Automatica 45 (2009) 2592–2596

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

A control design method for a class of switched linear systems[☆]

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ARTICLE INFO

Article history:
Received 28 August 2008
Received in revised form
17 February 2009
Accepted 6 July 2009
Available online 26 August 2009

Keywords:
Switched systems
Stability analysis
Control design

ABSTRACT

In this note we consider the stability properties of a system class that arises in the control design problem of switched linear systems. The control design we are studying is based on a classical pole-placement approach. We analyse the stability of the resulting switched system and develop analytic conditions which reduce the complexity of the stability problem. We further consider two special cases for which strongly simplified conditions are obtained that support the analytic controller design.

1. Introduction

In this paper we consider the control design problem for switched linear systems. Such systems arise in many engineering applications (Liberzon & Morse, 1999; Shorten, Wirth, Mason, Wulff, & King, 2007). However, while the stability analysis of this system class has been the subject of many publications (see De-Carlo, Branicky, Pettersson, & Lennartson, 2000; Liberzon & Morse, 1999; Shorten et al., 2007), a pressing need remains for analytic tools to support the design of stable switched systems. Our objective here is to do this and develop useful, classically inspired, design methods. Much recent work in the control systems community is Lyapunov based where a designer uses a Lyapunov function to inform the feedback design. This approach is widespread in the switched systems community where LMI based design methods are popular, see e.g. Daafouz and Bernussou (2002); Daafouz, Riedinger, and Iumg (2002). While such techniques are very effective, they lack transparency and interpretability that was a feature of classical techniques.

In view of this, our principal contribution is to develop, at the expense of studying a restrictive but important system class (see Solmaz, Shorten, Wulff, and Cairbre 2008 and the references therein), methods for the design of stable switched systems. Our main result shows that the stability analysis of such systems can be reduced to the study of lower dimensional systems. In some important situations this result leads to elegant design rules. Finally, while we present theoretical results, the control approach has been applied to design practical control systems for switched systems (Wulff, Wirth, & Shorten, 2005).

2. Problem statement

We consider the control of processes whose dynamics are governed by equations of the form:

$$y^{(n_p)} = \sum_{l=0}^{n_p-1} q_l(t) y^{(l)} + p_0(t) u \qquad (1)$$

where $y^{(n_p)}$ denotes the n_pth derivative of $y(t)$ and $p_0(t)$, $q_l(t)$ are piecewise constant functions taking on values in the finite sets $p_0(t) \in \{p_{01}, \ldots, p_{0N}\}$, and $q_l(t) \in \{q_{l1}, \ldots, q_{lN}\}$ $\forall$ $l = 0, \ldots, n_p - 1$. We assume that the discontinuities occur simultaneously such that $p_0(t) = p_{0k}$ whenever $q_l(t) = q_{lk}$ for all $l = 0, \ldots, n_p - 1$ where $k \in \mathcal{I} = \{1, \ldots, N\}$ denotes the plant mode.[1] Thus at any time instant the plant dynamics correspond to exactly one of the N linear systems

[1] Equations of this form describe many real world processes. Despite this, their stability properties remain unresolved for systems of dimension greater than three (Pyatnitskii & Rapoport, 1991).

☆ This work was supported by SFI PI Award 07/IN.1/1901. This paper was partially presented at CDC-ECC 2005 Sevilla. This paper was recommended for publication in revised form by Associate Editor Yasumasa Fujisaki under the direction of Editor Roberto Tempo.

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0005-1098/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2009.07.010

$$\dot{x}_p(t) = A_k x_p(t) + b_k u(t), \quad k \in \mathcal{I} = \{1, \dots, N\} \tag{2a}$$

$$y(t) = c^T x_p(t) \tag{2b}$$

where $x_p \in \mathbb{R}^{n_p}$ denotes the continuous state vector of the process,

$$A_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -q_{0,k} & -q_{1,k} & \cdots & \cdots & -q_{n_p-1,k} \end{pmatrix},$$

and $c = (1\ 0\ \cdots\ 0)^T$, $b_k = (0\ \cdots\ 0\ p_{0k})^T$.

