(\omega)\mathrm{e}^{j\left(\phi-\arctan\frac{h\omega}{1-\alpha}\right)}}{\sqrt{h^2\omega^2+(1-\alpha)^2}}}\right|. \end{aligned} \tag{3.83}$$

Using the fact that $\left|\mathrm{e}^{j\phi}\right| = 1$ for all ϕ, the bound yields

$$\begin{aligned} |z_1(j\omega)| \leq & \frac{1-2\alpha}{2}\frac{r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}} \\ & + \sqrt{\frac{(1-2\alpha)^2}{4}\frac{r^2(\omega)}{h^2\omega^2+(1-\alpha)^2}+\frac{\alpha r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}}. \end{aligned} \tag{3.84}$$

Since it is required that $|z_{1,2}(j\omega)| < 1$ for all ω we can derive from (3.84)

$$\frac{(1-2\alpha)^2}{4}\frac{r^2(\omega)}{h^2\omega^2+(1-\alpha)^2}+\frac{\alpha r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}} < \left(1-\frac{1-2\alpha}{2}\frac{r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}\right)^2 \tag{3.85}$$

and thus

$$\frac{\alpha r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}} < 1 - \frac{(1-2\alpha)\,r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}. \tag{3.86}$$

Simplifying even further yields for the infimal headway for a communication range of two

$$h_{0,2} = (1-\alpha)\sup_{\omega}\frac{\sqrt{r^2(\omega)-1}}{\omega} = (1-\alpha)h_0. \tag{3.87}$$

Note that for $\alpha = 1/2$ equation (3.87) becomes $h_{0,2} = 1/2 \cdot h_0$. However, for $\alpha > 1/2$ the infimal time headway becomes $h_{0,2} = \sup_\omega \sqrt{(3\alpha-1)^2 r^2(\omega) - (1-\alpha)^2)}/\omega$. For $\omega \to 0$ the absolute value of $|T(j\omega)| = r(\omega)$ will approach 1 and square root will go to a constant value $c(\alpha) \neq 0$. Therefore, $h_{0,2} \to \infty$ and the string is not string stable.

As in the unidirectional case we need to examine the limit for $(\omega,\theta) \to (0,0)$ closely since both the numerator and the denominator tend to zero at the same time and the system has a NSSK on the stability boundary. We will show that $\lim_{(\omega,\theta)\to(0,0)}\left|H_{\hat{e},\hat{d}}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$ is bounded:

$$\begin{aligned} \left|H_{\hat{e},\hat{d}}(s,z)\right| &= \left|\frac{(1-2\alpha)z^{-1}+\alpha z^{-2}-(Q(s)-\alpha)}{1-(1-2\alpha)\Gamma_\alpha(s)z^{-1}-\alpha\Gamma_\alpha(s)z^{-2}}\Gamma_\alpha(s)C_{\mathrm{h},\alpha}^{-1}(s)\right| \\ &= \left|\frac{1-(Q(s)-\alpha)\Gamma_\alpha(s)}{1-((1-2\alpha)z^{-1}+\alpha z^{-2})\Gamma_\alpha(s)}-1\right|\left|C_{\mathrm{h},\alpha}^{-1}(s)\right| \\ &\leq \left(\left|\frac{1-(Q(s)-\alpha)\Gamma_\alpha(s)}{1-((1-2\alpha)z^{-1}+\alpha z^{-2})\Gamma_\alpha(s)}\right|+1\right)\left|C_{\mathrm{h},\alpha}^{-1}(s)\right|. \end{aligned} \tag{3.88}$$

Since for $(\omega, \theta) \to (0,0)$ it is true that

$$\left|1 - ((1-2\alpha)z^{-1} + \alpha z^{-2})\Gamma_\alpha(s)\right| \geq 1 - \left|(1-2\alpha)z^{-1} + \alpha z^{-2}\right| |\Gamma_\alpha(s)| \tag{3.89}$$

and

$$\left|(1-2\alpha)z^{-1} + \alpha z^{-2}\right| \leq (1-2\alpha)\left|z^{-1}\right| + \alpha\left|z^{-2}\right| \leq 1-\alpha \tag{3.90}$$

we can rewrite (3.88) and get

$$\limsup_{(\omega,\theta)\to(0,0)} \left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right| \leq \limsup_{\omega\to 0}\left(\frac{|1-(Q(j\omega)-\alpha)\Gamma_\alpha(j\omega)|}{1-(1-\alpha)\,|\Gamma_\alpha(j\omega)|} + 1\right)\left|C_{\mathrm{h},\alpha}^{-1}(j\omega)\right|. \tag{3.91}$$

Using the fact that $\Gamma_\alpha(s) = \frac{T(s)}{Q(s)-\alpha}$, $T(s) = \frac{\tilde{L}(s)}{s^2+\tilde{L}(s)}$ and the following approximation for $\tilde{L}(j\omega)$ for small ω, $\tilde{L}(j\omega) = a_0 + a_2\omega^2 + a_4\omega^4 + \ldots (a_1\omega + a_3\omega^3 + \ldots)j$, we can simplify the first term of the right hand side of (3.91), use l'Hôpital's Rule and get

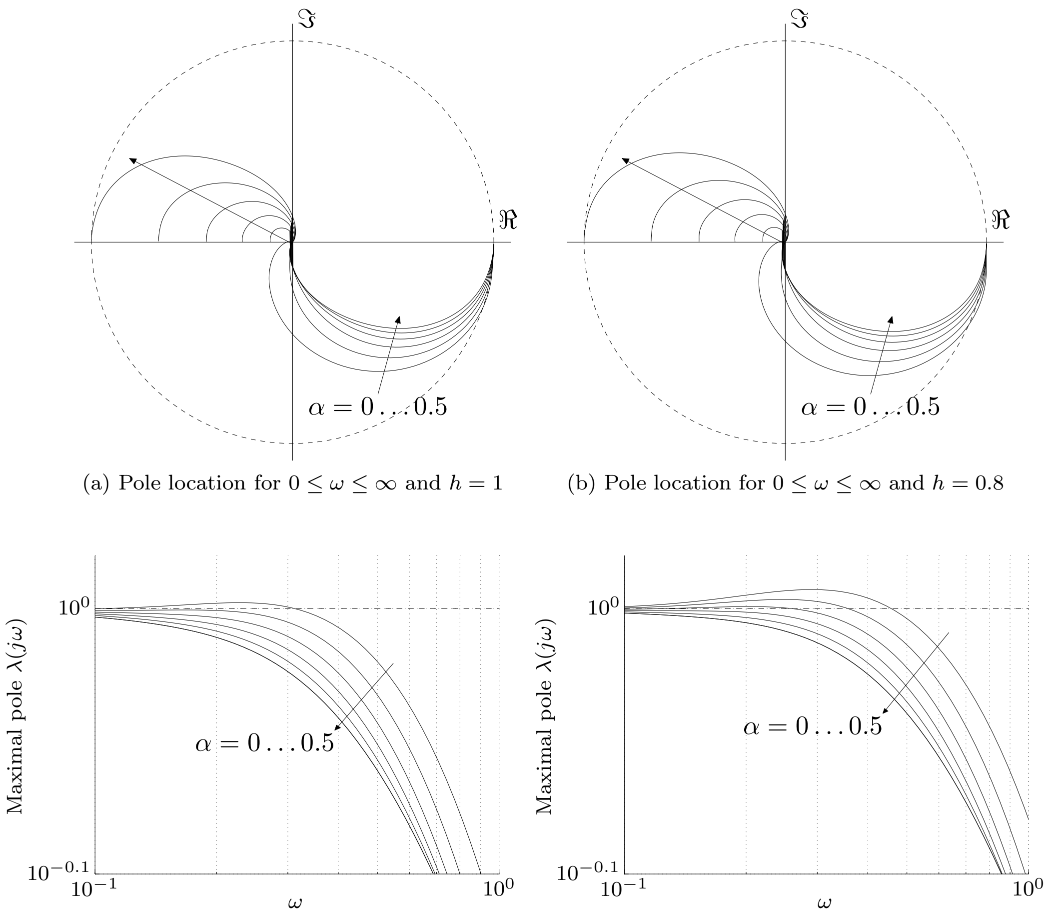
$$\begin{aligned}
&\limsup_{\omega\to 0}\frac{|1-(Q(j\omega)-\alpha)\Gamma_\alpha(j\omega)|}{1-(1-\alpha)\,|\Gamma_\alpha(j\omega)|} \\
&\quad= \limsup_{\omega\to 0}\frac{\frac{\omega^2}{\left|-\omega^2+\tilde{L}(j\omega)\right|}}{1-(1-\alpha)\frac{1}{\sqrt{h^2\omega^2+(1-\alpha)^2}}\frac{\left|\tilde{L}(j\omega)\right|}{\left|-\omega^2+\tilde{L}(j\omega)\right|}} \\
&\quad= \limsup_{\omega\to 0}\frac{\sqrt{h^2\omega^2+(1-\alpha)^2}\cdot\omega^2}{\sqrt{h^2\omega^2+(1-\alpha)^2}\cdot\left|-\omega^2+\tilde{L}(j\omega)\right| - (1-\alpha)\left|\tilde{L}(j\omega)\right|} \\
&\quad= \frac{1}{\frac{h^2 a_0}{2(1-\alpha)^2}-1}.
\end{aligned} \tag{3.92}$$

For any $h > h_{0,2} = (1-\alpha)h_0 = (1-\alpha)\sqrt{2/\left|\tilde{L}(0)\right|}$, (3.92) is bounded. Note that when choosing $\alpha = 1/2$ the system admits a second NSSK at $s=0$ and $z=-1$. The same argument as above can be followed to assure $\left|H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is bounded for $\omega = 0$ and $\theta = \pi$. Therefore, with $h > h_{0,2}$, $H_{\hat{e},\hat{d}}\left(j\omega, \mathrm{e}^{j\theta}\right)$ is bounded for all ω and θ.

3.5.4 Example and Simulations

To illustrate our results we simulate a platoon of forty vehicles with the same transfer functions $P(s)$ and $C(s)$ as above. As we have seen, to guarantee string stability for $\alpha = 0$ (unidirectional case with communication range 1) the time headway has to be greater than $h_0 \approx 1.18$.

The poles in the z-plane for different values of α and h are displayed in Figure 3.7. For $h = 1.5$ according to (3.87) both poles are within the closed unit circle for $\alpha \in [0, 0.5]$. That means that even with communication range 1 the system is string stable. However, using a communication range of 2 with a very small $\alpha = 0.1$ the performance of the string



(a) Pole location for $0 \leq \omega \leq \infty$ and $h = 1$ (b) Pole location for $0 \leq \omega \leq \infty$ and $h = 0.8$

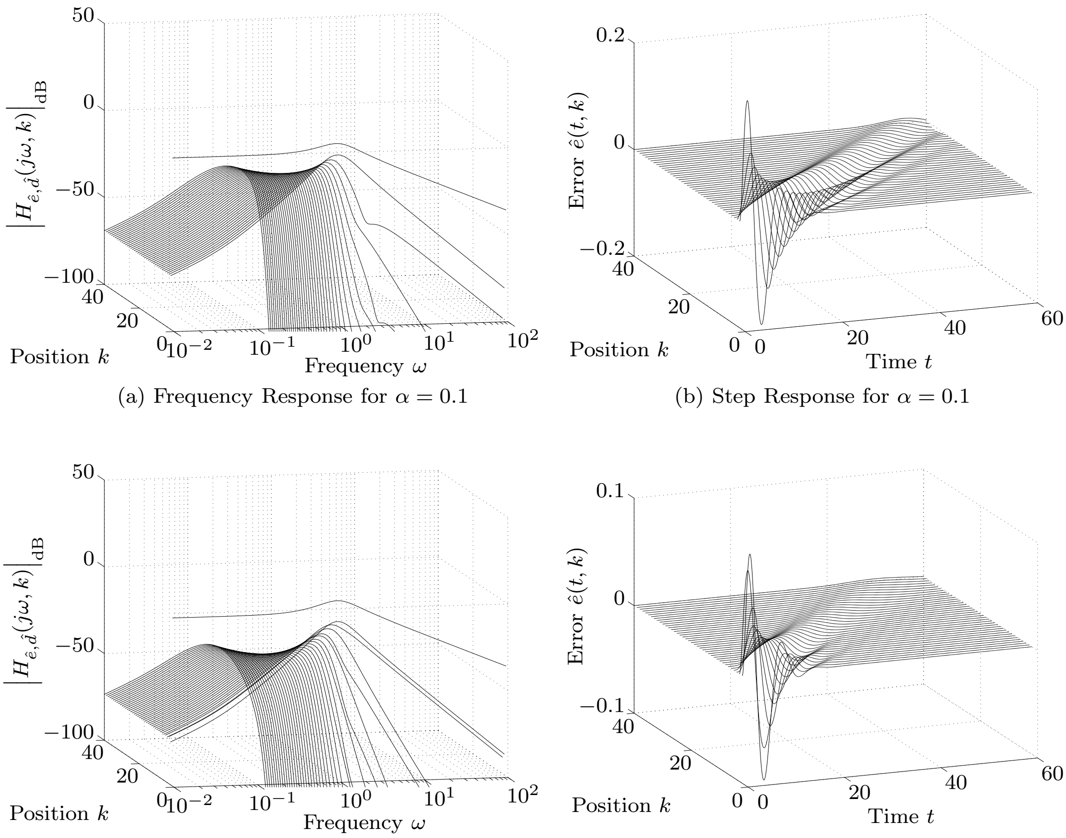
(c) Magnitude of maximal pole for $h = 1$ (d) Magnitude of maximal pole for $h = 0.8$

Figure 3.7: *Location of poles of $H_{\hat{e},\hat{d}}(s, z)$ for communication range 2*

can be improved significantly. Simulations for $\alpha = 0.1$ and $\alpha = 0.3$ are shown in Figure 3.8. To compare the results with the unidirectional case with communication range 1 please see Figure 3.6 on page 40.

For $h = 1$ according to (3.87) the minimal α to guarantee string stability is approximately 0.15. So choosing $\alpha = 0.1$ will lead to a system which is not string stable, since the maximal magnitude of one pole will be greater than 1. However, with $h = 1$ increasing α to 0.3 will guarantee string stability. Simulations for $\alpha = 0.1$ and $\alpha = 0.3$ are shown in Figure 3.9.

If $h = 0.8$ the minimal α to guarantee string stability according to (3.87) is approximately 0.32. Thus choosing $h = 0.8$ the system is string unstable for $\alpha = 0.3$ and string stable for $\alpha = 0.4$, see Figure 3.7. Both simulations are shown in Figure 3.10.



(a) Frequency Response for $\alpha = 0.1$ (b) Step Response for $\alpha = 0.1$

(c) Frequency Response for $\alpha = 0.3$ (d) Step Response for $\alpha = 0.3$

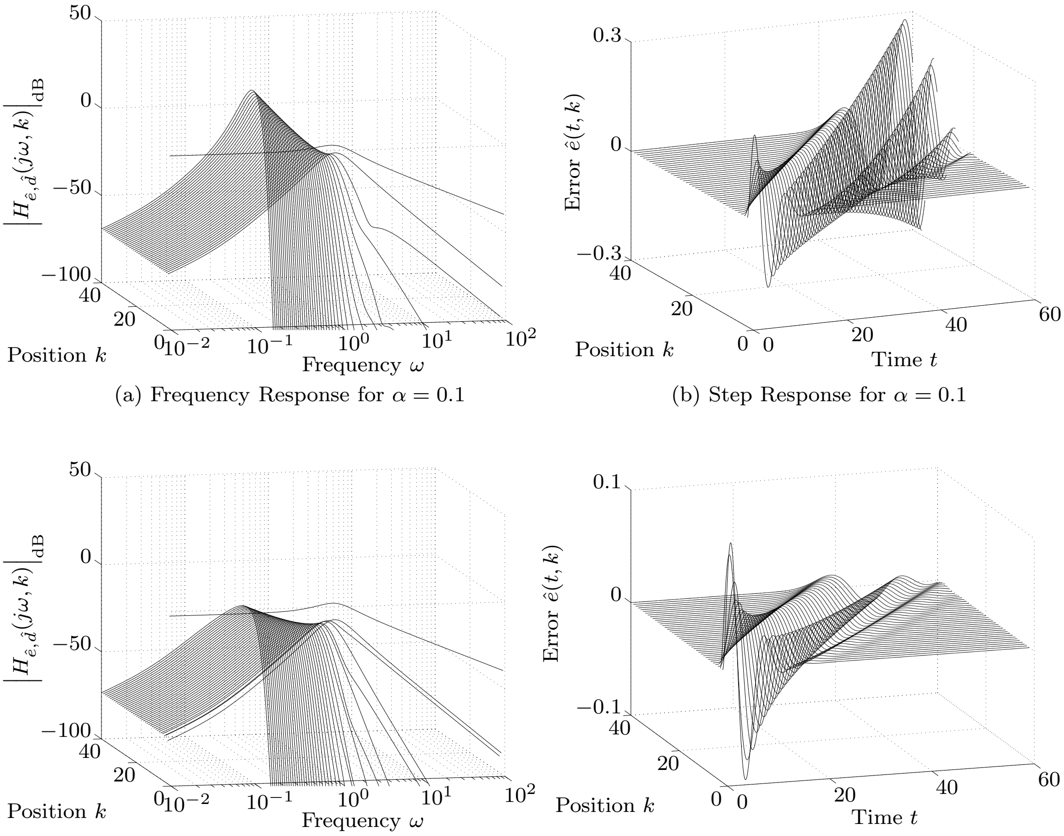
Figure 3.8: *Unidirectional string with communication range 2, $h = 1.5$, $\alpha = 0.1$ and 0.3*

3.6 Conclusion

Throughout this chapter we studied the BIBO stability of linear continuous-discrete two-dimensional systems. The systems were analysed in the frequency domain after applying the combined Laplace-Z transform and the corresponding L_2 induced operator norm was derived and enabled the string stability analysis of a linear homogeneous string of vehicles with unidirectional control.

This approach allows the study of such string stability problems in a generalised fashion for all unidirectional strings regardless of their communication range. The communication range only determines the degree of the complex variable z in the transfer function $H_{\hat{e},\hat{d}}(s,z)$ but does not require a change in the methods used to analyse the stability of the system.

However, every such system derived from a vehicle string model features an unavoidable structural nonessential singularity of the second kind on the stability boundary and special care is required to guarantee that the induced operator norm remains bounded in this



(a) Frequency Response for $\alpha = 0.1$

(b) Step Response for $\alpha = 0.1$

(c) Frequency Response for $\alpha = 0.3$

(d) Step Response for $\alpha = 0.3$

Figure 3.9: *Unidirectional string with communication range 2, $h = 1$, $\alpha = 0.1$ and 0.3*

region. Hence, the advantages of a generalised stability analysis for unidirectional vehicle strings are partly relativised due to this required additional step in the analysis.

Another disadvantage of this frequency domain approach is the fact that the poles $z(s)$ of $H_{\hat{e},\hat{d}}(s,z)$ are functions of the complex variable s. Thus inspecting the poles requires an infinite number of numerical calculations or a more advanced criterion to test the location of the poles.

Therefore, it will be the aim of the following chapter to derive different stability conditions for linear two-dimensional systems including singularities on the stability boundary.

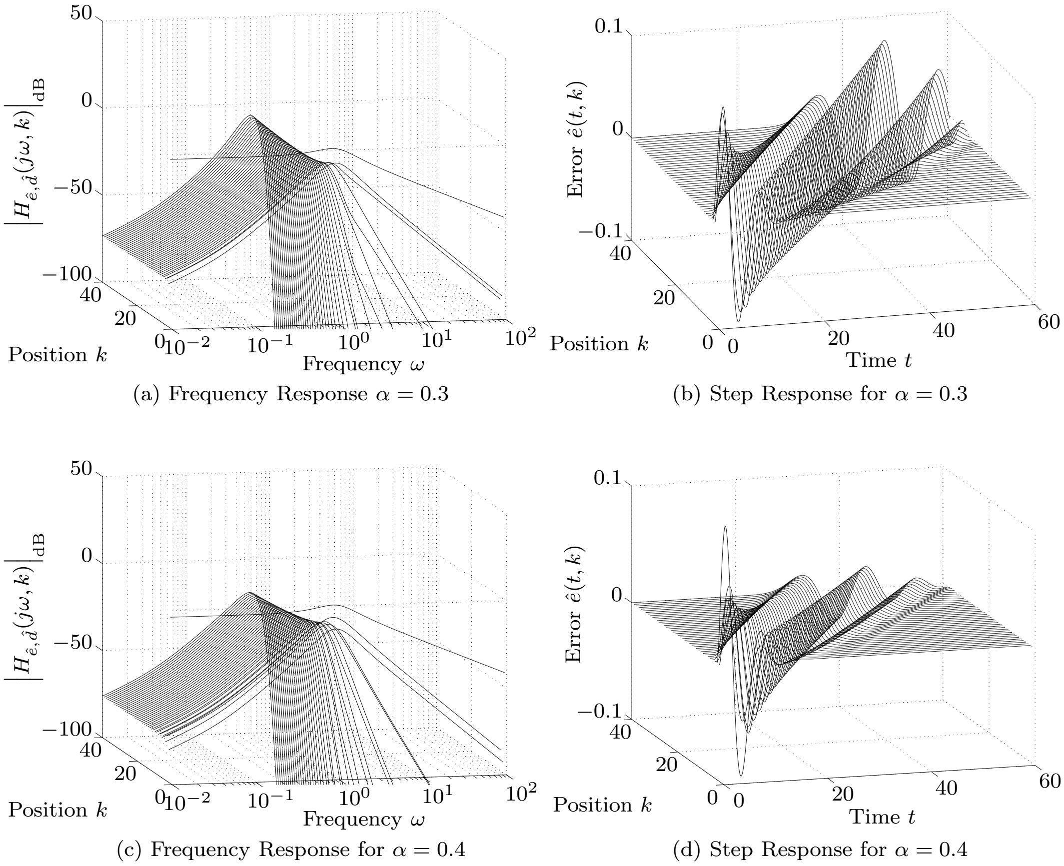


Figure 3.10: *Unidirectional string with communication range 2, $h = 0.8$, $\alpha = 0.3$ and 0.4*

3.A Chapter Appendix

3.A.1 Parameter Choice for the Disturbance Signal

To ensure $\left\|\hat{d}_\epsilon(\cdot,\cdot)\right\|_2 = 1$ we have to choose α_{ω_0} and α_{θ_0} appropriately. Note that

$$\begin{aligned}\left\|\hat{d}_\epsilon(\cdot,\cdot)\right\|_2^2 &= \sum_{k=0}^{\infty}\int_0^{\infty}\left(\alpha_{\omega_0}\mathrm{e}^{-\epsilon t}\cos\omega_0 t\cdot\alpha_{\theta_0}\mathrm{e}^{-\epsilon k}\cos\theta_0 k\right)^2\mathrm{d}t\\ &= \sum_{k=0}^{\infty}\alpha_{\theta_0}^2\mathrm{e}^{-2\epsilon k}\cos^2\theta_0 k\cdot\int_0^{\infty}\alpha_{\omega_0}^2\mathrm{e}^{-2\epsilon t}\cos^2\omega_0 t\mathrm{d}t.\end{aligned} \tag{3.93}$$

We will start by examining the part of $\hat{d}_\epsilon(t,k)$ depending on t. Assume first that $0 < |\omega_0| < \infty$. Solving the integral on the right hand side of (3.93) yields

$$\begin{aligned}\int_0^\infty \alpha_{\omega_0}^2 \mathrm{e}^{-2\epsilon t}\cos^2\omega_0 t \mathrm{d}t\\ &= \frac{\alpha_{\omega_0}^2}{4\epsilon^2+4\omega_0^2}\left[\mathrm{e}^{-2\epsilon t}\cos\omega_0 t\left(-2\epsilon\cos\omega_0 t + 2\omega_0\sin\omega_0 t\right) - \frac{\omega_0^2}{\epsilon}\mathrm{e}^{-2\epsilon t}\right]_{t=0}^\infty\\ &= \frac{\alpha_{\omega_0}^2}{4\epsilon^2+4\omega_0^2}\left(2\epsilon + \frac{\omega_0^2}{\epsilon}\right). \end{aligned} \tag{3.94}$$

Thus, $\alpha_{\omega_0}^2$ has to be

$$\alpha_{\omega_0}^2 = \frac{4\epsilon^2+4\omega_0^2}{2\epsilon+\omega_0^2/\epsilon}. \tag{3.95}$$

In case $\omega_0 = 0$ solving the integral on the right hand side of (3.93) yields

$$\int_0^\infty \alpha_{\omega_0}^2 \mathrm{e}^{-2\epsilon t}\mathrm{d}t = \frac{\alpha_{\omega_0}^2}{2\epsilon}\left[\mathrm{e}^{-2\epsilon t}\right]_{t=0}^\infty = \frac{\alpha_{\omega_0}^2}{2\epsilon}. \tag{3.96}$$

Thus, set $\alpha_{\omega_0}^2 = 2\epsilon$. Solving the summation on the right hand side of (3.93) for $0 < \theta_0 < 2\pi$ yields

$$\begin{aligned}\sum_{k=0}^\infty \alpha_{\theta_0}^2 \mathrm{e}^{-2\epsilon k}\cos^2\theta_0 k &= \sum_{k=0}^\infty \frac{\alpha_{\theta_0}^2 \mathrm{e}^{-2\epsilon k}}{4}\left(\mathrm{e}^{j2\theta_0 k} + 2 + \mathrm{e}^{-j2\theta_0 k}\right)\\ &= \frac{\alpha_{\theta_0}^2}{2}\left(\frac{1}{1-\mathrm{e}^{-2\epsilon}} + \frac{1-\mathrm{e}^{-2\epsilon}\cos 2\theta_0}{1-2\mathrm{e}^{-2\epsilon}\cos 2\theta_0 + \mathrm{e}^{-4\epsilon}}\right). \end{aligned} \tag{3.97}$$

So α_{θ_0} has to be chosen appropriately

$$\alpha_{\theta_0}^2 = \frac{2}{\frac{1}{1-\mathrm{e}^{-2\epsilon}} + \frac{1-\mathrm{e}^{-2\epsilon}\cos 2\theta_0}{1-2\mathrm{e}^{-2\epsilon}\cos 2\theta_0+\mathrm{e}^{-4\epsilon}}}. \tag{3.98}$$

In case $\theta_0 = 0$ observe that the summation on the right hand side of (3.93) simplifies to

$$\sum_{k=0}^\infty \alpha_{\theta_0}^2 \mathrm{e}^{-2\epsilon k} = \frac{\alpha_{\theta_0}^2}{1-\mathrm{e}^{-2\epsilon}}. \tag{3.99}$$

Hence, choose $\alpha_{\theta_0}^2 = 1-\mathrm{e}^{-2\epsilon}$ to guarantee that the L_2 norm of $\hat{d}_\epsilon(t,k)$ is 1.

