Stability and Spectral Properties in the Max Algebra

with Applications in Ranking Schemes

A dissertation submitted for the degree of
Doctor of Philosophy

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For my husband Kenan, my mother, father, and sister.

Sevgili eşim Kenan Gürsoy, canım annem Bedriye Benek, babam Mehmet Benek ve ablam Demet Balcı'ya.

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy from the Hamilton Institute is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: ___________________________

Date: 28th February, 2013.

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"The harder I practice, the luckier I get."
Gary Player

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Abstract

This thesis is concerned with the correspondence between the max algebra and non-negative linear algebra. It is motivated by the Perron-Frobenius theory as a powerful tool in ranking applications. Throughout the thesis, we consider max-algebraic versions of some standard results of non-negative linear algebra. We are specifically interested in the spectral and stability properties of non-negative matrices. We see that many well-known theorems in this context extend to the max algebra. We also consider how we can relate these results to ranking applications in decision making problems. In particular, we focus on deriving ranking schemes for the Analytic Hierarchy Process (AHP).

We start by describing fundamental concepts that will be used throughout the thesis after a general introduction. We also state well-known results in both non-negative linear algebra and the max algebra.

We are next interested in the characterisation of the spectral properties of matrix polynomials. We analyse their relation to multi-step difference equations. We show how results for matrix polynomials in the conventional algebra carry over naturally to the max-algebraic setting. We also consider an extension of the so-called Fundamental Theorem of Demography to the max algebra. Using the concept of a multigraph, we prove that a number of inequalities related to the spectral radius of a matrix polynomial are also true for its largest max eigenvalue.

We are next concerned with the asymptotic stability of non-negative matrices in the context of dynamical systems. We are motivated by the relation of P-matrices and positive stability of non-negative matrices. We discuss how equivalent conditions connected with this relation echo similar results over the max algebra. Moreover, we consider extensions of the properties of sets of P-matrices to the max algebra. In this direction, we highlight the central role of the max version of the generalised spectral radius.

We then focus on ranking applications in multi-criteria decision making problems. In particular, we consider the Analytic Hierarchy Process (AHP) which is a method to deal with these types of problems. We analyse the classical Eigenvalue Method (EM) for the AHP and its max-algebraic version for the single criterion case. We discuss how to treat multiple criteria within the max-algebraic framework. We address this generalisation by considering the multi-criteria AHP as a multi-objective optimisation problem. We consider three approaches within the framework of multi-objective optimisation, and use the optimal solution to provide an overall ranking scheme in each case.

We also study the problem of constructing a ranking scheme using a combinatorial approach. We are inspired by the so-called Matrix Tree Theorem for Markov Chains. It connects the spectral theory of non-negative matrices with directed spanning trees. We prove that a similar relation can be established over the max algebra. We consider its possible applications to decision making problems.

Finally, we conclude with a summary of our results and suggestions for future extensions of these.

Notations and Abbreviations

Notations:

Symbol	Meaning
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	The set of all n-tuples of real numbers
$\mathbb{R}^{n\times n}$	The set of all $n \times n$ matrices with real entries
$\mathbb{R}_+$	The set of non-negative real numbers
$\mathbb{R}_+^n$	The set of all n-tuples of non-negative real numbers
$\mathbb{R}_+^{n\times n}$	The set of all $n \times n$ matrices with non-negative real entries
$\mathrm{int}(\mathbb{R}_+^n)$	The set of all n-tuples of positive real numbers
$\oplus$	The maximum operation
$\otimes$	The usual multiplication operation
$\mathbb{R}_{\max,\times}$	The max algebra semiring
A^T	The transpose of A
A^k	The usual k^{th} power of A
$A_\otimes^k$	The k^{th} max power of A
A^C	The critical matrix of A
A^*	The Kleene star of A
I	The $n \times n$ identity matrix
P	An $n \times n$ permutation matrix
X	An $n \times n$ diagonal matrix
$\mathbf{1}_n$	An $n \times 1$ vector with every entry equals to one
$\mathbf{1}_n\mathbf{1}_n^T$	An $n \times n$ matrix with every entry equals to one
T	An $n \times n$ transitive matrix (*Chapter 5*)
x_i	The i^{th} entry of the vector x
a_{ij}	The $(i,j)^{\text{th}}$ entry of the matrix A
a_{ij}^p	The $(i,j)^{\text{th}}$ entry of the matrix A_p
$a_{ij}^{(k)}$	The $(i,j)^{\text{th}}$ entry of the matrix A^k
$A_{i.}$	The i^{th} row of A
$A_{.i}$	The i^{th} column of A
$\rho(A)$	The spectral radius of A
$\mu(A)$	The largest max eigenvalue of A
$D(A)$	The weighted directed graph of A
$N(A)$	The set of vertices in $D(A)$

$E(A)$	The set of edges in $D(A)$
$D^C(A)$	The critical graph of A
$N^C(A)$	The set of critical vertices in $D(A)$
$E^C(A)$	The set of critical edges in $D(A)$
$M(\Psi)$	The multigraph associated with the set Ψ
Γ	A cycle in a graph
$\pi(\Gamma)$	Weight of the cycle Γ
$l(\Gamma)$	Length of the cycle Γ
$d(D(A))$	The diameter of $D(A)$
$\exp(A)$	The exponent of A
$\mathrm{cyc}(A)$	The cyclicity index of A
$V(A)$	The set of max eigenvectors of A associated with $\mu(A)$
$V^*(A)$	The set of subeigenvectors of A associated with $\mu(A)$
T	A directed spanning tree of $D(A)$ (*Chapter 6*)
E_T	The set of edges in T
$\mathcal{T}_i$	The set of all directed spanning trees of $D(A)$ rooted at i

Abbreviations:

Symbol	**Meaning**
AHP	Analytic Hierarchy Process
g.c.d.	greatest common divisor
l.c.m.	least common multiple
PC-matrix	Pairwise comparison matrix
SR-matrix	Symmetrically reciprocal matrix
EM	Eigenvalue Method
RST	Rooted Directed Spanning Tree

CHAPTER 1

Introduction

In this chapter, we briefly discuss non-negative matrices and their relevance to applications. We then proceed to describe the max algebra, which is the main focus of our later work, and discuss some motivating examples. After outlining the structure of the thesis, we conclude by summarising the principal contributions of the work.

1.1 Introductory Remarks

Throughout this thesis, we shall deal with non-negative matrices over the max algebra. We first discuss non-negative matrices and then describe informally some aspects of the theory and applications of the max algebra.

1.1.1 The Class of Non-negative Matrices

Non-negative matrices are square matrices in which all the elements are greater than or equal to zero. Non-negativity appears naturally in applications in science and engineering. We list a few examples below.

Probability theory. Non-negative matrices are frequently used in the theory of Markov chains. Markov chains are applied widely in communications, economics, population dynamics and information retrieval [Sen06]. A Markov chain is characterised by a transition probability matrix. This is usually a *row stochastic matrix*, a non-negative matrix all of whose rows sum to one. For more background on this theory and applications, see [Sen06].

Economics. Non-negative matrices also arise in mathematical models of economics. A very classical example of this is the Leontief input-output economic model [Mey00]. The relationships between industries in an economy are represented by the consumption matrix. This is usually a *column stochastic matrix*, a non-negative matrix all of whose columns sum to one. For a discussion of the use of non-negative matrices in Economics, see [BP94].

Demography. In Demography, the famous Leslie distribution model is used to predict the future female population in a given society [Mey00]. The Leslie matrix in this model is non-negative and can be made *irreducible and primitive*[1] under certain conditions. For an analysis of the model in the context of non-negative matrix theory, see [Mey00].

Mathematical Programming. Non-negative matrix theory has an important role in certain types of Linear Complementarity Problems (LCPs). The LCP is a powerful framework for combinatorial optimisation with numerous applications in game theory, economics and numerical analysis [BP94]. In particular, *P-matrices*, matrices all of whose principal minors are positive, play a key role in characterising the solution of an LCP. Look at [BP94] for more information on the connection of LCPs with non-negative matrices.

Numerical analysis. Non-negative matrix theory is closely related with iterative numerical methods, such as those used in the solution of Dirichlet boundary value problems (see Varga [Var62] for example). In the numerical solution of such boundary value problems, discretisation methods typically lead to a large set of linear equations. A typical approach to

[1] A matrix A is said to be irreducible if and only if its directed graph is strongly connected. A is said to be primitive if and only if, for some positive integer k , $a_{ij}^{(k)}$ is positive for all i, j.

the solution of these is to consider an iterative method in which the iteration matrix is transformed to be non-negative.

Web search engines. The most common example demonstrating the importance of non-negativity is provided by Google's PageRank algorithm. The algorithm constructs the Google matrix by using the hyperlink structure of the web. By applying two modifications to this matrix, it is made row stochastic, irreducible and primitive. [LM06] is among the many references that describe the relation between the PageRank algorithm and non-negative matrix theory.

Operations research. Non-negative matrix theory plays a key role in connection with the *Analytic Hierarchy Process* (AHP) [Saa77a, Saa80, Saa88]. The AHP is a widely used technique in multi-criteria decision making problems. In the AHP, pairwise comparison matrices are *positive matrices*, i.e., matrices with positive elements. We dedicate Chapter 5 to the application of the max algebra in ranking schemes in the multi-criteria AHP.

Since the class of non-negative matrices is important in applications, the properties of this type of a matrix have attracted the interest of many researchers. In this direction, the *Perron-Frobenius theory* plays a central role in the characterisation of the dominant eigenvalues and eigenvectors of non-negative and related matrices. Moreover, it has an important role in characterising the asymptotic behaviour of the matrix powers.

The classical Perron-Frobenius theory was derived by Oscar Perron in 1907 [Per07]. It establishes results on the spectral and asymptotic properties of positive matrices. Shortly after that, these results were generalised to certain type of non-negative matrices by Georg F. Frobenius [Fro12]. In the past decades a wide range of books have been written on non-negative matrix theory [Gan59, LT85, HJ90, BP94, BR97, Mey00, Sen06]. Given the extensive usage of the classical Perron-Frobenius theory, the basic theory was generalised to infinite dimensional operators [Jen12] and nonlinear maps [Nus88, Lem06]. In a sense, the max-algebraic spectral theory is a special case of nonlinear Perron-Frobenius theory [Gun03, GG04, AGL11].

1.1.2 The Max Algebra

The max algebra is the main focus of this thesis. By the max algebra, we mean an algebraic structure on the non-negative numbers under the maximum and multiplication operations. It is isomorphic to the max-plus algebra via the natural isomorphism $x \to \log(x)$. In this thesis, we only deal with the max algebra setting. While we later describe key results in detail, we now provide a concise historical overview of the development of the field.