With each mode $k \in \mathcal{I}$ we associate the proper transfer function $P_k(s) = c^T(sI - A_k)^{-1}b_k$. For each set of fixed parameters the control design objectives shall be similar; such applications arise frequently in automotive control where the system dynamics are gear dependent, but the performance objectives may be gear independent (Shorten & ÓCairbre, 2002). We assume that these mode switches of the process are immediately detectable as is also the case in many applications. Given this, the objective is to design a controller such that the closed-loop system: (1) has the target poles $\Lambda_t \subset \mathbb{C}_-$, specified independently of mode $k \in \mathcal{I}$; (2) is asymptotically stable for arbitrary switching signals; and (3) has little or no output transients induced by the switching action of the system. In the next section we propose a control design method that is suitable to achieve these requirements. While this note deals with the task of achieving objectives 1 and 2, it is shown in Wulff et al. (2005) that the proposed methodology can be adopted to guarantee transient-free switching between the subsystems under certain minor additional conditions. In this sense our present work extends the work in Shorten and ÓCairbre (2002) and in Paxman and Vinnicombe (2003).

3. Preliminary discussion: Basic ideas

The controller structure considered is depicted in Fig. 1. For each plant mode k a controller $C_k(s)$ is designed to achieve the specified objectives. Each controller is realised as an LTI system. At any switching instant, the appropriate controller is deployed by switching the process input to the respective controller output. We shall further assume that there is no time-delay between the switching of the process and switching of the controller output. Further, we do not have a controller state reset as considered in Hespanha and Morse (2002) and Paxman and Vinnicombe (2003).

The dynamics of the individual controllers are

$$\dot{x}_k(t) = K_k x_k(t) + l_k e(t) \tag{3a}$$

$$u_k(t) = m_k^T x_k(t) + j_k e(t) \tag{3b}$$

where $x_k(t) \in \mathbb{R}^{n_c}$ is the state vector of the controller associated with mode $k \in \mathcal{I}$; the input $e(t) \in \mathbb{R}$ is shared by all controllers and each controller has an individual control signal $u_k(t) \in \mathbb{R}$. For the realisation of the controllers we choose the control canonical form with $K_k \in \mathbb{R}^{n_c \times n_c}$, $l_k, m_k^T \in \mathbb{R}^{n_c}$ and $j_k \in \mathbb{R}$. The respective transfer functions are given by $C_k(s) = m_k^T(sI - K_k)^{-1}l_k + j_k$. As a design-law for the controllers we choose a set of stable target poles Λ_t and design the controllers using standard pole-placement techniques. Our results throughout this paper are based on the following assumption.

Assumption 3.1 (*Pole-placement*). For each process mode $k \in \mathcal{I}$ the controller $C_k(s)$ is designed such that the poles of the closed-loop transfer function

$$\frac{C_k(s)P_k(s)}{1 + C_k(s)P_k(s)}$$

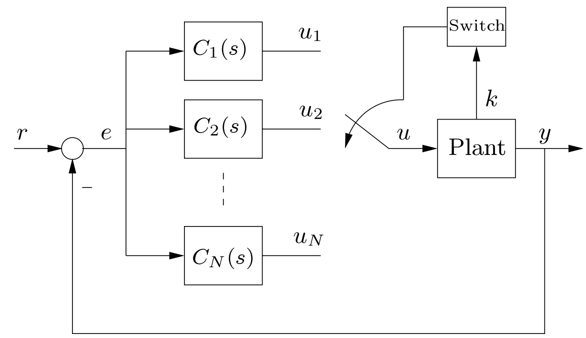

Fig. 1. Structure of the switched linear control system.

are simple and lie in the open left half-plane and are constant for all $k \in \mathcal{I}$. We denote the set of those target poles by $\Lambda_t = \{\lambda_1, \dots, \lambda_{n_p+n_c}\}$. The resulting controllers $C_k(s)$ have poles in the open left half-plane.

Comment: Stability of the switched system can only be achieved for the control architecture in Fig. 1 if the controllers are stable LTI systems.[2] This constitutes a limitation on our design procedure. However, it is easily verified that the parity interlacing property (Vidyasagar, 1987) is always satisfied and so stable controllers for each individual mode always exist.