3.A.2 The Limit of the Disturbance Signal Norm

Note that

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\iint\limits_{\mathcal{R}_\epsilon}\left|\hat{D}_\epsilon(j\omega,\mathrm{e}^{j\theta})\right|^2\mathrm{d}\omega\mathrm{d}\theta\\
&\quad=\lim_{\epsilon\to 0}\iint\limits_{\mathcal{R}_\epsilon}\left|\alpha_{\omega_0}\frac{j\omega+\epsilon}{(j\omega+\epsilon)^2+\omega_0^2}\cdot\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2\mathrm{d}\omega\mathrm{d}\theta\\
&\quad=\lim_{\epsilon\to 0}\int\limits_{\mathcal{R}_\epsilon(\omega)}\left|\alpha_{\omega_0}\frac{j\omega+\epsilon}{(j\omega+\epsilon)^2+\omega_0^2}\right|^2\mathrm{d}\omega\\
&\qquad\cdot\lim_{\epsilon\to 0}\int\limits_{\mathcal{R}_\epsilon(\theta)}\left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2\mathrm{d}\theta
\end{aligned}
\tag{3.100}
$$

where $\mathcal{R}_\epsilon(\omega)=\{\omega:|\omega\pm\omega_0|\leq\sqrt{\epsilon}\}$ and $\mathcal{R}_\epsilon(\theta)=\{\theta:|\theta\pm\theta_0|\leq\sqrt{\epsilon}\}$. To show that $\lim_{\epsilon\to 0}\int_{\mathcal{R}_\epsilon}|\hat{D}_\epsilon(j\omega,\mathrm{e}^{j\theta})|^2\mathrm{d}\omega\mathrm{d}\theta=(2\pi)^2$ we will start to evaluate the part of $\hat{D}_\epsilon$ depending on ω, i. e. the first limit on the right hand side of (3.100) and assume first that $0<|\omega_0|<\infty$. (Note that ln is the natural logarithm.) Thus

$$
\begin{aligned}
&\int\limits_{\mathcal{R}_\epsilon(\omega)}\left|\alpha_{\omega_0}\frac{j\omega+\epsilon}{(j\omega+\epsilon)^2+\omega_0^2}\right|^2\mathrm{d}\omega\\
&\quad=\int\limits_{\mathcal{R}_\epsilon(\omega)}\alpha_{\omega_0}^2\frac{\omega^2+\epsilon^2}{\omega^4-2(\omega_0^2-\epsilon^2)\omega^2+(\omega_0^2+\epsilon^2)^2}\mathrm{d}\omega\\
&\quad=\alpha_{\omega_0}^2\frac{1}{8}\frac{1}{(\epsilon^2+\omega_0^2)\epsilon}\Bigg[-\omega_0\epsilon\ln\left(\omega_0^2+\epsilon^2+2\omega_0\omega+\omega^2\right)\\
&\qquad+\left(4\epsilon^2+2\omega_0^2\right)\left(\arctan\left(\frac{\omega+\omega_0}{\epsilon}\right)+\arctan\left(\frac{\omega-\omega_0}{\epsilon}\right)\right)\\
&\qquad+\omega_0\epsilon\ln\left(\omega_0^2+\epsilon^2-2\omega_0\omega+\omega^2\right)\Bigg]_{\partial\mathcal{R}_\epsilon(\omega)}.
\end{aligned}
\tag{3.101}
$$

The next step is to evaluate the antiderivative for both peaks (around ω_0 and $-\omega_0$) and the limit for $\epsilon \to 0$:

$$
\begin{aligned}
&\lim_{\epsilon\to 0} \int\limits_{\mathcal{R}_\epsilon(\omega)} \left|\alpha_{\omega_0} \frac{j\omega+\epsilon}{(j\omega+\epsilon)^2+\omega_0^2}\right|^2 \mathrm{d}\omega \\
=&\lim_{\epsilon\to 0}\Bigg(\alpha_{\omega_0}^2 \frac{1}{8}\frac{1}{(\epsilon^2+\omega_0^2)\epsilon}\Bigg[-\omega_0\epsilon \ln\left(4\omega_0^2+\epsilon^2+4\omega_0\sqrt{\epsilon}+\epsilon\right) \\
&+\omega_0\epsilon\ln\left(\epsilon^2+\epsilon\right)+\left(4\epsilon^2+2\omega_0^2\right)\left(\arctan\left(\frac{2\omega_0+\sqrt{\epsilon}}{\epsilon}\right)+\arctan\left(\frac{1}{\sqrt{\epsilon}}\right)\right) \\
&+\omega_0\epsilon\ln\left(4\omega_0^2+\epsilon^2-4\omega_0\sqrt{\epsilon}+\epsilon\right)-\omega_0\epsilon\ln\left(\epsilon^2+\epsilon\right) \\
&-\left(4\epsilon^2+2\omega_0^2\right)\left(\arctan\left(\frac{2\omega_0-\sqrt{\epsilon}}{\epsilon}\right)+\arctan\left(\frac{1}{-\sqrt{\epsilon}}\right)\right)\Bigg]\mathrm{d}\theta \\
&+\lim_{\epsilon\to 0}\int\limits_{\theta\in\mathcal{R}_\epsilon} \alpha_{\omega_0}^2\frac{1}{8}\frac{1}{(\epsilon^2+\omega_0^2)\epsilon}\Bigg[\omega_0\epsilon\ln\left(4\omega_0^2+\epsilon^2-4\omega_0\sqrt{\epsilon}+\epsilon\right) \\
&-\omega_0\epsilon\ln\left(\epsilon^2+\epsilon\right)+\left(4\epsilon^2+2\omega_0^2\right)\left(\arctan\left(\frac{1}{\sqrt{\epsilon}}\right)+\arctan\left(\frac{-2\omega_0+\sqrt{\epsilon}}{\epsilon}\right)\right) \\
&+\omega_0\epsilon\ln\left(\epsilon^2+\epsilon\right)-\omega_0\epsilon\ln\left(4\omega_0^2+\epsilon^2+4\omega_0\sqrt{\epsilon}+\epsilon\right) \\
&-\left(4\epsilon^2+2\omega_0^2\right)\left(\arctan\left(\frac{1}{-\sqrt{\epsilon}}\right)+\arctan\left(\frac{-2\omega_0-\sqrt{\epsilon}}{\epsilon}\right)\right)\Bigg]\Bigg).
\end{aligned}
\tag{3.102}
$$

Note that with (3.95) we get $\frac{\alpha_{\omega_0}^2}{8(\epsilon^2+\omega_0^2)\epsilon} = \frac{1}{2(\epsilon^2+\omega_0^2)}$, and equation (3.102) becomes

$$
\begin{aligned}
&\lim_{\epsilon\to 0} \int\limits_{\mathcal{R}_\epsilon(\omega)} \left|\alpha_{\omega_0} \frac{j\omega+\epsilon}{(j\omega+\epsilon)^2+\omega_0^2}\right|^2 \mathrm{d}\omega \\
=&\frac{1}{2\omega_0^2}\left[-0+2\omega_0^2\left(\frac{\pi}{2}+\frac{\pi}{2}\right)+0-2\omega_0^2\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\right] \\
&+\frac{1}{2\omega_0^2}\left[+0+2\omega_0^2\left(\frac{\pi}{2}-\frac{\pi}{2}\right)-0-2\omega_0^2\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)\right] \\
=&2\pi.
\end{aligned}
\tag{3.103}
$$

In case $\omega_0 = 0$ the part of $\hat{d}_\epsilon(t,k)$ depending on t simplifies to $\alpha_{\omega_0}\mathrm{e}^{-\epsilon t}$. Its Laplace transform is $\alpha_{\omega_0}/(s+\epsilon)$. Setting $s = j\omega$ and examining the limit of the integral of

$|\alpha_{\omega_0}/(j\omega+\epsilon)|^2$ over $[-\sqrt{\epsilon},\sqrt{\epsilon}]$ yields

$$\begin{aligned}\lim_{\epsilon\to 0}\int\limits_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\left|\frac{\alpha_{\omega_0}}{j\omega+\epsilon}\right|^2\mathrm{d}\omega =&\lim_{\epsilon\to 0}\int\limits_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\frac{\alpha_{\omega_0}^2}{\omega^2+\epsilon^2}\mathrm{d}\omega\\ =&\lim_{\epsilon\to 0}\frac{\alpha_{\omega_0}^2}{\epsilon}\left[\arctan\frac{\omega}{\epsilon}\right]_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}}\\ =&\lim_{\epsilon\to 0}\frac{2\epsilon}{\epsilon}\left(\arctan\frac{1}{\sqrt{\epsilon}}-\arctan\frac{1}{-\sqrt{\epsilon}}\right)\\ =&2\pi.\end{aligned}\tag{3.104}$$

In case $|\omega_0|\to\infty$ choose the time dependent part of $\hat{d}_\epsilon(t,k)$ equal $\hat{d}_N(t)=\sqrt{2N}\mathrm{e}^{-Nt}$. Applying the Laplace transform and examining the limit of the integral of $\left|\hat{D}_N(\omega)\right|^2$ over the interval $\mathcal{R}_N=\left[-N^2,-\sqrt{N}\right]\cup\left[\sqrt{N},N^2\right]$ yields

$$\begin{aligned}&\lim_{N\to\infty}\int\limits_{\mathcal{R}_N}\left|\mathcal{L}\left\{\sqrt{2N}\mathrm{e}^{-Nt}\right\}\right|^2\mathrm{d}\omega\\ &\quad=\lim_{N\to\infty}\int\limits_{\mathcal{R}_N}\frac{2N}{\omega^2+N^2}\mathrm{d}\omega\\ &\quad=\lim_{N\to\infty}\frac{2N}{N}\left[\arctan\frac{\omega}{N}\right]_{\mathcal{R}_N}\\ &\quad=\lim_{N\to\infty}2\left(\arctan\frac{1}{-\sqrt{N}}-\arctan(-N)+\arctan(N)-\arctan\frac{1}{\sqrt{N}}\right)\\ &\quad=2\pi.\end{aligned}\tag{3.105}$$

Also, the limit of the interval over the second term on the right hand side of (3.100) has to be equal 2π. First, let us assume $0<\theta_0<2\pi$ and write $\left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2$ as

$$\begin{aligned}&\left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2\\ &=\alpha_{\theta_0}^2\frac{1-\frac{\mathrm{e}^{-\epsilon}}{2}\left(\mathrm{e}^{j\theta_0}-\mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{j\theta}-\mathrm{e}^{-j\theta}\right)+\mathrm{e}^{-2\epsilon}\cos^2\theta_0}{\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta}\right)\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta}\right)\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta}\right)\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta}\right)}\\ &=\alpha_{\theta_0}^2\left(M+\frac{A}{1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta}}+\frac{B}{1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta}}\right.\\ &\quad\left.+\frac{C}{1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta}}+\frac{D}{1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta}}\right)\end{aligned}\tag{3.106}$$

with

$$A = D = \frac{1 - \frac{\mathrm{e}^{-\epsilon}}{2}\left(\mathrm{e}^{j\theta_0} - \mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{-\epsilon+j\theta_0} - \mathrm{e}^{\epsilon-j\theta_0}\right) + \mathrm{e}^{2\epsilon}\cos^2\theta_0}{\left(1-\mathrm{e}^{-2\epsilon}\right)\left(1-\mathrm{e}^{-j2\theta_0}\right)\left(1-\mathrm{e}^{-2\epsilon+j2\theta_0}\right)}, \tag{3.107}$$

$$B = C = \frac{1 - \frac{\mathrm{e}^{-\epsilon}}{2}\left(\mathrm{e}^{j\theta_0} - \mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{\epsilon+j\theta_0} - \mathrm{e}^{-\epsilon-j\theta_0}\right) + \mathrm{e}^{2\epsilon}\cos^2\theta_0}{\left(1-\mathrm{e}^{-2\epsilon}\right)\left(1-\mathrm{e}^{j2\theta_0}\right)\left(1-\mathrm{e}^{-2\epsilon-j2\theta_0}\right)}, \tag{3.108}$$

$$M = \frac{\mathrm{e}^{-2\epsilon+4j\theta_0} - \mathrm{e}^{-2\epsilon+2j\theta_0}\cos^2\theta_0\left(1+\mathrm{e}^{-2\epsilon}\right) + \mathrm{e}^{-2\epsilon+2j\theta_0} - \mathrm{e}^{2j\theta_0} + \mathrm{e}^{-2\epsilon}}{\left(1-\mathrm{e}^{-2\epsilon}\right)\left(1-\mathrm{e}^{-2\epsilon+2j\theta_0}\right)\left(\mathrm{e}^{2j\theta_0}-\mathrm{e}^{-2\epsilon}\right)}. \tag{3.109}$$

The anti-derivative of (3.106) is

$$\begin{aligned}
&\int\limits_{\mathcal{R}_\epsilon(\theta)} \left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2 \mathrm{d}\theta \\
&\quad= \alpha_{\theta_0}^2\left[M\theta + A\left(\theta + \frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta}\right)\right)\right. \\
&\qquad + B\left(\theta + \frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta}\right)\right) + B\left(\theta - \frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta}\right)\right) \\
&\qquad \left.+ A\left(\theta - \frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta}\right)\right)\right]_{\partial\mathcal{R}_\epsilon(\theta)}.
\end{aligned} \tag{3.110}$$

Evaluating it at $\partial\mathcal{R}_\epsilon(\theta)$ yields

$$\begin{aligned}
&\int\limits_{\mathcal{R}_\epsilon(\theta)} \left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2\mathrm{d}\theta\\
&=\alpha_{\theta_0}^2\left[M\theta_0+M\sqrt{\epsilon}+A\left(\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta_0-j\sqrt{\epsilon}}\right)\right)\right.\\
&\quad+B\left(\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta_0-j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+A\left(\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad-M\theta_0+M\sqrt{\epsilon}+A\left(-\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(-\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(-\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta_0-j\sqrt{\epsilon}}\right)\right)\\
&\quad+A\left(-\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta_0-j\sqrt{\epsilon}}\right)\right)\\
&\quad-M\theta_0+M\sqrt{\epsilon}+A\left(-\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta_0-j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(-\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta_0-j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(-\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+A\left(-\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+M\theta_0+M\sqrt{\epsilon}+A\left(\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(\theta_0+\sqrt{\epsilon}-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{j\theta_0+j\sqrt{\epsilon}}\right)\right)\\
&\quad+B\left(\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon-j\theta_0}\mathrm{e}^{-j\theta_0-j\sqrt{\epsilon}}\right)\right)\\
&\quad\left.+\,A\left(\theta_0+\sqrt{\epsilon}+\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon+j\theta_0}\mathrm{e}^{-j\theta_0-j\sqrt{\epsilon}}\right)\right)\right].
\end{aligned}\tag{3.111}$$

Subsequently we will allow ϵ to go to 0. Note that A, B, and M contain the term $1-\mathrm{e}^{-2\epsilon}$ which is going to 0. However, it will cancel out with the same term in $\alpha_{\theta_0}^2$ so that the limits of $\alpha_{\theta_0}^2A$, $\alpha_{\theta_0}^2B$, and $\alpha_{\theta_0}^2M$ for $\epsilon\to 0$ exist and are bounded. For this reason the limit of the integral can be simplified significantly to

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int\limits_{\mathcal{R}_\epsilon(\theta)}\left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2\mathrm{d}\theta\\
&\quad=\lim_{\epsilon\to 0}2\alpha_{\theta_0}^2(A+B)\left(\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon}\mathrm{e}^{-j\sqrt{\epsilon}}\right)-\frac{1}{j}\ln\left(1-\mathrm{e}^{-\epsilon}\mathrm{e}^{j\sqrt{\epsilon}}\right)\right)\\
&\quad=\lim_{\epsilon\to 0}2\alpha_{\theta_0}^2(A+B)\frac{1}{j}\left(\ln\left(\sqrt{1-2\mathrm{e}^{-\epsilon}\cos\sqrt{\epsilon}+\mathrm{e}^{-2\epsilon}}\right)\right.\\
&\quad\quad-\ln\left(\sqrt{1-2\mathrm{e}^{-\epsilon}\cos\sqrt{\epsilon}+\mathrm{e}^{-2\epsilon}}\right)\\
&\quad\quad\left.+\,j\arctan\frac{\mathrm{e}^{-\epsilon}\sin\sqrt{\epsilon}}{1-\mathrm{e}^{-\epsilon}\cos\sqrt{\epsilon}}-j\arctan\frac{-\mathrm{e}^{-\epsilon}\sin\sqrt{\epsilon}}{1-\mathrm{e}^{-\epsilon}\cos\sqrt{\epsilon}}\right)\\
&\quad=\lim_{\epsilon\to 0}2\alpha_{\theta_0}^2(A+B)\left(\frac{\pi}{2}-\frac{-\pi}{2}\right).
\end{aligned}\tag{3.112}
$$

The limit of $\alpha_{\theta_0}^2 A$ is

$$
\begin{aligned}
\lim_{\epsilon\to 0}\alpha_{\theta_0}^2 A=&\lim_{\epsilon\to 0}\frac{1-\frac{\mathrm{e}^{-\epsilon}}{2}\left(\mathrm{e}^{j\theta_0}-\mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{-\epsilon+j\theta_0}-\mathrm{e}^{\epsilon-j\theta_0}\right)+\mathrm{e}^{2\epsilon}\cos^2\theta_0}{\left(1-\mathrm{e}^{-j2\theta_0}\right)\left(1-\mathrm{e}^{-2\epsilon+j2\theta_0}\right)}\\
&\cdot 2\frac{1-2\mathrm{e}^{-2\epsilon}\cos 2\theta_0+\mathrm{e}^{-4\epsilon}}{1-2\mathrm{e}^{-2\epsilon}\cos 2\theta_0+\mathrm{e}^{-4\epsilon}+(1-\mathrm{e}^{-2\epsilon})(1-\mathrm{e}^{-2\epsilon}\cos 2\theta_0)}\\
=&2\frac{1-\frac{1}{2}\left(\mathrm{e}^{j\theta_0}-\mathrm{e}^{-j\theta_0}\right)\left(\mathrm{e}^{j\theta_0}-\mathrm{e}^{-j\theta_0}\right)+\cos^2\theta_0}{\left(1-\mathrm{e}^{-j2\theta_0}\right)\left(1-\mathrm{e}^{j2\theta_0}\right)}\\
=&\frac{1-\frac{1}{2}(2\cos\theta_0)^2+\cos^2\theta_0}{1-\cos 2\theta_0}\\
=&\frac{1-\cos^2\theta_0}{2\sin^2\theta_0}\\
=&\frac{1}{2}.
\end{aligned}\tag{3.113}
$$

The limit of $\alpha_{\theta_0}^2 B$ for $\epsilon\to 0$ can be evaluated in the same way and is $1/2$ as well. Therefore (3.112) becomes

$$
\lim_{\epsilon\to 0}\int\limits_{\mathcal{R}_\epsilon(\theta)}\left|\alpha_{\theta_0}\frac{1-\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}}{1-2\mathrm{e}^{-\epsilon}\cos\theta_0\mathrm{e}^{-j\theta}+\mathrm{e}^{-2\epsilon}\mathrm{e}^{-2j\theta}}\right|^2\mathrm{d}\theta=2\pi.\tag{3.114}
$$

In case $\theta_0=0$ set the k dependent part of $\hat{d}_\epsilon(t,k)$ to $\alpha_{\theta_0}\mathrm{e}^{-\epsilon k}$. Applying the unilateral Z transform with respect to k, integrating the square of the Z transform over $\theta\in[-\sqrt{\epsilon},\sqrt{\epsilon}]$

and evaluating the limit as $\epsilon \to 0$ yields

$$
\begin{aligned}
\lim_{\epsilon\to 0} \int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} & \left| \frac{\alpha_{\theta_0} \mathrm{e}^{j\theta}}{\mathrm{e}^{j\theta} - \mathrm{e}^{-\epsilon}} \right|^2 \mathrm{d}\theta \\
&= \lim_{\epsilon\to 0} \int_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \frac{\alpha_{\theta_0}^2}{1 - 2\mathrm{e}^{-\epsilon}\cos\theta + \mathrm{e}^{-2\epsilon}} \mathrm{d}\theta \\
&= \lim_{\epsilon\to 0} \frac{2\alpha_{\theta_0}^2}{\sqrt{(1+\mathrm{e}^{-2\epsilon})^2 - 4\mathrm{e}^{-2\epsilon}}} \left[\arctan\left(\frac{1 + \mathrm{e}^{-2\epsilon} + 2\mathrm{e}^{-\epsilon}}{1 - \mathrm{e}^{-2\epsilon}} \tan\frac{\theta}{2} \right) \right]_{-\sqrt{\epsilon}}^{\sqrt{\epsilon}} \\
&= \lim_{\epsilon\to 0} 2 \left(\arctan\left(\frac{\left(1+\mathrm{e}^{2\epsilon}\right)^2}{1-\mathrm{e}^{-2\epsilon}} \tan\frac{\sqrt{\epsilon}}{2} \right) - \arctan\left(\frac{\left(1+\mathrm{e}^{2\epsilon}\right)^2}{1-\mathrm{e}^{-2\epsilon}} \tan\frac{\sqrt{-\epsilon}}{2} \right) \right) \\
&= 2 \lim_{\epsilon\to 0} \left(\arctan\left(\frac{\frac{1}{4}\epsilon^{-\frac{1}{2}}}{1 + \left(\frac{\sqrt{\epsilon}}{2}\right)^2} \frac{1}{2\mathrm{e}^{-2\epsilon}} \right) + \arctan\left(\frac{-\frac{1}{4}\epsilon^{-\frac{1}{2}}}{1 + \left(\frac{-\sqrt{\epsilon}}{2}\right)^2} \frac{1}{2\mathrm{e}^{-2\epsilon}} \right) \right) \\
&= 2\pi. \qquad (3.115)
\end{aligned}
$$

CHAPTER 4

Internal Stability of Linear 2D Systems

In this chapter we will discuss internal stability of linear two-dimensional systems described by the Roesser model. In particular sufficient conditions for stability, exponential stability and asymptotic stability will be derived. This will be done using linear matrix inequalities and a generalised notation studying continuous and discrete variables simultaneously. Special attention is paid to the analysis of systems with singularities on the stability boundary.

Chapter contents

4.1 Introduction

As discussed in the previous chapter a homogeneous platoon of vehicles with unidirectional control can be described as a continuous-discrete two-dimensional system. However, the stability analysis in the frequency domain yields some significant disadvantages. One drawback is the fact that every such system inherits an unavoidable structural nonessential singularity of the second kind (NSSK) on the stability boundary. Thus, the induced operator norm in the frequency domain has to be examined with special care around this singularity.

The aim of this chapter will be, therefore, to derive different conditions for the stability of two-dimensional systems in the time domain using linear matrix inequalities (LMI). Every system including a NSSK on the stability boundary in the frequency domain will also exhibit a singularity on the stability boundary (SSB) in the time domain description. Hence, as it will be shown later, no sign definite solution of the LMI can be found. Thus, previously derived conditions for stability of two-dimensional systems in the time domain presented in the literature cannot be applied as they require sign definite solutions.

Conditions for stability and asymptotic stability of linear two-dimensional in the time domain will be studied that only require semi-definite solutions to the LMI. Therefore, they are suitable to examine the stability of linear two-dimensional systems including SSB.

The conditions to guarantee exponential stability proposed here demand a strictly sign definite solution. Even though these conditions are only sufficient it will be shown that systems including SSB cannot be exponentially stable.

After clarifying the notation in Section 4.2 and giving some preliminary results in Section 4.3 we will prove exponential stability for two-dimensional systems in Section 4.4. Asymptotic stability of two-dimensional systems under suitable assumptions which include systems with SSB is studied in Section 4.5. All results in these sections will be given using a generalised system model that describes continuous, discrete and continuous-discrete systems simultaneously. The chapter concludes with string stability analysis for different vehicle platoon settings in Section 4.6.

A short version studying the stability of continuous two-dimensional systems has been accepted for publication in the Proceedings of the IEEE Conference on Decision and Control (CDC), Knorn and Middleton (2012b).

4.2 Notation

We will study stability of two-dimensional systems using a unified notation to describe the stability of the state variable $\boldsymbol{x}(t_1, t_2)$ where for $i \in \{1, 2\}$

$$t_i \in \mathbb{T}_i \quad \text{that is} \quad t_i \in \begin{cases} \mathbb{R}_{\geq 0} & \text{if } t_i \text{ continuous,} \\ \mathbb{N}_0 & \text{if } t_i \text{ discrete.} \end{cases} \tag{4.1}$$

We will use the generalised derivative operator δ_i for $i \in \{1, 2\}$ to represent either a derivative (continuous) or forward difference (discrete) with respect to t_i. For example:

$$\delta_1 \boldsymbol{x}(t_1, t_2) := \begin{cases} \frac{\partial}{\partial t_1} \boldsymbol{x}(t_1, t_2) & \text{if } t_1 \text{ continuous,} \\ \boldsymbol{x}(t_1 + 1, t_2) - \boldsymbol{x}(t_1, t_2) & \text{if } t_1 \text{ discrete.} \end{cases} \tag{4.2}$$

The generalised integration operator $\mathcal{S}$ is defined as regular integration in continuous time or left Riemann summation in discrete time. For example:

$$\overset{b}{\underset{a}{\mathcal{S}}} \boldsymbol{x}(t_1, t_2) \mathrm{d}t_1 := \begin{cases} \int\limits_a^b \boldsymbol{x}(t_1, t_2) \mathrm{d}t_1 & \text{if } t_1 \text{ continuous,} \\ \sum_{t_1=a}^{t_1=b-1} \boldsymbol{x}(t_1, t_2) & \text{if } t_1 \text{ discrete.} \end{cases} \tag{4.3}$$

We will consider autonomous two-dimensional systems of the following form (Roesser model, Roesser (1975))

$$\underbrace{\begin{pmatrix}\delta_1\boldsymbol{x}_1(t_1,t_2)\\ \delta_2\boldsymbol{x}_2(t_1,t_2)\end{pmatrix}}_{\delta\boldsymbol{x}(t_1,t_2)} = \underbrace{\begin{bmatrix}\boldsymbol{A}_{11} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22}\end{bmatrix}}_{\boldsymbol{A}}\underbrace{\begin{pmatrix}\boldsymbol{x}_1(t_1,t_2)\\ \boldsymbol{x}_2(t_1,t_2)\end{pmatrix}}_{\boldsymbol{x}(t_1,t_2)} \tag{4.4}$$

where $\boldsymbol{x}_1 \in \mathbb{R}^{n_1}$, $\boldsymbol{x}_2 \in \mathbb{R}^{n_2}$, with the initial conditions $\boldsymbol{x}_1(0,t_2) = \boldsymbol{x}_{10}(t_2)$ and $\boldsymbol{x}_2(t_1,0) = \boldsymbol{x}_{20}(t_1)$. The autonomous system (4.4) has a solution that must satisfy:

$$\boldsymbol{x}_1(t_1,t_2) = \mathrm{E}(\boldsymbol{A}_{11})^{t_1}\boldsymbol{x}_{10}(t_2) + \mathop{\mathcal{S}}_{0}^{t_1}\mathrm{E}(\boldsymbol{A}_{11})^{\tau}\boldsymbol{A}_{12}\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)\mathrm{d}\tau, \tag{4.5}$$

$$\boldsymbol{x}_2(t_1,t_2) = \mathrm{E}(\boldsymbol{A}_{22})^{t_2}\boldsymbol{x}_{20}(t_1) + \mathop{\mathcal{S}}_{0}^{t_2}\mathrm{E}(\boldsymbol{A}_{22})^{\tau}\boldsymbol{A}_{21}\boldsymbol{x}_1(t_1, t_2 - \mathbb{I}_2 - \tau)\mathrm{d}\tau, \tag{4.6}$$

where $\mathbb{I}_i$ for $i \in \{1,2\}$ denotes the indicator function

$$\mathbb{I}_i := \begin{cases} 0 & \text{if } t_i \text{ continuous,}\\ 1 & \text{if } t_i \text{ discrete,}\end{cases} \tag{4.7}$$

and

$$\mathrm{E}(\boldsymbol{A})^t := \begin{cases} \mathrm{e}^{\boldsymbol{A}t} & \text{if } t \text{ continuous,}\\ (\mathbf{I}+\boldsymbol{A})^t & \text{if } t \text{ discrete.}\end{cases} \tag{4.8}$$

We say $\boldsymbol{A}$ is stable to mean either $\boldsymbol{A}$ is Hurwitz stable (continuous case) or $\mathbf{I}+\boldsymbol{A}$ is Schur stable (discrete case). In either case, if $\boldsymbol{A}$ is stable, then there exist $\lambda > 0$ (and in addition $\lambda < 1$ in the discrete case) and $k < \infty$ such that

$$\left\|\mathrm{E}(\boldsymbol{A})^t\right\| \leq k\mathrm{E}(-\lambda)^t. \tag{4.9}$$

Note that

$$\mathop{\mathcal{S}}_{0}^{T}\mathrm{E}(-\lambda)^t\mathrm{d}t = \frac{1-\mathrm{E}(-\lambda)^T}{\lambda} \tag{4.10}$$

and for $-\lambda$ stable

$$\mathop{\mathcal{S}}_{T}^{\infty}\mathrm{E}(-\lambda)^t\mathrm{d}t = \frac{\mathrm{E}(-\lambda)^T}{\lambda}. \tag{4.11}$$

Moreover $\oplus$ denotes the direct sum of matrices, e. g. $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2 = \mathrm{diag}\{\boldsymbol{P}_1, \boldsymbol{P}_2\}$, $\mathbf{I}$ and $\mathbf{0}$ denote the identity matrix and the zero matrix of appropriate dimension, respectively, and the imaginary unit is denoted by j. Consider the two-dimensional vector Lyapunov function

$$\boldsymbol{V}(t_1,t_2) := \begin{bmatrix}\boldsymbol{x}_1^{\mathrm{T}}(t_1,t_2) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{x}_2^{\mathrm{T}}(t_1,t_2)\end{bmatrix}\boldsymbol{P}\boldsymbol{x}(t_1,t_2) = \begin{pmatrix}V_1(t_1,t_2)\\ V_2(t_1,t_2)\end{pmatrix} \tag{4.12}$$

with $\boldsymbol{P}_1 = \boldsymbol{P}_1^{\mathrm{T}} > 0$, $\boldsymbol{P}_2 = \boldsymbol{P}_2^{\mathrm{T}} > 0$, $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$ and

$$\operatorname{div} \boldsymbol{V}(t_1, t_2) = \delta_1 V_1(t_1, t_2) + \delta_2 V_2(t_1, t_2) \tag{4.13}$$

with $\delta_i V_i(t_1, t_2) = \boldsymbol{x}^{\mathrm{T}} \boldsymbol{Q}_i \boldsymbol{x}$ and

$$\boldsymbol{Q} = \boldsymbol{Q}_1 + \boldsymbol{Q}_2 \quad \text{where} \quad \boldsymbol{Q}_i = \boldsymbol{A}^{\mathrm{T}} \tilde{\boldsymbol{P}}_i + \tilde{\boldsymbol{P}}_i \boldsymbol{A} + \mathbb{I}_i \boldsymbol{A}^{\mathrm{T}} \tilde{\boldsymbol{P}}_i \boldsymbol{A} \quad \text{for } i \in \{1, 2\} \tag{4.14}$$

with $\tilde{\boldsymbol{P}}_1 = (\boldsymbol{P}_1 \oplus \mathbf{0})$ and $\tilde{\boldsymbol{P}}_2 = (\mathbf{0} \oplus \boldsymbol{P}_2)$.

Note that the generalised $\mathcal{T}_i$ transform for $i \in \{1, 2\}$ is

$$\mathcal{T}_i \{\boldsymbol{x}(t_i)\} = \boldsymbol{X}(\xi_i) = \begin{cases} \mathcal{L}\{\boldsymbol{x}(t_i)\} = \boldsymbol{X}(s_i) & \text{if } t_i \text{ continuous,} \\ \mathcal{Z}\{\boldsymbol{x}(t_i)\} = \boldsymbol{X}(z_i) & \text{if } t_i \text{ discrete,} \end{cases} \tag{4.15}$$

and ξ_i for $i \in \{1, 2\}$ is the Laplace variable s_i if t_i is continuous or the Z transform variable z_i if t_i is discrete.

Definition 4.1 (Singularity on the Stability Boundary (SSB))

The two-dimensional Roesser Model has a singularity on the stability boundary if there exits a set of ω_i (if t_i is continuous) or θ_i (if t_i is discrete) such that the matrix $((\xi_1 - \mathbb{I}_1)\mathbf{I} \oplus (\xi_2 - \mathbb{I}_2)\mathbf{I}) - \boldsymbol{A}$ is singular for $\xi_i = j\omega_i$ or $\xi_i = \mathrm{e}^{j\theta_i}$, respectively.

We will make use of the following different definitions of initial conditions.

Definition 4.2 (L_2 and L_∞ Bounded Initial Conditions)

The initial conditions of a two-dimensional Roesser Model are L_2 and L_∞ bounded if there exist $c_i, \zeta_i < \infty$ such that for $i \in \{1, 2\}$

$$\|\boldsymbol{x}_{i0}(\cdot)\|_2^2 = \int_0^\infty |\boldsymbol{x}_{i0}(t)|^2 \mathrm{d}t \le c_i, \quad \textit{and} \tag{4.16}$$

$$\|\boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t \ge 0} |\boldsymbol{x}_{i0}(t)| \le \zeta_i. \tag{4.17}$$

Definition 4.3 (L_2' and L_∞'' Smooth Bounded Initial Conditions)

The initial conditions of a two-dimensional Roesser Model are smooth bounded initial conditions if they are L_2 and L_∞ bounded according to Definition 4.2 and in addition there exist $c_i', \zeta_i', \zeta_i'' < \infty$ such that for $i \in \{1, 2\}$

$$\|\delta \boldsymbol{x}_{i0}(\cdot)\|_2^2 = \int_0^\infty |\delta \boldsymbol{x}_{i0}(t)|^2 \mathrm{d}t \le c_i', \tag{4.18}$$

$$\|\delta \boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t > 0} |\delta \boldsymbol{x}_{i0}(t)| \le \zeta_i', \quad \textit{and} \tag{4.19}$$

$$\|\delta^2 \boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t > 0} |\delta^2 \boldsymbol{x}_{i0}(t)| \le \zeta_i''. \tag{4.20}$$

Definition 4.4 (Exponentially Decaying Initial Conditions)

The initial conditions of a two-dimensional Roesser Model are exponentially decaying, if there exist $\mu_i > 0$ and $\kappa_i < \infty$ such that for $i, k \in \{1,2\}$ $i \neq k$

$$|\boldsymbol{x}_{i0}(t_k)| \leq \kappa_i \mathrm{e}^{-\mu_k t_k}. \tag{4.21}$$

We will discuss the stability of two-dimensional systems according to the following definitions.