One of the first appearances of using such algebraic structures is in Stephen C. Kleene's paper on the theory of finite automata in 1956 [Kle56]. The first extensive analysis on the fundamentals of the theory and applications is contained in the lecture notes of Raymond A. Cuninghame-Green [CG79]. In this context, another well-know manuscript is [BCOQ92], which specifically focuses on modelling and characterising of discrete event dynamical systems with the max-plus algebra. One of the recent textbooks in the field is Peter Butkovič's book [But10], which provides a broad analysis of the max-algebraic spectral theory with illustrative examples, as well as describing the latest developments. Several variations of the max algebra have been used in applications by different authors. Examples include the max-plus algebra [BCOQ92, GP97, HOvdW06, ABG07, But10], the min-plus algebra [BCOQ92, Pin98] and the max-min algebra [Gav97, Gav00]. Moreover, these variations have been rediscovered independently under different names such as extremal [Vor67, Zim77], tropical [Sim78], exotic [GP97], idempotent [KM97, LMS01, Kri05].

Such settings provide a natural framework for analysing a broad class of discrete event dynamical systems. Typical examples include the design and analysis of bus and railway timetables [HOvdW06], scheduling of high-throughput industrial processes [CDQV85], solution of combinatorial optimisation problems [Bap95, BB03, But03] and the analysis and improvement of flow systems in communication networks [DCMSM06]. So far, they have appeared in several branches of mathematics such as functional analysis, optimisation, stochastic systems and dynamic programming [BCOQ92, GP97, But03, ABG07]. In particular, tropical geometry is a recently evolving area [Mik06, BSS07, Ser09c, JK10].

The max algebra setting arises directly in applications such as the Viterbi algorithm [BCOQ92] and is used to construct ranking schemes for the AHP [EvdD04, EvdD10]. An attraction of the max algebra is that nonlinear problems can be described in a linear manner. Moreover, it provides us important tools to characterise the properties of non-negative matrices. Many key results in this context relate to extensions of the classical Perron-Frobenius theory over non-negative linear algebra to the max algebra setting. Very early results in this direction were obtained by [Vor67]. For a recent reference focussing specifically on the Perron-Frobenius theorem for this setting, see [Bap98], wherein several proofs of this fundamental theorem were presented. One of the proofs here highlights the key role played by the so-called critical graph. This connection with graph theory is an instance of the strong relationship between the max algebra and combinatorics [But03].

The classical power method was modified to obtain the max eigenvalue and max eigenvectors of non-negative irreducible matrices in [EvdD99, EvdD01]. Conditions guaranteeing convergence of the power method were also given in these papers. More detailed results on the behaviour of the max-algebraic powers are contained in [Ser09a, Ser09b, But10].

Problems including matrix scaling in the max algebra were described in [BS05, SSB09, Ser11] in connection with the max-algebraic spectral theory. Results relating classes of matrix norm, the maximal cycle geometric mean and asymptotic stability for a single matrix in the max algebra were presented in [Lur05]. This line of research was then further extended to sets of matrices in [Lur06, Pep08] and to infinite dimensional positive operators in [Pep09].

1.1.3 Motivating Example

We discuss the AHP to which both the classical Perron-Frobenius theory and the max-algebraic spectral theory was applied.

The AHP is a framework designed to deal with decision making problems involving more than one criterion. We give three basic examples below.

Buying a car: The criteria may be cost, reliability, speed, comfort and safety. How do we choose the best car among multiple alternatives with these in mind?

Choosing a university: The criteria may be teaching quality, research success, accommodation, social life and location. How do we decide the best university with respect to all these criteria?

Planning a vacation: The criteria may be cost of the trip, sight-seeing opportunities, entertainment, method of travel and eating places. Which one of these is the most important in selecting a vacation plan? (We will revisit this example in Chapter 5.)

In the AHP, a decision problem is represented in a treelike structure consisting of three layers. From top to bottom, these represent an overall goal, a set of criteria and alternatives. The general principle in the AHP is to compare alternatives in pairs with respect to each criterion and the set of criteria with respect to the main goal. At each step a positive matrix is constructed to represent these pairwise comparisons.

The AHP was introduced by Thomas L. Saaty in 1977 [Saa77a]. After this, it has successfully been used in several real life problems arising in manufacturing systems, finance, military, traffic, politics, education, business, industry and many others [Zah86, SV01, FG01, VK06, IL11]. It is a flexible and comprehensive framework due to its hierarchical structure. It is also straightforward to apply once pairwise comparisons are constructed. That's why the AHP applications are so extensive.

The main target in the AHP is to rank the alternatives depending on multiple criteria in a decision problem. Saaty suggested using the Perron vector [Saa77a, Saa80, Saa86a, Saa99] of the pairwise comparison matrices. Other approaches are also possible, for instance the Least Squares method [Chu98, FLR03] and the Logarithmic Least Squares method [WC80, Cra87]. These are largely based on the idea of approximating the pairwise comparison matrix by a *transitive matrix* [2]. A max-algebraic approach, using the max eigenvector, was proposed by Ludwig Elsner and Pauline van den Driessche in 2004 [EvdD04, EvdD10]. Recently, it has received a considerable attention as an alternative approach [Tra11].

[2] A positive matrix A is said to be transitive if $a_{ij}a_{jk} = a_{ik}$ for all i, j, k and $a_{ii} = 1$.

1.2 Thesis Overview

The overview of this thesis is as follows.

- In Chapter 2, we provide essential mathematical background on non-negative linear algebra and the max algebra. We formally define the eigenvalue problem and recall the Perron-Frobenius theory in each case. Also, we discuss numerical approaches to compute the eigenvalues and eigenvectors of a non-negative irreducible matrix.

- In Chapter 3, we consider the Perron-Frobenius theory for matrix polynomials over the max algebra. We study the connection between max matrix polynomials and multi-step difference equations. An important result of this chapter is a max version of the Fundamental Theorem of Demography (Theorem 3.1.3) which characterises the behaviour of the solution of multi-step difference equations in the max algebra. Further, we define a set of inequalities on the largest max eigenvalue of a max matrix polynomial in terms of an $n \times n$ non-negative matrix.

- In Chapter 4, we consider asymptotic stability over the max algebra. In this context, we introduce the class of P_{max}-matrices and discuss a number of its equivalent properties. Particularly, we discuss the relation between the P_{max}-property and the stability of delayed difference equations in the max algebra. Moreover, we extend these concepts to sets of non-negative matrices and introduce P_{max}-matrix sets. A main result of this chapter describes a number of equivalent results for P_{max}-matrix sets and answers some stability questions for a finite set of non-negative matrices and for discrete inclusions in the max algebra (Theorem 4.4.1).

- In Chapter 5, we explain the max algebra approach for the single criterion AHP. We discuss its extension to the multi-criteria case to derive an overall ranking scheme for the alternatives. Inspired by this, we introduce a novel approach in the framework of multi-objective optimisation. Basically, we are concerned with three types of optimal solutions for the considered optimisation problem. A key result of this chapter shows that Pareto optimal solutions are guaranteed to exist over the positive orthant (Corollary 5.5.8).

- In Chapter 6, we consider a max version of the Matrix Tree Theorem for Markov Chains. Specifically, we relate the max-algebraic spectral theory of an irreducible max-stochastic matrix to the weights of directed spanning trees in its associated digraph. A fundamental result in this Chapter establishes the relation between a max eigenvector of the matrix and the maximal weight of rooted directed spanning trees (Theorem 6.2.1). We also discuss possible applications of this result to ranking problems.

- In Chapter 7, we summarise our results and discuss a number of open questions related to the work of previous chapters.

1.3 The Contributions

This thesis contributes to the characterisation of the spectral and stability properties of non-negative matrices over the max algebra. The results are relevant to the analysis of multi-step difference equations, discrete inclusions and ranking schemes for decision making problems.

The following journal publications were prepared during the Ph.D. study.

1. B. Benek Gursoy, S. Kirkland, O. Mason and S. Sergeev, *On the Markov Chain Tree Theorem in the Max Algebra*, Electronic Journal of Linear Algebra 26 (2013) 15-27.

2. B. Benek Gursoy, O. Mason and S. Sergeev, *The Analytic Hierarchy Process, Max Algebra and Multi-objective Optimisation*, Linear Algebra Appl. 438 (2013) 2911-2928.

3. B. Benek Gursoy and O. Mason, *P^1_{max} and S_{max} properties and asymptotic stability in the max algebra*, Linear Algebra Appl. 435 (2011) 1008-1018.

4. B. Benek Gursoy and O. Mason, *Spectral properties of matrix polynomials in the max algebra*, Linear Algebra Appl. 435 (2011) 1626-1636.

The following conference and workshop abstracts were presented regarding the study in publications.

1. B. Benek Gursoy, O. Mason and S. Sergeev, *Application of the Max Algebra to Ranking Schemes in AHP*, Workshop of the LMS Joint Research Group: Tropical Mathematics and its Applications, 2012, Birmingham, UK.

2. B. Benek Gursoy and O. Mason, P^1_{max} *and* S_{max} *properties and asymptotic stability in the tropical linear algebra*, International Linear Algebra Society (ILAS), 2011, Braunschweig, Germany.

3. B. Benek Gursoy and O. Mason, *Matrix Polynomials in the Max Algebra; Eigenvalues, Eigenvectors and Inequalities*, International Linear Algebra Society (ILAS), 2010, Pisa, Italy.

CHAPTER 2

Fundamental Concepts

In this preliminary chapter, we introduce the general notation, definitions and some fundamental results that will be frequently used throughout the thesis. We first define certain types of non-negative matrices and recall the celebrated Perron-Frobenius theorems. Then, we introduce classical concepts from graph theory and present preliminary results on the relation between graphs and non-negative matrices. The chapter concludes with some formal definitions in the max algebra and extensions of some known results of non-negative linear algebra to the max algebra.

2.1 Non-negative Linear Algebra

To start with, we provide a basic mathematical background on the spectral theory of non-negative matrices. We also recall a number of necessary definitions concerning graph theory.

2.1.1 General Notation

$\mathbb{R}$ denotes the set of real numbers; $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers; $\mathbb{R}^{n\times n}$ stands for the space of $n\times n$ matrices with real entries. For $x \in \mathbb{R}^n$ and $1 \leq i \leq n$, x_i denotes the i^{th} component of x. Similarly, for $A \in \mathbb{R}^{n\times n}$ and $1 \leq i, j \leq n$, a_{ij} refers to the $(i, j)^{\text{th}}$ entry of A.