The resulting closed-loop system dynamics are then given by the switched linear system

$$\dot{x}(t) = H(t)x(t), \tag{4}$$

where $x \in \mathbb{R}^n$, $n = n_p + Nn_c$ consists of the process states x_p and the controller states x_k, $k \in \mathcal{I}$

$$x = \begin{pmatrix} x_p^T & x_1^T & \dots & x_N^T \end{pmatrix}^T,$$

and $H(\cdot)$ is a piecewise constant function $H : \mathbb{R} \to \mathcal{H} = \{H_1, \dots, H_N\} \subset \mathbb{R}^{n \times n}$. The constituent system matrices in each mode $k \in \mathcal{I}$ are given by

$$H_k = \begin{pmatrix} A_k - b_k j_k c^T & b_1 m_1^T \delta_{k1} & \cdots & b_N m_N^T \delta_{kN} \\ -l_1 c^T & K_1 & & 0 \\ \vdots & & \ddots & \\ -l_N c^T & 0 & & K_N \end{pmatrix} \tag{5}$$

where δ_{kj} denotes the Kronecker symbol.

Before we present the main results we note some preliminary observations.

Given the process (2) and controllers (3) in control canonical form, all closed-loop system matrices H_k are identical except for the n_pth row. Furthermore as all but one of the sets of Kronecker symbols are equal to 0, we have that $\sigma(H_k) \supset \sigma(K_l)$ for $l \neq k$. By design (Assumption 3.1) the remaining eigenvalues are given by Λ_t for all $k \in \mathcal{I}$. Thus the spectrum of H_k is given by

$$\sigma(H_k) = \Lambda_t \cup \bigcup_{l \neq k} \sigma(K_l),$$

accounting for multiplicities. Therefore the matrices H_k have pairwise $n_p + (N-1)n_c$ common eigenvalues.

From the design procedure in Assumption 3.1 it follows that the controller poles are distinct from the target poles, i. e. if $\lambda \in \Lambda_t$ then $\lambda \notin \sigma(K_k)$, $k \in \mathcal{I}$. We can show this fact by contradiction: consider the characteristic polynomial of the closed-loop system in mode k

[2] This restriction can be relaxed allowing for integrator control action by using variations of the controller architecture proposed in Wulff et al. (2005).

$$D_{Ck}(s)D_{Pk}(s) + N_{Ck}(s)N_{Pk}(s) \quad (6)$$

where $N(s)$, $D(s)$ denote the respective numerator and denominator of the transfer functions $C(s)$ and $P(s)$ in mode k. Let λ be a root of the characteristic polynomial (6). If λ is also an eigenvalue of K_k, i. e. $D_{Ck}(\lambda) = 0$, we require that either $N_{Ck}(\lambda) = 0$ or $N_{Pk}(\lambda) = 0$. However the root in N_{Ck} would immediately cancel with root of the controller denominator D_{Ck}, the latter contradicts our assumption about the plant model.

A useful consequence of this approach is that the subspace corresponding to the target poles do not depend on k given some mild conditions. This fact shall be useful in the following discussion and we state it formally.

Lemma 3.1. *Let $\lambda \in \Lambda_t$ be a simple eigenvalue of each H_k, then the H_k have a corresponding common eigenvector. That is, there exists a vector $v \neq 0$ such that for all $k \in \mathcal{I}$*

$$H_k v = \lambda v. \quad (7)$$

Proof. As $\lambda \in \sigma(H_k)$, $k \in \mathcal{I}$ the matrices $\lambda I - H_k$ are each singular. Thus the rows $\tilde{h}_{jk}$ of $\lambda I - H_k$ are linearly dependent for each k. On the other hand, using (5) and the definition of A_k and b_k we see that all the rows of H_k, but the n_pth are independent of k. By inspection the set of $n-1$ rows of $\lambda I - H_k$ obtained by omitting the n_pth row is linearly independent, since λ is not an eigenvalue of one of the controllers K_j, $j \in \mathcal{I}$. Thus for each k there are constants γ_{jk} such that

$$\tilde{h}_{n_p k} = \sum_{j \neq n_p} \gamma_{jk} \tilde{h}_{jk} \,. \quad (8)$$

Now by definition an eigenvector v of H_1 corresponding to the eigenvalue λ satisfies $\tilde{h}_{j1} v = 0, j = 1, \dots, n$. This implies that $\tilde{h}_{jk} v = 0, j = 1, \dots, n, j \neq n_p$ for each $k \in \mathcal{I}$. This, however, implies by (8) that also $\tilde{h}_{n_p k} v = 0$, so that we have $(\lambda I - H_k)v = 0$. This completes the proof. □

Hence, if the eigenvalues $\lambda \in \Lambda_t$ are simple, all closed-loop system matrices H_k have $n_p + n_c$ eigenvectors in common. This fact can be exploited to derive simple conditions for stability as we shall discuss in the following section.