Definition 4.5 (Stability of Two-Dimensional Roesser Model)

The autonomous two-dimensional Roesser Model (4.4) is stable if for each $M > 0$ there exists a set of $c_i(M), \zeta_i(M) > 0$ such that if the initial conditions are in L_2 and L_∞ with bounds c_i and ζ_i for $i \in \{1,2\}$, respectively, then

$$|\boldsymbol{x}(t_1, t_2)| \leq M \quad \textit{for all } t_1, t_2 > 0. \tag{4.22}$$

Definition 4.6 (Asymptotic Stability of Two-Dimensional Roesser Model with Smooth Bounded Initial Conditions)

The autonomous two-dimensional Roesser Model (4.4) is asymptotically stable, if for any Smooth Bounded Initial Conditions (according to Definition 4.3) it is stable, and the following limit holds

$$\lim_{t_1+t_2\to\infty} \boldsymbol{x}(t_1, t_2) = 0. \tag{4.23}$$

Note that asymptotic stability requires the states to tend to zero as $t_1 + t_2 \to \infty$. That includes the cases where $t_1 \to \infty$, $t_2 \to \infty$ and the double limit $\lim_{t_1,t_2\to\infty}$ where t_1 and t_2 tend to $+\infty$ at the same time but in any possible form and direction.

Definition 4.7 (Exponential Stability of Two-Dimensional Roesser Model)

The autonomous, two-dimensional Roesser Model (4.4) is exponentially stable, if for any exponentially decaying initial conditions there exist $\eta_i > 0$, and $M_i < \infty$ such that for $i \in \{1,2\}$ the following condition holds:

$$|\boldsymbol{x}_i(t_1, t_2)|_2^2 \leq M_i \mathrm{e}^{-\eta_1 t_1} \mathrm{e}^{-\eta_2 t_2}. \tag{4.24}$$

4.3 Mathematical Preliminaries

Before presenting our results concerning the asymptotic stability of two-dimensional systems we would like to show the connection between singularities on the stability boundary and the solution of the Lyapunov equation.

Lemma 4.1

Consider the autonomous two-dimensional system (4.4). If the system has a singularity on the stability boundary (SSB), then for every symmetric choice of $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, there exists a nonzero vector $\boldsymbol{v}$ such that $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = 0$ where $\boldsymbol{Q}$ is given in (4.14).

Proof The characteristic polynomial is equal to:

$$\operatorname{den}(\xi_1, \xi_2) = \det \begin{bmatrix} (\xi_1 - \mathbb{I}_1)\mathbf{I} - \boldsymbol{A}_{11} & -\boldsymbol{A}_{12} \\ -\boldsymbol{A}_{21} & (\xi_2 - \mathbb{I}_2)\mathbf{I} - \boldsymbol{A}_{22} \end{bmatrix}. \tag{4.25}$$

Since the system has a singularity at $\xi_i = j\omega_i$ or $\xi_i = \mathrm{e}^{j\theta_i}$, respectively, the matrix $((\xi_1 - \mathbb{I}_1)\mathbf{I} \oplus (\xi_2 - \mathbb{I}_2)\mathbf{I}) - \boldsymbol{A}$ has an eigenvalue at 0 for $\xi_i = j\omega_i$ or $\xi_i = \mathrm{e}^{j\theta_i}$, respectively. Therefore, there exists a vector $\boldsymbol{v} \in \mathbb{C}^n$ such that

$$\left(\begin{bmatrix} (\xi_1 - \mathbb{I}_1)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\xi_2 - \mathbb{I}_2)\mathbf{I} \end{bmatrix} - \boldsymbol{A} \right) \boldsymbol{v} = 0. \tag{4.26}$$

Using (4.26) we can rewrite $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = \boldsymbol{v}^{\mathrm{H}}(\boldsymbol{Q}_1 + \boldsymbol{Q}_2)\boldsymbol{v}$ and see from (4.14) that for instance if t_1 is continuous and t_2 is discrete

$$\begin{aligned}
\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} =& \boldsymbol{v}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_1\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\tilde{\boldsymbol{P}}_1\boldsymbol{A}\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_2\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\tilde{\boldsymbol{P}}_2\boldsymbol{A}\boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}}\boldsymbol{A}^{\mathrm{T}}\tilde{\boldsymbol{P}}_2\boldsymbol{A}\boldsymbol{v} \\
=& \boldsymbol{v}^{\mathrm{H}} \begin{bmatrix} j\omega_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} \boldsymbol{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}} \begin{bmatrix} \boldsymbol{P}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} j\omega_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \boldsymbol{v} \\
&+ \boldsymbol{v}^{\mathrm{H}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\mathrm{e}^{j\theta_1} - 1\right)\mathbf{I} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2 \end{bmatrix} \boldsymbol{v} + \boldsymbol{v}^{\mathrm{H}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\mathrm{e}^{j\theta_1} - 1\right)\mathbf{I} \end{bmatrix} \boldsymbol{v} \\
&+ \boldsymbol{v}^{\mathrm{H}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\mathrm{e}^{j\theta_1} - 1\right)\mathbf{I} \end{bmatrix}^{\mathrm{H}} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \left(\mathrm{e}^{j\theta_1} - 1\right)\mathbf{I} \end{bmatrix} \boldsymbol{v} \\
=& \boldsymbol{v}^{\mathrm{H}} \begin{bmatrix} \left(-j\omega_1 + j\omega_1\right)\boldsymbol{P}_1 & \mathbf{0} \\ \mathbf{0} & \left(\mathrm{e}^{-j\theta_2} - 1 + \mathrm{e}^{j\theta_2} - 1 + 1 - \mathrm{e}^{-j\theta_2} - \mathrm{e}^{j\theta_2} + 1\right)\boldsymbol{P}_2 \end{bmatrix} \boldsymbol{v} \\
=& 0.
\end{aligned} \tag{4.27}$$

If t_1 is discrete or t_2 is continuous it can be shown in a similar way that $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = 0$. Thus, $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{Q}\boldsymbol{v} = 0$ independently of $\boldsymbol{P}$. □

Note therefore that, for a system including SSB it is not possible to find positive definite matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{Q}$ is sign definite.

Even though for systems including SSB $\boldsymbol{Q}$ can never be sign definite, the existence of a negative semi-definite $\boldsymbol{Q}$ with some additional assumptions on $\boldsymbol{A}$ is sufficient for stability. Furthermore, with assumptions on the initial conditions we are able to guarantee asymptotic stability (with bounded smooth initial conditions).

Before we show stability we will first use some interesting properties of two-dimensional nonnegative vector fields with nonpositive divergence.

Lemma 4.2

Consider the two-dimensional space of two variables t_1 and t_2 and the two-dimensional nonnegative vector field $\boldsymbol{V}^{\mathrm{T}}(t_1,t_2) = (V_1(t_1,t_2), V_2(t_1,t_2))$. If the divergence of the vector field $\boldsymbol{V}(t_1,t_2)$ is nonpositive for every t_1 and t_2, then the generalised integral of $V_1(t_1,t_2)$ and $V_2(t_1,t_2)$ over $t_2 \in [0,T_2]$ and $t_1 \in [0,T_1]$, respectively, is bounded by the initial conditions $V_1(0,t_2)$ and $V_2(t_1,0)$, that is for all $T_1, T_2 > 0$:

$$\mathop{\mathcal{S}}_{0}^{T_2} V_1(T_1,t_2)\mathrm{d}t_2 \le \mathop{\mathcal{S}}_{0}^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1,0)\mathrm{d}t_1, \tag{4.28}$$

$$\mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1,T_2)\mathrm{d}t_1 \le \mathop{\mathcal{S}}_{0}^{T_2} V_1(0,t_2)\mathrm{d}t_2 + \mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1,0)\mathrm{d}t_1. \tag{4.29}$$

Proof To prove this lemma we will simply consider the generalised surface integral of the divergence of $\boldsymbol{V}(t_1,t_2)$ over the rectangular region $t_1 \in [0,T_1]$, $t_2 \in [0,T_2]$:

$$W(T_1,T_2) := \mathop{\mathcal{S}}_{0}^{T_2} \mathop{\mathcal{S}}_{0}^{T_1} \left(\delta_1 V_1(t_1,t_2) + \delta_2 V_2(t_1,t_2)\right) \mathrm{d}t_1 \mathrm{d}t_2. \tag{4.30}$$

Using the fundamental theorem of calculus or Gauss Divergence Theorem for continuous variables and simple arithmetic for discrete variables, (4.30) can be transformed into

$$\begin{aligned} W(T_1,T_2) = &\mathop{\mathcal{S}}_{0}^{T_2} V_1(T_1,t_2)\mathrm{d}t_2 - \mathop{\mathcal{S}}_{0}^{T_2} V_1(0,t_2)\mathrm{d}t_2 \\ &+ \mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1,T_2)\mathrm{d}t_1 - \mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1,0)\mathrm{d}t_1. \end{aligned} \tag{4.31}$$

Since the divergence is nonpositive for every t_1 and t_2, from (4.30) we get $W(T_1,T_2) \le 0$. Also, $V_2(t_1,t_2)$ is a nonnegative function of t_1 and t_2. Therefore (4.31) implies (4.28). The bound on of the integral of $V_2(t_1,t_2)$ in (4.29) follows equivalently. □

We now consider the two-dimensional Lyapunov function $\boldsymbol{V}(t_1,t_2)$ introduced above, to show that under some assumptions $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ in (4.4) are bounded and the system is therefore stable according to Definition 4.5.

Corollary 4.1 (Stability of Linear Two-Dimensional Systems)

Consider the autonomous two-dimensional system in (4.4). If the following conditions hold *(i)* $\boldsymbol{A}_{11}$ *and* $\boldsymbol{A}_{22}$ *are stable, and*

(ii) *there exist positive definite, symmetric matrices* $\boldsymbol{P}_1$ *and* $\boldsymbol{P}_2$ *such that* $\boldsymbol{Q} \leq 0$*, where* $\boldsymbol{Q}$ *is given in (4.14),*

then the system is stable according to Definition 4.5.

Proof Since $\boldsymbol{A}_{ii}$ is stable, there exist $k_i < \infty$ and $\lambda_i > 0$ (and $\lambda_i < 1$ in the discrete case) such that $\left\| \mathrm{E}(\boldsymbol{A}_{ii})^{t_i} \right\| \leq k_i \mathrm{E}(-\lambda_i)^{t_i}$. Therefore, using (4.5) we have

$$|\boldsymbol{x}_1(t_1,t_2)| \leq k_1 \mathrm{E}(-\lambda_1)^{t_1} |\boldsymbol{x}_{10}(t_2)| + \underset{0}{\overset{t_1}{\mathcal{S}}} k_1 \mathrm{E}(-\lambda_1)^{\tau} \|\boldsymbol{A}_{12}\| \, |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)| \, \mathrm{d}\tau. \tag{4.32}$$

We choose $\boldsymbol{P}_2$ as in condition (ii) and then define the Lyapunov function candidate $V_2(t_1,t_2) = \boldsymbol{x}_2^{\mathrm{T}}(t_1,t_2)\boldsymbol{P}_2\boldsymbol{x}_2(t_1,t_2)$. Using the definition of $V_2(t_1,t_2)$ and the Cauchy-Schwarz inequality, (4.32) becomes

$$\begin{aligned} |\boldsymbol{x}_1(t_1,t_2)| \leq & k_1 |\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}} \underset{0}{\overset{t_1}{\mathcal{S}}} \mathrm{E}(-\lambda_1)^{\tau} \sqrt{V_2(t_1 - \mathbb{I}_1 - \tau, t_2)} \mathrm{d}\tau \\ \leq & k_1 |\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}} \left(\underset{0}{\overset{t_1}{\mathcal{S}}} \mathrm{E}(-\lambda_1)^{2\tau} \mathrm{d}\tau \right)^{1/2} \left(\underset{0}{\overset{t_1}{\mathcal{S}}} V_2(\tau,t_2)\,\mathrm{d}\tau \right)^{1/2}. \end{aligned} \tag{4.33}$$

With (4.10), Lemma 4.2 and the fact that the initial conditions are in L_2, (4.33) becomes

$$\begin{aligned} |\boldsymbol{x}_1(t_1,t_2)| \leq & k_1 |\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}} \sqrt{\frac{1-\mathrm{E}(-\lambda_1)^{2t_1}}{2\lambda_1 - \lambda_1^2 \mathbb{I}_1}} \\ & \cdot \left(\underset{0}{\overset{t_2}{\mathcal{S}}} V_1(0,\tau)\mathrm{d}\tau + \underset{0}{\overset{t_1}{\mathcal{S}}} V_2(\tau,0)\mathrm{d}\tau \right)^{1/2} \\ \leq & k_1 |\boldsymbol{x}_{10}(t_2)| + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}\sqrt{2\lambda_1 - \lambda_1^2\mathbb{I}_1}} \left(\|\boldsymbol{P}_1\|c_1 + \|\boldsymbol{P}_2\|c_2\right)^{1/2}. \end{aligned} \tag{4.34}$$

Note that since for t_i discrete we have $\mathbb{I}_i = 1$ and $\lambda_i < 1$ we find that $2\lambda_i - \lambda_i^2\mathbb{I}_i > \lambda_i$. Thus, $1/(2\lambda_i - \lambda_i^2\mathbb{I}_i) < 1/\lambda_i$. Since the initial conditions are also in L_∞, we find that

$$|\boldsymbol{x}_1(t_1,t_2)| \leq M_1 =: k_1\zeta_1 + \frac{k_1\|\boldsymbol{A}_{12}\|}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)}\sqrt{\lambda_1}} \left(\|\boldsymbol{P}_1\|c_1 + \|\boldsymbol{P}_2\|c_2\right)^{1/2} \tag{4.35}$$

for all $t_1, t_2 > 0$. Note that the bound M_1 is scaled by the L_2 and L_∞ norms of the initial conditions, i. e. ζ_1, c_1, c_2. A similar bound for $\boldsymbol{x}_2(t_1,t_2)$ can be found in the same way. The system is therefore stable. □

The following corollary shows stability for systems whose matrices $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are not strictly Hurwitz or Schur stable but have singularities on the stability boundary and additional assumptions on $\boldsymbol{A}_{12}$ and $\boldsymbol{A}_{21}$. It will be used to show stability of the system presented in Example 4.3.

Corollary 4.2

Consider the autonomous two-dimensional system in (4.4). If the following conditions hold

(i) $\boldsymbol{A}_{11}$ *and* $\boldsymbol{A}_{22}$ *are marginally stable and for eigenvalues on the stability boundary the geometric multiplicity is equal to the algebraic multiplicity,*

(ii) the marginally stable modes of $\boldsymbol{A}_{11}$ *and* $\boldsymbol{A}_{22}$ *are not controllable by* $\boldsymbol{A}_{12}$ *and* $\boldsymbol{A}_{21}$*, respectively, and*

(iii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$ *and* $\boldsymbol{P}_2$ *such that* $\boldsymbol{Q} \leq 0$*, where* $\boldsymbol{Q}$ *is given in (4.14),*

then the system is stable according to Definition 4.5.

Proof The proof is similar to the proof of Corollary 4.1 above. Note that if $\boldsymbol{A}_{ii}$ is marginally stable and the geometric multiplicity of each marginally stable mode is equal to its algebraic multiplicity than we can find a $k_i < \infty$ such that $\left\| \mathrm{E}(\boldsymbol{A}_{ii})^{t_i} \right\| \leq k_i$. Since we assume that the algebraic and the geometric multiplicity of the marginally stable eigenvalues are equal we can furthermore find a state transformation matrix $\boldsymbol{T}$ and $\lambda_i^* > 0$ (and in addition $\lambda_i^* < 1$ in the discrete case) such that

$$\mathrm{E}(\boldsymbol{A}_{ii})^{t_i} \boldsymbol{A}_{ik} \leq \boldsymbol{T}^{-1} \left(\mathbf{I}_{n_{i0}} \oplus \mathrm{E}(-\lambda_i^*)^{t_i} \mathbf{I}_{n_i - n_{i0}} \right) \boldsymbol{T} \boldsymbol{A}_{ik} \tag{4.36}$$

where n_{i0} is the number of marginally stable eigenvalues of $\boldsymbol{A}_{ii}$. The marginally stable eigenvalue of $\boldsymbol{A}_{ii} + \mathbb{I}_i$ is not controllable by $\boldsymbol{A}_{ik}$ for $i \neq k$. Thus, (4.36) yields

$$\begin{aligned} \mathrm{E}(\boldsymbol{A}_{ii})^{t_i} \boldsymbol{A}_{ik} \leq & \boldsymbol{T}^{-1} \left(\mathbf{I} \oplus \mathrm{E}(-\lambda_i^*)^{t_i} \mathbf{I} \right) \begin{bmatrix} \mathbf{0} \\ \boldsymbol{A}_{ik}^* \end{bmatrix} \\ = & \boldsymbol{T}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathrm{E}(-\lambda_i^*)^{t_i} \boldsymbol{A}_{ik}^* \end{bmatrix}. \end{aligned} \tag{4.37}$$

Hence there exists a k_i^* such that $\left\| \mathrm{E}(\boldsymbol{A}_{ii})^{t_i} \boldsymbol{A}_{ik} \right\| \leq k_i^* \mathrm{E}(-\lambda_i^*)^{t_i}$. Therefore (4.5) is bounded by

$$\left| \boldsymbol{x}_1 (t_1, t_2) \right| \leq k_1 \left| \boldsymbol{x}_{10} (t_2) \right| + \underset{0}{\overset{t_1}{\mathcal{S}}} k_1^* \mathrm{E}(-\lambda_1^*)^{\tau} \left| \boldsymbol{x}_2 (t_1 - \mathbb{I}_1 - \tau, t_2) \right| \mathrm{d}\tau. \tag{4.38}$$

Using similar steps as in the proof of Corollary 4.1 above we can show that

$$\left| \boldsymbol{x}_1 (t_1, t_2) \right| \leq k_1 \zeta_1 + \frac{k_1^* \sqrt{\|\boldsymbol{P}_1\| c_1 + \|\boldsymbol{P}_2\| c_2}}{\sqrt{\sigma_{\min}(\boldsymbol{P}_2)} \sqrt{\lambda_1^*}}. \tag{4.39}$$

□

Under the same assumptions as in Corollary 4.1 we can further show that not only is $\boldsymbol{x}_i(t_1,t_2)$ bounded (that is in L_∞) but also the generalised integrals $\mathcal{S}_0^\infty |\boldsymbol{x}_1(t_1,t_2)|^2 \mathrm{d}t_1$ and $\mathcal{S}_0^\infty |\boldsymbol{x}_2(t_1,t_2)|^2 \mathrm{d}t_2$ are bounded. This will facilitate the proof of asymptotic stability later in Section 4.5.

Corollary 4.3

Consider the autonomous two-dimensional System in (4.4). If the following conditions hold

(i) the initial conditions are L_2 and L_∞ bounded according to Definition 4.2,

(ii) $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and

(iii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ such that $\boldsymbol{Q} \leq 0$, where $\boldsymbol{Q}$ is given in (4.14),

then there exist $\overline{M}_1, \overline{M}_2 < \infty$ independently of t_2 and t_1, respectively, such that

$$\mathop{\mathcal{S}}_{0}^{\infty} |\boldsymbol{x}_1(t_1,t_2)|^2 \,\mathrm{d}t_1 \leq \overline{M}_1 \quad \textit{and} \quad \mathop{\mathcal{S}}_{0}^{\infty} |\boldsymbol{x}_2(t_1,t_2)|^2 \,\mathrm{d}t_2 \leq \overline{M}_2. \tag{4.40}$$

Proof From (4.32), note that

$$\begin{aligned}\mathop{\mathcal{S}}_{0}^{\infty} |\boldsymbol{x}_1(t_1,t_2)|^2 \,\mathrm{d}t_1 \leq & 2k_1^2 \mathop{\mathcal{S}}_{0}^{\infty} \mathrm{E}(-\lambda_1)^{2t_1} |\boldsymbol{x}_{10}(t_2)|^2 \,\mathrm{d}t_1 \\ & + 2k_1^2 \|\boldsymbol{A}_{12}\|^2 \mathop{\mathcal{S}}_{0}^{\infty} \left(\mathop{\mathcal{S}}_{0}^{t_1} \mathrm{E}(-\lambda_1)^{\tau} |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)| \mathrm{d}\tau \right)^2 \mathrm{d}t_1.\end{aligned} \tag{4.41}$$

The first term of the right hand side of (4.41) can be bounded by

$$2k_1^2 \mathop{\mathcal{S}}_{0}^{\infty} \mathrm{E}(-\lambda_1)^{2t_1} |\boldsymbol{x}_{10}(t_2)|^2 \,\mathrm{d}t_1 \leq \frac{2k_1^2 \zeta_1^2}{\lambda_1}. \tag{4.42}$$

With the Cauchy-Schwarz inequality the second term of the right hand side of (4.41) allows a bound to be calculated as

$$\begin{aligned} & 2k_1^2 \|\boldsymbol{A}_{12}\|^2 \mathop{\mathcal{S}}_{0}^{\infty} \left(\mathop{\mathcal{S}}_{0}^{t_1} \mathrm{E}(-\lambda_1)^{\tau} |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)| \mathrm{d}\tau \right)^2 \mathrm{d}t_1 \\ & \leq 2k_1^2 \|\boldsymbol{A}_{12}\|^2 \mathop{\mathcal{S}}_{0}^{\infty} \left(\mathop{\mathcal{S}}_{0}^{t_1} \mathrm{E}(-\lambda_1)^{\tau} \mathrm{d}\tau \right) \left(\mathop{\mathcal{S}}_{0}^{t_1} \mathrm{E}(-\lambda_1)^{\tau} |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)|^2 \mathrm{d}\tau \right) \mathrm{d}t_1 \\ & \leq \frac{2k_1^2 \|\boldsymbol{A}_{12}\|^2}{\lambda_1} \mathop{\mathcal{S}}_{0}^{\infty} \mathop{\mathcal{S}}_{0}^{t_1} \mathrm{E}(-\lambda_1)^{\tau} |\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2)|^2 \,\mathrm{d}\tau \mathrm{d}t_1. \end{aligned} \tag{4.43}$$

Interchanging the order of integration in (4.43) yields

$$\begin{aligned}\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1}&\int_0^\infty\int_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}|\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau\mathrm{d}t_1\\ &\leq\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1}\int_0^\infty\int_{\tau+\mathbb{I}_1}^\infty\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}|\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}t_1\mathrm{d}\tau\\ &\leq\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1^2}\int_0^\infty|\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau.\end{aligned}\tag{4.44}$$

Taking the limit as $T_1\to\infty$ of (4.29) in Lemma 4.2 on page 63 we see that the generalised integral in (4.44) is bounded independently of t_2. Thus $\overline{M}_1$ exists. The existence of $\overline{M}_2$ can be shown in the same way. □

To facilitate the proof of asymptotic stability of two-dimensional systems in Section 4.5 we also need results on the state derivatives. We will show that under suitable assumptions the first generalised derivatives, i.e. $\delta_i\boldsymbol{x}_k(t_1,t_2)$, $i,k\in\{1,2\}$, are in both $L_2\ [0,\infty)\times[0,\infty)$ and $L_\infty\ [0,\infty)\times[0,\infty)$ and the second generalised derivatives, i.e. $\delta_i^2\boldsymbol{x}_k(t_1,t_2)$ and $\delta_i\delta_k\boldsymbol{x}_k(t_1,t_2)$ for $i,k\in\{1,2\}$, are in $L_\infty\ [0,\infty)\times[0,\infty)$.

Lemma 4.3

Consider the autonomous two-dimensional system in (4.4). If the following conditions hold

(i) the initial conditions are L_2' and L_∞'' smooth bounded according to Definition 4.3,

(ii) $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and

(iii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ and $\boldsymbol{R}$ such that $\boldsymbol{Q}=-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}$, where $\boldsymbol{Q}$ is given in (4.14),

then

(a) the first generalised derivatives of $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are in $L_\infty\ [0,\infty)\times[0,\infty)$ and $L_2\ [0,\infty)\times[0,\infty)$, i.e. there exist $M_{ik},\overline{M}_{ik}<\infty$ such that for $i,k\in\{1,2\}$,

$$\sup_{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2}|\delta_k\boldsymbol{x}_i(t_1,t_2)|\leq M_{ik}\tag{4.45}$$

$$\int_0^\infty\int_0^\infty|\delta_k\boldsymbol{x}_i(t_1,t_2)|^2\mathrm{d}t_1\mathrm{d}t_2\leq\overline{M}_{ik},\ \text{and}\tag{4.46}$$

(b) the second generalised derivatives of $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are in $L_\infty\ [0,\infty)\times[0,\infty)$, i.e. there exist $M_{ikl}<\infty$ such that for $i,k,l\in\{1,2\}$

$$\sup_{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2}|\delta_k\delta_l\boldsymbol{x}_i(t_1,t_2)|\leq M_{ikl}.\tag{4.47}$$

Proof (a): We will first show that $\delta_1\boldsymbol{x}_1(t_1,t_2)$ (and $\delta_2\boldsymbol{x}_2(t_1,t_2)$) are in $L_\infty\,[0,\infty)\times[0,\infty)$ using the state space description for $\delta_1\boldsymbol{x}_1(t_1,t_2)$ in (4.4) and transform it into

$$|\delta_1\boldsymbol{x}_1(t_1,t_2)| \leq \|\boldsymbol{A}_{11}\|\cdot|\boldsymbol{x}_1(t_1,t_2)| + \|\boldsymbol{A}_{12}\|\cdot|\boldsymbol{x}_2(t_1,t_2)|. \tag{4.48}$$

Since $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are stable (Corollary 4.1) there exist $M_i<\infty$ such that $|\boldsymbol{x}_i(t_1,t_2)|\leq M_i$ for all t_1,t_2 and $i\in\{1,2\}$. Thus, $M_{11}:=\|\boldsymbol{A}_{11}\|M_1+\|\boldsymbol{A}_{12}\|M_2$. The existence of M_{22} can be shown in the same way.

To prove that $\delta_2\boldsymbol{x}_1(t_1,t_2)$ and $\delta_1\boldsymbol{x}_2(t_1,t_2)$ are in $L_\infty\,[0,\infty)\times[0,\infty)$ as well, we will take δ_2 of (4.4)

$$\delta_1\left(\delta_2\boldsymbol{x}_1(t_1,t_2)\right) = \boldsymbol{A}_{11}\left(\delta_2\boldsymbol{x}_1(t_1,t_2)\right) + \boldsymbol{A}_{12}\left(\delta_2\boldsymbol{x}_2(t_1,t_2)\right). \tag{4.49}$$

That yields

$$\begin{aligned}|\delta_2\boldsymbol{x}_1(t_1,t_2)| \leq& k_1\mathrm{E}(-\lambda_1)^{t_1}|\delta_2\boldsymbol{x}_{10}(t_2)|\\ &+k_1\|\boldsymbol{A}_{12}\|\cdot\left|\int_0^{t_1}\mathrm{E}(-\lambda_1)^{\tau}\delta_2\boldsymbol{x}_2(t_1-\mathbb{I}_1-\tau,t_2)\mathrm{d}\tau\right|\\ &\leq k_1\zeta_1'+\frac{k_1\|\boldsymbol{A}_{12}\|M_{22}}{\lambda_1}=:M_{12}.\end{aligned} \tag{4.50}$$

The boundedness of $\delta_1\boldsymbol{x}_2(t_1,t_2)$ can be proven in the same way.

To show that the first generalised derivatives are also in $L_2\,[0,\infty)\times[0,\infty)$ we will use the Lyapunov function candidate $\boldsymbol{V}(t_1,t_2)$ from (4.12). Given the fact that $\boldsymbol{x}^{\mathrm{T}}(t_1,t_2)\boldsymbol{Q}\boldsymbol{x}(t_1,t_2)$ is the divergence of $\boldsymbol{V}(t_1,t_2)$ we can show with the fundamental theorem of calculus that

$$\begin{aligned}&\int_0^{T_2}\int_0^{T_1}\begin{pmatrix}\delta_1\boldsymbol{x}_1^{\mathrm{T}}(t_1,t_2) & \delta_2\boldsymbol{x}_2^{\mathrm{T}}(t_1,t_2)\end{pmatrix}\boldsymbol{R}\begin{pmatrix}\delta_1\boldsymbol{x}_1(t_1,t_2)\\ \delta_2\boldsymbol{x}_2(t_1,t_2)\end{pmatrix}\mathrm{d}t_1\mathrm{d}t_2\\ &\leq\int_0^{T_2}V_1(0,t_2)\mathrm{d}t_2+\int_0^{T_1}V_2(t_1,0)\mathrm{d}t_1.\end{aligned} \tag{4.51}$$

The limit for $T_1,T_2\to\infty$ of the left hand side of (4.51) can be bounded from below by

$$\begin{aligned}&\int_0^{\infty}\int_0^{\infty}\begin{pmatrix}\delta_1\boldsymbol{x}_1^{\mathrm{T}}(t_1,t_2) & \delta_2\boldsymbol{x}_2^{\mathrm{T}}(t_1,t_2)\end{pmatrix}\boldsymbol{R}\begin{pmatrix}\delta_1\boldsymbol{x}_1(t_1,t_2)\\ \delta_2\boldsymbol{x}_2(t_1,t_2)\end{pmatrix}\mathrm{d}t_1\mathrm{d}t_2\\ &\geq\sigma_{\min}(\boldsymbol{R})\int_0^{\infty}\int_0^{\infty}\left|\begin{pmatrix}\delta_1\boldsymbol{x}_1(t_1,t_2)\\ \delta_2\boldsymbol{x}_2(t_1,t_2)\end{pmatrix}\right|^2\mathrm{d}t_1\mathrm{d}t_2.\end{aligned} \tag{4.52}$$

The right hand side of (4.51) can be bounded by

$$\int_0^{T_2}V_1(0,t_2)\mathrm{d}t_2+\int_0^{T_1}V_2(t_1,0)\mathrm{d}t_1\leq\int_0^{\infty}V_1(0,t_2)\mathrm{d}t_2+\int_0^{\infty}V_2(t_1,0)\mathrm{d}t_1\leq\|\boldsymbol{P}_1\|c_1+\|\boldsymbol{P}_2\|c_2. \tag{4.53}$$

Hence,

$$\int_0^{\infty}\int_0^{\infty}|\delta_1\boldsymbol{x}_1(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2\leq\frac{\|\boldsymbol{P}_1\|c_1+\|\boldsymbol{P}_2\|c_2}{\sigma_{\min}(\boldsymbol{R})}=:\overline{M}_{11}, \tag{4.54}$$

$$\int_0^{\infty}\int_0^{\infty}|\delta_2\boldsymbol{x}_2(t_1,t_2)|^2\,\mathrm{d}t_1\mathrm{d}t_2\leq\frac{\|\boldsymbol{P}_1\|c_1+\|\boldsymbol{P}_2\|c_2}{\sigma_{\min}(\boldsymbol{R})}=:\overline{M}_{22}. \tag{4.55}$$

To show the existence of $\overline{M}_{12}$ we will transform the solution given in (4.5) into

$$\begin{aligned}\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty|\delta_2\boldsymbol{x}_1(t_1,t_2)|^2\mathrm{d}t_1\mathrm{d}t_2 \leq{}& 2k_1^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty \mathrm{E}(-\lambda_1)^{2t_1}|\delta_2\boldsymbol{x}_{10}(t_2)|^2\mathrm{d}t_1\mathrm{d}t_2\\ &+2k_1^2\|\boldsymbol{A}_{12}\|^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left|\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}\delta_2\boldsymbol{x}_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2.\end{aligned}\tag{4.56}$$

Since the initial conditions are L_2' smooth the first term on the right side of (4.56) can be bounded by

$$2k_1^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty \mathrm{E}(-\lambda_1)^{2t_1}|\delta_2\boldsymbol{x}_{10}(t_2)|^2\mathrm{d}t_1\mathrm{d}t_2 \leq \frac{2k_1^2c_1'}{\lambda_1}.\tag{4.57}$$

The second term can be transformed using the Cauchy Schwarz inequality

$$\begin{aligned}2k_1^2\|\boldsymbol{A}_{12}\|^2&\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left|\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}\delta_2\boldsymbol{x}_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2\\ &\leq 2k_1^2\|\boldsymbol{A}_{12}\|^2\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\Bigg(\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{\tau}\mathrm{d}\tau\\ &\qquad\cdot\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}|\delta_2\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau\Bigg)\mathrm{d}t_1\mathrm{d}t_2.\end{aligned}\tag{4.58}$$

We will now solve the first inner generalised integral and change the order of (generalised) integration of the remaining part. Thus (4.58) becomes

$$\begin{aligned}2k_1^2\|\boldsymbol{A}_{12}\|^2&\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\left|\mathop{\mathcal{S}}_0^{t_1}\mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}\delta_2\boldsymbol{x}_2(\tau,t_2)\mathrm{d}\tau\right|^2\mathrm{d}t_1\mathrm{d}t_2\\ \leq&\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1}\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_{\tau_1+\mathbb{I}_1}^\infty \mathrm{E}(-\lambda_1)^{t_1-\mathbb{I}_1-\tau}|\delta_2\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}t_1\mathrm{d}\tau\mathrm{d}t_2\\ \leq&\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1^2}\mathop{\mathcal{S}}_0^\infty\mathop{\mathcal{S}}_0^\infty|\delta_2\boldsymbol{x}_2(\tau,t_2)|^2\mathrm{d}\tau\mathrm{d}t_2\\ \leq&\frac{2k_1^2\|\boldsymbol{A}_{12}\|^2}{\lambda_1^2}\overline{M}_{22}=:\overline{M}_{12}.\end{aligned}\tag{4.59}$$

(b): To complete the proof we will show that the second generalised derivatives are in L_∞ $[0,\infty)\times[0,\infty)$. First the norm of the generalised derivatives $\delta_1^2\boldsymbol{x}_1(t_1,t_2)$ and $\delta_1\delta_2\boldsymbol{x}_1(t_1,t_2)$ will be considered. Taking the generalised derivative of the first part of the state space description (4.4) with respect to t_1 or t_2, respectively, yields

$$\delta_1^2\boldsymbol{x}_1(t_1,t_2)=\boldsymbol{A}_{11}\delta_1\boldsymbol{x}_1(t_1,t_2)+\boldsymbol{A}_{12}\delta_1\boldsymbol{x}_2(t_1,t_2),\tag{4.60}$$

$$\delta_1\delta_2\boldsymbol{x}_1(t_1,t_2)=\boldsymbol{A}_{11}\delta_2\boldsymbol{x}_1(t_1,t_2)+\boldsymbol{A}_{12}\delta_2\boldsymbol{x}_2(t_1,t_2).\tag{4.61}$$

Thus $M_{111}:=\|\boldsymbol{A}_{11}\|M_{11}+\|\boldsymbol{A}_{12}\|M_{12}$ and $M_{112}=M_{121}:=\|\boldsymbol{A}_{11}\|M_{12}+\|\boldsymbol{A}_{12}\|M_{22}$. To show that $|\delta_2^2\boldsymbol{x}_1(t_1,t_2)|$ is bounded, follow a similar argument as in (4.50), so that M_{122} becomes

$$M_{122}:=k_1\zeta_1''+\frac{k_1\|\boldsymbol{A}_{12}\|M_{222}}{\lambda_1}.\tag{4.62}$$

The existence of $M_{211}, M_{212}, M_{221}$ and M_{222} can be proven in the same manner. □

We will now prove a two-dimensional version of Barbalat's Lemma, (Logemann and Ryan, 2004, Lemma 3.1), which will enable the proof of asymptotic stability of two-dimensional systems.