Further, $\mathbb{R}_+ = [0, \infty)$ denotes the set of all non-negative real numbers; $\mathbb{R}^n_+$ denotes the set of all n-tuples of non-negative real numbers such that

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}. \tag{2.1}$$

$\mathbb{R}^n_+$ is called *the non-negative orthant*. $\mathbb{R}^{n\times n}_+$ stands for the set of all $n \times n$ matrices with non-negative real entries such that

$$\mathbb{R}^{n\times n}_+ = \{A \in \mathbb{R}^{n\times n} \mid a_{ij} \geq 0, 1 \leq i, j \leq n\}. \tag{2.2}$$

We use x^T and A^T for the transpose of $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ respectively. For $1 \leq i \leq n$, $A_{i.}$ denotes the i^{th} row of A and $A_{.i}$ denotes the i^{th} column of A. A^k represents the k^{th} power of A for some $k \in \mathbb{R}$. We use I for the $n \times n$ identity matrix and $X = \text{diag }(x_1, x_2, ..., x_n)$ for an $n \times n$ diagonal matrix given by

$$X = \begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_n \end{bmatrix}$$

where $x_i \in \mathbb{R}$ $(i = 1, 2, ..., n)$. Moreover, $\mathbf{1}_n \mathbf{1}_n^T$ represents a matrix with every entry equals to one where $\mathbf{1}_n \in \mathbb{R}^n_+$ is the vector of all ones.

A vector $x \in \mathbb{R}^n$ is said to be *positive* if $x_i > 0$ for $1 \leq i \leq n$. This is denoted by $x > 0$. If $x_i \geq 0$ for $1 \leq i \leq n$, we say that x is *non-negative* and write $x \geq 0$ or $x \in \mathbb{R}^n_+$.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be *positive* if $a_{ij} > 0$ for $1 \leq i, j \leq n$. This is denoted by $A > 0$. We say that A is *non-negative* and write $A \geq 0$ or $A \in \mathbb{R}^{n\times n}_+$ if $a_{ij} \geq 0$ for $1 \leq i, j \leq n$. Notice that $A > 0$ doesn't mean $A \in \mathbb{R}^{n\times n}_+$ and $A \neq 0$. In particular, we adopt the following notation.

(i) For $x, y \in \mathbb{R}^n$, we define $x \geq y$ if $x - y \geq 0$ and $x > y$ if $x - y > 0$;

(ii) For $A, B \in \mathbb{R}^{n\times n}$, we define $A \geq B$ if $A - B \geq 0$ and $A > B$ if $A - B > 0$.

2.1.2 Some Special Types of Non-negative Matrices

Here, we briefly define a number of special types of matrices in non-negative linear algebra. We will recall these matrices throughout the section in the context of graph theory.

A matrix $P \in \mathbb{R}^{n\times n}$ is said to be a *permutation matrix* if it is obtained from the identity matrix, $I \in \mathbb{R}^{n\times n}$, by interchanging its rows and columns. An example of a permutation matrix can be given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

By multiplying a general 3×3 matrix A by P, we get

$$PA = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}.$$

Note that multiplication of A by P from right swaps the columns of A in a similar fashion.

For $n \geq 2$, a matrix $A \in \mathbb{R}^{n\times n}$ is said to be *reducible* if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

where B and D are square matrices. If B or D is reducible, we can apply proper permutations to these matrices until we obtain the following form

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ 0 & A_{22} & \dots & A_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{rr} \end{bmatrix} \tag{2.3}$$

so that each block matrix $A_{ii}(1 \leq i \leq r)$ can not be reduced further. (2.3) is called the *Frobenius normal form* of A [BR91].

If A is not reducible, it is said to be an *irreducible matrix*. The following is an immediate result on irreducible matrices [HJ90, BP94].

Theorem 2.1.1. *For $A \in \mathbb{R}_+^{n\times n}$, the following are equivalent.*

(i) A is irreducible;

(ii) A^T *is irreducible;*

(iii) $(I + A)^{n-1} > 0$.

A matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be *primitive* if and only if there exists some positive number k such that A^k is positive. In this context, we define the *exponent* of A as follows [HJ90, BR91, BP94].

$$\exp(A) = \min\{k \mid A^k > 0, k \in \mathbb{R}_+\} \tag{2.4}$$

The following result describes an upper bound for $\exp(A)$ which is also known as the Wielandt bound [WM67].

Theorem 2.1.2. *If* $A \in \mathbb{R}_+^{n\times n}$ *is primitive, then exp*$(A) \leq n^2 - 2n + 2$.

2.1.3 The Perron-Frobenius Theory

We next recall the Perron-Frobenius theory for non-negative matrices. It provides a characterisation for the spectral properties of non-negative matrices. We start by introducing the spectral radius of a matrix.

For an $n \times n$ matrix A, the eigenequation is given by

$$Ax = \lambda x \tag{2.5}$$

where λ is an *eigenvalue* and x is an eigenvector of A ($x \neq 0$). We call x a *right eigenvector* associated with λ. Moreover, y satisfying $y^T A = \lambda y^T$ is called a *left eigenvector* associated with λ. Any nonzero linear combination of eigenvectors corresponding to λ is also an eigenvector of A.

The eigenvalues of A can be determined by finding the roots of the *characteristic equation* of A:

$$\det(A - \lambda I) = 0. \tag{2.6}$$

We can represent (2.6) as an n^{th} degree polynomial of the form $\prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i}$ for $k \leq n$. Here, m_i is said to be the *algebraic multiplicity* of λ_i for $i = 1, ..., k$. The number of linearly independent eigenvectors corresponding to λ_i is said to be the *geometric multiplicity* of λ_i for $i = 1, ..., k$. The set of eigenvalues is called the *spectrum* of A and denoted by $\sigma(A)$. The *maximum absolute value* of the elements in the spectrum is called the *spectral radius* of A:

$$\rho(A) = \max_i \{|\lambda_i| \mid \lambda_i \in \sigma(A)\}. \tag{2.7}$$

$\rho(A)$ plays an important role in characterising the convergence of matrix powers in the sense of dynamical systems. In this context, it is well-known that $\lim_{k\to\infty} A^k = 0$ if and only if $\rho(A) < 1$ [HJ90].

Next, we describe some bounds and inequalities for the spectral radius of $A \in \mathbb{R}^{n\times n}$ that are taken from [HJ90]. It is well-known that $\rho(A) \leq ||A||$ for any matrix norm on $\mathbb{R}^{n\times n}$ and the following hold.

$$\begin{aligned} \rho(A) &= \lim_{k\to\infty} \rho(A^k)^{\frac{1}{k}} \\ \rho(A) &= \lim_{k\to\infty} ||A^k||^{\frac{1}{k}} \end{aligned} \tag{2.8}$$

The second equation was introduced by Israel M. Gelfand $(1913 - 2009)$ in 1941 [Gel41]. It is known as the *Gelfand formula*. It provides a link between powers of the matrix A and $\rho(A)$.

Next, we state the so-called *Perron Theorem*. It was proven by Oscar Perron $(1880 - 1975)$ in 1907 [Per07]. It provides a characterisation of eigenvalues and eigenvectors of positive matrices [HJ90, Mey00].

Theorem 2.1.3. *(**The Perron Theorem**) For $A > 0$, the following are true.*

(i) $\rho(A) > 0$;

(ii) $\rho(A) \in \sigma(A)$;

(iii) *$\rho(A)$ has algebraic and geometric multiplicity one;*

(iv) *There exist positive right and left eigenvectors $x, y > 0$ such that $Ax = \rho(A)x$ and $y^T A = \rho(A)y^T$;*

(v) $\lim_{k\to\infty} \left(\frac{A}{\rho(A)}\right)^k = xy^T$ *where* $Ax = \rho(A)x$, $y^T A = \rho(A)y^T$ *and* $x^T y = 1$.

Briefly, if A is positive, then $\rho(A)$ is the only eigenvalue of A in the spectrum and there is a positive and unique eigenvector (up to a scalar multiple) corresponding to it. $\rho(A)$ is called the *Perron root* and x is called the *Perron vector*.

Theorem 2.1.4. ***(The Perron Theorem for Non-negative Matrices)*** *For $A \in \mathbb{R}_+^{n\times n}$, the following are true.*

(i) $\rho(A) \in \sigma(A)$*;*

(ii) There exist non-negative and nonzero right and left eigenvectors $x, y \in \mathbb{R}_+^n$ such that $Ax = \rho(A)x$ and $y^T A = \rho(A)y^T$.

The Perron Theorem was extended to irreducible matrices by Ferdinand G. Frobenius (1849−1917) in 1912 [Fro12]. In the following, we state the so-called *Perron-Frobenius theorem* [HJ90, Mey00].

Theorem 2.1.5. ***(The Perron-Frobenius Theorem)*** *For an irreducible $A \in \mathbb{R}_+^{n\times n}$, the following are true.*

(i) $\rho(A) > 0$*;*

(ii) $\rho(A) \in \sigma(A)$*;*

(iii) $\rho(A)$ has algebraic and geometric multiplicity one;

(iv) There exist positive right and left eigenvectors $x, y > 0$ such that $Ax = \rho(A)x$ and $y^T A = \rho(A)y^T$;

(v) If there exist h distinct eigenvalues $\{\lambda_0, ..., \lambda_{h-1}\}$ each with algebraic multiplicity one and absolute value $\rho(A)$, then $\lambda_j = \rho(A)e^{i\frac{2\pi j}{h}}$ for $j = 0, 1, ..., h-1$;

(vi) If $h > 1$, then there exists a permutation matrix P such that A can be reduced to the following form:

$$P^T A P = \begin{bmatrix} 0 & A_{12} & 0 & \dots & 0 \\ 0 & 0 & A_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_{h-1h} \\ A_{h1} & 0 & 0 & \dots & 0 \end{bmatrix}.$$

The term h in Theorem 2.1.5 is said to be the *index of imprimitivity* of A. It equals to the number of eigenvalues whose absolute value equals to $\rho(A)$. If $h = 1$, then A is a primitive matrix. For primitive matrices, the result (v) of Theorem 2.1.3 holds in addition to Theorem 2.1.5 (i)-(iv) [HJ90].

Theorem 2.1.6. *If $A \in \mathbb{R}_+^{n\times n}$ is primitive, then*

$$\lim_{k\to\infty}\left(\frac{A}{\rho(A)}\right)^k = xy^T > 0$$

where $Ax = \rho(A)x$, $y^T A = \rho(A)y^T$, $x, y > 0$ and $x^T y = 1$.

2.1.4 Graph Theory

In this section, we recall various concepts in graph theory that will be used later on.

An *undirected graph* is a pair denoted by $G = (N, E)$ that consists of

- N : finite set of vertices;
- E : a set of edges between vertices in N.

If there exists an edge between the vertices $u, v \in N$, we write $e = \{u, v\}$. In this notation, u and v are called *endpoints*. If two end points are same, then the edge connecting them is called a *loop*.

A *directed graph* (*digraph*) is an ordered pair $G = (N, E)$ consisting of a finite set of vertices N and a set of directed edges E between vertices in N. The edges in a directed graph are given by ordered pairs of vertices $e = (u, v)$.