4. Main results

In this section we derive simplified stability conditions for the switched system resulting from the control approach described above. We first consider the most general case where the process consists of N subsystems of n_pth order. Based on this result we subsequently consider two special cases: (i) processes with N subsystems and first-order controllers and (ii) processes with two subsystems and controllers of arbitrary order. For each case we obtain simplified stability conditions.

4.1. Stability condition for N subsystems of arbitrary order

Assume that we are given N matrices of the form (5) and that the poles of the individual systems have been placed so that Lemma 3.1 is applicable. Let the columns of $V_t \in \mathbb{C}^{n\times(n_p+n_c)}$ form a basis of the common subspace of all matrices $H_k \in \mathcal{H}$ and consider the matrix

$$T := \begin{pmatrix} V_t & e_{n_p+n_c+1} & \cdots & e_n \end{pmatrix}. \quad (9)$$

Note that T is invertible as the vectors $e_{(n_p+n_c+1)}, \dots, e_n$ form a basis of an invariant subspace of H_1, which does not intersect span V_t as $\Lambda_t \cap \sigma(K_k) = \emptyset \forall\ k \in \mathcal{I}$. Applying the similarity transformation T we obtain

$$T^{-1}H_1T = \operatorname{diag}(D_t, K_2, \dots, K_N),$$

$$T^{-1}H_2T = \operatorname{diag}(D_t, K_2, \dots, K_N) + T^{-1}e_n\tilde{h}_2^{\mathrm{T}}T,$$

up to

$$T^{-1}H_NT = \operatorname{diag}(D_t, K_2, \dots, K_N) + T^{-1}e_n\tilde{h}_N^{\mathrm{T}}T,$$

where $\sigma(D_t) = \Lambda_t$ and $\tilde{h}_k := h_{kn_p} - h_{1n_p}$ denotes the differences between the n_pth rows of H_k and H_1. As implied by our construction the differences between the matrices are all multiples of the same columns. Furthermore inspection of the n_pth rows of the matrices H_k shows that $\tilde{h}_k$ can only have nonzero entries in its first $n_p + n_c$ positions and in the positions $n_p + (k-1)n_c + 1, \dots, n_p + kn_c$. Hence, in the lower block corresponding to the controllers only the controller K_k is perturbed. So that for $k = 2, \dots, N$ the matrices after similarity transformation are of the form

$$T^{-1}H_kT = \begin{pmatrix} D_t & 0 & \dots & U_{1k} & 0 \\ 0 & K_2 & 0 & U_{2k} & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ & & & K_k + U_{kk} & \\ & & & & 0 \\ & & & U_{Nk} & K_N \end{pmatrix} \quad (10)$$

where $U_k = \begin{pmatrix} U_{1k}^{\mathrm{T}} & U_{2k}^{\mathrm{T}} & \dots & U_{Nk}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\times n_c}$ denotes the perturbation term of the kth system. Since $\operatorname{rank}\{H_j - H_k\} = 1$ for all $j \neq k$ and $j, k \in \mathcal{I}$ the perturbation term U_k has rank 1. We denote

$$R_1 := \operatorname{diag}(K_2, \dots, K_N),$$

and for $k = 2, \dots, N$ the lower right $(N-1)n_c \times (N-1)n_c$-block of $T^{-1}H_kT$ by

$$R_k := \begin{pmatrix} K_2 & 0 & U_{2k} & & 0 \\ & \ddots & \vdots & & \vdots \\ & & K_k + U_{kk} & & \\ & & \dots & \ddots & 0 \\ 0 & & U_{Nk} & & K_N \end{pmatrix}.$$

It follows that the closed-loop system is exponentially stable if and only if the switched system formed by the matrices R_k, $k \in \mathcal{I}$ is exponentially stable.

Lemma 4.1. *Consider the switched process (2) and let Assumption 3.1 be satisfied. Then the following statements are equivalent:*

(1) *The switched linear system (4) with $H(t) \in \mathcal{H}$ is exponentially stable.*
(2) *The switched linear system $\dot{x} = R(t)x$ with $R : \mathbb{R} \to \{R_1, \dots, R_N\}$ is exponentially stable.*

Proof. The transformed system matrices $T^{-1}H_kT$ in (10) are in block triangular form with homogeneous dimensions for all $k \in \mathcal{I}$. It is well known that switched systems of this structure are exponentially stable if and only if the switched systems formed from the diagonal blocks are stable.[3] Since D_t is a Hurwitz matrix by Assumption 3.1, the switched system with system matrices (10), $k \in \mathcal{I}$ is exponentially stable if and only if the switched system $\{R_1, \dots, R_N\}$ is exponentially stable. □

The above lemma reduces the stability analysis of the switched system of dimension $n_p + Nn_c$ to the stability of a system of dimension $(N-1)n_c$. Note, that result does not resort to any specific type of Lyapunov function. However, there exists *some* common Lyapunov function for the reduced system if and only if there exists some common Lyapunov function for the original system. We now present two situations where the above results are particularly useful.