Lemma 4.4

Consider the function $f : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$. If $f(t_1, t_2)$ is both in L_p $[0,\infty) \times [0,\infty)$ and L_∞ $[0,\infty) \times [0,\infty)$ and both its generalised derivatives $\delta_1 f(t_1, t_2)$ and $\delta_2 f(t_1, t_2)$ are in L_∞ $[0,\infty) \times [0,\infty)$, then $\lim_{t_1+t_2\to\infty} f(t_1, t_2) = 0$ and $f(t_1, t_2)$ is uniformly convergent in both directions, i.e. for all $\epsilon > 0$ there exists a $T(\epsilon) < \infty$ such that

$$\forall (t_1, t_2) \in \{\mathbb{T}_1 \times [T(\epsilon), \infty)\} \cup \{[T(\epsilon), \infty) \times \mathbb{T}_2\} : |f(t_1, t_2)| < \epsilon. \tag{4.63}$$

Proof Define the supremum of $f(t_1, t_2)$ and the supremum over the maximum of both generalised derivatives in the complete quadrant as

$$\overline{f} := \sup_{t_1,t_2 \in \mathbb{T}_1\times\mathbb{T}_2} |f(t_1, t_2)| \quad \text{and} \tag{4.64}$$

$$\overline{f'} := \sup_{t_1,t_2 \in \mathbb{T}_1\times\mathbb{T}_2} \{\max\{|\delta_1 f(t_1, t_2)|, |\delta_2 f(t_1, t_2)|\}\} \tag{4.65}$$

and the region R_l as

$$R_l := \{[0, l+1) \times [l, l+1)\} \cup \{[l, l+1) \times [0, l)\}. \tag{4.66}$$

Note then that

$$\|f(\cdot,\cdot)\|^p_{L_p\ [0,\infty)\times[0,\infty)} = \sum_{l=0}^{\infty} \underset{R_l}{\mathcal{SS}} |f(t_1, t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 < \infty. \tag{4.67}$$

where $\mathcal{SS}_{R_l} \cdot \mathrm{d}t_1 \mathrm{d}t_2$ refers to the two-dimensional general integration over the region R_l. Therefore,

$$\lim_{l\to\infty} \underset{R_l}{\mathcal{SS}} |f(t_1, t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 = 0. \tag{4.68}$$

Let the supremum of f within R_l be defined as

$$\overline{f_l} := \sup_{(t_1,t_2)\in R_l} |f(t_1, t_2)|. \tag{4.69}$$

Then if t_1 is continuous

$$\sup_{(t_1,t_2)\in R_l} \frac{\mathrm{d}}{\mathrm{d}t_1} |f(t_1, t_2)|^p \le \sup_{(t_1,t_2)\in R_l} \left(p|f(t_1, t_2)|^{p-1} \left| \frac{\mathrm{d}}{\mathrm{d}t_1} f(t_1, t_2) \right| \right) \le p\overline{f_l}^{p-1}\overline{f'}. \tag{4.70}$$

We will now bound the double generalised integral $\mathcal{SS}_{R_l}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ from below using the geometric form of $f(t_1,t_2)$ depending on the nature of t_1 and t_2.

If both independent variables t_1 and t_2 are continuous, $\mathcal{SS}_{R_l}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ is the double integral over a L-shaped surface. It can be bounded from below by the smallest possible pyramid with height $\overline{f}_l^p$, where the base is bounded by $\frac{\overline{f}_l}{p\overline{f'}}$ or the dimensions of the region R_l.

In case one variable is continuous and one is discrete (mixed case) $\mathcal{SS}_{R_l}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ is a summation of l line integrals. It can be bounded from below by the smallest possible triangle with height $\overline{f}_l^p$, where the base is bounded by $\frac{\overline{f}_l}{p\overline{f'}}$ or the smallest possible length of any line fragment in R_l.

If both variables are discrete $\mathcal{SS}_{R_l}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2$ is a summation with $2l+1$ summands. Thus it can be bounded from below by a single summand. (Here we will take the maximal summand $\overline{f}_l^p$.)

$$\underset{R_l}{\mathcal{SS}}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2 \geq \begin{cases} \frac{1}{6}\overline{f_l}^p\cdot\min\left\{\frac{\overline{f_l}}{p\overline{f'}}, l+1\right\}\cdot\min\left\{\frac{\overline{f_l}}{p\overline{f'}},1\right\} & \text{if } t_1,t_2 \text{ continuous,}\\ \frac{1}{2}\overline{f_l}^p\cdot\min\left\{\frac{\overline{f_l}}{p\overline{f'}},1\right\} & \text{for mixed case,}\\ \overline{f_l}^p & \text{if } t_1,t_2 \text{ discrete.}\end{cases} \tag{4.71}$$

If both t_1 and t_2 are discrete the result follows immediately from (4.68). In the continuous case we can transform (4.71) into

$$\begin{aligned}\underset{R_l}{\mathcal{SS}}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2 \geq&\frac{1}{6}\overline{f_l}^p\cdot\min\left\{\frac{\overline{f_l}}{p\overline{f'}},(l+1)\frac{\overline{f_l}}{\overline{f}}\right\}\cdot\min\left\{\frac{\overline{f_l}}{p\overline{f'}},\frac{\overline{f_l}}{\overline{f}}\right\}\\ =&\frac{1}{6}\overline{f_l}^{p+2}\cdot\min\left\{\frac{1}{p\overline{f'}},\frac{l+1}{\overline{f}}\right\}\cdot\min\left\{\frac{1}{p\overline{f'}},\frac{1}{\overline{f}}\right\}.\end{aligned} \tag{4.72}$$

Thus

$$\begin{aligned}\overline{f_l}^{p+2} \leq&6\max\left\{p\overline{f'},\frac{\overline{f}}{l+1}\right\}\cdot\max\left\{p\overline{f'},\overline{f}\right\}\cdot\underset{R_l}{\mathcal{SS}}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2\\ \leq&6\left(\max\left\{p\overline{f'},\overline{f}\right\}\right)^2\cdot\underset{R_l}{\mathcal{SS}}\,|f(t_1,t_2)|^p\mathrm{d}t_1\mathrm{d}t_2.\end{aligned} \tag{4.73}$$

As $\overline{f'}$ and $\overline{f}$ are bounded $\overline{f_l}$ tends to zero as l grows without bound. Hence from the definition of $\overline{f}_l$ (4.69), $f(t_1,t_2)$ for $(t_1,t_2)\in R_l$ tends to zero as l grows without bound.

In case one variable is continuous and one is discrete a similar argument can be made. □

Most of the results derived above (such as Corollary 4.3, Lemma 4.3 and Lemma 4.4) will be used in Section 4.5 to prove asymptotic stability for systems with nonpositive divergence and smooth bounded initial conditions. They are not needed, however, for the following proof of exponential stability.

4.4 Exponential Stability

Before discussing asymptotic stability we will now present a theorem regarding exponential stability of two-dimensional Roesser models. If a system with exponentially decaying initial conditions admits a two-dimensional quadratic Lyapunov function with a strictly negative definite divergence, it is exponentially stable.

Theorem 4.1 (Exponential Stability of Linear Two-Dimensional Systems)

The two-dimensional system (4.4) is exponentially stable if the following conditions hold

(i) the matrices $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable, and

(ii) there exist two positive definite, symmetric matrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ and scalars $\alpha_1, \alpha_2 > 0$ such that $\boldsymbol{Q} \leq -(\alpha_1 \boldsymbol{P}_1 \oplus \alpha_2 \boldsymbol{P}_2)$, where $\boldsymbol{Q}$ is given in (4.14).

Proof We will show that the two-dimensional system (4.4) with exponentially decaying initial conditions is exponentially stable by showing that there exist $\eta_1, \eta_2 > 0$ such that the system

$$\tilde{\boldsymbol{x}}_1(t_1, t_2) = \mathrm{e}^{\eta_1 t_1} \mathrm{e}^{\eta_2 t_2} \boldsymbol{x}_1(t_1, t_2) \tag{4.74}$$

$$\tilde{\boldsymbol{x}}_2(t_1, t_2) = \mathrm{e}^{\eta_1 t_1} \mathrm{e}^{\eta_2 t_2} \boldsymbol{x}_2(t_1, t_2) \tag{4.75}$$

is stable.

The dynamical equation for the new, autonomous system is

$$\delta \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{A}} \tilde{\boldsymbol{x}} \quad \text{with} \tag{4.76}$$

$$\begin{aligned} \tilde{\boldsymbol{A}} = & \begin{bmatrix} \mathrm{e}^{\eta_1 \mathbb{I}_1} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathrm{e}^{\eta_2 \mathbb{I}_2} \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \\ & + \begin{bmatrix} (\eta_1 (1 - \mathbb{I}_1) + (\mathrm{e}^{\eta_1} - 1) \, \mathbb{I}_1) \, \mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\eta_2 (1 - \mathbb{I}_2) + (\mathrm{e}^{\eta_2} - 1) \, \mathbb{I}_2) \, \mathbf{I} \end{bmatrix} . \end{aligned} \tag{4.77}$$

Since the matrices $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are stable we can choose a $\eta_i > 0$ small enough to guarantee that $\boldsymbol{A}_{ii} + \eta_i \mathbf{I}$ is Hurwitz stable (in case t_i is continuous, $\eta_i < -\max_k \Re\{\lambda_k(\boldsymbol{A}_{ii})\}$) or $\mathrm{e}^{\eta_i}(\boldsymbol{A}_{ii} + \mathbf{I})$ is Schur stable (in case t_i is discrete, $\eta_i < -\ln(\max_k |\lambda_k(\boldsymbol{A}_{ii})|)$). Note that $\lambda_k(\boldsymbol{A})$ is the kth eigenvalue of $\boldsymbol{A}$.

Using the new Lyapunov function

$$\tilde{\boldsymbol{V}}(t_1, t_2) = \begin{bmatrix} \tilde{\boldsymbol{x}}_1^{\mathrm{T}}(t_1, t_2) & \mathbf{0} \\ \mathbf{0} & \tilde{\boldsymbol{x}}_2^{\mathrm{T}}(t_1, t_2) \end{bmatrix} \boldsymbol{P} \begin{bmatrix} \tilde{\boldsymbol{x}}_1(t_1, t_2) \\ \tilde{\boldsymbol{x}}_2(t_1, t_2) \end{bmatrix} \tag{4.78}$$

with $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$, (4.14), (4.76) and (4.77) we see that

$$\begin{aligned}\operatorname{div}\tilde{\boldsymbol{V}}(t_1,t_2) \leq & \tilde{\boldsymbol{x}}^{\mathrm{T}}(t_1,t_2)\left(\boldsymbol{Q} + \begin{bmatrix}\tilde{\eta}_1\boldsymbol{P}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \tilde{\eta}_2\boldsymbol{P}_2\end{bmatrix}\right)\tilde{\boldsymbol{x}}(t_1,t_2)\\ \leq & \tilde{\boldsymbol{x}}^{\mathrm{T}}(t_1,t_2)\begin{bmatrix}(\tilde{\eta}_1-\alpha_1)\mathbf{I} & \boldsymbol{0}\\ \boldsymbol{0} & (\tilde{\eta}_2-\alpha_2)\mathbf{I}\end{bmatrix}\boldsymbol{P}\tilde{\boldsymbol{x}}(t_1,t_2)\end{aligned} \tag{4.79}$$

where $\tilde{\eta}_i = 2\eta_i(1-\mathbb{I}_i) + (\mathrm{e}^{2\eta_i}-1)\|\boldsymbol{A}+\mathbf{I}\|^2\mathbb{I}_i$ for $i \in \{1,2\}$. For any $\alpha_i > 0$ it is possible to find a $\eta_i \leq \alpha_i/2$ in case t_i is continuous and $\eta_i \leq \frac{1}{2}\ln\left(\frac{\alpha_i}{\|\boldsymbol{A}+\mathbf{I}\|^2}+1\right)$ in case t_i is discrete to ensure that the divergence of $\tilde{\boldsymbol{V}}(t_1,t_2)$ is always nonpositive.

Finally the initial conditions $\tilde{\boldsymbol{x}}_{10}(t_2)$ and $\tilde{\boldsymbol{x}}_{20}(t_1)$ need to be bounded. Using the fact that $\boldsymbol{x}_{10}(t_2)$ and $\boldsymbol{x}_{20}(t_1)$ are exponentially decaying we see that

$$|\tilde{\boldsymbol{x}}_{10}(t_2)| \leq \kappa_1\mathrm{e}^{(\eta_2-\mu_2)t_2} \quad \text{and} \quad |\tilde{\boldsymbol{x}}_{20}(t_1)| \leq \kappa_2\mathrm{e}^{(\eta_1-\mu_1)t_1} \tag{4.80}$$

are bounded if $\eta_1 \leq \mu_1$ and $\eta_2 \leq \mu_2$. Thus, for

$$\eta_i \leq \begin{cases}\min\{-\max_k \Re\{\lambda_k(\boldsymbol{A}_{ii})\}, \quad \alpha_i, \quad \mu_i\} & \text{if } t_i \text{ continuous,}\\ \min\left\{-\ln(\max_k|\lambda_k(\boldsymbol{A}_{ii})|), \quad \frac{1}{2}\ln\left(\frac{\alpha_i}{\|\boldsymbol{A}+\mathbf{I}\|^2}+1\right), \quad \mu_i\right\} & \text{if } t_i \text{ discrete.}\end{cases} \tag{4.81}$$

Hence, $\tilde{\boldsymbol{x}}_1(t_1,t_2)$ and $\tilde{\boldsymbol{x}}_2(t_1,t_2)$ are bounded for every $t_1,t_2 > 0$ using Corollary 4.1 on page 63. Therefore from (4.74) and (4.75) the two-dimensional system (4.4) with exponentially decaying initial conditions is exponentially stable. □

Although the Lyapunov type argument above is only sufficient for exponential stability we will now present a lemma proving that a system with a SSB cannot be exponentially stable.

Lemma 4.5

If the two-dimensional system (4.4) includes singularities on the stability boundary (SSB) it cannot be exponentially stable.

Proof The proof follows by contradiction. We will show that there exists no set $\eta_1,\eta_2 > 0$ such that the system with $\tilde{\boldsymbol{x}}(t_1,t_2) = \mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2 t_2}\boldsymbol{x}(t_1,t_2)$ is stable. Note that if t_i is continuous the $\mathcal{T}_i$ transform of $\delta_i\boldsymbol{x}_i(t_i)$ is

$$\mathcal{T}_i\{\delta_i\boldsymbol{x}_i(t_i)\} = \mathcal{L}\{\dot{\boldsymbol{x}}(t_i)\} = s_i\boldsymbol{X}(s_i) - \boldsymbol{x}(0). \tag{4.82}$$

Thus in this case for $\dot{\boldsymbol{x}}(t_i) = \boldsymbol{A}\boldsymbol{x}(t_i)$ we have

$$\boldsymbol{X}(s_i) = (s_i\mathbf{I} - \boldsymbol{A})^{-1}x(0). \tag{4.83}$$

For t_i discrete the $\mathcal{T}_i$ transform of $\delta_i\boldsymbol{x}_i(t_i)$ is

$$\mathcal{T}_i\{\delta_i\boldsymbol{x}_i(t_i)\} = \mathcal{Z}\{\Delta\boldsymbol{x}(t_i)\} = z_i\boldsymbol{X}(z_i) - z_i\boldsymbol{x}(0) - \boldsymbol{X}(z_i). \tag{4.84}$$

Thus in this case for $\Delta\boldsymbol{x}(t_i) = \boldsymbol{A}\boldsymbol{x}(t_i)$ we have

$$\boldsymbol{X}(z_i) = ((z_i - 1)\mathbf{I} - \boldsymbol{A})^{-1} z_i x(0). \tag{4.85}$$

Hence, we can transform the system description given in (4.76)-(4.77) into

$$\tilde{\boldsymbol{X}}(\xi_1,\xi_2) = \left(\begin{bmatrix}(\xi_1 - \mathbb{I}_1)\,\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (\xi_2 - \mathbb{I}_2)\,\mathbf{I}\end{bmatrix} - \tilde{\boldsymbol{A}}\right)^{-1} \tag{4.86}$$
$$\cdot\begin{bmatrix}(1-\mathbb{I}_1+\xi_1\mathbb{I}_1)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (1-\mathbb{I}_2+\xi_2\mathbb{I}_2)\mathbf{I}\end{bmatrix}\begin{pmatrix}\mathcal{T}_2\{\tilde{\boldsymbol{x}}_1(0,t_2)\}\\ \mathcal{T}_1\{\tilde{\boldsymbol{x}}_2(t_1,0)\}\end{pmatrix}$$
$$= \left(\begin{bmatrix}(\xi_1 - \eta_1(1-\mathbb{I}_1) - \mathrm{e}^{\eta_1}\mathbb{I}_1)\,\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (\xi_2 - \eta_2(1-\mathbb{I}_2) - \mathrm{e}^{\eta_2}\mathbb{I}_2)\,\mathbf{I}\end{bmatrix} - \begin{bmatrix}\mathrm{e}^{\eta_1\mathbb{I}_1}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathrm{e}^{\eta_2\mathbb{I}_2}\mathbf{I}\end{bmatrix}\boldsymbol{A}\right)^{-1}$$
$$\cdot\begin{bmatrix}(1-\mathbb{I}_1+\xi_1\mathbb{I}_1)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (1-\mathbb{I}_2+\xi_2\mathbb{I}_2)\mathbf{I}\end{bmatrix}\begin{pmatrix}\mathcal{T}_2\{\tilde{\boldsymbol{x}}_1(0,t_2)\}\\ \mathcal{T}_1\{\tilde{\boldsymbol{x}}_2(t_1,0)\}\end{pmatrix}. \tag{4.87}$$

Inserting $\left(\mathrm{e}^{\eta_1\mathbb{I}_1}\mathbf{I}\oplus\mathrm{e}^{\eta_2\mathbb{I}_2}\mathbf{I}\right)\cdot\left(\mathrm{e}^{-\eta_1\mathbb{I}_1}\mathbf{I}\oplus\mathrm{e}^{-\eta_2\mathbb{I}_2}\mathbf{I}\right)$ and using the fact that $\mathbb{I}_i$ can only be either 0 or 1 for $i\in\{1,2\}$, (4.87) yields

$$\tilde{\boldsymbol{X}}(\xi_1,\xi_2) = \left(\begin{bmatrix}(\xi_1 - \mathbb{I}_1)\,\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (\xi_2 - \mathbb{I}_2)\,\mathbf{I}\end{bmatrix} - \tilde{\boldsymbol{A}}\right)^{-1}$$
$$\cdot\begin{bmatrix}(1-\mathbb{I}_1+\xi_1\mathbb{I}_1)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (1-\mathbb{I}_2+\xi_2\mathbb{I}_2)\mathbf{I}\end{bmatrix}\begin{pmatrix}\mathcal{T}_2\{\tilde{\boldsymbol{x}}_1(0,t_2)\}\\ \mathcal{T}_1\{\tilde{\boldsymbol{x}}_2(t_1,0)\}\end{pmatrix}$$
$$= \left(\begin{bmatrix}\left(\mathrm{e}^{-\eta_1\mathbb{I}_1}\xi_1 - \eta_1(1-\mathbb{I}_1) - \mathbb{I}_1\right)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \left(\mathrm{e}^{-\eta_2\mathbb{I}_2}\xi_2 - \eta_2(1-\mathbb{I}_2) - \mathbb{I}_2\right)\mathbf{I}\end{bmatrix} - \boldsymbol{A}\right)^{-1}$$
$$\cdot\begin{bmatrix}(1-\mathbb{I}_1+\mathrm{e}^{-\eta_1}\xi_1\mathbb{I}_1)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (1-\mathbb{I}_2+\mathrm{e}^{-\eta_2}\xi_2\mathbb{I}_2)\mathbf{I}\end{bmatrix}\begin{pmatrix}\mathcal{T}_2\{\tilde{\boldsymbol{x}}_1(0,t_2)\}\\ \mathcal{T}_1\{\tilde{\boldsymbol{x}}_2(t_1,0)\}\end{pmatrix}. \tag{4.88}$$

From (4.86) we see that if there exists a set ξ_1 and ξ_2 with $\Re\{\xi_i\} > 0$ (for t_i continuous) or $|\xi_i| > 1$ (for t_i discrete) such that $\det(((\xi_1 - \mathbb{I}_1)\mathbf{I}\oplus(\xi_2 - \mathbb{I}_2)\mathbf{I}) - \tilde{\boldsymbol{A}}) = 0$ the system (4.76)-(4.77) is unstable.

Choosing $\xi_i = s_i = \eta_1 + j\omega_i$ if t_i is continuous or $\xi_i = z_i = \mathrm{e}^{\eta_2}\mathrm{e}^{j\theta_i}$ if t_i is discrete it becomes clear that

$$\begin{bmatrix}(\xi_1 - \mathbb{I}_1)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (\xi_2 - \mathbb{I}_2)\mathbf{I}\end{bmatrix} - \tilde{\boldsymbol{A}}$$
$$= \begin{bmatrix}(j\omega_1(1-\mathbb{I}_1) + (\mathrm{e}^{j\theta_1}-1)\mathbb{I}_1)\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (j\omega_2(1-\mathbb{I}_2) + (\mathrm{e}^{j\theta_2}-1)\mathbb{I}_2)\mathbf{I}\end{bmatrix} - \boldsymbol{A}. \tag{4.89}$$

Since the system with matrix $\boldsymbol{A}$ includes at least one singularity on the stability boundary, there exists a set of ω_i (if t_i is continuous) or θ_i (if t_i is discrete) for $i\in\{1,2\}$ such that the determinant of the right hand side of (4.89) is 0.

Thus the system given in (4.76)-(4.77) with a singularity outside the stability region is unstable and the system (4.4) with a singularity on the stability boundary cannot be exponentially stable. □

4.5 Asymptotic Stability

In this section we will present our theorem on asymptotic stability of two-dimensional systems described by the Roesser model using intermediate results presented in the Section 4.3.

Even though systems with SSB cannot be exponentially stable as shown above, asymptotic stability can be guaranteed if the divergence of the Lyapunov function is negative semi-definite. However, the initial conditions have to fulfil stricter requirements.

Theorem 4.2 (Asymptotic Stability of Two-Dimensional Roesser Models)

The two-dimensional system (4.4) is asymptotically stable with smooth bounded initial conditions according to Definition 4.6 if the following conditions hold

(i) $\boldsymbol{A}_{11}$ *and* $\boldsymbol{A}_{22}$ *are stable, and*

(ii) there exist positive definite, symmetric matrices $\boldsymbol{P}_1$, $\boldsymbol{P}_2$ *and* $\boldsymbol{R}$ *such that* $\boldsymbol{Q} = -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}$, *where* $\boldsymbol{Q}$ *is given in (4.14).*

Proof Consider the two-dimensional Lyapunov function $\boldsymbol{V}(t_1, t_2)$ given in (4.12) and the integral of $V_1(t_1,t_2) + V_2(t_1,t_2)$ along the line $\Omega(l) := (t_1,t_2) \in \{[0,l] \times \{l\}\} \cup \{\{l\} \times [0,l]\}$ for $l \in \mathbb{R}_+$ or $l \in \mathbb{N}$, respectively, and $l > 0$ defined as:

$$\begin{aligned} U(l) :=& \underset{\Omega(l)}{\mathcal{S}} \left(V_1(t_1,t_2) + V_2(t_1,t_2)\right) \mathrm{d}s \\ =& \overset{l}{\underset{0}{\mathcal{S}}} \left(V_1(t_1,l) + V_2(t_1,l)\right) \mathrm{d}t_1 + \overset{l}{\underset{0}{\mathcal{S}}} \left(V_1(l,t_2) + V_2(l,t_2)\right) \mathrm{d}t_2. \end{aligned} \tag{4.90}$$

Using the results in Lemma 4.2 and Corollary 4.3 we see that there exists a C such that for all l: $U(l) \le C$. Since the first generalised derivatives of $\boldsymbol{x}(t_1,t_2)$ with respect to t_1 and t_2 are L_∞ bounded (Lemma 4.3) we can find $d_{11}(l)$, $d_{12}(l)$, $d_{21}(l)$ and $d_{22}(l)$ such that

$$d_{11}(l) := \sup_{0\le t_1\le l} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2\,, \qquad d_{12}(l) := \sup_{0\le t_2\le l} |\delta_2 \boldsymbol{x}_1(l,t_2)|_2\,, \tag{4.91}$$

$$d_{21}(l) := \sup_{0\le t_1\le l} |\delta_1 \boldsymbol{x}_2(t_1,l)|_2\,, \quad \text{and} \qquad d_{22}(l) := \sup_{0\le t_2\le l} |\delta_2 \boldsymbol{x}_2(l,t_2)|_2\,. \tag{4.92}$$

Note that $d_{11}(l) \le \sup_{t_1\ge 0} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2$. Making use of the version of Barbalat's Lemma in Lemma 4.4, we can conclude that the first generalised derivatives tend to zero as $t_1, t_2 \to \infty$

and are uniformly convergent in both directions. That allows us to interchange the order of supremum and limit and thus we conclude that

$$\begin{aligned}\lim_{l\to\infty} d_{11}(l) \leq & \lim_{l\to\infty} \sup_{t_1\geq 0} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2 \\ =& \sup_{t_1\geq 0} \lim_{l\to\infty} |\delta_1 \boldsymbol{x}_1(t_1,l)|_2 \\ =&0. \end{aligned} \tag{4.93}$$

It can be shown in a similar way that the limits of $d_{12}(l)$, $d_{21}(l)$, and $d_{22}(l)$ for $l \to \infty$ are 0.

Thus for t_1, t_2 continuous we can bound the derivatives of $V_1(t_1,t_2)$ and $V_2(t_1,t_2)$ by

$$\forall t_1 \leq l : \frac{\mathrm{d}}{\mathrm{d}t_1} V_1(t_1,l) \leq 2d_{11}(l)\|\boldsymbol{P}_1\|M_1, \tag{4.94}$$

$$\forall t_2 \leq l : \frac{\mathrm{d}}{\mathrm{d}t_2} V_1(l,t_2) \leq 2d_{12}(l)\|\boldsymbol{P}_1\|M_1, \tag{4.95}$$

$$\forall t_1 \leq l : \frac{\mathrm{d}}{\mathrm{d}t_1} V_2(t_1,l) \leq 2d_{21}(l)\|\boldsymbol{P}_2\|M_2, \tag{4.96}$$

$$\forall t_2 \leq l : \frac{\mathrm{d}}{\mathrm{d}t_2} V_2(l,t_2) \leq 2d_{22}(l)\|\boldsymbol{P}_2\|M_2 \tag{4.97}$$

where M_1 and M_2 are bounds on $|\boldsymbol{x}_1(t_1,t_2)|$ and $|\boldsymbol{x}_2(t_1,t_2)|$ (as introduced in the proof of Lemma 4.3). Note that in fact the same bounds apply for t_1 or t_2 discrete because for t_1 discrete we have for $(t_1,t_2) \in \Omega(l)$

$$\delta_1\left(\boldsymbol{x}_1^{\mathrm{T}} \boldsymbol{P}_1 \boldsymbol{x}_1\right) \leq \boldsymbol{x}_1^{\mathrm{T}} \boldsymbol{P}_1 \boldsymbol{x}_1 - (\boldsymbol{x}_1 - d_{11}(l)\mathbf{1})^{\mathrm{T}} \boldsymbol{P}_1 (\boldsymbol{x}_1 - d_{11}(l)\mathbf{1}) \leq 2d_{11}(l)\|\boldsymbol{P}_1\|M_1 \tag{4.98}$$

where $\mathbf{1}$ is a vector of 1s of appropriate length.