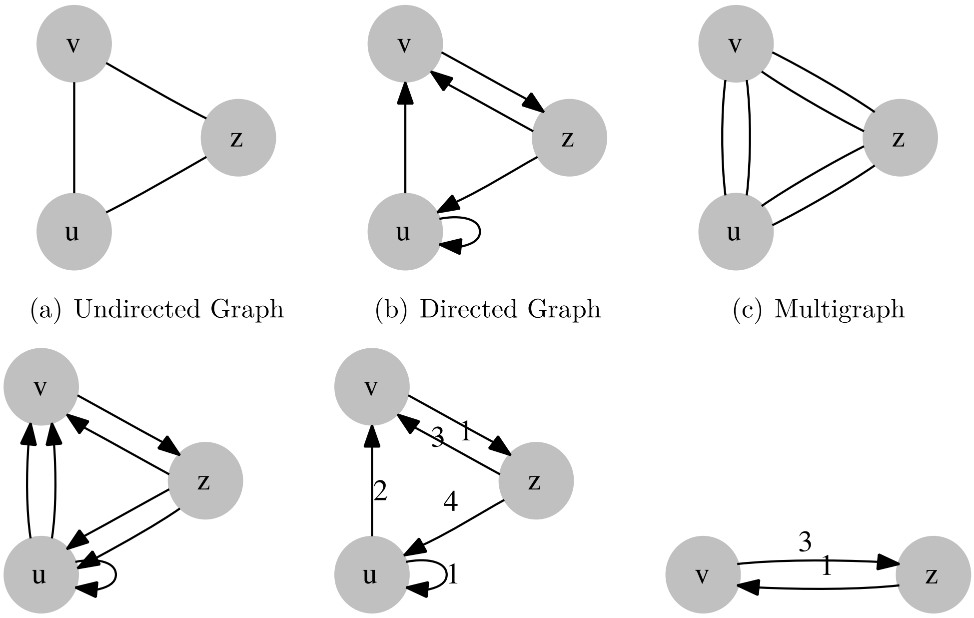
A *multigraph* is undirected or directed graph where there are multiple edges between vertices. The edges in the multigraph are called *multiedges*.

G is called a *weighted directed graph* if some positive numbers are assigned to the edges in E. Each number is said to be the *weight* of the corresponding edge.

Finally, a graph consisting of a subset of vertices N and edges E of G is said to be a *subgraph* of G. We illustrate these concepts in Figure 2.1.

Non-negative Matrices and Digraphs:

For $A \in \mathbb{R}_+^{n\times n}$, we denote the *weighted directed graph associated with* A by $D(A) = (N(A), E(A))$. Formally, $D(A)$ consists of the finite set of vertices $\{1, 2, ..., n\}$ and there is a directed edge (i, j) from i to j if and only if $a_{ij} > 0$.



(a) Undirected Graph (b) Directed Graph (c) Multigraph

(d) Multigraph (e) Weighted Digraph (f) Subgraph of (e)

Figure 2.1: Types of Graphs

A *directed path* $(i = i_1, i_2, \dots, i_k = j)$ is a sequence of distinct vertices between any two vertices i, j in $D(A)$, where (i_p, i_{p+1}) is an edge in $D(A)$ for $p = 1, \dots, k-1$. The *length* of a path is the number of edges on the path. The *weight* of a path $(i = i_1, i_2, \dots, i_k = j)$ of length $k-1$ is given by $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$. A *cycle* of length k in $D(A)$ is a closed path of the form $(i_1, i_2, ..., i_k, i_1)$ where $i_1, i_2, ..., i_k$ are in $\{1, 2, ..., n\}$ and distinct. Note that a loop (i_j, i_j) $(1 \le j \le n)$ is a cycle of length 1 with weight $a_{i_j i_j}$. Moreover, the cycles $(i_1, i_2, ..., i_k, i_1), (i_2, i_3, ..., i_1, i_2), ..., (i_k, i_1, ..., i_{k-1}, i_k)$ are identical.

Let Γ denote the cycle $(i_1, i_2, ..., i_k, i_1)$. Then, we adopt the notation

i. $\pi(\Gamma) = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$ for the weight of the cycle Γ;

ii. $l(\Gamma) = k$ for the length of the cycle Γ.

For such a cycle the *cycle geometric mean* is given by $\sqrt[k]{a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}}$.

Two vertices $i, j \in N(A)$ is strongly connected if there exists a directed path from i to j and from j to i. This defines an equivalence relation on the set of vertices. The corresponding equivalence classes are called the *strongly connected components* of $D(A)$ [BR91].

$D(A)$ is called *strongly connected* if and only if there is a directed path between any two vertices i, j in $D(A)$. It is standard that A is an irreducible matrix if and only if $D(A)$ is strongly connected [HJ90]. For an irreducible matrix, the index of imprimitivity of A corresponds to the greatest common divisor (g.c.d.) of the lengths of all of the cycles in $D(A)$. Specifically, if it is one, then we say that A is a primitive matrix [HJ90]. If A is not irreducible, then there exist more than one strongly connected components in $D(A)$.

The *distance* between any two vertices i and j is the length of the shortest path between i and j. Formally,

$$d(i, j) = \min\{l \mid \text{there exists a path of length } l \text{ from } i \text{ to } j\}.$$

In particular, $d(i, i) = 0$. For an irreducible $A \in \mathbb{R}_+^{n \times n}$, the *diameter* of $D(A)$ is the maximum value of the distance over all pairs of its vertices given by

$$d(D(A)) = \max\{d(i, j) \mid i, j \in N(A), i \neq j\}. \tag{2.9}$$

Non-negative Matrices and Multigraphs:

Given $\Psi = \{A_1, \ldots, A_m\} \subset \mathbb{R}_+^{n \times n}$, we write a_{ij}^p for the (i, j) entry of A_p for $1 \le p \le m$. We write $M(\Psi)$ for the *multigraph* associated with the set Ψ. Thus $M(\Psi)$ consists of the vertices $\{1, \ldots, n\}$ with an edge of weight a_{ij}^p from i to j for every p for which $a_{ij}^p > 0$. In an abuse of notation we shall identify the edge with its weight a_{ij}^p in this case. A path in the multigraph $M(\Psi)$ is then a sequence of vertices $(i_1, i_2, \ldots, i_k)$ and edges $a_{i_j i_{j+1}}^{p_j} > 0$, $1 \le j \le k$ such that $i_1, \ldots, i_k$ are in $\{1, 2, ..., n\}$ and distinct and $p_1, \ldots, p_k$ are in $\{1, \ldots m\}$. A cycle in the multigraph $M(\Psi)$ is a closed path of the form $(i_1, i_2, ..., i_k, i_1)$ where $i_1, i_2, ..., i_k$ are in $\{1, 2, ..., n\}$ and distinct. Definitions of the path length, weight and the cycle geometric mean are analogous to the case of a directed weighted graph.

2.2 The Max Algebra

We next provide necessary mathematical background in the max algebra to understand the results presented in the following chapters.

2.2.1 General Notation

In this section, we recall general notation and basic algebraic properties of the max algebra. The max algebra denoted by $\mathbb{R}_{\max,\times}$ is an algebraic structure consisting of the set of non-negative numbers equipped with the two basic operations:

$$a \oplus b = \max(a, b) \tag{2.10}$$
$$a \otimes b = ab$$

$\mathbb{R}_{\max,\times} = (\mathbb{R}_+, \oplus, \otimes)$ forms a semiring (a ring with no additive inverse) of non-negative numbers with the operations: $\oplus$ and $\otimes$. The identity element for $\oplus$ is 0 i.e., $a \oplus 0 = 0 \oplus a = a$ for all $a \in \mathbb{R}_+$. Moreover, $\oplus$ is idempotent, i.e., $a \oplus a = a$ and there is a usual order on the max algebra semiring such that $a \oplus b = b$ implies $a \leq b$ for all $a, b \in \mathbb{R}_+$. The multiplication $\otimes$ is commutative, i.e., $a \otimes b = b \otimes a$ and has an identity element 1, i.e., $a \otimes 1 = 1 \otimes a = a$ for all $a, b \in \mathbb{R}_+$.

For a brief list of well-known semirings, see Table 2.1 [GP97].

$\mathbb{R}_{\max,\times}$	$(\max, \times)$ semiring	$(\mathbb{R}_+, \max, \times)$
$\mathbb{R}_{\max}$	$(\max, +)$ semiring	$(\mathbb{R} \cup \{-\infty\}, \max, +)$
$\mathbb{R}_{\min}$	$(\min, +)$ semiring	$(\mathbb{R} \cup \{+\infty\}, \min, +)$
$\mathbb{R}_{\max,\min}$	$(\max, \min)$ semiring	$(\mathbb{R} \cup \{\pm\infty\}, \max, \min)$
$\mathbb{R}_h$	Maslov semiring	$(\mathbb{R} \cup \{-\infty\}, \oplus_h, +)$ $a \oplus_h b = h \log(\exp(a/h) + \exp(b/h))$

Table 2.1: Some well-known semirings

The max algebra is isomorphic to the max-plus algebra, which is an algebraic structure on $\mathbb{R} \cup \{-\infty\}$ together with max and $+$, by the map $a \to \ln(a)$ (See Table 2.2). The name was used for the max-plus algebra setting in several works [BSvdD93, MP00, But03, BCGG09]. However, we will only be concerned with the max and $\times$ setting in this thesis and follow the conventions in [Bap98].

Operations of the max algebra, $\oplus$ and $\otimes$, extend to vectors and matrices in the same way as in conventional linear algebra [CG79, BCOQ92, Bap98, But10]. For $A, B \in \mathbb{R}_+^{n\times n}$, we denote the sum by $(A \oplus B)_{ij} = \max(a_{ij}, b_{ij})$ and the

	Max algebra	Max-plus algebra
	a, b	$\ln(a), \ln(b)$
$a \oplus b$:	$\max(a, b)$	$\max(\ln(a), \ln(b))$
$a \otimes b$:	ab	$\ln(a) + \ln(b)$
identity for $\oplus$:	0	$-\infty$
identity for $\otimes$:	1	0

Table 2.2: The isomorphism between the max and max-plus algebra

product by $(A \otimes B)_{ij} = \max\limits_{1 \leq k \leq n} (a_{ik} b_{kj})$ for $1 \leq i, j \leq n$. For instance, consider the following matrices

$$A = \begin{bmatrix} 8 & 3 & 1 \\ 5 & 4 & 2 \\ 4 & 5 & 9 \end{bmatrix}, B = \begin{bmatrix} 7 & 1 & 9 \\ 6 & 2 & 6 \\ 8 & 4 & 1 \end{bmatrix}.$$

Then,

$$A \oplus B = \begin{bmatrix} 8 & 3 & 9 \\ 6 & 4 & 6 \\ 8 & 5 & 9 \end{bmatrix}, A \otimes B = \begin{bmatrix} 56 & 8 & 72 \\ 35 & 8 & 45 \\ 72 & 36 & 36 \end{bmatrix}.$$

For $x \in \mathbb{R}_+^n$, the matrix-vector product is defined by $(A \otimes x)_i = \max\limits_{1 \leq j \leq n} (a_{ij} x_j)$ for $1 \leq i \leq n$. Moreover, we denote the k^{th} max power of $A \in \mathbb{R}_+^{n \times n}$ by

$$A_\otimes^k = \underbrace{A \otimes A \otimes ... \otimes A}_{k \text{ times}}$$

in the max algebra. Multiplication by a scalar $\alpha \in \mathbb{R}_+$ is given by $(\alpha A)_{ij} = \alpha a_{ij}$ for $1 \leq i, j \leq n$. The identity matrix is the same as in conventional linear algebra.