A. *Processes with N subsystems and first-order controllers applied:* We shall now consider the special case where the controllers C_k are of first order. Thus for Assumption 3.1 to hold, the process dynamics

3 This follows e.g. from Exercise 4.13 in Rugh (1996).

have to be of order strictly less than three. We now employ Theorem 3.1 in Shorten and ÓCairbre (2001). Essentially, the theorem establishes asymptotic stability of the class of switched systems (4) with the following properties:

- every matrix in $\mathcal{H}$ is Hurwitz and diagonalisable;
- the eigenvectors of any matrix in $\mathcal{H}$ are real;
- every pair of matrices in $\mathcal{H}$ share at least $n-1$ linearly independent common eigenvectors.

Let the target poles Λ_t be distinct and real. With the assumption that the pole-placement is feasible for all modes $k \in \mathcal{I}$, the resulting closed-loop system matrices H_k have pairwise $n-1$ real distinct eigenvalues. By Lemma 3.1 the matrices H_k, $k \in \mathcal{I}$, have n_p+1 common eigenvectors. Moreover, since each pair of closed-loop system matrices H_k share $N-2$ of the remaining inactive controllers they have pairwise $n-1$ common eigenvectors. Thus the requirements for Theorem 3.1 in Shorten and ÓCairbre (2001) are met and the closed-loop system is exponentially stable for arbitrary switching sequences. In other words, the switched system (4) is stable for arbitrary switching if we choose arbitrary real negative target poles Λ_t such that the design-law in Assumption 3.1 is satisfied by first-order controllers (Shorten & ÓCairbre, 2002).

Lemma 4.1 can be used to extend this result for systems with non-real target poles Λ_t. Choosing a modal-basis for V_t in (9) we obtain a transformation matrix T with real entries. It follows that the system matrices R_k of the reduced system are in $\mathbb{R}^{N-1\times N-1}$. Further, $\sigma(R_k) = \cup_{l\neq k}\sigma(K_l)$. Since the controllers are of first order, it follows that the matrices R_k also satisfy the requirement of Theorem 3.1 in Shorten and ÓCairbre (2001).

Corollary 4.1. *The switched system* (4) *with system matrices* (5) *where Assumption 3.1 is satisfied using N stable first-order controllers is asymptotically stable.*

B. *Two subsystems of arbitrary order:* Consider now the special case where $N=2$ and the controllers are of arbitrary order n_c. Due to the pole-placement requirement (Assumption 3.1) we obtain for the respective spectra $\sigma(H_1) = \Lambda_t \cup \sigma(K_2), \sigma(H_2) = \Lambda_t \cup \sigma(K_1)$. Applying the similarity transformation T of (9) to our two system matrices we obtain

$$T^{-1}H_1T = \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} \tag{11a}$$

$$T^{-1}H_2T = \begin{pmatrix} D_t & 0 \\ 0 & K_2 \end{pmatrix} + \begin{pmatrix} 0 & U_1 \\ 0 & U_2 \end{pmatrix} \tag{11b}$$

where $\left(U_1^{\mathrm{T}}\ U_2^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{2n_c\times n_c}$ and $\sigma(D_t) = \Lambda_t$. Note that $\operatorname{rank}\{U_2\} = 1$ as we have $\operatorname{rank}\{H_1 - H_2\} = 1$. Further it follows from the spectrum of H_2 that $\sigma(K_2+U_2) = \sigma(K_1)$. The following theorem reduces the stability problem of the switched system defined by $\{H_1, H_2\}$ to a stability problem only involving the controllers.