To find a lower bound on $U(l)$ we will use a similar trick as in the proof of Lemma 4.4 above: If t_1 is continuous and the maximum of $V_i(t_1,t_2)$

$$\overline{V}_i(l) := \max_{(t_1,t_2)\in\Omega(l)} V_i(t_1,t_2) \tag{4.99}$$

for $i \in \{1,2\}$) along $\Omega(l)$ occurs along the part of $\Omega(l)$ where $(t_1,t_2) \in [0,l] \times \{l\}$ we can bound the integral of $V_i(t_1,t_2)$ over $\Omega(l)$ from below by a triangle with the base equal to $\min\left\{\overline{V}_i(l)/(2d_{i1}(l)\|\boldsymbol{P}_i\|M_i), l\right\}$ and $\overline{V}_i(l)$ as the height of the triangle. Both possible triangles are illustrated in Figure 4.1.

In case t_1 is discrete and $\overline{V}_i(l)$ occurs at $(t_1,t_2) \in [0,l] \times \{l\}$ the summation of $V_i(t_1,t_2)$ along t_1 for $l > \overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i$ can be bounded by

$$\begin{aligned}\sum_{t_1=0}^{l} V_i(t_1,t_2) \geq& \overline{V}_i(l) + \left(\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i\right) + \left(\overline{V}_i(l) - 4d_{i1}(l)\|\boldsymbol{P}_i\|M_i\right) + \dots \\ =&(\nu+1)\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i \sum_{n=1}^{\nu} n \end{aligned} \tag{4.100}$$

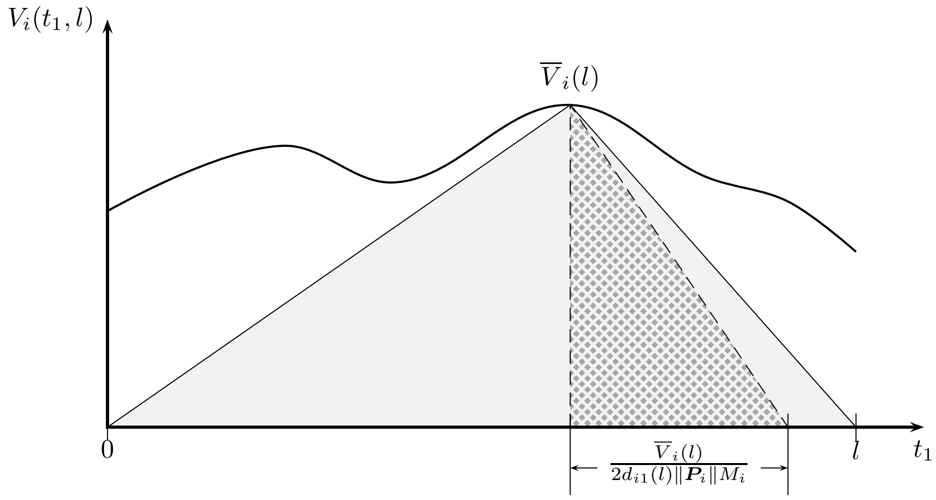


Figure 4.1: *Approximating $U(l)$ from below by a triangle*

where $\nu = \lfloor \overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i \rfloor$. Resolving the summation on the right hand side of (4.100) yields

$$\begin{aligned}\sum_{t_1=0}^{l} V_i(t_1, t_2) \geq &(\nu+1)\left(\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i\frac{\nu}{2}\right)\\ \geq &(\nu+1)\left(\overline{V}_i(l) - 2d_{i1}(l)\|\boldsymbol{P}_i\|M_i\frac{\overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i}{2}\right)\\ = &(\nu+1)\frac{\overline{V}_i(l)}{2}\\ \geq &\frac{\overline{V}_i^2(l)}{d_{i1}(l)\|\boldsymbol{P}_i\|M_i}.\end{aligned} \tag{4.101}$$

For $l \leq \overline{V}_i(l)/2d_{i1}(l)\|\boldsymbol{P}_i\|M_i$ the summation can be bounded by

$$\sum_{t_1=0}^{l} V_i(t_1, t_2) \geq \frac{\overline{V}_i(l)l}{2}. \tag{4.102}$$

Thus, $U(l)$ can be bounded from below by

$$\begin{aligned}U(l) \geq &\min\left\{\frac{\overline{V}_1^2(l)}{4d_{11}(l)\|\boldsymbol{P}_1\|M_1}, \quad \frac{\overline{V}_1^2(l)}{4d_{12}(l)\|\boldsymbol{P}_1\|M_1}, \quad \frac{\overline{V}_1(l)l}{2}\right\}\\ &+\min\left\{\frac{\overline{V}_2^2(l)}{4d_{21}(l)\|\boldsymbol{P}_2\|M_2}, \quad \frac{\overline{V}_2^2(l)}{4d_{22}(l)\|\boldsymbol{P}_2\|M_2}, \quad \frac{\overline{V}_2(l)l}{2}\right\}.\end{aligned} \tag{4.103}$$

Since $\overline{V}_i(l) \leq M_i^2\|\boldsymbol{P}_i\|$, this implies

$$\overline{V}_i^2(l) \leq C \cdot \max\left\{4d_{i1}(l)\|\boldsymbol{P}_i\|M_i, \quad 4d_{i2}(l)\|\boldsymbol{P}_i\|M_i, \quad \frac{2M_i^2\|\boldsymbol{P}_i\|}{l}\right\} \tag{4.104}$$

Note that as l tends to infinity each component of the maximum in (4.104) goes to zero and, hence, $\lim_{t_1,t_2\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$. Note that the limits $\lim_{t_1\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$ and $\lim_{t_2\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$ exist as well. □

4.6 Examples

To illustrate our result on asymptotic stability of two-dimensional systems we will discuss a simple 'platooning' problem. Consider a group or platoon of vehicles driving on a straight road one after each other. Every vehicle is equipped with a controller and aims to maintain a specified distance to its predecessor while the first vehicle is following a given reference signal. Only locally measurable data such as the distance to the predecessor shall be used.

Example 4.1 We will use a simplified, linearised second order model for each vehicle introduced in (3.64) and the simple PID controller introduced in (3.65) to minimize the local spacing error. (A more detailed discussion of the system can be found in (Klinge, 2008, p. 7).) It can be shown that using a fixed distance policy will lead to string instability or the 'slinky effect', where disturbances are amplified while traveling through the string, Klinge (2008).

One known possibility, Chien and Ioannou (1992), to avoid that problem is to introduce a time headway h and maintain a velocity depending distance between each vehicle and its predecessor rather than a fixed distance. Hence, the new local error is

$$\hat{e}(t,k) = \hat{x}(t,k-1) - \hat{x}(t,k) - h\hat{v}(t,k). \tag{4.105}$$

Note that $t_1 = t$ is continuous and $t_2 = k$ is discrete. In order to maintain the same closed loop poles of the kth vehicle an additional pole at $-\frac{1}{h}$ is added to each local controller. A block diagram of the system can be found in Figure 3.2 on page 33. Thus the system can be described as a two-dimensional Roesser model as $\delta\boldsymbol{x}(t,k) = \boldsymbol{A}\boldsymbol{x}(t,k)$ with

$$\boldsymbol{A} = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2C_\mathrm{d}v_0 & 1 & 0 & 0 & 0 \\ -\frac{1}{h}\left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right) & -\left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right) & -\frac{1}{h} & \frac{1}{h} & -\frac{k_\mathrm{d}}{hT^2} & \frac{1}{h}\left(k_\mathrm{p} + \frac{k_\mathrm{d}}{T}\right) \\ -k_\mathrm{i} & -hk_\mathrm{i} & 0 & 0 & 0 & k_\mathrm{i} \\ -1 & -h & 0 & 0 & -\frac{1}{T} & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 \end{array}\right] \tag{4.106}$$

where $\boldsymbol{x}_1(t,k)$ is the state vector of the kth vehicle including its position $\hat{x}(t,k)$, velocity $\hat{v}(t,k)$, and three controller states $\hat{x}_{\mathrm{ctr.}_i}(t,k)$ for $i \in \{1,2,3\}$ and $\boldsymbol{x}_2(t,k)$ is the position of the predecessor to vehicle k at time t, $\hat{x}(t,k-1)$.

It can be shown that choosing a time headway $h > 1.18$ the system is string stable, Klinge and Middleton (2009b), see also Section 3.4 and Figure 3.3 on page 38. For $h = 2$

the eigenvalues of the upper left part of $\boldsymbol{A}$ are -25.1, -4.5, -0.5, -0.25 and -0.18. Thus $\boldsymbol{A}_{11}$ is Hurwitz stable. $\boldsymbol{A}_{22} + \mathbf{I} = 0$ is Schur stable.

Note that as discussed in Section 3.4, the transfer function of the corresponding input-output system has a NSSK at $s = 0$ and $z = 1$, or $\omega = 0$ and $\theta = 0$. Thus, $(j\omega\mathbf{I} \oplus (\mathrm{e}^{j\theta} - 1)) - \boldsymbol{A} = -\boldsymbol{A}$ has one eigenvalue at 0 for $\omega = 0$ and $\theta = 0$. Therefore, there exists no $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$ such that $\boldsymbol{Q}$ is sign definite.

Matlab finds two symmetric, positive definite matrices:

$$\boldsymbol{P}_1 = \begin{bmatrix} 1.72 \cdot 10^3 & 0 & 0 & 5.05 \cdot 10^3 & 0 \\ 0 & 2.65 \cdot 10^3 & 2.09 \cdot 10^3 & -1.69 \cdot 10^4 & -1.27 \cdot 10^5 \\ 0 & 2.09 \cdot 10^3 & 5.92 \cdot 10^3 & -2.16 \cdot 10^4 & -3.65 \cdot 10^5 \\ 5.05 \cdot 10^3 & -1.69 \cdot 10^4 & -2.16 \cdot 10^4 & 1.77 \cdot 10^5 & 1.32 \cdot 10^6 \\ 0 & -1.27 \cdot 20^5 & -3.65 \cdot 10^5 & 1.32 \cdot 10^6 & 2.25 \cdot 10^7 \end{bmatrix} \tag{4.107}$$

with eigenvalues at $2.26 \cdot 10^7$, 10^5, $1.78 \cdot 10^3$, 711 and 7.84 and $\boldsymbol{P}_2 = 859$ such that the eigenvalues of $\boldsymbol{Q}$ are $-4.5 \cdot 10^6$, $-1.78 \cdot 10^4$, $-4.48 \cdot 10^3$, -587, -106 and 0. There exists a positive definite matrix

$$\boldsymbol{R} = \begin{bmatrix} 3.01 \cdot 10^3 & -7.08 \cdot 10^3 & -1.15 \cdot 10^4 & 6.98 \cdot 10^4 & 7.08 \cdot 10^5 & 0 \\ -7.08 \cdot 10^3 & 2.96 \cdot 10^4 & 4.95 \cdot 10^4 & -2.95 \cdot 10^5 & 3.04 \cdot 10^6 & 0 \\ -1.15 \cdot 10^4 & 4.95 \cdot 10^4 & 8.64 \cdot 10^4 & -4.93 \cdot 10^5 & -5.31 \cdot 10^6 & 0 \\ 6.98 \cdot 10^4 & -2.95 \cdot 10^5 & -4.93 \cdot 10^5 & 2.96 \cdot 10^6 & 3.02 \cdot 10^7 & 0 \\ 7.08 \cdot 10^5 & -3.04 \cdot 10^6 & -5.31 \cdot 10^6 & 3.02 \cdot 10^7 & 3.26 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.108}$$

such that $\boldsymbol{Q} = -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A}$. Thus, the system with L_2' and L_∞'' smooth bounded initial conditions is asymptotically stable in the two-dimensional sense and hence string stable.

Simulation results are displayed in Figure 4.2 and Figure 4.3. In Figure 4.2 we see that the local error signal $\hat{e}(t,k)$ tends to zero for $t \to \infty$. Hence, every single subsystem is asymptotically stable. Also the maximum of $\hat{e}(t,k)$ over time and the L_2 norm with respect to time decreases when k grows. See Figure 4.3 for details. $*$

After demonstrating an affirming example, where asymptotic stability can be shown using Theorem 4.2, we will choose two examples, where one condition for asymptotic stability in Theorem 4.2 is violated each time and the system is not asymptotically stable. In this way we show that there is no trivial relaxation of the conditions for Theorem 4.2 that produces the same result.

Example 4.2 Consider the same system structure as presented in Example 4.1. However, choosing a time headway of $h = 0.5$ that is clearly less than the infimal time headway required, will lead to a string unstable system.

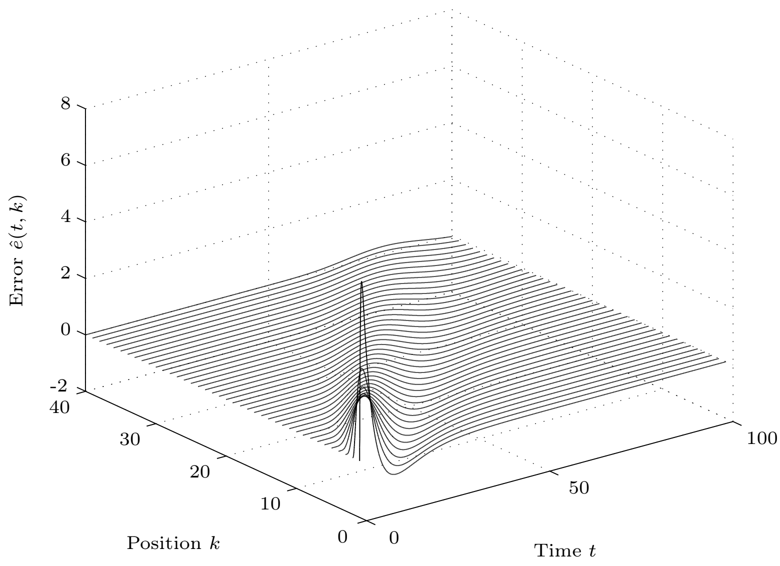


Figure 4.2: *String stable system with* $h = 2$*: error* $\hat{e}(t, k)$

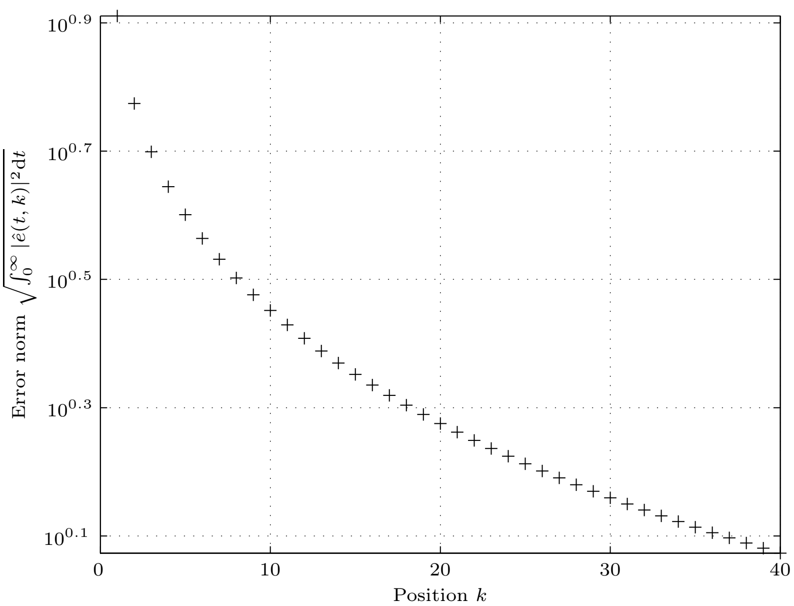


Figure 4.3: *String stable system with* $h = 2$*:* L_2 *norm of error* $\hat{e}(t, k)$

Even though $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}+\mathbf{I}$ are Hurwitz and Schur stable, respectively, it is not possible to find a symmetric, positive definite matrix $\boldsymbol{P}$ such that $\boldsymbol{Q} \leq 0$. Suppose that there exist $\boldsymbol{P}_1, \boldsymbol{P}_2 > 0$ such that $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2$. Since $\boldsymbol{P}_2$ and $\boldsymbol{x}_2$ are scalars, we can set $\boldsymbol{P}_2 = 1$ without loss of generality. It yields

$$\begin{aligned}\boldsymbol{Q} =& \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}} \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 1 \end{array}\right] \boldsymbol{A} \\ =& \begin{bmatrix} \boldsymbol{A}_{11}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A}_{11} + \boldsymbol{A}_{21}^{\mathrm{T}}\boldsymbol{A}_{21} & \boldsymbol{P}_1\boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{P}_1 & -1 \end{bmatrix}. \end{aligned} \tag{4.109}$$

Using the Schur compliment we see that $\boldsymbol{Q} \leq 0$ is equivalent to

$$\boldsymbol{A}_{11}^{\mathrm{T}}\boldsymbol{P}_1 + \boldsymbol{P}_1\boldsymbol{A}_{11} + \boldsymbol{A}_{21}^{\mathrm{T}}\boldsymbol{A}_{21} + \boldsymbol{P}_1\boldsymbol{A}_{12}\boldsymbol{A}_{12}^{\mathrm{T}}\boldsymbol{P}_1 \leq 0. \tag{4.110}$$

Using the Bounded Real Lemma this yields

$$\left\|\boldsymbol{A}_{21}\left(j\omega\mathbf{I} - \boldsymbol{A}_{11}\right)^{-1}\boldsymbol{A}_{12}\right\|_{\infty} \leq 1. \tag{4.111}$$

Note that $\Gamma(j\omega) = \boldsymbol{A}_{21}\left(j\omega\mathbf{I} - \boldsymbol{A}_{11}\right)^{-1}\boldsymbol{A}_{12}$ is the transfer function from the position of the kth vehicle to the position of the $k+1$th vehicle. However, when choosing a time headway that is less than the infimal headway $h_0 = 1.18$ we know that $\|\Gamma(j\omega)\|_{\infty} > 1$. Therefore, generally, any string unstable system of this type ($\|\Gamma(j\omega)\|_{\infty} > 1$), does not permit a solution with $\boldsymbol{Q} \leq 0$.

In the simulation (displayed in Figure 4.4) we observe that the system is not stable in the two-dimensional sense and thus not string stable because a small perturbation at the beginning of the string is amplified while traveling through the string. The local error $\hat{e}(t,k)$ goes to zero for every fixed k as $t \to \infty$. However, the maximal error over time for each subsystem grows with k and the double limit $\lim_{t,k\to\infty} \hat{e}(t,k)$ does not exist. Also the L_2 norm of $\hat{e}(t,k)$ with respect to time grows as k grows, Figure 4.5. $\ast$

Also when relaxing the first condition for asymptotic stability in Theorem 4.2 and allowing $\boldsymbol{A}_{11}$ or $\boldsymbol{A}_{22}$ not to be stable the system might not be asymptotically stable.

Example 4.3 Consider the system described in (4.106) with the general error $\hat{e}_{\mathrm{g}}(t,k)$ (that is the general error of the predecessor plus the local error) as an additional state in $\boldsymbol{x}_2(t,k)$ (and $h=2$) such that the system matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \left[\begin{array}{ccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2C_{\mathrm{d}}v_0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{h}\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & -\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & -\frac{1}{h} & \frac{1}{h} & -\frac{k_{\mathrm{d}}}{hT^2} & \frac{1}{h}\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & 0 \\ -k_{\mathrm{i}} & -hk_{\mathrm{i}} & 0 & 0 & 0 & k_{\mathrm{i}} & 0 \\ -1 & -h & 0 & 0 & -\frac{1}{T} & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]. \tag{4.112}$$

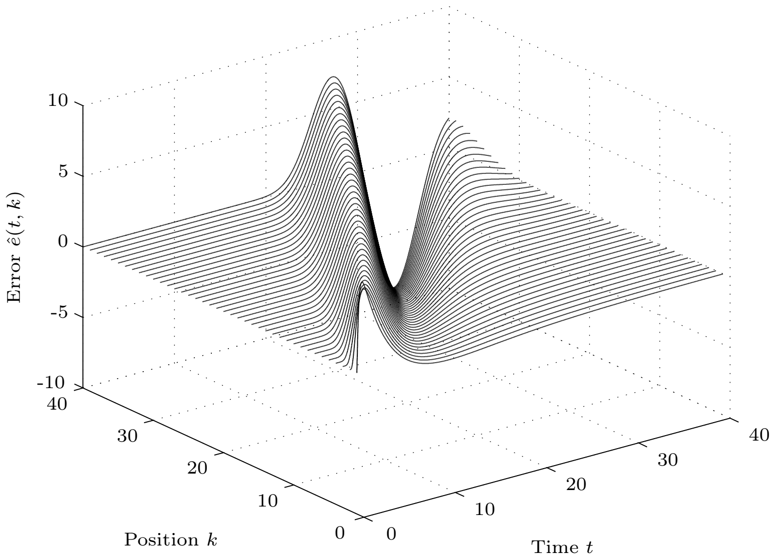


Figure 4.4: *String unstable system with $h = 0.5$: error $\hat{e}(t,k)$*

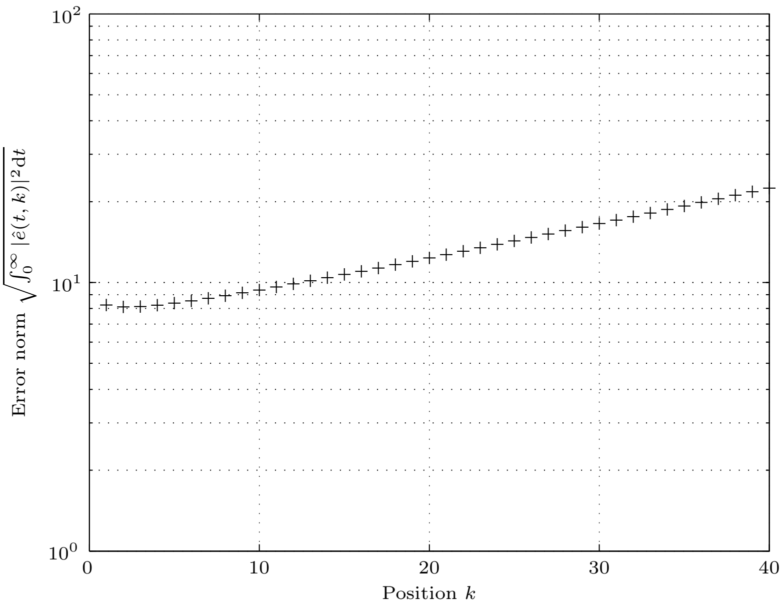


Figure 4.5: *String unstable system with $h = 0.5$: L_2 norm of error $\hat{e}(t,k)$*

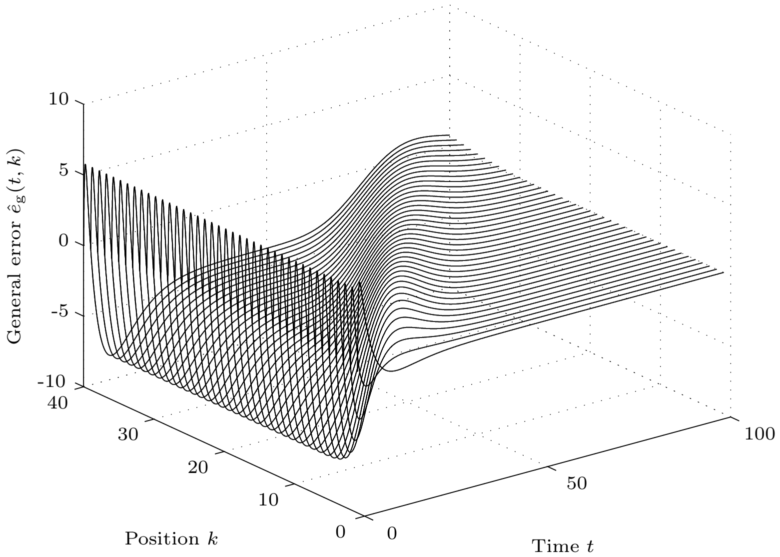


Figure 4.6: *String unstable system: general error* $\hat{e}_{\mathrm{g}}(t,k)$

While $\boldsymbol{A}_{11}$ is still Hurwitz stable, $\boldsymbol{A}_{22} + \mathbf{I}$ has one eigenvalue at 1. Thus it is not Schur stable. The second condition is not violated as Matlab can find strictly positive matrices $\boldsymbol{P}_1$ with eigenvalues at 130, $7.72 \cdot 10^3$, $2.56 \cdot 10^4$, $7.71 \cdot 10^5$ and $1.58 \cdot 10^8$ and $\boldsymbol{P}_2$ with eigenvalues at $6.66 \cdot 10^3$ and $1.08 \cdot 10^6$ such that $\boldsymbol{Q}$ has eigenvalues at $-1.75 \cdot 10^7$, $-1.22 \cdot 10^5$,$-4.42 \cdot 10^4$, $-3.31 \cdot 10^3$, $-1.25 \cdot 10^3$ and two at 0. Also $\boldsymbol{A}$ has two eigenvalues at 0 and there exists a positive definite matrix $\boldsymbol{R}$ such that $-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{A} = \boldsymbol{Q}$.

So, even though $\boldsymbol{Q}$ is negative semidefinite and there exists a suitable $\boldsymbol{R}$ the system is not asymptotically stable, as the simulation in Figure 4.6 and Figure 4.7 demonstrate. However, since the marginally stable mode of $\boldsymbol{A}_{22} + \mathbf{I}$ is not controllable by $\boldsymbol{A}_{21}$ the system is still stable as we have shown in Corollary 4.2. $*$

In our last linear example for stability of two-dimensional systems we consider again the linear, unidirectional string with communication range 2 studied in Section 3.5.

Example 4.4 Consider a string of vehicles with the same plant and controller transfer function as used in Example 4.1 and with communication range 2 with

$$\hat{e}(t,k) = (1-\alpha)\left(\hat{x}(t,k-1) - \hat{x}(t,k)\right) + \alpha\left(\hat{x}(t,k-2) - \hat{x}(t,k-1)\right) - h\hat{v}(t,k). \tag{4.113}$$

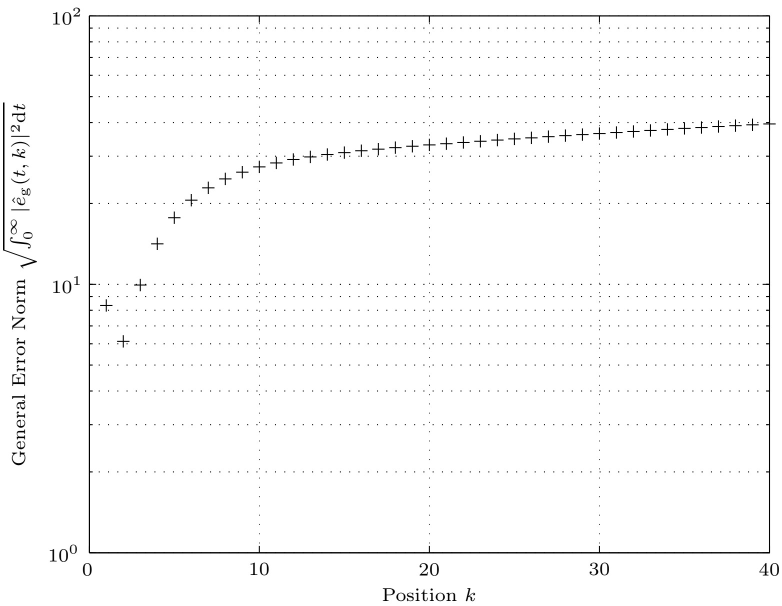


Figure 4.7: *String unstable system: L_2 norm of general error $\hat{e}_{\mathrm{g}}(t,k)$*

Thus, the corresponding continuous-discrete two-dimensional system description is $\delta \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}$ with

$$\boldsymbol{A} = \left[\begin{array}{ccccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2C_{\mathrm{d}}v_0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1-\alpha}{h}\phi & -\phi & -\frac{1}{h} & \frac{1}{h} & -\frac{k_{\mathrm{d}}}{hT^2} & \frac{(1-2\alpha)\phi}{h} & \frac{\alpha\phi}{h} \\ -(1-\alpha)k_{\mathrm{i}} & -hk_{\mathrm{i}} & 0 & 0 & 0 & (1-2\alpha)k_{\mathrm{i}} & \alpha k_{\mathrm{i}} \\ -1+\alpha & -h & 0 & 0 & -\frac{1}{T} & (1-2\alpha) & \alpha \\ \hline 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \tag{4.114}$$

where $\phi = k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}$. $\boldsymbol{A}_{11}$ is Hurwitz stable and $\boldsymbol{A}_{22} + \mathbf{I}$ is Schur stable. From (3.87) in Section 3.5.3 we know that the minimal time headway $h_{0,2}$ is equal to $(1-\alpha)h_0$. Thus, considering that $h_0 = 1.18$ the string is stable for $\alpha = 0.3$ and $h = 1$.

Matlab finds two symmetric, positive matrices

$$\boldsymbol{P}_1 = \begin{bmatrix} 1.39 \cdot 10^4 & 0 & 0 & 8.198 \cdot 10^4 & 0 \\ 0 & 5.36 \cdot 10^4 & 4.28 \cdot 10^4 & -5.13 \cdot 10^5 & -5.24 \cdot 10^6 \\ 0 & 4.28 \cdot 10^4 & 4.61 \cdot 10^4 & -4.19 \cdot 10^5 & -5.67 \cdot 10^6 \\ 8.198 \cdot 10^4 & -5.13 \cdot 10^5 & -4.19 \cdot 10^5 & 6.18 \cdot 10^6 & 5.12 \cdot 10^7 \\ 0 & -5.24 \cdot 10^6 & -5.67 \cdot 10^6 & 5.12 \cdot 10^7 & 6.98 \cdot 10^8 \end{bmatrix} \tag{4.115}$$

with eigenvalues at 54.3, 4571, 14031, $2.42 \cdot 10^6$ and $7.02 \cdot 10^8$, and

$$\boldsymbol{P}_2 = \begin{bmatrix} 8483 & 1258 \\ 1258 & 2917 \end{bmatrix} \tag{4.116}$$

with eigenvalues at 2646 and 8754 such that the eigenvalues of $\boldsymbol{Q}$ are $-4.98 \cdot 10^7$, $-6.05 \cdot 10^5$, -27427, -2998, -955, -310 and 0. Hence, the system is asymptotically stable. For simulations see Figure 3.10 on page 48.