2.2.2 Max Version of the Perron-Frobenius Theory

Next, we describe an extension of the Perron-Frobenius theory to the max algebra.

For an $n \times n$ non-negative matrix A, the eigenequation in the max algebra is given by

$$A \otimes x = \lambda x \tag{2.11}$$

where $\lambda \geq 0$ is a *max eigenvalue*, $x \geq 0$ $(x \neq 0)$ is a max eigenvector. We call x a *right max eigenvector* associated with λ. Moreover, y satisfying $y^T \otimes A = \lambda y^T$ is called a *left max eigenvector* associated with λ.

For A in $\mathbb{R}_+^{n\times n}$, the *largest max eigenvalue* is denoted by $\mu(A)$. It is the *maximum cycle geometric mean* in $D(A)$ over all possible cycles [CG79, BCOQ92, Bap98, Lur05, But10]:

$$\mu(A) = \max\{\sqrt[k]{\pi(\Gamma)} \mid \Gamma \in D(A) \text{ with } l(\Gamma) = k, 1 \leq k \leq n\}. \tag{2.12}$$

In a sense, this is still true in the case where $D(A)$ contains no cycles and $\mu(A) = 0$.

A cycle in $D(A)$ whose cycle geometric mean equals to $\mu(A)$ is called a *critical cycle*. Vertices that lie on some critical cycle are known as *critical vertices*. The set of edges belonging to critical cycles are said to be *critical edges*. The *critical matrix* [EvdD99, EvdD01] of $A \in \mathbb{R}^{n\times n}$, denoted by A^C, is formed from the submatrix of A consisting of the rows and columns of A corresponding to critical vertices as follows.

$$a_{ij}^C = \begin{cases} a_{ij} & \text{if } i, j \text{ lies on a critical cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

$D^C(A)$ is used to designate the weighted directed graph of the critical matrix and denoted by $D^C(A) = (N^C(A), E^C(A))$ where $N^C(A)$ stands for critical vertices while $E^C(A)$ stands for critical edges. It is said to be the *critical graph* of A. We now illustrate these concepts with an example.

Example 2.2.1. Consider the following matrix

$$A = \begin{bmatrix} 1 & 0.25 & 0.2 & 0.5 \\ 4 & 0.2 & 0 & 1 \\ 1 & 0 & 0.5 & 0 \\ 0 & 0.25 & 1 & 1 \end{bmatrix}.$$

There are ten cycles in $D(A)$ as follows.

- $\Gamma_1 = (1, 1)$ with $\pi(\Gamma_1) = a_{11} = 1$ and $l(\Gamma_1) = 1$
- $\Gamma_2 = (2, 2)$ with $\pi(\Gamma_2) = a_{22} = 0.2$ and $l(\Gamma_2) = 1$

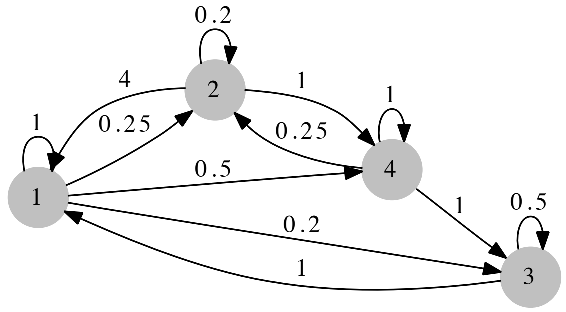


Figure 2.2: $D(A)$

- $\Gamma_3 = (3, 3)$ with $\pi(\Gamma_3) = a_{33} = 0.5$ and $l(\Gamma_3) = 1$
- $\Gamma_4 = (4, 4)$ with $\pi(\Gamma_4) = a_{44} = 1$ and $l(\Gamma_4) = 1$
- $\Gamma_5 = (1, 2, 1)$ with $\pi(\Gamma_5) = a_{12}a_{21} = 1$ and $l(\Gamma_5) = 2$
- $\Gamma_6 = (1, 3, 1)$ with $\pi(\Gamma_6) = a_{13}a_{31} = 0.2$ and $l(\Gamma_6) = 2$
- $\Gamma_7 = (2, 4, 2)$ with $\pi(\Gamma_7) = a_{24}a_{42} = 0.25$ and $l(\Gamma_7) = 2$
- $\Gamma_8 = (1, 4, 2, 1)$ with $\pi(\Gamma_8) = a_{14}a_{42}a_{21} = 0.5$ and $l(\Gamma_8) = 3$
- $\Gamma_9 = (1, 4, 3, 1)$ with $\pi(\Gamma_9) = a_{14}a_{43}a_{31} = 0.5$ and $l(\Gamma_9) = 3$
- $\Gamma_{10} = (1, 2, 4, 3, 1)$ with $\pi(\Gamma_{10}) = a_{12}a_{24}a_{43}a_{31} = 0.25$ and $l(\Gamma_10) = 4$

Then, $\mu(A) = \max(1, 0.2, 0.5, \sqrt{0.2}, \sqrt{0.25}, \sqrt[3]{0.5}, \sqrt[4]{0.25}) = 1$. Thus, Γ_1, Γ_4 and Γ_5 are critical cycles. We get the following critical matrix and critical graph of A (Figure 2.3).

$$A^C = \begin{bmatrix} 1 & 0.25 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

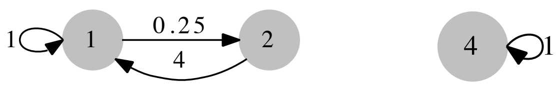


Figure 2.3: $D^C(A)$

Next, we state the max version of the Perron-Frobenius theorems [CG79, BCOQ92, Bap98, But10].

Theorem 2.2.1. *For $A \in \mathbb{R}_+^{n\times n}$, the following are true.*

(i) $\mu(A) \geq 0$*;*

(ii) There exist non-negative and nonzero right and left eigenvectors $x, y \in \mathbb{R}_+^n$ *such that* $A \otimes x = \mu(A)x$ *and* $y^T \otimes A = \mu(A)y^T$*.*

Theorem 2.2.2. *For an irreducible matrix $A \in \mathbb{R}_+^{n\times n}$, the following are true.*

(i) $\mu(A) > 0$*;*

(ii) $\mu(A)$ *is the unique max eigenvalue of A;*

(iii) There exists positive right and left eigenvectors $x, y > 0$ *such that* $A \otimes x = \mu(A)x$ *and* $y^T \otimes A = \mu(A)y^T$*;*

(iv) x and y are unique (up to a scalar multiple) if and only if $D^C(A)$ is strongly connected.

Briefly, if A is irreducible, then $\mu(A)$ is the unique max eigenvalue of A and there is a positive max eigenvector corresponding to it. The eigenvector may not be unique if the critical matrix of A is not irreducible.

Maximum Cycle Geometric Mean:

The maximum cycle geometric mean, $\mu(A)$, can be characterised as follows:

$$\max\{\lambda \in \mathbb{R}_+ \mid \exists x \in \mathbb{R}_+^n, x \neq 0 \text{ such that } A \otimes x = \lambda x\}.$$

We have the following immediate results for $A, B \in \mathbb{R}_+^{n\times n}$:

(i) If $A \leq B$, then $\mu(A) \leq \mu(B)$ [EvdD04];

(ii) $\mu(A) \oplus \mu(B) \leq \mu(A \oplus B)$;

(iii) $\mu(\alpha A) = \alpha\mu(A)$ for $\alpha \in \mathbb{R}_+$ [Ser09b, But10].

If $\mu(A) = 1$, then A is called a *definite matrix* [But03]. If $\mu(A) > 0$, we can normalise A to obtain a definite matrix. We will adopt the following notation.

$$\hat{A} = \frac{A}{\mu(A)} \tag{2.13}$$

Theorem 2.2.3. *Let x be a max eigenvector of $A \in \mathbb{R}_+^{n\times n}$ associated with $\mu(A)$. Let $\mu(A)$ be positive. Then, x is a max eigenvector of $\hat{A}$ associated with the max eigenvalue* 1.

Proof: It is straightforward from the max eigenequation of A that

$$A \otimes x = \mu(A)x \text{ if and only if } \hat{A} \otimes x = x$$

since $\mu(A) > 0$. $\square$

Similar to non-negative linear algebra, the max eigenvalue is closely related to the behaviour of the max powers of a non-negative matrix. In keeping with [Lur05], $A \in \mathbb{R}_+^{n\times n}$ is said to be *asymptotically stable* if $\lim\limits_{k\to\infty} A_\otimes^k = 0$. The following is a well-known result on the stability of non-negative matrices in the max algebra [Car71, Lur05].

Theorem 2.2.4. *For $A \in \mathbb{R}_+^{n\times n}$, $\lim\limits_{k\to\infty} A_\otimes^k = 0$ if and only if $\mu(A) < 1$.*

In the following, we present a series of inequalities for $\mu(A)$ that describe its relation with $\rho(A)$ and matrix norms. For the remainder of this section, note that A^k denotes the usual k^{th} power of a matrix and $A_\otimes^k$ denotes the k^{th} max power of a matrix for some $k \in \mathbb{R}_+$. Further, we denote the k^{th} *Hadamard power* of the matrix A by $A_\odot^k$ such that $(A_\odot^k)_{ij} = (a_{ij})^k$ for all i, j.

Let $||.||$ be a vector norm on $\mathbb{R}_+^n$. Following [Lur05], define a matrix norm associated with $||.||$ and $A \in \mathbb{R}_+^{n\times n}$ over the max algebra as follows.

$$\eta_{||.||}(A) = \sup_{x\neq 0} \frac{||A \otimes x||}{||x||} = \max_{||x||=1} ||A \otimes x|| \tag{2.14}$$

It is shown in [Lur05] that $\mu(A) \leq \eta_{||.||}(A)$ and the following hold.

$$\begin{aligned} \mu(A) &= \lim_{k\to\infty} \mu(A_\otimes^k)^{\frac{1}{k}} \\ \mu(A) &= \limsup_{k\to\infty} (\eta_{||.||}(A_\otimes^k))^{\frac{1}{k}} \\ \mu(A) &= \lim_{k\to\infty} ||A_\otimes^k||^{\frac{1}{k}} \end{aligned} \tag{2.15}$$

Note that the third equation in (2.15) is a max version of the Gelfand formula in (2.8) and it is true for any matrix norm on $\mathbb{R}_+^{n\times n}$ [EvdD99, Lur05].