Theorem 4.1. *Consider the matrices H_1, H_2 in* (5) *and let Assumption* 3.1 *be satisfied such that $\sigma(H_k) = \Lambda_t \cup \sigma(K_l)$ for $k, l = 1, 2$, $k \neq l$. Assume furthermore that $\Lambda_t \cap \sigma(K_k) = \emptyset$, $k = 1, 2$. Then the following statements are equivalent:* (i) *The switched system given by the set of matrices $\{H_1, H_2\}$ is asymptotically stable for arbitrary switching signals;* (ii) *The switched system given by the set of matrices $\{K_2, K_2+U_2\}$ is asymptotically stable for arbitrary switching signals;* (iii) *The switched system given by the set of matrices $\{K_1, K_2\}$ is asymptotically stable for arbitrary switching signals.*

Proof. The equivalence of (i) and (ii) can be seen as follows. Firstly, the matrices in (5) and (11) are obtained from one another by simultaneous similarity. Thus the set $\{H_1, H_2\}$ defines an asymptotically stable switched system if and only if $\{T_1^{-1}H_1T_1, T_1^{-1}H_2T_1\}$ does. On the other hand $\sigma(D_t) = \Lambda_t \subset \mathbb{C}_-$, so that the exponential stability of $\{T_1^{-1}H_1T_1, T_1^{-1}H_2T_1\}$ is equivalent to that of the lower diagonal block $\{K_2, K_2+U_2\}$. The equivalence (ii) $\Leftrightarrow$ (iii) follows if we find a similarity transformation that transforms K_2 and K_2+U_2 into K_2 and K_1 respectively. Note first, that since $\operatorname{rank}\{H_2-H_1\} = 1$, the perturbation $(U_1^{\mathrm{T}}, U_2^{\mathrm{T}})^{\mathrm{T}}$ is also of rank one. Consider now the matrices K_2^{T} and $K_2^{\mathrm{T}}+U_2^{\mathrm{T}}$ and define

$$x_m := \left(K_2^{\mathrm{T}}\right)^m x = \left(K_2^{\mathrm{T}} + U_2^{\mathrm{T}}\right)^m x, \tag{12}$$

for $m \in \{0, \ldots, n_c-1\}$ and some $x \in \mathbb{R}^{n_c}$. If we can find a vector x such that the sequence x_m, $m = 0, \ldots, n_c-1$ is well defined and linearly independent, then the similarity transformation $S = \left(x_0 \ \cdots \ x_{n_c-1}\right)$ yields

$$S^{-1}K_2^{\mathrm{T}}S = K_2^{\mathrm{T}}, \quad \text{and} \quad S^{-1}(K_2^{\mathrm{T}}+U_2^{\mathrm{T}})S = K_1^{\mathrm{T}}.$$

This assertion follows since the assumption (12) guarantees that both matrices are brought simultaneously in transposed companion form (sometimes also known as second companion form) and because the companion form of K_2+U_2 is K_1 since $\sigma(K_1) = \sigma(K_2+U_2)$. By taking transposes of the previous equations we have found the desired transformation that concludes the proof in case that (12) holds. Consider the sequence of conditions for $m = 1, 2, \ldots$:

$$\begin{aligned} K_2^{\mathrm{T}}x &= \left(K_2^{\mathrm{T}} + U_2^{\mathrm{T}}\right)x, \\ \left(K_2^{\mathrm{T}}\right)^2 x &= \left(\left(K_2^{\mathrm{T}}\right)^2 + K_2^{\mathrm{T}}U_2^{\mathrm{T}} + U_2^{\mathrm{T}}K_2^{\mathrm{T}} + \left(U_2^{\mathrm{T}}\right)^2\right)x, \\ \vdots &= \vdots \end{aligned}$$

By induction these conditions require that

$$U_2^{\mathrm{T}}\left(K_2^{\mathrm{T}}\right)^m x = 0, \quad \text{for } m = 0, \ldots, n_c-2.$$

Consider now the intersection of the kernels of $U_2^{\mathrm{T}}\left(K_2^{\mathrm{T}}\right)^m$ for $m = 0, \ldots, n_c-2$

$$V := \bigcap_{m=0}^{n_c-2} \ker U_2^{\mathrm{T}}\left(K_2^{\mathrm{T}}\right)^m.$$