4.7 Conclusion

Sufficient conditions for stability, exponential stability and asymptotic stability of linear two-dimensional systems have been derived in this chapter. They are given in linear matrix inequalities. The conditions for stability and asymptotic stability require – in stark contrast to the results presented in the literature – only *semi*-definite solutions. Thus, they are suitable to analyse linear two-dimensional systems with singularities of the stability boundary in general and string stability of vehicle platoons in particular.

The smoothness constraints on the initial conditions to guarantee asymptotic stability are rather restrictive. However, it was shown in Fornasini and Marchesini (1978) that for merely bounded initial conditions the system is asymptotically stable if and only if the system is devoid of singularities in the closed stability region. Thus, some additional constraints on the initial conditions have to be expected.

Exponential stability, however, can only be shown if there exists a strictly negative definite solution of the LMI and the system, therefore, is devoid of singularities on the stability boundary.

As all results presented in this chapter are only suitable to study the stability of *linear* two-dimensional systems it will be the aim of the following chapter to extend some results to *nonlinear* two-dimensional systems.

CHAPTER 5

Internal Stability of Nonlinear 2D Systems

Stability, asymptotic stability and exponential stability of general nonlinear two-dimensional systems will be studied. The sufficient conditions for stability proposed require nonpositive divergence of the two-dimensional Lyapunov function. If in addition some other regularity conditions are satisfied, asymptotic stability can be guaranteed. Exponential stability is proven if the system admits a Lyapunov function with strictly negative divergence.

Chapter contents

5.1 Introduction

After studying bounded-input bounded-output stability of linear two-dimensional systems in the frequency domain in Chapter 3 and stability, asymptotic stability and exponential stability of linear two-dimensional systems in the time domain in Chapter 4 we seek to extend some of these results to more general nonlinear two-dimensional systems.

First, the notation and some mathematical preliminaries are discussed in Section 5.2 and Section 5.3. This includes a proof of stability for general nonlinear two-dimensional systems in Corollary 5.1 based on the assumption that there exists a suitable two-dimensional Lyapunov function with a negative *semi*-definite divergence.

The stability proof will then be used to show exponential stability of a class of nonlinear two-dimensional systems in Theorem 5.1 if the divergence of the Lyapunov function is *strictly* negative definite.

In Section 5.5 it will be shown, that — similar to the results in Section 4.5 — asymptotic stability of general nonlinear two-dimensional systems can also be guaranteed if the divergence is negative *semi*-definite, rather than negative definite. In addition to the smoothness of the initial conditions it will also be required that the state space equations and the Lyapunov function fulfil certain smoothness criteria.

Throughout the chapter the notion of input-to-state stability (ISS) and integral input-to-state stability (iISS) and related results will be used. Continuing with the notation used in Chapter 4, the results will be given using the generalised form for continuous and discrete systems.

Two examples will be discussed in Section 5.6 to illustrate the results. In Example 5.2 the string stability of a nonlinear homogeneous string with variable time headway will be discussed.

5.2 Notation

Similar to the previous chapter the results of this chapter will be given in a general notation describing continuous and discrete systems at the same time. The notation of $\mathbb{T}_i$, δ_i, $\mathcal{S} \cdot \mathrm{d}t_i$, $\mathbb{I}_i$, and $\mathrm{E}(\boldsymbol{A})^t$ given in (4.1), (4.2), (4.3), (4.7), and (4.8), respectively, remain in place.

Combining the general nonlinear continuous one-dimensional system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{5.1}$$

and the general nonlinear discrete one-dimensional system

$$\Delta \boldsymbol{x}(t) = \boldsymbol{x}(t+1) - \boldsymbol{x}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{5.2}$$

consider the autonomous general nonlinear two-dimensional system of the form

$$\delta_1 \boldsymbol{x}_1(t_1, t_2) = \boldsymbol{f}_1(\boldsymbol{x}_1(t_1, t_2), \boldsymbol{x}_2(t_1, t_2)), \tag{5.3}$$

$$\delta_2 \boldsymbol{x}_2(t_1, t_2) = \boldsymbol{f}_2(\boldsymbol{x}_1(t_1, t_2), \boldsymbol{x}_2(t_1, t_2)). \tag{5.4}$$

Note that in contrast to the one-dimensional systems in (5.1) and (5.2), the general nonlinear two-dimensional model (5.3)-(5.4) is autonomous. Instead of an input signal $\boldsymbol{u}$ each state derivative depends on the states $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. This system, however, is not equivalent to the cascade system with feedback given for example in Sontag (2008) since both states here evolve in two different independent variables instead of one common variable.

Note that requiring the existence of a two-dimensional Lyapunov function V as in Definition 5.1 (later used for the proof of stability of nonlinear two-dimensional systems in Corollary 5.1) implicitly requires that the origin is an equilibrium of the system, i. e. $\boldsymbol{f}_i(0,0) = 0$ for $i \in \{1,2\}$. However, for two-dimensional systems describing a vehicle platoon the origin is part of an invariant set $(\overline{\boldsymbol{x}}_1(\gamma), \overline{\boldsymbol{x}}_2(\gamma))$ where $\gamma \in \mathbb{D} \subseteq \mathbb{R}$. As this implies that for each choice of two-dimensional Lyapunov function $\operatorname{div} V\,(\overline{\boldsymbol{x}}_1(\gamma), \overline{\boldsymbol{x}}_2(\gamma)) = 0$ for all $\gamma \in \mathbb{D}$, it is not possible to show global (asymptotic) stability of the origin. A two-dimensional version of Krasovskii-LaSalle principle would be needed to show that the invariant set is globally attractive - but this is out of the scope of this thesis. Here, we will focus on local (asymptotic) stability. Restricting the initial or boundary conditions to the set of L_V and L_∞ bounded initial conditions (defined in Definition 5.4) local stability of the origin can be shown. Assuming further that the initial conditions are also L'_p and L''_∞ smooth bounded (as in Definition 5.5) local asymptotic stability of the origin can be shown.

To facilitate the proof of asymptotic stability later in this chapter two additional constraints on the nonlinear system (5.3)-(5.4) are needed: First, the nonlinear two-dimensional system in (5.3)-(5.4) is called C^m smooth if the first m derivatives of $\boldsymbol{f}_i$ are bounded. That is for every $\kappa > 0$ there exists a $K(\kappa) > 0$ such that for all $|\boldsymbol{x}_{1s}|, |\boldsymbol{x}_{2s}| < K$ the derivatives

$$\left| \boldsymbol{F}_{i\,\underbrace{1\ldots1}_{m_1\text{ times}}\,\underbrace{2\ldots2}_{m_2\text{ times}}}(t_1,t_2) \right| := \left| \frac{\partial^{m_1+m_2} \boldsymbol{f}_i(\boldsymbol{x}_1,\boldsymbol{x}_2)}{\partial \boldsymbol{x}_1^{m_1} \partial \boldsymbol{x}_2^{m_2}} \bigg|_{\boldsymbol{x}_1=\boldsymbol{x}_{1s},\boldsymbol{x}_2=\boldsymbol{x}_{2s}} \right| < \kappa \quad \forall t_1, t_2 > 0 \tag{5.5}$$

where $i \in \{1,2\}$, $0 \leq m_1, m_2 \leq m$ and $m_1 + m_2 \leq m$. Note that the case $m_1 = m_2 = 0$ is included as well. Thus, $|\boldsymbol{F}_i| = |\boldsymbol{f}_i(\boldsymbol{x}_{1s}, \boldsymbol{x}_{2s})| < \kappa$ also for $|\boldsymbol{x}_{1s}|, |\boldsymbol{x}_{2s}| < K$.

Furthermore, we require that the nonlinear system (5.3)-(5.4) has exponentially stable Jacobian matrices, i. e. the solution of the time varying differential / difference equation $\delta_i \boldsymbol{x}(t_1,t_2) = \boldsymbol{F}_{ii}(t_1,t_2)\boldsymbol{x}(t_1,t_2)$ is exponentially decaying for bounded initial conditions.

Different sufficient conditions exist to guarantee that a time varying system is (exponentially) stable: If t_i is continuous the system $\delta_i \boldsymbol{x}(t_1,t_2) = \boldsymbol{F}_{ii}(t_1,t_2)\boldsymbol{x}(t_1,t_2)$ is exponentially stable, if there exist $\lambda_i > 0$ such that

$$\boldsymbol{F}_{ii}(t_1,t_2) + \boldsymbol{F}_{ii}^{\mathrm{T}}(t_1,t_2) \leq -\lambda_i \mathbf{I} \quad \forall t_1, t_2 > 0. \tag{5.6}$$

If t_i is discrete the system $\delta_i \boldsymbol{x}(t_1,t_2) = \boldsymbol{F}_{ii}(t_1,t_2)\boldsymbol{x}(t_1,t_2)$ is exponentially stable, if there exist $\lambda_i \in (0,1]$ such that

$$\boldsymbol{F}_{ii}(t_1,t_2)\boldsymbol{F}_{ii}^{\mathrm{T}}(t_1,t_2) - \mathbf{I} \leq -\lambda_i \mathbf{I} \quad \forall t_1, t_2 > 0. \tag{5.7}$$

(Note that this is equivalent to requiring the existence of a $\tilde{\lambda}_i$ with $0 \leq \tilde{\lambda}_i < 1$ such that $\boldsymbol{F}_{ii}(t_1,t_2)\boldsymbol{F}_{ii}^{\mathrm{T}}(t_1,t_2) < \tilde{\lambda}_i \mathbf{I}$. We will adhere the notation introduced in (5.7) enabling us to still use the notion of $\mathrm{E}(-\lambda_i)^{t_i}$.)

These sufficient conditions (5.6)-(5.7) are equivalent to the existence of the Lyapunov function $V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. This is a very strong condition and sometimes cannot be satisfied even though the system is exponentially stable. Different sufficient conditions can be found in Rosenbrock (1963); Desoer (1969, 1970); Ilchmann *et al.* (1987); Zhang (1993); Hill and Ilchmann (2011). If the matrix $\boldsymbol{F}_{ii}(t_1,t_2)$ satisfies any of these sufficient stability conditions, it will be denoted as “exponentially stable”.

Throughout this chapter the notion of "positive definite", $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions will be used: A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite if it is continuous and satisfies $f(0) = 0$ and $f(x) > 0$ for all $x > 0$. A function α is of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is positive definite and strictly increasing. A function α is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is of class $\mathcal{K}$ and in addition $\alpha(x) \to +\infty$ as $x \to +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if $\beta(\cdot, t)$ is of class $\mathcal{K}_\infty$ for all $t \geq 0$ and satisfies $\beta(r, t) \to 0$ as $t \to \infty$ for each $r \geq 0$.

Combining (and slightly altering) the definitions for iISS-Lyapunov functions for continuous systems given in (Angeli *et al.*, 2000, Definition II.2) and discrete systems given in Angeli (1999), consider this notion of "two-dimensional Lyapunov function":

Definition 5.1 (Two-Dimensional Lyapunov Function)

A two-dimensional function $\boldsymbol{V}^{\mathrm{T}} = \begin{pmatrix} V_1(\boldsymbol{x}_1) & V_2(\boldsymbol{x}_2) \end{pmatrix}$ *is called a two-dimensional Lyapunov function for system (5.3)-(5.4) if* $V_i(\boldsymbol{x}_i)$ *is an iISS-Lyapunov function for subsystem* $\delta_i \boldsymbol{x}_i(t_1, t_2) = \boldsymbol{f}_i(\boldsymbol{x}_1(t_1, t_2), \boldsymbol{x}_2(t_1, t_2))$*, that is there exist functions* $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$*, positive definite functions* α_i *and constants* $0 \leq b_i < \infty$ *such that for* $i, k \in \{1, 2\}$*,* $i \neq k$

$$\underline{\alpha}_i(|\boldsymbol{x}_i|) \leq V_i(\boldsymbol{x}_i) \leq \overline{\alpha}_i(|\boldsymbol{x}_i|), \tag{5.8}$$
$$\delta_i V_i(\boldsymbol{x}_i) \leq -\alpha_i(V_i(\boldsymbol{x}_i)) + b_i V_k(\boldsymbol{x}_k) \tag{5.9}$$

and in addition

$$\operatorname{div} \boldsymbol{V} = \delta_1 V_1(\boldsymbol{x}_1) + \delta_2 V_2(\boldsymbol{x}_2) \leq 0 \quad \textit{for all } t_1, t_2 > 0. \tag{5.10}$$

Note that according to the definitions of iISS-Lyapunov functions in Angeli *et al.* (2000) and Angeli (1999) $V_i(\boldsymbol{x}_i)$ needs to be continuously differentiable if t_i is continuous and merely continuous if t_i is discrete.

The definitions for iISS-Lyapunov functions from Angeli *et al.* (2000) and Angeli (1999) have been altered in the way that the last term in (5.9) explicitly contains $V_k(\boldsymbol{x}_k)$ instead of a general class $\mathcal{K}_\infty$ function $\gamma_k(\boldsymbol{x}_k)$.

In order to prove asymptotic stability later in this chapter, the definition above needs to be strengthened:

Definition 5.2 (Regular Two-Dimensional Lyapunov Function)

A two-dimensional function $\boldsymbol{V}$ *of two smooth functions* $V_1(\boldsymbol{x}_1)$ *and* $V_2(\boldsymbol{x}_2)$ *is called a regular two-dimensional Lyapunov function for system (5.3)-(5.4) if there exist functions* $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ *and constants* $0 < a_i, b_i < \infty$ *(and in addition* $a_i < 1$ *if* t_i *is discrete) such that for* $i, k \in \{1, 2\}$*,* $i \neq k$

$$\underline{\alpha}_i(|\boldsymbol{x}_i|) \leq V_i(\boldsymbol{x}_i) \leq \overline{\alpha}_i(|\boldsymbol{x}_i|), \tag{5.11}$$
$$\delta_i V_i(\boldsymbol{x}_i) \leq -a_i V_i(\boldsymbol{x}_i) + b_i V_k(\boldsymbol{x}_k) \tag{5.12}$$

and

$$\operatorname{div} \boldsymbol{V} = \delta_1 V_1(\boldsymbol{x}_1) + \delta_2 V_2(\boldsymbol{x}_2) \leq 0 \quad \textit{for all } t_1, t_2 > 0. \tag{5.13}$$

Note that the main difference between these two definitions for two-dimensional Lyapunov functions is that in Definition 5.2 the first term on the right hand side of (5.12) is $a_i \cdot V_i(\boldsymbol{x}_i)$ instead of the more general form $\alpha_i(V_i(\boldsymbol{x}_i))$ in Definition 5.1.

This definition can be strengthened further by requiring that the divergence of the Lyapunov function is strictly negative and can be bounded by the Lyapunov function components. This will enable the proof of exponential stability.

Definition 5.3 (Strict Two-Dimensional Lyapunov Function)

A two-dimensional function $\boldsymbol{V}$ of two smooth functions $V_1(\boldsymbol{x}_1)$ and $V_2(\boldsymbol{x}_2)$ is called a strict two-dimensional Lyapunov function for system (5.3)-(5.4) if there exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ and constants $0 < a_i, b_i, \alpha_i < \infty$ (and in addition $a_i, \alpha_i < 1$ if t_i is discrete) such that for $i, k \in \{1, 2\}$, $i \neq k$

$$\underline{\alpha}_i(|\boldsymbol{x}_i|) \leq V_i(\boldsymbol{x}_i) \leq \overline{\alpha}_i(|\boldsymbol{x}_i|), \tag{5.14}$$

$$\delta_i V_i(\boldsymbol{x}_i) \leq -a_i V_i(\boldsymbol{x}_i) + b_i V_k(\boldsymbol{x}_k) \tag{5.15}$$

and

$$\operatorname{div} \boldsymbol{V} \leq -\alpha_1 V_1(\boldsymbol{x}_1) - \alpha_2 V_2(\boldsymbol{x}_2) < 0 \quad \textit{for all } t_1, t_2 > 0. \tag{5.16}$$

Similar to Definition 4.2 and Definition 4.3 in Chapter 4 we will use the following definitions for initial conditions:

Definition 5.4 (L_V and L_∞ Bounded Initial Conditions)

Given positive definite functions V_i, the initial conditions of the nonlinear two-dimensional system (5.3)-(5.4) are L_V and L_∞ bounded, if there exist $c_i, \zeta_i < \infty$ such that for $i \in \{1, 2\}$

$$\|\boldsymbol{x}_{i0}(\cdot)\|_V := \mathop{S}_{0}^{\infty} V_i\left(\boldsymbol{x}_{i0}(t)\right) \mathrm{d}t \leq c_i, \quad \textit{and} \tag{5.17}$$

$$\|\boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t \geq 0} |\boldsymbol{x}_{i0}(t)| \leq \zeta_i. \tag{5.18}$$

Definition 5.5 (L'_p and L''_∞ Smooth Bounded Initial Conditions)

Given positive definite functions V_i and an integer $1 \leq p < \infty$, the initial conditions of the nonlinear two-dimensional system (5.3)-(5.4) are Smooth Bounded Initial Conditions *if they are L_V and L_∞ bounded according to Definition 5.4 and in addition there exist $c_i', \zeta_i', \zeta_i'' < \infty$ such that for $i \in \{1, 2\}$*

$$\|\delta \boldsymbol{x}_{i0}(\cdot)\|_p^p = \int_0^\infty |\delta \boldsymbol{x}_{i0}(t)|^p \,\mathrm{d}t \le c_i', \tag{5.19}$$

$$\|\delta \boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t>0} |\delta \boldsymbol{x}_{i0}(t)| \le \zeta_i' \quad \textit{and} \tag{5.20}$$

$$\|\delta^2 \boldsymbol{x}_{i0}(\cdot)\|_\infty = \sup_{t>0} |\delta^2 \boldsymbol{x}_{i0}(t)| \le \zeta_i''. \tag{5.21}$$

The proof of exponential stability in Section 5.4 requires the definition of exponentially decaying initial conditions, similar to Definition 4.4.

Definition 5.6 (Exponentially Decaying Initial Conditions)

Given positive definite functions V_i, the initial conditions of the nonlinear two-dimensional system (5.3)-(5.4) are exponentially decaying, if there exist $\mu_i > 0$ and $\kappa_i < \infty$ such that for $i, k \in \{1, 2\}$, $i \neq k$

$$V_i(\boldsymbol{x}_{i0}(t_k)) \le \kappa_i \mathrm{e}^{-\mu_k t_k}. \tag{5.22}$$

Similar to Definition 4.5, Definition 4.6 and Definition 4.7 we define stability, asymptotic stability and exponential stability of nonlinear two-dimensional systems:

Definition 5.7 (Stability of Nonlinear Two-Dimensional Systems)

The autonomous nonlinear two-dimensional system (5.3)-(5.4) is stable if for each $M > 0$ there exists a set of $c_i(M), \zeta_i(M) > 0$ such that if the initial conditions are L_V and L_∞ bounded with bounds $c_i(M)$ and $\zeta_i(M)$ for $i \in \{1, 2\}$, respectively, then

$$|\boldsymbol{x}(t_1, t_2)| \le M \quad \textit{for all } t_1, t_2 > 0. \tag{5.23}$$

Definition 5.8 (Asymptotic Stability of Smooth Nonlinear Two-Dimensional Systems with Smooth Bounded Initial Conditions)

The autonomous nonlinear two-dimensional system (5.3)-(5.4) is asymptotically stable, if for any L_p' and L_∞'' Smooth Bounded Initial Conditions (according to Definition 5.5) it is stable, and the following limit holds

$$\lim_{t_1+t_2\to\infty} \boldsymbol{x}(t_1, t_2) = 0. \tag{5.24}$$

Definition 5.9 (Exponential Stability of Nonlinear Two-Dimensional Systems)

The autonomous nonlinear two-dimensional system (5.3)-(5.4) is exponentially stable, if for any exponentially decaying initial conditions there exist $\eta_1, \eta_2 > 0$, and $M_i < \infty$ such that for $i \in \{1, 2\}$ the following condition holds:

$$|\boldsymbol{x}_i(t_1, t_2)| \leq M_i \mathrm{e}^{-\eta_1 t_1} \mathrm{e}^{-\eta_2 t_2}. \tag{5.25}$$

5.3 Mathematical Preliminaries

The following lemma will be needed for the proof of stability of two-dimensional systems. The continuous version was proposed in (Angeli *et al.*, 2000, Corollary IV.3) and the proof of the discrete version can be found in (Angeli, 1999, Proof of Theorem 2, p. 301).

Lemma 5.1

Given any continuous positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists a $\mathcal{KL}$-function β with the following property. For any $0 < \tilde{t} \leq \infty$, and for any (locally) absolutely continuous function $V : [0, \tilde{t}) \to \mathbb{R}_{\geq 0}$ and any measurable, locally essentially bounded function $\gamma : [0, \tilde{t}) \to \mathbb{R}_{\geq 0}$, if

$$\delta V(t) \leq -\alpha(V(t)) + \gamma(t) \tag{5.26}$$

holds for almost all $t \in [0, \tilde{t})$, then the following estimate holds:

$$V(t) \leq \beta(V(0), t) + \mathop{\mathcal{S}}_{0}^{t} 2\gamma(s)\mathrm{d}s \tag{5.27}$$

for all $t \in [0, \tilde{t})$.

Lemma 5.1 enables the proof of stability of general nonlinear two-dimensional systems below.

Corollary 5.1 (Stability of Nonlinear Two-Dimensional Systems)

The nonlinear two-dimensional system (5.3)-(5.4) is stable if there exists a two-dimensional Lyapunov function according to Definition 5.1.

Proof Since the divergence of $\boldsymbol{V}$ is nonpositive for all $t_1, t_2 > 0$ we get again the results from (4.28)-(4.29) (in Lemma 4.2 on page 63)

$$\mathop{\mathcal{S}}_{0}^{T_2} V_1(T_1, t_2)\mathrm{d}t_2 \leq \mathop{\mathcal{S}}_{0}^{T_2} V_1(0, t_2)\mathrm{d}t_2 + \mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1, 0)\mathrm{d}t_1, \tag{5.28}$$

$$\mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1, T_2)\mathrm{d}t_1 \leq \mathop{\mathcal{S}}_{0}^{T_2} V_1(0, t_2)\mathrm{d}t_2 + \mathop{\mathcal{S}}_{0}^{T_1} V_2(t_1, 0)\mathrm{d}t_1. \tag{5.29}$$

Together with the fact that the initial conditions are L_V bounded we get

$$\mathcal{S}_0^{T_2} V_1(T_1, t_2)\mathrm{d}t_2 \leq c_1 + c_2 \quad \text{and} \quad \mathcal{S}_0^{T_1} V_2(t_1, T_2)\mathrm{d}t_1 \leq c_1 + c_2. \tag{5.30}$$

Applying Lemma 5.1 we can guarantee that there exists a function $\beta_1 \in \mathcal{KL}$ such that

$$V_1(\boldsymbol{x}_1(t_1, t_2)) \leq \beta_1(V_{10}(t_2), t_1) + \mathcal{S}_0^{t_1} 2b_1 V_2(\boldsymbol{x}_2(\tau, t_2))\mathrm{d}\tau. \tag{5.31}$$

Using the fact that the initial conditions are in L_∞ and (5.30), equation (5.31) yields

$$V_1(\boldsymbol{x}_1(t_1, t_2)) \leq \beta_1(\zeta_1, t_1) + 2b_1(c_1 + c_2). \tag{5.32}$$

Since there exists a class $\mathcal{K}_\infty$ function $\underline{\alpha}_1(|\boldsymbol{x}_1|) \leq V_1(\boldsymbol{x})$ we find that

$$|\boldsymbol{x}_1(t_1, t_2)| \leq M_1 := \underline{\alpha}_1^{-1}\left(\beta_1(\zeta_1, 0) + 2b_1(c_1 + c_2)\right) < \infty \tag{5.33}$$

for all t_1, t_2. Note that the bound M_1 depends on the norm of the initial conditions, i. e. ζ_1, c_1, c_2. Thus, the maximal value of $|\boldsymbol{x}_1|$ for all t_1, t_2 is determined by the norm of the initial conditions. Furthermore, if ζ_1, c_1 and c_2 tend to zero, then M_1 also tends to zero. A similar bound $M_2 < \infty$ for the norm of $\boldsymbol{x}_2$ can also be found and thus the system is stable according to Definition 5.7. □

Similar to Corollary 4.3 we will further show that if a suitable regular two-dimensional Lyapunov function exists, the generalised integral $\mathcal{S}_0^\infty V_i(\boldsymbol{x}_i(t_1, t_2))\mathrm{d}t_i$ is bounded for $i \in \{1, 2\}$.

Corollary 5.2

Consider the nonlinear two-dimensional system in (5.3)-(5.4). If there exists a regular two-dimensional Lyapunov function $\boldsymbol{V}$ according to Definition 5.2 and the initial conditions are L_V and L_∞ bounded according to Definition 5.4, then there exist $\overline{M}_1, \overline{M}_2 < \infty$ independently of t_2 and t_1, respectively, such that

$$\mathcal{S}_0^{\infty} V_1(\boldsymbol{x}_1(t_1, t_2))\mathrm{d}t_1 \leq \overline{M}_1, \quad \textit{and} \quad \mathcal{S}_0^{\infty} V_2(\boldsymbol{x}_2(t_1, t_2))\mathrm{d}t_2 \leq \overline{M}_2. \tag{5.34}$$

Proof From the definition of the regular two-dimensional Lyapunov function we derive

$$V_1(\boldsymbol{x}_1(t_1, t_2)) \leq \mathrm{E}(-a_1)^{t_1} V_1(\boldsymbol{x}_1(0, t_2)) + b_1 \mathcal{S}_0^{t_1} \mathrm{E}(-a_1)^{\tau} V_2(\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2))\mathrm{d}\tau \tag{5.35}$$

and thus

$$\begin{aligned}\mathcal{S}_0^{\infty} V_1(\boldsymbol{x}_1(t_1, t_2))\mathrm{d}t_1 \leq& V_1(\boldsymbol{x}_1(0, t_2)) \mathcal{S}_0^{\infty} \mathrm{E}(-a_1)^{t_1}\mathrm{d}t_1 \\ &+ b_1 \mathcal{S}_0^{\infty} \mathcal{S}_0^{t_1} \mathrm{E}(-a_1)^{\tau} V_2(\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2))\mathrm{d}\tau\mathrm{d}t_1.\end{aligned} \tag{5.36}$$

Since the initial conditions are L_∞ bounded and $a_1 > 0$ (and in addition $a_1 < 1$ if t_1 is discrete), the first term of the right hand side of (5.36) can be bounded by

$$V_1(\boldsymbol{x}_1(0,t_2)) \mathop{\mathcal{S}}\limits_0^\infty \mathrm{E}(-a_1)^{t_1}\mathrm{d}t_1 \leq \frac{V_1(\zeta_1)}{a_1}. \tag{5.37}$$

Using the fact that the convolution is commutative and interchanging the order of integration at the second term of the right hand side of (5.36) yields

$$\begin{aligned}
&b_1 \mathop{\mathcal{S}}\limits_0^\infty \mathop{\mathcal{S}}\limits_0^{t_1} \mathrm{E}(-a_1,)^{\tau} V_2(\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2))\mathrm{d}\tau \mathrm{d}t_1 \\
&\quad \leq b_1 \mathop{\mathcal{S}}\limits_0^\infty \mathop{\mathcal{S}}\limits_{\tau+\mathbb{I}_1}^\infty \mathrm{E}(-a_1)^{t_1-\mathbb{I}_1-\tau} V_2(\boldsymbol{x}_2(\tau,t_2))\mathrm{d}t_1\mathrm{d}\tau \\
&\quad \leq b_1 \mathop{\mathcal{S}}\limits_0^\infty V_2(\boldsymbol{x}_2(\tau,t_2)) \cdot \left(\mathop{\mathcal{S}}\limits_{\tau+\mathbb{I}_1}^\infty \mathrm{E}(-a_1)^{t_1-\mathbb{I}_1-\tau}\mathrm{d}t_1 \right) \mathrm{d}\tau \\
&\quad \leq \frac{b_1}{a_1} \mathop{\mathcal{S}}\limits_0^\infty V_2(\boldsymbol{x}_2(t_1,t_2))\mathrm{d}t_1.
\end{aligned} \tag{5.38}$$

Since the divergence is nonpositive we can use the results in (5.28)-(5.29), and equation (5.38) yields

$$\begin{aligned}
&b_1 \mathop{\mathcal{S}}\limits_0^\infty \mathop{\mathcal{S}}\limits_0^{t_1} \mathrm{E}(-a_1)^{\tau} V_2(\boldsymbol{x}_2(t_1 - \mathbb{I}_1 - \tau, t_2))\mathrm{d}\tau \mathrm{d}t_1 \\
&\quad \leq \frac{b_1}{a_1} \left(\mathop{\mathcal{S}}\limits_0^\infty V_1(\boldsymbol{x}_1(0,t_2))\mathrm{d}t_2 + \mathop{\mathcal{S}}\limits_0^\infty V_2(\boldsymbol{x}_2(t_1,0))\mathrm{d}t_1 \right)
\end{aligned} \tag{5.39}$$

As the initial conditions are in L_V, the bound $\overline{M}_1$ is (independently of t_2)

$$\overline{M}_1 := \frac{V_1(\zeta_1)}{a_1} + \frac{b_1}{a_1}(c_1 + c_2). \tag{5.40}$$

The existence of $\overline{M}_2$ can be shown in the same way. □

Similar to Lemma 4.3 we will show that the first derivatives of the states $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ with respect to t_1 and t_2 are bounded if the state space equations, the initial conditions and the Lyapunov function fulfils certain smoothness criteria.

Lemma 5.2

Consider the C^2 smooth nonlinear two-dimensional system (5.3)-(5.4) with exponentially stable Jacobian matrices $\boldsymbol{F}_{ii}$ for $i \in \{1,2\}$. If there exists a regular two-dimensional Lyapunov function according to Definition 5.2 and in addition there exist scalars $1 \leq p < \infty$ and $0 < a_1', a_2' < \infty$ such that the initial conditions are L_p' and L_∞'' smooth bounded according to Definition 5.5 and

$$\operatorname{div} \boldsymbol{V}(t_1,t_2) \leq -a_1' \left|\delta_1 \boldsymbol{x}_1(t_1,t_2)\right|_p^p - a_2' \left|\delta_2 \boldsymbol{x}_2(t_1,t_2)\right|_p^p \tag{5.41}$$

for all $t_1, t_2 > 0$, then

(a) the first generalised derivatives of $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are in $L_\infty\ [0,\infty)\times[0,\infty)$ and $L_p\ [0,\infty)\times[0,\infty)$, i.e. there exist $M_{ik}, \overline{M}_{ik} < \infty$ such that for $i,k \in \{1,2\}$,

$$\sup_{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2} |\delta_k \boldsymbol{x}_i(t_1,t_2)| \leq M_{ik}, \tag{5.42}$$

$$\int_0^\infty \int_0^\infty |\delta_k \boldsymbol{x}_i(t_1,t_2)|^p \mathrm{d}t_1 \mathrm{d}t_2 \leq \overline{M}_{ik} \text{ and} \tag{5.43}$$

(b) the second generalised derivatives of $\boldsymbol{x}_1(t_1,t_2)$ and $\boldsymbol{x}_2(t_1,t_2)$ are in $L_\infty\ [0,\infty)\times[0,\infty)$, i.e. there exist $M_{ikl} < \infty$ such that for $i,k,l \in \{1,2\}$

$$\sup_{(t_1,t_2)\in\mathbb{T}_1\times\mathbb{T}_2} |\delta_k \delta_l \boldsymbol{x}_i(t_1,t_2)| \leq M_{ikl}. \tag{5.44}$$

Proof (a): The system is stable by Corollary 5.1 and therefore the states are bounded (there exist $M_i < \infty$ such that $|\boldsymbol{x}_i(t_1,t_2)| \leq M_i$ for all t_1, t_2 for $i \in \{1,2\}$). The bounds M_{11} and M_{22} are equal to the bounds on $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ for all $|\boldsymbol{x}_1| \leq M_1$ and $|\boldsymbol{x}_2| \leq M_2$.