Moreover, it is shown in [EvdD99] that

$$\mu(A) = \lim_{k\to\infty} \rho(A_{\otimes}^{k})^{\frac{1}{k}} = \lim_{k\to\infty} \mu(A_{\odot}^{k})^{\frac{1}{k}}. \tag{2.16}$$

It is known from [EJDdS88] that $\mu(A) \leq \rho(A) \leq n\mu(A)$ and from [Fri86] that $\lim_{k\to\infty} \rho(A_{\odot}^{k})^{\frac{1}{k}} = \mu(A)$ for $A \in \mathbb{R}_{+}^{n\times n}$. Further information on the relation between $\rho(A)$ and $\mu(A)$ is given by the following fact from [Bap98].

$$\rho(A) = \lim_{k\to\infty} \mu(A^{k})^{\frac{1}{k}}. \tag{2.17}$$

2.2.3 Kleene Star and Kleene Cone

In this section, we introduce two important matrices that are used in optimal path problems in graph theory such as finding the shortest path in a given graph [BCOQ92, HOvdW06]. Moreover, they enable us to characterise max eigenvectors and subeigenvectors in the max algebra.

Kleene Star:

For $A \in \mathbb{R}_{+}^{n\times n}$, consider the following series.

$$\begin{aligned} A^{*} &= I \oplus A \oplus A_{\otimes}^{2} \oplus ... \oplus A_{\otimes}^{n} \oplus ... \\ A^{+} &= A \otimes A^{*} \end{aligned} \tag{2.18}$$

For $\mu(A) \leq 1$, the series in (2.18) converge to the following matrices respectively.

$$\begin{aligned} A^{*} &= I \oplus A \oplus A_{\otimes}^{2} \oplus ... \oplus A_{\otimes}^{n-1} \\ A^{+} &= A \oplus A^{2} \oplus A_{\otimes}^{3} \oplus ... \oplus A_{\otimes}^{n} \end{aligned} \tag{2.19}$$

From a graph theoretic view, the $(i,j)^{\text{th}}$ entry of $A_{\otimes}^{k}$ denotes the maximum weight of a path in $D(A)$ from i to j of length k $(k \geq 1)$ [Car71, Bap98, HOvdW06]. It equals to 0 if no such path exists. It follows that, a_{ij}^{*} is the maximum weight of a path from i to j of any length for $1 \leq i,j \leq n$ when $i \neq j$ and a_{ij}^{+} is the maximum weight of a path from i to j of any length for all i,j.

A^{*} in (2.19) is said to be the *Kleene star of A* when $\mu(A) \leq 1$. It was introduced by Stephen C. Kleene $(1909 - 1994)$ in 1956 [Kle56]. It is also called the strong transitive closure of A [But10]. A^+ in (2.19) is said to be a metric matrix [CG79], the weak transitive closure of A [But10] or the shortest path matrix [BCOQ92].

The Kleene star of A has the following fundamental properties.

(i) $a^*_{ij} < \infty$ for all i, j since $\mu(A) \leq 1$;

(ii) $I \leq A^*$ and $a^*_{ii} = 1$ for $1 \leq i \leq n$;

(iii) $A^* \otimes A^* = A^*$ for $\mu(A) = 1$;

(iv) $(A^*)^* = A^*$ for $\mu(A) = 1$;

(v) If $A \in \mathbb{R}^{n\times n}_+$ is irreducible, then $A^* > 0$.

The following result is well-known from [CG79, BCOQ92, Bap98, EvdD01, EvdD04, HOvdW06, But10].

Theorem 2.2.5. *Let $A \in \mathbb{R}^{n\times n}_+$ be an irreducible matrix with $\mu(A) = 1$. Assume that $D^C(A)$ has r strongly connected components. Then, the following are true.*

(i) $A^*_{.i}$ is a right max eigenvector associated with $\mu(A)$ for $i \in N^C(A)$ ($A^+_{.i}$ is a right max eigenvector associated with $\mu(A)$ for $a^+_{ii} = 1$);

(ii) For $i, j \in N^C(A)$ ($i \neq j$), $A^*_{.i}$ and $A^*_{.j}$ are scalar multiples of each other if i and j belong to the same strongly connected component in $D^C(A)$;

(iii) There exist r linearly independent (in a max-algebraic sense) right max eigenvectors of A associated with $\mu(A)$.

We next illustrate Theorem 2.2.5.

Example 2.2.2. Consider the matrix given in Example 2.2.1. Then, the Kleene star of A is given by

$$A^* = \begin{bmatrix} 1 & 0.25 & 0.5 & 0.5 \\ 4 & 1 & 2 & 2 \\ 1 & 0.25 & 1 & 0.5 \\ 1 & 0.25 & 1 & 1 \end{bmatrix}.$$

It follows from Figure 2.3 that there are 2 strongly connected components in the critical graph and $N^C(A) = \{1, 2, 4\}$. The vertices 1 and 2 belong to the same strongly connected component. Thus, $A^*_{.1}$ and $A^*_{.2}$ are scalar multiples of each other. They are scaled so that the diagonal entry equals to one. Thus, the first max eigenvector is $\begin{bmatrix} 1 & 4 & 1 & 1 \end{bmatrix}^T$. The vertex 4 belongs to the second strongly connected component. Thus, $A^*_{.4}$ corresponds to the second max eigenvector: $\begin{bmatrix} 0.5 & 2 & 0.5 & 1 \end{bmatrix}^T$.

Kleene Cone:

Next, we define max convex cones which are analogous to convex cones in the conventional algebra. Following the notation in [BSS07], let $S = \{x_1, x_2, ..., x_m\}$ be a subset of $\mathbb{R}^n_+$. A vector $y \in \mathbb{R}^n_+$ is a called a *max combination* of $\{x_1, x_2, ..., x_m\}$ if

$$y = \bigoplus_{i=1}^{m} \alpha_i x_i, \quad \alpha_i \in \mathbb{R}_+, \quad x_i \in S. \tag{2.20}$$

The set of vectors $\{x_1, x_2, ..., x_m\}$ is called *linearly independent* in a max-algebraic sense if none of them can be expressed as a max combination of others. The set of all max combinations of the vectors $\{x_1, x_2, ..., x_m\}$ is denoted by *Span*(S). In particular, we adopt the notation

$$\text{span}_{\oplus}(A) = \bigoplus_{i=1}^{n} \alpha_i A_{.i}, \quad \alpha_i \in \mathbb{R}_+, \quad A \in \mathbb{R}^{n\times n}_+ \tag{2.21}$$

for the max-algebraic column span of A. $y \in \mathbb{R}^n_+$ is a called a *max convex combination* of $\{x_1, x_2, ..., x_m\}$ if

$$y = \bigoplus_{i=1}^{m} \alpha_i x_i, \quad \alpha_i \in \mathbb{R}_+, \quad \bigoplus_{i=1}^{m} \alpha_i = 1, \quad x_i \in S. \tag{2.22}$$

S is called a *max convex cone* if it is closed under the max and $\times$ operations as follows

$$\bigoplus_{i=1}^{m} \alpha_i x_i \in S, \quad \alpha_i \in \mathbb{R}_+, \quad x_i \in S. \tag{2.23}$$

The set of all eigenvectors of $A \in \mathbb{R}^{n\times n}_+$ associated with $\mu(A)$ is called an *eigencone* of A [SSB09, Ser09a, Ser09b]. It is denoted by $V(A)$ and given by

$$V(A) = \{x \in \mathbb{R}^n_+ \mid A \otimes x = \mu(A)x\}. \tag{2.24}$$

For $A \in \mathbb{R}_+^{n\times n}$ with $\mu(A) = 1$, $V(A)$ is described by

$$V(A) = \left\{ \bigoplus_{i\in r(A)} \alpha_i A^*_{.i} \mid \alpha_i \in \mathbb{R}_+ \right\} \tag{2.25}$$

where $r(A)$ contains exactly one index from each strongly connected component in $D^C(A)$ [SSB09]. (Recall from Theorem 2.2.5 (ii) that if $i, j \in N^C(A)$ belong to the same strongly connected component in $D^C(A)$, then $A^*_{.i}$ and $A^*_{.j}$ are scalar multiples of each other.)

For an $n \times n$ non-negative matrix A, consider the following inequality

$$A \otimes v \leq \mu(A)v. \tag{2.26}$$

$v \in \mathbb{R}^n_+$ is called a *subeigenvector* of A associated with $\mu(A)$ [Gau92, SSB09]. The role of subeigenvectors in resource optimisation has been discussed in [Gau95b]. In addition, they have applications in discrete max-plus spectral theory [AGW05]. The set of subeigenvectors of $A \in \mathbb{R}_+^{n\times n}$ associated with $\mu(A)$ is called a *subeigencone* of A [SSB09, Ser09a, Ser09b]. It is denoted by $V^*(A)$ and given by

$$V^*(A) = \left\{ v \in \mathbb{R}^n_+ \mid A \otimes v \leq \mu(A)v \right\}. \tag{2.27}$$

It was shown in [SSB09] that if $\mu(A) = 1$, each column of A^* is a subeigenvector of A. For $A \in \mathbb{R}_+^{n\times n}$ with $\mu(A) = 1$, $V^*(A)$ is described by

$$V^*(A) = \left\{ \bigoplus_{i\in r(A)} \alpha_i A^*_{.i} \oplus \bigoplus_{j\in \overline{N^C(A)}} \alpha_j A^*_{.j} \mid \alpha_i, \alpha_j \in \mathbb{R}_+ \right\} \tag{2.28}$$

where $\overline{N^C(A)}$ is a set of non-critical vertices in $D(A)$. Theorem 2.2.3 implies that $V(A) = V(\hat{A})$ and $V^*(A) = V^*(\hat{A})$. The following are known results on some geometric properties of $V(A)$ and $V^*(A)$ [SSB09, Ser09c].

Proposition 2.2.6. *For $A \in \mathbb{R}_+^{n\times n}$, the following are true.*

(i) $V^*(A)$ and $V(A)$ are max convex cones;

(ii) $V(A)$ is a max subcone of $V^*(A)$;

(ii) $V^*(A)$ is a convex cone in the conventional algebra while $V(A)$ is not.

Proposition 2.2.7. *Let $A \in \mathbb{R}_+^{n\times n}$ have $\mu(A) = 1$. Then, $V^*(A) = V(A^*) = span_\oplus(A^*)$.*

A max convex cone is called a *Kleene cone* if it can be represented as a max-algebraic column span of A^* [SSB09, Ser09c]. In the light of Proposition 2.2.7, the authors of [SSB09] call $V^*(A)$ a Kleene cone.

2.2.4 Visualisation Scaling

Next, we recall diagonal similarity scaling of non-negative matrices and its connection with the max algebra. *A diagonal similarity scaling* of $A \in \mathbb{R}_+^{n\times n}$ is a matrix given by $B = X^{-1}AX$ where $X = \text{diag}\,(x_1, x_2, ..., x_n)$ and $x_1, x_2, ..., x_n$ are some positive numbers. This scaling preserves many spectral properties of $A \in \mathbb{R}_+^{n\times n}$. In particular, $\mu(A) = \mu(B)$ and there is a one-to-one correspondence between $V(A)(V^*(A))$ and $V(B)(V^*(B))$ [BS05, SSB09, But10].