As $\operatorname{rank}\left\{U_2^{\mathrm{T}}\right\} = 1$, the kernel of $U_2^{\mathrm{T}}\left(K_2^{\mathrm{T}}\right)^m$ has dimension n_c-1 for $m = 0, \ldots, n_c-2$ and so by dimensionality reasons we find that $\dim V \geq 1$. Choose an $x \in V$, $x \neq 0$. If the set of vectors $\{x_m, m = 0, \ldots, n_c-1\}$ is linearly independent, then (12) holds and we are done. If this is not the case this means that the lower dimensional subspace

$$W := \operatorname{span}\{x_m \mid m = 0, \ldots, n_c-1\}$$

is K_2^{T}-invariant and by definition is contained in the kernel of U_2^{T}. Hence on this lower dimensional subspace K_2^{T} is not perturbed by U_2^{T}. We may then repeat the argument on the restriction of K_2^{T} to an invariant subspace complementary to W. This procedure can be iterated until (12) holds on one any of these lower dimensional complementary subspaces. For reasons of dimensionality this procedure terminates and the assertion follows. □

Theorem 4.1 reduces the complexity of the stability analysis of the switched system considerably. To guarantee asymptotic stability of the switched system (4) with $N=2$ we only need to consider the asymptotic stability of the switched system given by

$$\dot{x} = K(t)x, \quad K(t) \in \{K_1, \ldots, K_N\} \subset \mathbb{R}^{n_c\times n_c} \tag{13}$$

for arbitrary switching signals. Thus, the stability problem of the switched system (4) of order n_p+2n_c is reduced to the stability problem of a switched system of order n_c.

Comment: This finding is useful as it implies that control design methods for lower dimensional systems can sometimes be applied to higher dimensional ones. For example, in Wulff et al. (2005) we use a third-order design procedure to guarantee stability of a sixth-order switched system.

Finally, we note that the equivalence of the asymptotic stability of the system (4) and (13) is less obvious than intuition might suggest. In this context it is worth noting that the switched system (13) is not explicitly part of the closed-loop system (4). For the switched system (13) the controller dynamics K_k act on the same state-space; however the controllers in the closed-loop system (4) are realised as individual LTI systems and therefore do not share the states. Finally, the above algebra suggests that the switched closed-loop system (4) is stable if and only if the switched system (13) consisting of the controllers form a stable system. We conclude our paper by noting that, unfortunately, that is generally not true as the following example shows.

Example 4.1. Consider the switched process (2) with $N = 3$, where

$$A_1 = \begin{pmatrix} 0 & 1 \\ -11.84 & -2.4 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ -34.28 & -11.6 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 1 \\ -29.7 & -11 \end{pmatrix}$$

and $b_k = (0 \;\; 1)^{\mathrm{T}}$, $c_k^{\mathrm{T}} = (1 \;\; 0)$ for $k = 1, 2, 3$, and let the requested target poles be given by $\Lambda_t = \{-1 \pm 3\mathrm{i}, -1.8, -8\}$. It can be verified that the pole-placement requirement is satisfied by the following set of controllers (3) with

$$K_1 = \begin{pmatrix} 0 & 1 \\ -9.6 & -9.4 \end{pmatrix}, \qquad K_2 = \begin{pmatrix} 0 & 1 \\ -7.4 & -0.2 \end{pmatrix},$$
$$K_3 = \begin{pmatrix} 0 & 1 \\ -5.5 & -0.8 \end{pmatrix},$$

and $m_1^{\mathrm{T}} = (30.34 \;\; -7.536)$, $m_2^{\mathrm{T}} = (-109.7 \;\; 34.1)$, $m_3^{\mathrm{T}} = (-19.35 \;\; 42.54)$, and $l_k^{\mathrm{T}} = (0 \;\; 1)$, $j_k = 0$ for $k = 1, 2, 3$. It can be numerically verified that the lower dimensional system is quadratically stable. However, the periodic switching signal associated with the monodromy matrix $\Phi(t+T, t) = e^{H_3T_3}e^{H_2T_2}e^{H_1T_1}$ is unstable for $T = T_1+T_2+T_3$ and $T_1 = 0.72$, $T_2 = 0.32$, $T_3 = 0.22$.

5. Conclusions

In this paper we consider feedback design for SISO switched linear systems based on analytic control design methods. We investigate the stability properties of the resulting closed-loop switched system and show in our main results that the complexity of this analytic design process can be significantly reduced by exploiting the structure of the closed-loop system. This reduction expands the range of applications of available analytic tools for switched systems of low order. The application of our results may e.g. allow for the analysis of switched control systems with standard controllers such as PID controllers, by resorting to stability conditions for second-order switched systems. The analytic nature of these conditions supports the classical control design.

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