Combining assumption (5.41) and the fundamental theorem of calculus yields

$$\begin{aligned}
a_1' \left\|\delta_1 \boldsymbol{x}_1(\cdot,\cdot)\right\|_p^p &+ a_2' \left\|\delta_2 \boldsymbol{x}_2(\cdot,\cdot)\right\|_p^p \\
&= \int_0^\infty \int_0^\infty a_1' \left|\delta_1 \boldsymbol{x}_1(t_1,t_2)\right|_p^p + a_2' \left|\delta_2 \boldsymbol{x}_2(t_1,t_2)\right|_p^p \mathrm{d}t_1 \mathrm{d}t_2 \\
&\leq -\int_0^\infty \int_0^\infty \operatorname{div} \boldsymbol{V}(t_1,t_2) \mathrm{d}t_1 \mathrm{d}t_2 \\
&= \int_0^\infty V_1\left(\boldsymbol{x}_1(0,t_2)\right) \mathrm{d}t_2 - \lim_{T_1\to\infty} \int_0^\infty V_1\left(\boldsymbol{x}_1(T_1,t_2)\right) \mathrm{d}t_2 \\
&\quad + \int_0^\infty V_2\left(\boldsymbol{x}_2(t_1,0)\right) \mathrm{d}t_1 - \lim_{T_2\to\infty} \int_0^\infty V_2\left(\boldsymbol{x}_2(t_1,T_2)\right) \mathrm{d}t_1 \\
&\leq \int_0^\infty V_1\left(\boldsymbol{x}_{10}(t_2)\right) \mathrm{d}t_2 + \int_0^\infty V_2\left(\boldsymbol{x}_{20}(t_1)\right) \mathrm{d}t_1 \\
&\leq c_1 + c_2.
\end{aligned} \tag{5.45}$$

Thus, $\delta_i \boldsymbol{x}_i(t_1,t_2)$ is also in $L_p\ [0,\infty)\times[0,\infty)$ and $\overline{M}_{ii}$ exists for $i \in \{1,2\}$.

To show that the first mixed derivatives $\delta_1 \boldsymbol{x}_2$ and $\delta_2 \boldsymbol{x}_1$ are also in L_p and L_∞, note that $\delta_1 \boldsymbol{x}_1(t_1,t_2) = \boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_2)$ yields

$$\begin{aligned}
\delta_2 \delta_1 \boldsymbol{x}_1(t_1,t_2) &= \delta_2 \boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_2) \\
\delta_1\left(\delta_2 \boldsymbol{x}_1(t_1,t_2)\right) &= \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{x}_1} \delta_2 \boldsymbol{x}_1(t_1,t_2) + \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{x}_2} \delta_2 \boldsymbol{x}_2(t_1,t_2) \\
&= \boldsymbol{F}_{11}(t_1,t_2) \delta_2 \boldsymbol{x}_1(t_1,t_2) + \boldsymbol{F}_{12}(t_1,t_2) \boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2).
\end{aligned} \tag{5.46}$$

Since $\boldsymbol{F}_{11}(t_1,t_2)$ is exponentially stable and $\boldsymbol{F}_{12}(t_1,t_2)$ and $\boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2)$ are bounded, the system (5.46) is exponentially stable. Thus, since $\boldsymbol{f}_2(\boldsymbol{x}_1,\boldsymbol{x}_2)$ is in L_p, $\delta_2\boldsymbol{x}_1(t_1,t_2)$ treated as a function of t_1 is both in L_p and L_∞.

(b): To show that M_{iii} and M_{iik} exist for $i,k\in\{1,2\}$, $i\neq k$ observe that

$$\begin{aligned}\delta_i^2\boldsymbol{x}_i(t_1,t_2) =& \boldsymbol{F}_{ii}(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{f}_i(\boldsymbol{x}_1(t_1,t_2),\boldsymbol{x}_2(t_1,t_2))+\boldsymbol{F}_{ik}(\boldsymbol{x}_1,\boldsymbol{x}_2)\delta_i\boldsymbol{x}_k(t_1,t_2), &(5.47)\\ \delta_i\delta_k\boldsymbol{x}_i(t_1,t_2) =& \boldsymbol{F}_{ii}(\boldsymbol{x}_1,\boldsymbol{x}_2)\delta_i\boldsymbol{x}_k(t_1,t_2)+\boldsymbol{F}_{ik}(\boldsymbol{x}_1,\boldsymbol{x}_2)\boldsymbol{f}_k(\boldsymbol{x}_1(t_1,t_2),\boldsymbol{x}_2(t_1,t_2)). &(5.48)\end{aligned}$$

Since $\boldsymbol{F}_{ii}$, $\boldsymbol{f}_i$ and $\delta_i\boldsymbol{x}_k$ are bounded for all $t_1,t_2>0$ and $i,k\in\{1,2\}$, $\delta_i^2\boldsymbol{x}_i$ and $\delta_i\delta_k\boldsymbol{x}_i$ are in L_∞ $[0,\infty)\times[0,\infty)$ for $i,k\in\{1,2\}$.

To show that $\delta_k^2\boldsymbol{x}_i$ is in L_∞ for $i,k\in\{1,2\}, i\neq k$ consider the same trick as above using the fact that

$$\begin{aligned}\delta_2^2\delta_1\boldsymbol{x}_1(t_1,t_2) =& \delta_2^2\boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_2)\\ \delta_1\left(\delta_2^2\boldsymbol{x}_1(t_1,t_2)\right) =& \delta_2\left(\boldsymbol{F}_{11}(t_1,t_2)\delta_2\boldsymbol{x}_1(t_1,t_2)+\boldsymbol{F}_{12}(t_1,t_2)\delta_2\boldsymbol{x}_2(t_1,t_2)\right)\\ =& \boldsymbol{F}_{11}(t_1,t_2)\delta_2^2\boldsymbol{x}_1(t_1,t_2)+\boldsymbol{F}_{111}(t_1,t_2)\left(\delta_2\boldsymbol{x}_1(t_1,t_2)\right)^2\\ &+\boldsymbol{F}_{12}(t_1,t_2)\delta_2^2\boldsymbol{x}_2(t_1,t_2)+2\boldsymbol{F}_{112}(t_1,t_2)\delta_2\boldsymbol{x}_1(t_1,t_2)\delta_2\boldsymbol{x}_2(t_1,t_2)\\ &+\boldsymbol{F}_{122}(t_1,t_2)\left(\delta_2\boldsymbol{x}_2(t_1,t_2)\right)^2. \qquad (5.49)\end{aligned}$$

Since the system is C^2 smooth and the derivatives $\delta_i\boldsymbol{x}_k$ and $\delta_i\boldsymbol{x}_i$ for $i,k\in\{1,2\}$, $i\neq k$ are bounded, all but the first term on the right hand side of (5.49) are bounded. As $\boldsymbol{F}_{11}(t_1,t_2)$ is exponentially stable, $\delta_2^2\boldsymbol{x}_1(t_1,t_2)$ is in L_∞. □

Similar to the results in Chapter 4, the findings in Corollary 5.2 and Lemma 5.2 (together with Lemma 4.4) will be used in Section 5.5 to prove asymptotic stability. The proof of exponential stability below only requires the results of Corollary 5.1.

5.4 Exponential Stability

Assuming the divergence of the Lyapunov function is strictly negative, exponential stability can be shown. The initial conditions also need to be exponentially decaying.

Theorem 5.1 (Exponential Stability of Nonlinear Two-Dimensional Systems)

The nonlinear two-dimensional system (5.3)-(5.4) is exponentially stable, if there exist a strict two-dimensional Lyapunov function of the form Definition 5.3 and $p_i\in(0,\infty)$ for $i\in\{1,2\}$ such that $\underline{\alpha}_i(|\boldsymbol{x}_i|)\geq|\boldsymbol{x}_i|^{p_i}$.

Proof First, consider the Lyapunov function candidate

$$\tilde{\boldsymbol{V}}=\begin{pmatrix}\tilde{V}_1(\boldsymbol{x}_1)\\ \tilde{V}_2(\boldsymbol{x}_2)\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{\eta_1(t_1-\mathbb{I}_1)}\mathrm{e}^{\eta_2 t_2}V_1\\ \mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2(t_2-\mathbb{I}_2)}V_2\end{pmatrix} \qquad (5.50)$$

with positive constants η_1 and η_2. Note that

$$\delta_i\tilde{V}_i = \mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2 t_2}\delta_i V_i + \tilde{\eta}_i\tilde{V}_i \tag{5.51}$$

where $\tilde{\eta}_i = \eta_i(1-\mathbb{I}_i) + (\mathrm{e}^{\eta_i}-1)\,\mathbb{I}_i$. Applying condition (5.15) of the definition for a strict two-dimensional Lyapunov function yields

$$\begin{aligned}\delta_i\tilde{V}_i \le& -\mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2 t_2}a_i V_i + \mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2 t_2}b_i V_k + \tilde{\eta}_i\tilde{V}_i\\ \le& \left((\eta_i - a_i)(1-\mathbb{I}_i) + \mathrm{e}^{\eta_i}\left(1-\mathrm{e}^{-\eta_i}-a_i\right)\mathbb{I}_i\right)\tilde{V}_i + b_i\mathrm{e}^{\eta_k\mathbb{I}_k}\tilde{V}_k.\end{aligned} \tag{5.52}$$

Choosing $\eta_i < a_i$ in case t_i is continuous and $\eta_i < -\ln(1-a_i)$ in case t_i is discrete guarantees that $\tilde{V}_i$ satisfies condition (5.9). (Note that ln is the natural logarithm.) Consider now the divergence and condition (5.16)

$$\begin{aligned}\operatorname{div}\tilde{\boldsymbol{V}} =& \mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2 t_2}\operatorname{div}\boldsymbol{V} + \tilde{\eta}_1\tilde{V}_1 + \tilde{\eta}_2\tilde{V}_2\\ \le& \mathrm{e}^{\eta_1 t_1}\mathrm{e}^{\eta_2 t_2}\left(-\alpha_1 V_1 - \alpha_2 V_2\right) + \tilde{\eta}_1\tilde{V}_1 + \tilde{\eta}_2\tilde{V}_2\\ =& \left(\tilde{\eta}_1 - \mathrm{e}^{\eta_1\mathbb{I}_1}\alpha_1\right)\tilde{V}_1 + \left(\tilde{\eta}_2 - \mathrm{e}^{\eta_2\mathbb{I}_2}\alpha_2\right)\tilde{V}_2.\end{aligned} \tag{5.53}$$

If $\eta_i < \alpha_i$ in case t_i is continuous and $\eta_i < -\ln(1-\alpha_i)$ in case t_i is discrete, $\tilde{\boldsymbol{V}}$ satisfies (5.10). For $i,k \in \{1,2\}$ and $i \ne k$ note that the initial conditions of $\boldsymbol{V}$ are exponentially decaying and thus $\tilde{V}_i(\boldsymbol{x}_{i0})$ is bounded by

$$\tilde{V}_i(\boldsymbol{x}_{i0}(t_k)) = \mathrm{e}^{\eta_k(t_k-\mathbb{I}_k)}V_i(\boldsymbol{x}_{i0}(t_k)) \le \mathrm{e}^{\eta_k(t_k-\mathbb{I}_k)}\kappa_i\mathrm{e}^{-\mu_k t_k}. \tag{5.54}$$

So choosing $\eta_k < \mu_i$ also guarantees that the initial conditions of $\tilde{\boldsymbol{V}}$ are in L_∞ and L_V. Hence, choosing

$$\eta_i < \begin{cases}\min\{a_i, \quad \alpha_i, \quad \mu_i\} & \text{if } t_i \text{ continuous,}\\ \min\{-\ln(1-a_i), \quad -\ln(1-\alpha_i), \quad \mu_i\} & \text{if } t_i \text{ discrete,}\end{cases} \tag{5.55}$$

for $i \in \{1,2\}$ allows us to follow the same argumentation as in the proof of Corollary 5.1 up to the equivalent of (5.32) guaranteeing that there exists a $C < \infty$ such that

$$\tilde{V}_i \le C. \tag{5.56}$$

Since $|\boldsymbol{x}_i|^{p_i} \le \underline{\alpha}_i(|\boldsymbol{x}_i|) \le V_i(\boldsymbol{x}_i)$ we can conclude that

$$|\boldsymbol{x}_1| \le \mathrm{e}^{-\frac{\eta_1}{p_1}(t_1-\mathbb{I}_1)}\mathrm{e}^{-\frac{\eta_2}{p_1}t_2}C^{1/p_1} \tag{5.57}$$

$$|\boldsymbol{x}_2| \le \mathrm{e}^{-\frac{\eta_1}{p_2}t_1}\mathrm{e}^{-\frac{\eta_2}{p_2}(t_2-\mathbb{I}_2)}C^{1/p_2}. \tag{5.58}$$

□

Note that the rate with which $|\boldsymbol{x}_i|$ decays depends on the Lyapunov function V_i, $\underline{\alpha}_i$, p_1 and p_2.

5.5 Asymptotic Stability

In this section asymptotic stability is shown for general nonlinear two-dimensional systems. In contrast to the conditions known in the literature, the sufficient conditions proposed here can be applied to systems that only allow a Lyapunov function with a nonpositive divergence — rather than a strictly negative divergence. In contrast to the theorem on exponential stability in the previous section, additional smoothness assumptions on the initial conditions and the state space system have to be made.

Theorem 5.2 (Asymptotic Stability of Nonlinear Two-Dimensional Systems)

Consider the C^2 smooth nonlinear two-dimensional system (5.3)-(5.4) with exponentially stable Jacobian matrices $\boldsymbol{F}_{11}$ and $\boldsymbol{F}_{22}$. If there exists a regular two-dimensional Lyapunov function according to Definition 5.2 and in addition there exist scalars $1 \leq p < \infty$ and $0 < a_1', a_2' < \infty$ such that the initial conditions are L_V' and L_∞'' smooth bounded according to Definition 5.5, and

$$\operatorname{div} \boldsymbol{V}(t_1, t_2) \leq -a_1' \left|\delta_1 \boldsymbol{x}_1(t_1, t_2)\right|^p - a_2' \left|\delta_2 \boldsymbol{x}_2(t_1, t_2)\right|^p \tag{5.59}$$

for all $t_1, t_2 > 0$, then the system is asymptotically stable according to Definition 5.8.

Proof The proof of asymptotic stability for nonlinear two-dimensional systems is very similar to the proof for linear systems in Section 4.5:

Consider the integral of $V_1(t_1,t_2)+V_2(t_1,t_2)$ along the line $\Omega(l) := (t_1, t_2) \in \{[0,l] \times \{l\}\} \cup \{\{l\} \times [0,l]\}$ for $l \in \mathbb{R}_+$ or $l \in \mathbb{N}$, and $l > 0$ as:

$$\begin{aligned} U(l) &:= \underset{\Omega(l)}{\mathcal{S}} V_1(\boldsymbol{x}_1(t_1,t_2)) + V_2(\boldsymbol{x}_2(t_1,t_2)) \mathrm{d}s \\ &= \overset{l}{\underset{0}{\mathcal{S}}} \left(V_1(\boldsymbol{x}_1(t_1,l)) + V_2(\boldsymbol{x}_2(t_1,l))\right) \mathrm{d}t_1 + \overset{l}{\underset{0}{\mathcal{S}}} \left(V_1(\boldsymbol{x}_1(l,t_2)) + V_2(\boldsymbol{x}_2(l,t_2))\right) \mathrm{d}t_2. \end{aligned} \tag{5.60}$$

Similarly to the linear case there exists a C such that $U(l) \leq C$ for all l due to the results in Lemma 4.2 and Corollary 5.2.

Since the first generalised derivatives of $\boldsymbol{x}(t_1,t_2)$ with respect to t_1 and t_2 are L_∞ bounded (Lemma 5.2) and the fact that the Lyapunov function components V_1 and V_2 are smooth, we can define

$$d_{11}(l) := \sup_{0 \leq t_1 \leq l} \left|\delta_1 V_1(\boldsymbol{x}_1(t_1,l))\right|, \qquad d_{12}(l) := \sup_{0 \leq t_2 \leq l} \left|\delta_2 V_1(\boldsymbol{x}_1(l,t_2))\right|, \tag{5.61}$$

$$d_{21}(l) := \sup_{0 \leq t_1 \leq l} \left|\delta_1 V_2(\boldsymbol{x}_2(t_1,l))\right|, \quad \text{and} \quad d_{22}(l) := \sup_{0 \leq t_2 \leq l} \left|\delta_2 V_2(\boldsymbol{x}_2(l,t_2))\right|. \tag{5.62}$$

Note that for t_i continuous the above bounds follow immediately from the results in Lemma 5.2 and the chain rule of differentiation: $\mathrm{d}V_k/\mathrm{d}t_i = \partial V_k/\partial \boldsymbol{x}_k \cdot \mathrm{d}\boldsymbol{x}_k/\mathrm{d}t_i$. In case t_i is discrete, observe that we can interpolate V_k between $t_i + 1$ and t_i and for each time instance t_i there exists a $t_i \leq s \leq t_i + 1$ such that

$$\Delta_i V_k(\boldsymbol{x}_k) = V_k(\boldsymbol{x}_k(t_i+1)) - V_k(\boldsymbol{x}_k(t_i)) = \frac{\mathrm{d}V_k(\boldsymbol{x}_k(s))}{\mathrm{d}s} \tag{5.63}$$

with $\boldsymbol{x}_k(s) := \boldsymbol{x}_k(t_i) + (s - t_i)(\boldsymbol{x}_k(t_i+1) - \boldsymbol{x}_k(t_i))$. (For convenience, the second variable t_k is neglected here.) Thus

$$\Delta_i V_k(\boldsymbol{x}_k) = \frac{\mathrm{d}V_k(\boldsymbol{x}_k(s))}{\mathrm{d}s} = \left.\frac{\mathrm{d}V_k(\boldsymbol{x}_k)}{\mathrm{d}\boldsymbol{x}_k}\right|_{\boldsymbol{x}_k=\boldsymbol{x}_k(s)} \frac{\mathrm{d}\boldsymbol{x}_k(s)}{\mathrm{d}s} = \left.\frac{\mathrm{d}V_k(\boldsymbol{x}_k)}{\mathrm{d}\boldsymbol{x}_k}\right|_{\boldsymbol{x}_k=\boldsymbol{x}_k(s)} \Delta_i \boldsymbol{x}_k(t_i). \tag{5.64}$$

Note that $d_{11}(l) \leq \sup_{t_1 \geq 0} |\delta_1 V_1(\boldsymbol{x}_1(t_1, l))|$. Using again the version of Barbalat's Lemma in Lemma 4.4, we can conclude that the first generalised derivatives tend to zero as $t_1, t_2 \to \infty$ and are uniformly convergent in both directions. That allows us to interchange the order of supremum and limit and thus we conclude that

$$\begin{aligned}\lim_{l\to\infty} d_{11}(l) \leq & \lim_{l\to\infty} \sup_{t_1\geq 0} |\delta_1 V_1(\boldsymbol{x}_1(t_1, l))| \\ &= \sup_{t_1\geq 0} \lim_{l\to\infty} |\delta_1 V_1(\boldsymbol{x}_1(t_1, l))| \\ &= 0.\end{aligned} \tag{5.65}$$

To find a lower bound on $U(l)$ we will use a similar trick as in the proof of Lemma 4.4 and Theorem 4.2 on page 75.

If t_1 is continuous and the maximum of $V_i(t_1, t_2)$

$$\overline{V}_i(l) := \max_{(t_1,t_2)\in\Omega(l)} V_i(t_1, t_2) \tag{5.66}$$

for $i \in \{1, 2\}$) along $\Omega(l)$ occurs along the part of $\Omega(l)$ where $(t_1, t_2) \in [0, l] \times \{l\}$ we can bound the integral of $V_i(t_1, t_2)$ over $\Omega(l)$ from below by a triangle with the base equal to $\min\left\{\overline{V}_i(l)\big/d_{i1}(l), l\right\}$ and $\overline{V}_i(l)$ as the height of the triangle. In case t_1 is discrete and $\overline{V}_i(l)$ occurs at $(t_1, t_2) \in [0, l] \times \{l\}$ a similar argument as in Section 4.5 can be followed.

Thus, $U(l)$ can be bounded from below by

$$U(l) \geq \min\left\{\frac{\overline{V}_1^2(l)}{2d_{11}(l)}, \quad \frac{\overline{V}_1^2(l)}{2d_{12}(l)}, \quad \frac{\overline{V}_1(l)l}{2}\right\} + \min\left\{\frac{\overline{V}_2^2(l)}{2d_{21}(l)}, \quad \frac{\overline{V}_2^2(l)}{2d_{22}(l)}, \quad \frac{\overline{V}_2(l)l}{2}\right\}. \tag{5.67}$$

Since $\overline{V}_i(l) \leq V_i(M_i)$ where $\overline{V}_i(l)$ is the maximum of V_i in the region $\Omega(l)$ and $V_i(M_i)$ is the maximal possible value of V_i if $|\boldsymbol{x}_i| \leq M_i$ for all $t_1, t_2 > 0$, this implies

$$\overline{V}_i^2(l) \leq C \cdot \max\left\{2d_{i1}(l), \quad 2d_{i2}(l), \quad \frac{2V_i(M_i)}{l}\right\}. \tag{5.68}$$

Note that as l tends to infinity each component of the maximum in (5.68) goes to zero and, hence, $\lim_{t_1,t_2\to\infty} |V_i(t_1, t_2)| = 0$ and therefore $\lim_{t_1,t_2\to\infty} |\boldsymbol{x}_i(t_1, t_2)| = 0$. The limits

$\lim_{t_1\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$ and $\lim_{t_2\to\infty} |\boldsymbol{x}_i(t_1,t_2)| = 0$ exist as well. $\square$

It should be noted that the assumptions used to guarantee asymptotic stability seem to be rather restrictive. However, requiring smoothness of the initial conditions, the state space description and the Lyapunov functions allows to formulate sufficient conditions for asymptotic stability even if the divergence, $\operatorname{div}\boldsymbol{V}$, is negative *semi-definite*.

5.6 Examples

After proving exponential stability of systems which admit a strict two-dimensional Lyapunov function (with a strictly negative divergence), asymptotic stability was shown for systems that admit a regular two-dimensional Lyapunov function and satisfy certain smoothness criteria.

Two examples will be discussed to illustrate the result on asymptotic stability. Note that in both examples the divergence of the Lyapunov function is merely nonpositive. Thus, they are not suitable to guarantee exponential stability. The second example discusses the stability of a nonlinear vehicle string with a variable time headway.

Example 5.1 Consider the continuous-discrete two-dimensional system

$$\dot{x}_1(t,k) = -\,\phi^2(x_1)x_1(t,k) + \phi(x_1)x_2(t,k) \tag{5.69}$$

$$\Delta x_2(t,k) = \phi(x_1)x_1(t,k) - x_2(t,k) \tag{5.70}$$

with the bounded function $0 < \underline{\Phi} \le \phi(x_1) \le \overline{\Phi} < \infty$. Also assume the first two derivatives of $\phi(x_1)$ are bounded: $|\mathrm{d}\phi(x_1)/\mathrm{d}x_1| \le \Phi' < \infty$ and $|\mathrm{d}^2\phi(x_1)/\mathrm{d}x_1^2| \le \Phi'' < \infty$.

Stability Consider the Lyapunov function

$$\boldsymbol{V} = \begin{pmatrix} V_1(x_1) \\ V_2(x_2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_1^2(t,k) \\ \frac{1}{2}x_2^2(t,k) \end{pmatrix}. \tag{5.71}$$

Thus

$$\dot{V}_1(x_1) = -\phi^2(x_1)x_1^2(t,k) + \phi(x_1)x_1(t,k)x_2(t,k), \tag{5.72}$$

$$\Delta V_2(x_2) = \frac{1}{2}\phi^2(x_1)x_1^2(t,k) - \frac{1}{2}x_2^2(t,k) \quad \text{and} \tag{5.73}$$

$$\operatorname{div}\boldsymbol{V} = -\frac{1}{2}\left(\phi(x_1)x_1(t,k) - x_2(t,k)\right)^2 \le 0. \tag{5.74}$$

Equation (5.72) yields

$$\begin{aligned} \dot{V}_1(x_1) &= -\frac{1}{2}\phi^2(x_1)x_1^2(t,k) - \frac{1}{2}\left(\phi(x_1)x_1(t,k) - x_2(t,k)\right)^2 + \frac{1}{2}x_2^2(t,k) \\ &\le -\frac{1}{2}\phi^2(x_1)x_1^2(t,k) + \frac{1}{2}x_2^2(t,k) \\ &\le -\underline{\Phi}^2 V_1(x_1) + V_2(x_2). \end{aligned} \tag{5.75}$$

Equation (5.73) becomes

$$\Delta V_2(x_2) \leq -V_2(x_2) + \overline{\Phi}^2 V_1(x_1). \tag{5.76}$$

Hence, the system with L_V and L_∞ bounded initial conditions is stable and there exists an upper bound on $|x_1|$ and $|x_2|$ for all t and k.

Asymptotic Stability Note that the system is C^2 smooth since the first three derivatives of $\phi(x_1)$ are bounded, and the Lyapunov functions V_1 and V_2 are also smooth.

Furthermore, the current proof for asymptotic stability requires $\boldsymbol{F}_{ii}(t_1, t_2)$ to be exponentially stable. Since x_1 and x_2 are scalar, condition (5.6) simplifies to $\boldsymbol{F}_{ii}(t,k) < 0$ for all t and k with

$$\boldsymbol{F}_{11}(t,k) = -\phi^2(x_1) - 2\phi(x_1)\frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}x_1(t,k) + \frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}x_2(t,k), \tag{5.77}$$

$$\boldsymbol{F}_{22}(t,k) = -1. \tag{5.78}$$

Assume that $\phi(x_1)$ is chosen such that

$$\left.\frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}\right|_{x_1=0} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}x_1 > 0 \quad \text{for } x_1 \neq 0. \tag{5.79}$$

Thus (5.77) yields

$$\boldsymbol{F}_{11}(t,k) < -\phi^2(x_1) + \frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}x_2(t,k). \tag{5.80}$$

Since the system is stable there exists a $M_2 < \infty$ such that $|x_2(t,k)| < M_2$ for all t and k. Hence, we require

$$\boldsymbol{F}_{11}(t,k) < -\phi^2(x_1) + \left|\frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}\right| M_2 \leq 0. \tag{5.81}$$

Hence, choosing $\phi(x_1)$ such that its derivative satisfies

$$\left|\frac{\mathrm{d}\phi(x_1)}{\mathrm{d}x_1}\right| \leq \frac{1}{M_2}\phi^2(x_1) \tag{5.82}$$

guarantees that $\boldsymbol{F}_{11}$ is exponentially stable. One such choice is for instance $\phi(x_1) = x_1^2 + 1$. With a set of initial conditions sufficiently small such that for instance $M_2 = |x_2| = 1$, the system is asymptotically stable for all $\boldsymbol{x}_1$.

Further observe that a_i and b_i from (5.12) are given by

$$a_1 = \overline{\Phi}^2, \quad b_1 = 1, \quad a_2 = 1, \quad b_2 = \overline{\Phi}^2. \tag{5.83}$$

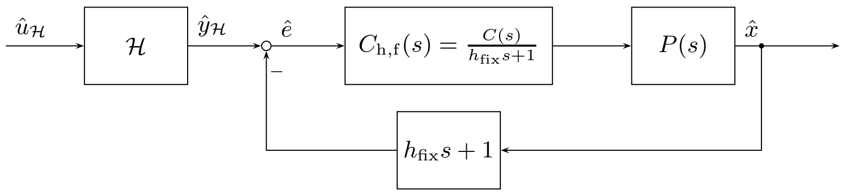


Figure 5.1: *Block diagram of subsystem with variable time headway*

Finally, note that condition (5.59) is satisfied since

$$\begin{aligned}\operatorname{div} \boldsymbol{V} =& -\frac{1}{2}\phi^2(x_1)x_1^2(t,k) - \frac{1}{2}x_2^2(t,k) - \frac{1}{2}\phi(x_1)x_1(t,k)x_2(t,k)\\ =& -\frac{1}{4\phi^2(x_1)}\left(\phi^4(x_1)x_1^2(t,k) + \phi^2(x_1)x_2^2(t,k) - 2\phi^3(x_1)x_1(t,k)x_2(t,k)\right)\\ &-\frac{1}{4}\left(\phi^2(x_1)x_1^2(t,k) + x_2^2(t,k) - 2\phi(x_1)x_1(t,k)x_2(t,k)\right)\\ \leq& -\frac{1}{4\overline{\Phi}^2}\left(\dot{x}_1(t,k)\right)^2 - \frac{1}{4}\left(\Delta x_2(t,k)\right)^2. \end{aligned} \tag{5.84}$$

Thus, the system is locally asymptotically stable. $*$

In our second nonlinear example we will study a nonlinear extension of Example 4.1. Instead of a fixed time headway a varying time headway is considered. The form of h_{var} given in (5.95) was proposed in Yanakiev and Kanellakopoulos (1995) (including an additional upper saturation bound). Yet, string stability of the system has not been shown analytically but was demonstrated through simulations.

Example 5.2 Consider the plant model and the PID controller given in (3.64) and (3.65). Instead of a fixed time headway h we will now use a variable time headway

$$h_{\mathrm{var}} = h_{\mathrm{fix}} + \Delta h_{\mathrm{var}}(\boldsymbol{x}) \tag{5.85}$$

where h_{fix} is a constant greater than the critical time headway $h_0 = 1.18$ and $\Delta h_{\mathrm{var}}(\boldsymbol{x}) \geq 0$ is the variable part of the time headway which depends on the state $\boldsymbol{x}$. An additional pole at $-\frac{1}{h_{\mathrm{fix}}}$ is added to each subsystem.