Diagonal similarity scaling has been studied by a large number of authors since the sixties [Afr63, Afr74, FP67, FP69, ES73, ES75]. It has motivated many works on matrix scaling problems in non-negative linear algebra and the max algebra. In this context, the characterisation of max-balanced flows on networks was considered in [SS90, SS91, RSS92]. In [BS05] the role of the max algebra in finding solutions of a number of matrix scaling problems was presented. In this direction, visualisation scaling over the max algebra was introduced in [SSB09, But10].

A matrix $A \in \mathbb{R}_+^{n\times n}$ is called *visualised* (*strictly visualised*) if

$$\begin{aligned} &a_{ij} = \mu(A) \text{ for all } (i,j) \in E^C(A) \\ &a_{ij} \leq \mu(A)(a_{ij} < \mu(A)) \text{ for all } (i,j) \neq E^C(A). \end{aligned} \tag{2.29}$$

Visualisation scaling can be thought of as a *Fiedler-Ptak scaling* in a way that the scaled matrix $X^{-1}AX$ is visualised if x is a subeigenvector associated with $\mu(A)$ [Ser11]. It is well-known from [Afr63, Afr74, FP67, FP69, BS05, SSB09, But10] that max combination of columns of A^* can be used to obtain a visualisation of a definite matrix $A \in \mathbb{R}_+^{n\times n}$. Further, positive linear combination of columns of A^* can be used to obtain a strict visualisation of a definite matrix $A \in \mathbb{R}_+^{n\times n}$. A well-known work on these results in the max algebra in connection with the Kleene cone and its relative interior is [SSB09]. In particular, we state the following result taken from this paper.

Proposition 2.2.8. *For $A \in \mathbb{R}_+^{n\times n}$ and $x \in V^*(A)$, $a_{ij}x_j = \mu(A)x_i$ for all $(i, j) \in E^C(A)$.*

2.2.5 Max Powers of an Irreducible Matrix

We briefly discuss the behaviour of the powers of an irreducible matrix in the max algebra. In this context, we recall necessary and sufficient conditions for the convergence of the max-algebraic powers. We say that the sequence of matrix powers $\{A_\otimes^k\}_{k\geq 0}$ converges to a matrix $\bar{A}$ when

$$\lim_{k\to\infty} |a_{ij}^{(k)} - \bar{a}_{ij}| = 0 \text{ for all } i, j.$$

The Cyclicity Index of A:

Let $D^C(A)$ have r strongly connected components. If $r = 1$, then A^C is irreducible. Further, let h be the index of imprimitivity of A^C. Then, the *cyclicity index* of A: $\text{cyc}(A) = h$. In this case, $(A^C)^h$ is a direct sum of h primitive matrices [EvdD99, EvdD01, HOvdW06]. If $r > 1$, each strongly connected component in $D^C(A)$ corresponds to an irreducible block matrix. Assume that each matrix has the index of imprimitivity equal to h_i, $i = 1, 2, ..., r$. Then, the cyclicity index of A is the least common multiple (l.c.m.) of these indices:

$$\text{cyc}(A) = \text{l.c.m.}(h_1, h_2, ..., h_r). \tag{2.30}$$

After a suitable permutation, $(A^C)^{\text{cyc}(A)}$ can be written as a direct sum of $\sum_{i=1}^{r} h_i$ primitive matrices [EvdD99, EvdD01, HOvdW06].

The following result highlights the role of the cyclicity index of an irreducible matrix in the behaviour of the max-algebraic powers.

The Cyclicity Theorem:

It is well-known that the max-algebraic powers of irreducible matrices are ultimately periodic. This fact is known as the *cyclicity theorem* [CDQV85]. See also [BCOQ92, Sch00, HOvdW06, BCG07, Ser09b]. The cyclicity theorem establishes a relation between the cyclicity index of an irreducible matrix and the periodicity of its max powers.

Theorem 2.2.9. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible matrix. Then, there exist positive integers c and k_0 such that*

$$A_\otimes^{k+c} = \mu(A)^c A_\otimes^k \text{ for all } k \geq k_0. \tag{2.31}$$

The least $c > 0$ satisfying (2.31) is said to be the *period* of A. It is standard that the ultimate period equals to $\operatorname{cyc}(A)$. Moreover, the smallest value for k_0 in (2.31) is called a *transient time* of A [HOvdW06, Ser09b].

The following results on the behaviour of the max powers of irreducible matrices are taken from [EvdD99, EvdD01].

Theorem 2.2.10. *Consider $A \in \mathbb{R}_+^{n\times n}$ with $\mu(A) = 1$. Assume that its critical matrix, A^C, is the direct sum of primitive matrices. If either A is irreducible or $N^C(A) = N(A)$, then $\lim_{k\to\infty} A_\otimes^k$ exists and $A_\otimes^k = \bar{A}$ for sufficiently large k.*

Theorem 2.2.11. *Consider an irreducible matrix $A \in \mathbb{R}_+^{n\times n}$ with $\mu(A) = 1$ satisfying the following conditions:*

(i) $N^C(A) = N(A)$;

(ii) A^C *is the direct sum of primitive matrices.*

Then, $\bar{A} = \lim_{k\to\infty} A_\otimes^k = A_\otimes^p$ where p is given by

$$p = r - 1 + \sum_{i=1}^{r} d_i + \max_{1\leq i\leq r}\{\sigma_i - d_i \mid d_i \geq 2\}. \tag{2.32}$$

Here, r is the number of strongly connected components in $D^C(A)$, d_i is the diameter of the i^{th} strongly connected component (See (2.9)) and σ_i is the exponent of the corresponding irreducible block matrix (See (2.4)) for $i = 1, 2, ..., r$.

Note that Theorem 2.2.11 (ii) is enough to ensure the convergence. However, (i) is needed for the definition of p in (2.32) [EvdD01]. The following result, which is Corollary 2.6 of [EvdD01], describes an upper bound for p.

Corollary 2.2.12. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible matrix with $\mu(A) = 1$ satisfying the conditions (i) and (ii) in Theorem 2.2.11. Then, $\bar{A} = A_\otimes^p$ where $p \leq n^2 - 1$.*

We next illustrate Theorem 2.2.11.

Example 2.2.3. Consider the following irreducible matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1 & 1/4 \\ 1/3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 5/6 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

For A, $\mu(A) = 1$, there are three strongly connected components in $D^C(A)$ and $N^C(A) = N(A)$. (See Figure 2.4)

Figure 2.4: $D^C(A)$

Denote the index of imprimitivity of each strongly connected component by h_i $(i = 1, 2, 3)$. Then, $h_1 = \text{g.c.d.}(1, 2) = 1$ and $h_2 = h_3 = 1$. Using that, we get $\text{cyc}(A) = \text{l.c.m.}(1, 1, 1) = 1$. By applying the permutation matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ to } A^C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we get the following matrix written as a direct sum of three primitive matrices, each of which corresponds to the i^{th} strongly connected component in $D^C(A)$.

$$PA^CP^T = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \dotplus 1 \dotplus 1.$$

Since both conditions in Theorem 2.2.11 are satisfied, we find $p = 5$ as $r = 3$ and $d_i = 1$ for all i. Thus, $\bar{A} = A_{\otimes}^5$. It follows from Theorem 2.2.9 that the least value for k at which $A_{\otimes}^{k+1} = A_{\otimes}^k$ is the transient time ($c = \text{cyc}(A) = 1$). It is easy to obtain by matrix multiplication that it equals to 3.

2.2.6 Numerical Computation of Max Eigenpairs

Next, we recall some numerical algorithms to find the largest max eigenvalue and max eigenvectors of an irreducible matrix. In the conventional algebra, *the power method* is one of the most popular algorithms to find the spectral radius and the eigenvector corresponding to it [Var62, Ste98, BF05]. Convergence of the power method is guaranteed when the matrix is primitive.

The power method was extended to the max algebra by [BO93, EvdD99, HOvdW06]. Basically, we start with an initial vector $x(0)$ and repeatedly multiply $x(k)$ by A such that $x(k+1) = A \otimes x(k)$ $(k = 0, 1, ...)$. We search for some c where $x(k+c)$ is a multiple of $x(k)$. c is the period of A in the cyclicity theorem. In particular, if c is one, then $x(k)$ is a max eigenvector for some $k \in \mathbb{R}_+$. Otherwise, $x(k)$ enters into a *periodic regime:* $\{x(k), x(k+1), ..., x(k+c-1)\}$ for some $k \in \mathbb{R}_+$ [HOvdW06]. We require the critical matrix to be a direct sum of primitive matrices to conclude convergence.

We summarise the max power method in Algorithm 1 below. Note that the algorithm has $O(n^4)$ time complexity to calculate $\mu(A)$ and a max eigenvector corresponding to it. We only get one max eigenvector which depends on the selection of the initial vector [EvdD99]. We illustrate Algorithm 1 in Example 2.2.4.

Algorithm 1 Calculate $\mu(A)$ and a max eigenvector associated with $\mu(A)$

```
A ∈ R_+^{n×n}, irreducible, x(0) ∈ R_+^n, x(0) ≠ 0
for k = 0, 1, 2, ... do
  x(k+1) = A ⊗ x(k)
  for c = 1 to k do
    if x(k+1) = αx(k+1−c) for some α > 0 then
      μ(A) = α^{1/c}
      t = k + 1 − c
      x = x(t) ⊕ ⊕_{i=1}^{c−1} x(t+i)/μ(A)^i
    end if
  end for
end for
```

Example 2.2.4. Consider the following irreducible matrix

$$A = \begin{bmatrix} 1 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1 \\ 1 & 1/2 & 1 & 1 \\ 1/4 & 1 & 5/6 & 1/4 \end{bmatrix}.$$

For $x(0) = \begin{bmatrix} 0.091 & 0.202 & 0.460 & 0.707 \end{bmatrix}^T$, we obtain the following sequence:

$$x(1) = \begin{bmatrix} 0.460 \\ 0.707 \\ 0.707 \\ 0.383 \end{bmatrix} \quad x(2) = \begin{bmatrix} 0.707 \\ 0.383 \\ 0.707 \\ 0.707 \end{bmatrix} \quad x(3) = \begin{bmatrix} 0.707 \\ 0.707 \\ 0.707 \\ 0.589 \end{bmatrix}$$

$$x(4) = \begin{bmatrix} 0.707 \\ 0.589 \\ 0.707 \\ 0.707 \end{bmatrix} \quad x(5) = \begin{bmatrix} 0.707 \\ 0.707 \\ 0.707 \\ 0.589 \end{bmatrix}$$

The algorithm stops at the 4^{th} iteration when $x(3) = x(5)$. Then, $\operatorname{cyc}(A) = 2$, $\mu(A) = 1$ and the max eigenvector is $x(3) \oplus x(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$.