In order to analyse the stability of the system we will transform the system in Figure 3.2 into the scheme with the additional abstract block $\mathcal{H}$, see Figure 5.1, where the position of the kth vehicle is the input for $\mathcal{H}$ of subsystem $k+1$, i. e. $\hat{x}(t,k) = \hat{u}_{\mathcal{H}}(t,k+1)$. We will use the following state space description for the additional state $x_{1_2}(t)$ of the system $\mathcal{H}$:

$$\dot{x}_{1_2}(t) = -\frac{1}{h_{\mathrm{var}}}x_{1_2}(t) + \frac{\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}}{h_{\mathrm{var}}}u_{\mathrm{H}}(t), \tag{5.86}$$

$$y_{\mathrm{H}}(t) = \frac{\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}}{h_{\mathrm{var}}}x_{1_2}(t) + \frac{h_{\mathrm{fix}}}{h_{\mathrm{var}}}u_{\mathrm{H}}(t). \tag{5.87}$$

Note that with h_{var} fixed, the frozen system $\mathcal{H}$ is linear, time invariant with transfer function $H(s) = \frac{h_{\mathrm{fix}}s+1}{h_{\mathrm{var}}s+1}$. In general, we allow h_{var} to be any time varying function that satisfies $h_{\mathrm{var}} \geq h_{\mathrm{fix}}$. Thus, the system is described by

$$\begin{pmatrix} \dot{\boldsymbol{x}}_{1_1}(t,k) \\ \dot{x}_{1_2}(t,k) \\ \Delta x_2(t,k) \end{pmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A}_0 & \boldsymbol{b}_0\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & \boldsymbol{b}_0 h_{\mathrm{fix}}/h_{\mathrm{var}} \\ \boldsymbol{0} & -1/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \\ \boldsymbol{c} & 0 & -1 \end{bmatrix}}_{\boldsymbol{A}(t)} \begin{pmatrix} \boldsymbol{x}_{1_1}(t,k) \\ x_{1_2}(t,k) \\ x_2(t,k) \end{pmatrix} \tag{5.88}$$

where $\boldsymbol{x}_{1_1}(t,k)$ are the existing states of the controller $C_{\mathrm{h,f}}(s)$ and the vehicle model $P(s)$ and, therefore, $\boldsymbol{A}_0$ and $\boldsymbol{b}_0$ are equal to $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{12}$ in (4.106) for $h = h_{\mathrm{fix}}$:

$$\boldsymbol{A}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -2C_{\mathrm{d}}v_0 & 1 & 0 & 0 \\ -\frac{1}{h_{\mathrm{fix}}}\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & -\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & -\frac{1}{h_{\mathrm{fix}}} & \frac{1}{h_{\mathrm{fix}}} & -\frac{k_{\mathrm{d}}}{h_{\mathrm{fix}}T^2} \\ -k_{\mathrm{i}} & -h_{\mathrm{fix}}k_{\mathrm{i}} & 0 & 0 & 0 \\ -1 & -h_{\mathrm{fix}} & 0 & 0 & -\frac{1}{T} \end{bmatrix}, \tag{5.89}$$

$$\boldsymbol{b}_0 = \begin{pmatrix} 0 & 0 & \frac{1}{h_{\mathrm{fix}}}\left(k_{\mathrm{p}} + \frac{k_{\mathrm{d}}}{T}\right) & k_{\mathrm{i}} & 1 \end{pmatrix}^{\mathrm{T}} \tag{5.90}$$

and $\boldsymbol{c} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. Note that the eigenvalues of $\boldsymbol{A}_{11}$ have negative real parts for $h_{\mathrm{fix}}, h_{\mathrm{var}} > 0$.

Consider the Lyapunov function candidate $\boldsymbol{V}$ with $V_1(\boldsymbol{x}_1) = \boldsymbol{x}_{1_1}^{\mathrm{T}}(t,k)\boldsymbol{P}\boldsymbol{x}_{1_1}(t,k) + x_{1_2}^2$ and $V_2(x_2) = x_2^{\mathrm{T}}(t,k)x_2(t,k)$. The divergence then is $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}$ where

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} & \boldsymbol{P}\boldsymbol{b}_0\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & \boldsymbol{P}\boldsymbol{b}_0 h_{\mathrm{fix}}/h_{\mathrm{var}} \\ \boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & -2/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \\ \boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}h_{\mathrm{fix}}/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & -1 \end{bmatrix}. \tag{5.91}$$

Using the Schur complement, the requirement $\boldsymbol{Q} \leq 0$ yields

$$\begin{aligned} &\begin{bmatrix} \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} & \boldsymbol{P}\boldsymbol{b}_0\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \\ \boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} & -2/h_{\mathrm{var}} \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{P}\boldsymbol{b}_0 h_{\mathrm{fix}}/h_{\mathrm{var}} \\ \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}h_{\mathrm{fix}}/h_{\mathrm{var}} & \sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} + \boldsymbol{P}\boldsymbol{b}_0\boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}h_{\mathrm{fix}}^2/h_{\mathrm{var}}^2 & \boldsymbol{P}\boldsymbol{b}_0(h_{\mathrm{var}} + h_{\mathrm{fix}})\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}}^2 \\ \boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}(h_{\mathrm{var}} + h_{\mathrm{fix}})\sqrt{h_{\mathrm{var}} - h_{\mathrm{fix}}}/h_{\mathrm{var}}^2 & -(h_{\mathrm{var}} + h_{\mathrm{fix}})/h_{\mathrm{var}} \end{bmatrix} \\ &\leq 0. \end{aligned} \tag{5.92}$$

Applying the Schur complement once again it becomes

$$\begin{aligned}
&\boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} + \boldsymbol{P}\boldsymbol{b}_0\boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}h_{\mathrm{fix}}^2/h_{\mathrm{var}}^2 + \boldsymbol{P}\boldsymbol{b}_0\boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}(h_{\mathrm{var}} + h_{\mathrm{fix}})(h_{\mathrm{var}} - h_{\mathrm{fix}})/h_{\mathrm{var}}^2 \\
&\quad = \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} + \boldsymbol{P}\boldsymbol{b}_0\boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}h_{\mathrm{fix}}^2/h_{\mathrm{var}}^2 + \boldsymbol{P}\boldsymbol{b}_0\boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P}(h_{\mathrm{var}}^2 - h_{\mathrm{fix}}^2)/h_{\mathrm{var}}^2 \\
&\quad = \boldsymbol{A}_0^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} + \boldsymbol{P}\boldsymbol{b}_0\boldsymbol{b}_0^{\mathrm{T}}\boldsymbol{P} \\
&\quad \leq 0.
\end{aligned} \tag{5.93}$$

Applying the Bounded Real Lemma we can show that existence of a $\boldsymbol{P} > 0$ satisfying (5.93), is equivalent to the condition

$$\left\| \boldsymbol{c}\left(j\omega\mathbf{I} - \boldsymbol{A}_0\right)^{-1}\boldsymbol{b}_0 \right\|_\infty \leq 1. \tag{5.94}$$

As we have seen before $\Gamma_0(j\omega) = \boldsymbol{c}\left(j\omega\mathbf{I} - \boldsymbol{A}_0\right)^{-1}\boldsymbol{b}_0$ is the transfer function from the kth to the $k+1$th vehicle for $h_{\mathrm{var}} = h_{\mathrm{fix}}$. Since the time headway h_{fix} is greater than the infimal time headway $h_0 = 1.18$, $|\Gamma(j\omega)| = 1$ for all ω and $|\Gamma(j\omega)| < 1$ for $\omega \neq 0$. Thus, a positive definite Matrix $\boldsymbol{P}$ exists such that $\boldsymbol{Q}$ is negative semi-definite independently of h_{var} (for $h_{\mathrm{fix}} > h_0$, and $\Delta h_{\mathrm{var}}(\boldsymbol{x}) \geq 0$) and the system is stable.

Note that using the current proof for asymptotic stability, it is required to ensure that the Jacobian matrices $\boldsymbol{F}_{ii}(t_1, t_2)$ are exponentially stable. This, however, requires more work as h_{var} can depend explicitly on $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and therefore conditions restricting $\partial h_{\mathrm{var}}/\partial \boldsymbol{x}_1$ might be necessary.

Consider the variable time headway

$$h_{\mathrm{var}}(t,k) = \begin{cases} h_{\mathrm{ss}} + k_{\mathrm{h}}\left(\hat{v}(t,k) - \hat{v}(t,k-1)\right) & \text{for } h_{\min} \leq h_{\mathrm{var}}(t,k), \\ h_{\min} & \text{else}, \end{cases} \tag{5.95}$$

where the time headway in steady state is $h_{\mathrm{ss}} = 1.4$, $k_{\mathrm{h}} = 0.05$ and the variable time headway is saturated at $h_{\mathrm{fix}} = h_{\min} = 1.2$. The motivation for the choice (5.95) is that in case the vehicle is driving slower than its predecessor, the variable time headway decreases and the vehicle thus accelerates faster and therefore can reach its desired position faster. A string of forty vehicles has been simulated. The local error is shown in Figure 5.2 and the variable time headway $h_{\mathrm{var}}(t,k)$ in Figure 5.3.

As shown in Figure 5.2 the error for the first vehicle increases to a maximal value that is twice as high as the maximal value of the local error of the first vehicle in a string with a constant time headway ($h = 2$) in Figure 4.2. This is because of the decreased time headway, and consequently the desired distance between the first vehicle and reference position decreases temporarily and thus the error increases.

Note that the simulations suggest that the system is not only stable but also asymptotically stable. As mentioned above, a rigorous proof for asymptotic stability in the current form, however, requires more work and restrictions on $\partial h_{\mathrm{var}}/\partial \boldsymbol{x}_1$ or a different proof for asymptotic stability that requires less restrictive conditions on $\boldsymbol{F}_{ii}$. $*$

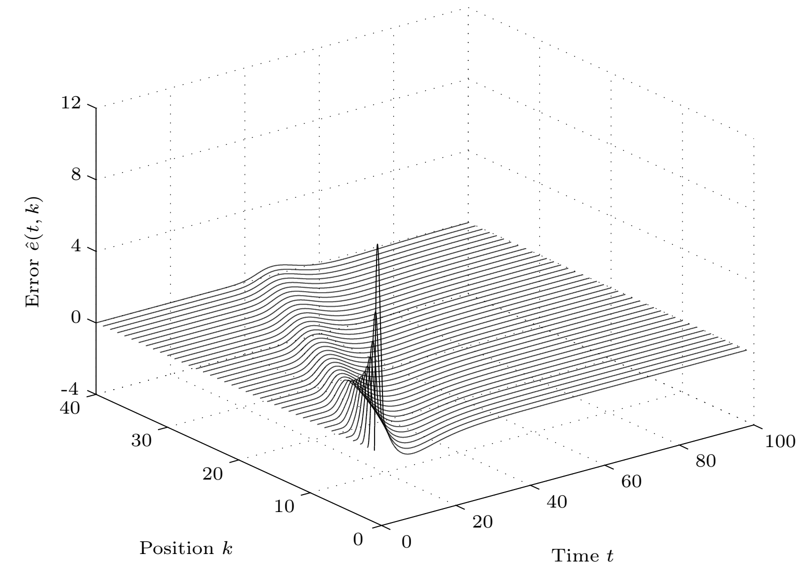


Figure 5.2: *String with variable time headway: error* $\hat{e}(t,k)$

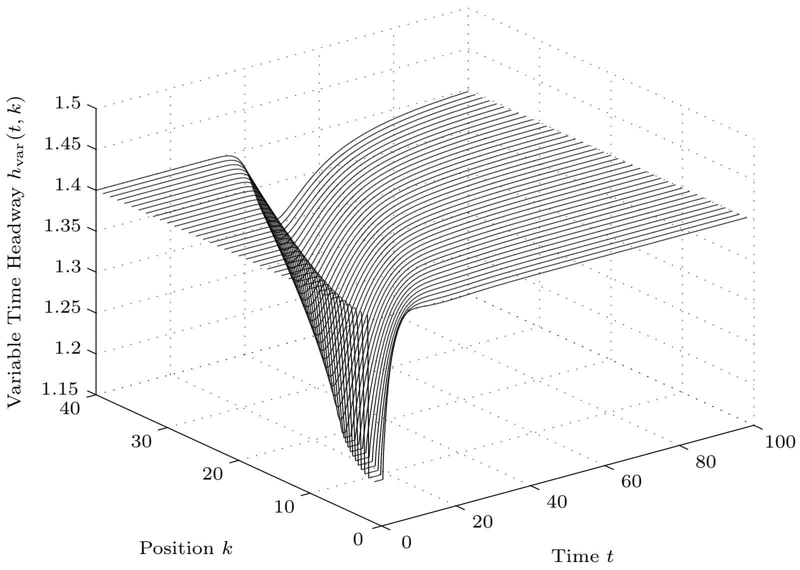


Figure 5.3: *String with variable time headway:* $h_{\mathrm{var}}(t,k)$

5.7 Conclusion

The sufficient conditions guaranteeing stability, exponential stability and asymptotic stability of linear two-dimensional systems given in Chapter 4 have been expanded. Instead of using linear matrix inequalities, here the theory of (integral) input-to-state stability has been employed to derive sufficient stability conditions for general nonlinear two-dimensional systems.

As the divergence of the two-dimensional Lyapunov function is only required to be nonpositive, additional assumptions have been made to prove stability and asymptotic stability. In order to guarantee stability of general nonlinear two-dimensional systems, the only additional assumption is that the iISS-Lyapunov function derivative $\delta_i V_i$ depends on the second part of the Lyapunov function V_k in a certain form. However, the proof for asymptotic stability in Section 5.5 also requires certain smoothness conditions on the initial conditions, the state space equations and the Lyapunov function. In some ways this had to be expected as it was noted in (Zhu and Hu, 2011, Remark 3) in order to show global asymptotic stability for nonpositive differences, at least the assumptions on the initial conditions need to be stronger than merely boundedness.

However, if a Lyapunov function with a strictly negative divergence can be found, exponential stability can be guaranteed. This extends a similar result given in Kurek (1995) where $\operatorname{div} \boldsymbol{V} < 0$ is required to ensure *asymptotic* stability.

To the best of our knowledge, the stability discussion in Example 5.2 is the first rigorous proof of string stability for a nonlinear platoon system with variable time headway. However, in order to guarantee asymptotic stability with the current proof, additional conditions on the derivatives of h_{var} might be required. Alternatively, a different proof with less conservative conditions is needed.

CHAPTER 6

Conclusion

In this last chapter we summarise the contributions of this thesis and suggest possible future directions for continued research in the relevant areas.

Chapter contents

6.1 Summary

In Chapter 1 we motivated the study of two-dimensional systems in general and continuous-discrete two-dimensional systems in particular by introducing the platooning problem: In order to achieve tight spacing between vehicles travelling in a group ("string" or "platoon") one behind the other along one single direction, the vehicles are equipped with an automatic controller for longitudinal position. The aim is to maintain a specified distance towards the predecessor while the first vehicle follows a given trajectory using only locally measurable data and distributed control (i. e. one independent controller for each vehicle rather than one controller for the entire platoon).

It is a known issue that in some settings even if all subsystems are stable and the local error coordinates are bounded and tend to zero for time t going to infinity, the norm of the local error might grow exponentially with the position in the string: A small disturbance at the beginning of the string is propagated through the string and amplified from each vehicle to its follower. Additional constraints have to be derived in order to guarantee the system's stability in the usual sense and also to bound the error independently of the string length. This much stronger stability requirement is referred to as "string stability".

A vehicle string model can also be written as a two-dimensional system with the two independent variables time t and the position within the string k. Even though this description yields some significant advantages and simplifications in the (string) stability analysis, a linear two-dimensional system modelling a vehicle platoon also inherits an unavoidable singularity at the stability boundary, which then requires special attention when analysing stability.

Relevant literature has been discussed in Chapter 2. Some concepts and notation in the field of platooning were introduced before reviewing the most important findings regarding string stability.

Related literature on the stability of linear two-dimensional systems was studied. The emphasis here was to note which of the stability conditions known in the literature are suitable to guarantee stability of systems with singularities at the stability boundary. Most researchers explicitly or implicitly exclude this marginal case and only few results studying the bounded-input bounded-output stability of such systems in the frequency domain are available.

To the best of the authors' knowledge there are no published conditions suitable to guarantee stability of such systems with a singularity at the stability boundary in the time domain. Often, stability conditions are given as linear matrix inequalities (LMI) and require a negative definite solution of the inequality. We have shown, however, that systems with a singularity at the stability boundary can never admit a sign definite solution of such an LMI. Hence, previous LMI methods do not apply to the situation studied here.

Compared to the vast amount of papers on the stability of linear two-dimensional systems very few stability conditions for general nonlinear two-dimensional systems have been published so far.

Chapter 3 investigated the bounded-input bounded-output (BIBO) stability of linear two-dimensional systems in the frequency domain. A combination of the well known Laplace transform (with respect to the continuous time t) and the Z transform (with respect to the discrete position within the sting k) was introduced. It was shown that a version of Parseval's Theorem also holds for two-dimensional systems. Hence, the L_2 norm of a two-dimensional signal in the time domain is equal to the L_2 norm of the Laplace-Z transform of the signal in the frequency domain. This was followed by the derivation of the corresponding L_2 induced operator norm.

These results were used to analyse the BIBO stability of a linear continuous-discrete two-dimensional system describing a vehicle platoon. Since the communication range (the number of vehicles ahead whose information is considered by each vehicle controller) only determines the order of z, systems with different communication settings may be studied using the same approach. The two-dimensional description of a vehicle string with communication range 1 and 2 was analysed and an infimal time headway to guarantee string stability was derived. However, due to the singularity at the stability boundary, it turns out that the magnitude of the operator is discontinuous at this singularity and thus the operator norm had to be examined with special care.

As the stability analysis of linear two-dimensional systems with nonessential singularities of the second kind (NSSK) at the stability boundary in the frequency domain yields some disadvantages, sufficient conditions for stability in the time domain were proposed in Chapter 4. Using a two-dimensional quadratic Lyapunov function and linear matrix inequalities (Lyapunov) stability can be guaranteed if the divergence of the Lyapunov function is nonpositive. This is in contrast to the findings proposed in the literature so far as they require a strictly negative divergence. If the linear two-dimensional system with exponentially decaying initial conditions admits a suitable Lyapunov function with a strictly negative divergence, exponential stability can be guaranteed.

Moreover, it was shown that a system including a singularity at the stability boundary cannot be exponentially stable since in that case the divergence of the Lyapunov function cannot be sign definite. Provided the initial conditions fulfil certain smoothness criteria, asymptotic stability can, however, be guaranteed even if the divergence is merely nonpositive. This sufficient condition for asymptotic stability is thus suitable to analyse the stability of two-dimensional systems including singularities at the stability boundary, such as two-dimensional models of vehicle strings.

All proofs regarding stability, exponential stability and asymptotic stability of linear two-dimensional systems were given in a generalised notation, allowing systems with continuous and discrete time to be studied in a unified manner.

The same vehicle string with communication range 1 discussed as an example for BIBO stability in Chapter 3 was also used to illustrate the findings in Chapter 4. It was shown that if we restrict attention to quadratic Lyapunov functions a time headway greater than the same infimal time headway derived before is necessary and sufficient to guarantee stability and asymptotic stability of the continuous-discrete two-dimensional system.

In Chapter 5 the results for linear two-dimensional systems presented in Chapter 4 were extended to *nonlinear* two-dimensional systems. Instead of demanding a two-dimensional quadratic Lyapunov function, here more general forms of two-dimensional Lyapunov functions were allowed. These functions are similar to (one-dimensional) Lyapunov functions used to study (integral) input-to-state stability of nonlinear systems and the proofs for stability, exponential stability and asymptotic stability of nonlinear two-dimensional systems are partly based on (integral) input-to-state stability theory.

Similar to the sufficient conditions for linear systems, it was proven that exponential stability can be guaranteed if the divergence of the Lyapunov function is strictly negative.

An analogous result to that for linear two-dimensional systems gives stability of nonlinear two-dimensional systems if a Lyapunov function with nonpositive divergence exists. Additionally assuming some extra smoothness conditions on the initial conditions, the state space functions and the Lyapunov function, asymptotic stability was also guaranteed.

These results were used to analyse a two-dimensional system description of a nonlinear string of vehicles: Instead of a fixed time headway, the *variable* time headway depends on the states of the vehicle and its predecessor, hence varies over time and leads to an overall nonlinear system. To the best of our knowledge this is the first rigorous proof for string stability of a nonlinear vehicle string with variable time headway.

6.2 Future Directions

As this work is drawing to a close let us discuss possible future directions to extend the results proposed in this thesis.

The proof for asymptotic stability of nonlinear two-dimensional systems in Chapter 5 can be used to analyse the stability and string stability of different vehicle string settings with nonlinear controller or vehicle dynamics. To apply it to the string with a variable time headway it is necessary, however, to show that the Jacobian matrices are exponentially stable. This will most certainly require bounds on the derivative of the variable time headway.

Finding less conservative conditions on the Jacobian matrices $\boldsymbol{F}_{ii}$ guaranteeing asymptotic stability of nonlinear two-dimensional systems might enable a proof for asymptotic stability of a string with variable time headway without restricting the derivative of the time headway. Alternatively, finding another way to prove asymptotic stability of general nonlinear two-dimensional systems without conditions on the Jacobian matrices at all might not only allow us to show asymptotic stability of a string with variable time headway with even less restrictions but also apply the results to a wider range of applications.

Another example is a saturated actuator where the vehicles cannot accelerate or decelerate faster than some maximal value. This would allow us to guarantee string stability for a more realistic setting. In order to do so, however, it is desirable to extend the findings. For instance, when using a saturated actuator signal, at least in the current setting , it cannot be guaranteed that the system is globally stable as this would require that the vehicle string can follow any given trajectory. Thus, deriving sufficient conditions for *local* stability and asymptotic stability given that the parameter and the trajectory lie within a certain region would be necessary.

If possible, it would also be useful to loosen the rather strict requirements on the initial condition enabling us to apply the results to a wider range of applications. This can be done by easing the smoothness requirements on the initial conditions for asymptotic stability or by showing that initial conditions of a different kind are also suitable to assure asymptotic stability. In particular, it would be worth investigating how initial conditions that are zero after some time T could be incorporated. Also allowing piecewise continuous initial conditions might extend the possible range of applications.

If, however, the smoothness of the initial conditions or the state space functions and the exponential stability of the Jacobian matrices in the nonlinear case are necessary (as well as sufficient), it would be helpful to show such necessity as it would contribute to a better understanding of the underlying dynamics.

Since disturbances in real world problems cannot always be neglected it would be worth extending the results on BIBO stability for linear two-dimensional systems and establish a theory of two-dimensional input-to-state stability for nonlinear two-dimensional systems.

Furthermore it would be worth to extend the stability results to derive controller design strategies for linear two-dimensional systems with SSB based on LMIs, or for nonlinear two-dimensional systems that only admit Lyapunov functions with nonpositive divergences based on the theory of ISS and iISS.

Another possible future direction is to investigate the nonlinear analogue of the singularity at the stability boundary and whether the two-dimensional description of a nonlinear vehicle string can never admit a Lyapunov function with a strictly negative divergence.

Applying the proposed stability criteria to other two-dimensional systems — other than vehicle strings — might contribute to the stability analysis of these fields. It would be particularly interesting to study systems with singularities at the stability boundary, that only admit Lyapunov functions with nonpositive divergences, as it was not possible to study their stability with the previously known results on two-dimensional systems in the literature.

It should also be noted here, that it is not possible to directly study heterogeneous or bidirectional vehicle strings with the methods proposed in this work as they cannot be modelled as two-dimensional systems. In heterogeneous strings the dynamics of each vehicle depend on the position within the string and in general it is therefore not possible to find a general state space equation for all subsystems, i. e. vehicles. In a bidirectional vehicle string each vehicle measures at least the distance towards the preceding and the following vehicle. This implies that this string of N vehicles is not just a truncation of an infinite string. As the two-dimensional systems proposed in this thesis are defined for $t_1, t_2 \to \infty$ bidirectional vehicle strings cannot be modelled as such. Thus, different ways of describing these systems and suitable stability criteria have to be developed.

Notation

Basic Notation

a Scalars; lowercase letters, p. 4

$\boldsymbol{x}$ Vectors and vector valued functions in the time domain $\boldsymbol{x}(\cdot)$; lowercase bold letters; note that the ith element of $\boldsymbol{x}$ is denoted by x_i, p. 4

$\boldsymbol{A}$ Matrices; uppercase bold letters, p. 4

$\boldsymbol{X}(\cdot)$ Transforms in the frequency domain of vector valued functions in the time domain such as $\boldsymbol{X}(s) = \mathcal{L}\{\boldsymbol{x}(t)\}$; uppercase bold letters depending on a complex variable, p. 24

$\mathbf{I}$ Identity matrix, p. 59

$\mathbf{0}$ Zero matrix, p. 59

j Imaginary unit, p. 24

ξ_i Generalised frequency domain variable; s_i or z_i, p. 60

Functions and Function Classes

ln natural logarithm; $\ln x = \log_{\mathrm{e}} x$, p. 50

$V(\cdot)$ Lyapunov function, p. 59

$\mathbb{I}_i$ Indicator function, p. 59

$\mathrm{E}(\boldsymbol{A})^t$ Generalised exponential; $\mathrm{e}^{\boldsymbol{A}t}$ or $(\mathbf{I} + \boldsymbol{A})^t$, p. 59

p.d. α is positive definite if it is continuous, $\alpha(0) = 0$ and $\alpha(x) > 0$ for all $x > 0$, p. 89

$\mathcal{K}$ $\alpha \in \mathcal{K}$ if it is positive definite and strictly increasing, p. 89

$\mathcal{K}_\infty$ $\alpha \in \mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and in addition $\alpha(x) \to +\infty$ as $x \to +\infty$, p. 89

$\mathcal{KL}$ $\beta \in \mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}_\infty$ for all $t \geq 0$ and satisfies $\beta(r, t) \to 0$ as $t \to \infty$ for all $r \geq 0$, p. 89

Local variables for subsystems

$\hat{x}(t,k)$ Position of the kth vehicle in the string, given in metres m, p. 33

$\hat{v}(t,k)$ Velocity of the kth vehicle in the string, given in metres per second m/s, p. 33

$\hat{e}(t,k)$ (Local) Error of the kth vehicle in the string with the corresponding Laplace-Z transform $\hat{E}(s,z)$, given in metres m, p. 33

$\hat{d}(t,k)$ (Local) Disturbance of the kth vehicle in the string with the corresponding Laplace-Z transform $\hat{D}(s,z)$, given in metres per square second m/s^2, p. 33

$\hat{u}(t,k)$ (Local) Control input of the kth vehicle in the string, given in metres per square second m/s^2, p. 33

$\hat{x}(t,0)$ Reference signal for the first vehicle in the string, given in metres m, p. 32

$\hat{x}_{\mathrm{d}}$ Required distance between two vehicles in the string, given in metres m, p. 32

h Time headway, given in seconds s, p. 33

Operators and Transformations

$\mathcal{L}$ Laplace transform, p. 24

$\mathcal{Z}$ Z transform (unilateral), p. 24

$\mathcal{LZ}$ Lapace-Z transform (unilateral), p. 25

$\mathcal{T}_i$ Generalised transform; Laplace transform or Z transform, p. 60

δ_i Generalised derivative operator; $\frac{\mathrm{d}}{\mathrm{d}t_i}$ or Δ_i, p. 58

$\mathcal{S}$ Generalised integration operator; $\int \cdot \mathrm{d}t_i$ or $\sum_{t_i}$, p. 58

div Generalised divergence operator; $\mathrm{div} = \begin{pmatrix} \delta_1 & \delta_2 & \dots \end{pmatrix}$, p. 60

$\oplus$ Direct matrix sum; e. g. $\boldsymbol{P} = \boldsymbol{P}_1 \oplus \boldsymbol{P}_2 = \mathrm{diag}\{\boldsymbol{P}_1, \boldsymbol{P}_2\}$, p. 15

Norms

$\|\boldsymbol{A}\|$ Matrix norm; maximal singular value; $\|\boldsymbol{A}\| = \sqrt{\lambda_{\max}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})} = \sigma_{\max}(\boldsymbol{A})$, p. 59

$|\boldsymbol{x}(t)|_p$ pointwise L_p norm; $\left(\sum_i |x_i(t)|^p\right)^{1/p}$; note that for scalar $x(t)$ all L_p norms are equal, if p is not specified we assume $p = 2$, p. 24

$|\boldsymbol{x}(t)|_\infty$ pointwise L_∞ norm; $\max_i |x_i(t)|$, p. 24

$\|\boldsymbol{x}(\cdot)\|_p$ (one-dimensional) L_p norm; $\left(\mathcal{S}_0^\infty |\boldsymbol{x}(t)|_p^p \mathrm{d}t\right)^{1/p}$; note that we say $\boldsymbol{x}(t)$ is in L_p $[0,\infty)$ if $\|\boldsymbol{x}(\cdot)\|_p < \infty$, p. 60

$\|\boldsymbol{x}(\cdot)\|_\infty$ (one-dimensional) L_∞ norm; $\sup_{t>0} |\boldsymbol{x}(t)|_\infty$; note that we say $\boldsymbol{x}(t)$ is in $L_\infty\,[0,\infty)$ if $\|\boldsymbol{x}(\cdot)\|_\infty < \infty$, p. 60

$\|\boldsymbol{x}(\cdot,\cdot)\|_p$ (two-dimensional) L_p norm; $\left(\mathcal{S}_0^\infty \mathcal{S}_0^\infty |\boldsymbol{x}(t_1,t_2)|_p^p \mathrm{d}t_1 \mathrm{d}t_2\right)^{1/p}$; note that we say $\boldsymbol{x}(t_1,t_2)$ is in $L_p\,[0,\infty)\times[0,\infty)$ if $\|\boldsymbol{x}(\cdot,\cdot)\|_p < \infty$, p. 27

$\|\boldsymbol{x}(\cdot,\cdot)\|_\infty$ (two-dimensional) L_∞ norm; $\sup_{t_1,t_2>0} |\boldsymbol{x}(t_1,t_2)|_\infty$; note that we say $\boldsymbol{x}(t_1,t_2)$ is in $L_\infty\,[0,\infty)\times[0,\infty)$ if $\|\boldsymbol{x}(\cdot,\cdot)\|_\infty < \infty$, p. 69

$\|\boldsymbol{X}(\cdot,\cdot)\|_2$ (two-dimensional) 2-norm in the frequency domain, p. 27

$\|\boldsymbol{x}(\cdot)\|_V$ (one-dimensional) V-norm; $\mathcal{S}_0^\infty V_i\left(\boldsymbol{x}_{i0}(t)\right) \mathrm{d}t$; note that we say $\boldsymbol{x}(t)$ is in L_V if $\|\boldsymbol{x}(\cdot)\|_V < \infty$, p. 91

Abbreviations

2D Two-**D**imensional, p. 3

BIBO **B**ounded-**I**nput **B**ounded-**O**utput, p. 14

BRL **B**ounded **R**eal **L**emma, p. 14

FM1 **F**ornasini **M**archesini's first model, p. 13

FM2 **F**ornasini **M**archesini's second model, p. 13

iISS **I**ntegral **I**nput-to-**S**tate **S**tability (or **S**table), p. 20

ISS **I**nput-to-**S**tate **S**tability (or **S**table), p. 19

LMI **L**inear **M**atrix **I**nequality, p. 14

NSSK **N**onessential **S**ingularity of the **S**econd **K**ind, p. 17

SSB **S**ingularity on the **S**tability **B**oundary, p. 5

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