In [EvdD01], Elsner and van den Driessche propose a new algorithm to calculate $\mu(A)$ and the max eigenvectors corresponding to it. In order to find $\mu(A)$, they implement the well-known *Karp algorithm* which has $O(n^3)$ time complexity. The key result is the following [Kar78, BCOQ92, EvdD01, HOvdW06].

Theorem 2.2.13. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible matrix. Then,*

$$\mu(A) = \max_{i=1,\dots,n} \min_{k=1,\dots,n} \left(\frac{(A_\otimes^{n+1})_{ij}}{(A_\otimes^{k})_{ij}} \right)^{\frac{1}{n+1-k}}. \tag{2.33}$$

The algorithm in [EvdD01] can be used to obtain all max eigenvectors associated with $\mu(A)$. However, it doesn't have an optimal time complexity.

The *Floyd-Warshall algorithm* is preferred to find max eigenvectors of an irreducible matrix [GM84, PS98, ORE99, EvdD04]. Basically, the Floyd-Warshall algorithm constructs paths of maximum weight from i to j in $D(A)$ for all i, j. It has $O(n^3)$ time complexity. We summarise it in Algorithm 2 and illustrate in Example 2.2.5. Remark that the algorithm computes the A^+ matrix.

Algorithm 2 Calculate A^+

$A \in \mathbb{R}_+^{n \times n}, irreducible$
Calculate $\mu(A)$
if $\mu(A) < 1$ **then**
  $A_0 = A$
else
  $A_0 = A/\mu(A)$
end if
for $k = 1, ..., n$ **do**
  for $i = 1, ..., n$ **do**
    for $j = 1, ..., n$ **do**
      $a_{ij}^{(k)} = \max(a_{ij}^{(k-1)}, a_{ik}^{(k-1)} a_{kj}^{(k-1)})$
    end for
  end for
end for

Example 2.2.5. Consider the matrix given in Example 2.2.4. By using Algorithm 2, we get

$$A^+ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.833 & 1 & 0.833 & 1 \\ 1 & 1 & 1 & 1 \\ 0.833 & 1 & 0.833 & 1 \end{bmatrix}.$$

It follows from Theorem 2.2.5 (i) that the first max eigenvector is $A_{.1}^+ = A_{.3}^+ = \begin{bmatrix} 1 & 0.833 & 1 & 0.833 \end{bmatrix}^T$ and the second max eigenvector is $A_{.2}^+ = A_{.4}^+ = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$. Other max eigenvectors can be obtained by max combinations of these.

2.3 Concluding Remarks

In this chapter, we introduced fundamental concepts and preliminary results in non-negative linear algebra and the max algebra, thereby provided a mathematical background for the following chapters. In particular,

- we recalled the celebrated Perron-Frobenius theorems;
- we described a number of definitions from graph theory and highlighted some connections with non-negative matrices;
- we discussed fundamentals of the max-algebraic spectral theory.

CHAPTER 3

Spectral Properties of Max Matrix Polynomials

In this chapter, we consider max matrix polynomials of the form $P(\lambda) = A_0 \oplus \lambda A_1 \oplus \cdots \oplus \lambda^{m-1} A_{m-1}$ *where* $A_0, A_1, \ldots, A_{m-1} \in \mathbb{R}_+^{n \times n}$*. We show how the Perron-Frobenius theory for the max algebra extends to such polynomials. Applications of this result to the convergence properties of multi-step difference equations over the max algebra are also described. Additionally, we present a number of inequalities, echoing similar results over the conventional algebra, for the largest max eigenvalue of a matrix polynomial.*

3.1 Motivation and Mathematical Background

In this section, we briefly provide mathematical background on matrix polynomials with non-negative coefficients over the conventional algebra. In particular, we investigate their spectral properties and discuss some applications in the context of dynamical systems.

3.1.1 Introduction

Matrix polynomials are polynomials with real or complex matrix coefficients [GLR82]. They have important applications in the study of higher order differential/difference equations arising in wide variety of fields such as systems theory [Fuh87], economic modelling [RD00], queueing theory [BLM02] and population dynamics [LS02]. In particular, the spectral theory of matrix polynomials was extensively studied. In [TM00], the relevance of eigenvalues of quadratic matrix polynomials to problems in acoustics, electronics and electrical circuits was highlighted.

Matrix polynomials were considered in connection with scheduling problems and timetable analysis over the max-plus algebra [HOvdW06]. The spectral properties of such polynomials have important implications for the stability of timetables with respect to the propagation of delays. This theory was applied to the modelling of the Dutch railway system [Gov05, HOvdW06].

The motivation of our study of matrix polynomials comes from their close connection with multi-step difference equations. In this direction, the layout of this chapter is as follows. First, we briefly review results over the conventional algebra. Inspired by the work of Psarrakos and Tsatsomeros [PT04], in Section 3.2, we are concerned with extending spectral properties of matrix polynomials with non-negative coefficient matrices to the max algebra. In Section 3.3, we consider their relation with multi-step difference equations. Finally, in Section 3.4, we discuss the characterisation of the largest max eigenvalue of a max matrix polynomial using an $n \times n$ non-negative matrix. The work contained in this chapter has resulted in the publication: [BGM11b].

3.1.2 Perron Polynomials

Following the notation of Psarrakos and Tsatsomeros [PT04], consider a matrix polynomial of the form

$$L(\lambda) = I\lambda^m - A_{m-1}\lambda^{m-1} - \dots - A_1\lambda - A_0 \tag{3.1}$$

where $A_i \in \mathbb{R}_+^{n\times n}$ for $0 \leq i \leq m$. $L(\lambda)$ is called *Perron polynomial of degree m*. The eigenvalue problem for $L(\lambda)$ is defined by

$$L(\lambda)x = 0 \tag{3.2}$$

where λ is an *eigenvalue* of $L(\lambda)$ and x is a *right eigenvector* associated with it. A vector y satisfying $y^T L(\lambda) = 0$ is a *left eigenvector* associated with λ. The set of eigenvalues is called the *spectrum* of $L(\lambda)$ and denoted by $\sigma(L(\lambda))$. The eigenvalues can be determined by finding the roots of the following equation

$$\det(L(\lambda)) = 0. \tag{3.3}$$

A classical approach to obtain the spectrum of $L(\lambda)$ is called linearisation [GLR82]. A key idea of linearisation is to associate $L(\lambda)$ with a linear matrix polynomial $I\lambda - C_L$ where C_L is called the *companion matrix* of $L(\lambda)$ and has the following form

$$C_L = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & I \\ A_0 & A_1 & \dots & A_{m-2} & A_{m-1} \end{bmatrix} \in \mathbb{R}_+^{mn \times mn}. \tag{3.4}$$

An important result shown in [PT04] is that $\sigma(L(\lambda)) = \sigma(C_L)$. Moreover, there is a one to one correspondence between the eigenvectors of $L(\lambda)$ and C_L [PT04].

Next, we state the Perron-Frobenius theorems for Perron polynomials [PT04]. Remark that the *spectral radius* of $L(\lambda)$ can be defined in the same way as for a matrix. (Recall (2.7).)

$$\rho(L(\lambda)) = \max_i \{|\lambda_i| \mid \lambda_i \in \sigma(L(\lambda))\} \tag{3.5}$$

In this section, we will denote it by ρ.

Theorem 3.1.1. *Let $L(\lambda)$ be a Perron polynomial defined in (3.1). Then, the following are true.*

(i) $\rho \in \sigma(L(\lambda))$;

(ii) There exist non-negative and nonzero right and left eigenvectors $x, y \in \mathbb{R}_+^n$ such that $L(\rho)x = 0$ and $y^T L(\rho) = 0$.

The next result is for the irreducible case [PT04].

Theorem 3.1.2. *Let $L(\lambda)$ be a Perron polynomial defined in (3.1) and C_L be the corresponding companion matrix. Suppose that C_L is irreducible. Then, the following are true.*

(i) $\rho > 0$;

(ii) $\rho \in \sigma(L(\lambda))$;

(iii) The algebraic and geometric multiplicity of ρ are one;

(iv) There exists positive right and left eigenvectors $x, y > 0$ such that $L(\rho)x = 0$ and $y^T L(\rho) = 0$.

3.1.3 Multi-step Difference Equations

Matrix polynomials are closely related to multi-step difference equations. In [PT04], this relationship was exploited to derive a multi-step version of the Fundamental Theorem of Demography for the conventional algebra. For a reference highlighting the role played by the classical Perron-Frobenius theory in the Fundamental Theorem of Demography and other key results of population dynamics, see [LS02].

Consider the multi-step difference equation of the form

$$u_{k+m} = A_{m-1}u_{k+m-1} + \cdots + A_1 u_{k+1} + A_0 u_k \quad (k = 0, 1, ...) \tag{3.6}$$

where $A_i \in \mathbb{R}_+^{n\times n}$ for all $0 \leq i \leq m-1$.

For an initial condition $v(0) = \begin{bmatrix} u_0 & u_1 & \cdots & u_{m-1} \end{bmatrix}^T \in \mathbb{R}_+^{nm}$, the solution is given by $u_k = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} C_L^k v(0), k \geq 0$ [PT04]. Motivated by this, a generalisation of the Fundamental Theorem of Demography is stated as follows [PT04].

Theorem 3.1.3. *Let $L(\lambda)$ be a Perron polynomial defined in (3.1) and C_L be the corresponding companion matrix in (3.4). Suppose that C_L is primitive. Further, let $x, y > 0$ be right and left eigenvectors of $L(\lambda)$ corresponding to ρ,* *normalised as follows*

$$\begin{bmatrix} y^T E_1(\rho) & \cdots & y^T E_{m-1}(\rho) & y^T \end{bmatrix} \begin{bmatrix} x \\ \rho x \\ \vdots \\ \rho^{m-1} x \end{bmatrix} = 1$$

where $E_m(\lambda) = I$ *and* $E_{i-1}(\lambda) = \lambda E_i(\lambda) - A_{i-1}$ *for* $i = m, m-1, ..., 2$ *and* $\lambda > 0$. *Let* $\{u_0, u_1, ...\}$ *be the solution of (3.6). Then,*

$$\lim_{k\to\infty} \frac{u_k}{\rho^k} = y^T \left(\sum_{i=1}^{m} E_i(\rho) u_{i-1} \right) x.$$

3.1.4 Rational Matrix Functions

Another concept discussed in [PT04] is the rational matrix function associated with $L(\lambda)$ of the form

$$S_L(\lambda) = A_{m-1} + \frac{1}{\lambda} A_{m-2} + \cdots + \frac{1}{\lambda^{m-1}} A_0 \tag{3.7}$$

where $A_i \in \mathbb{R}_+^{n\times n}$ for all $0 \le i \le m-1$ and $\lambda > 0$. The next result presents a number of bounds for the spectral radius of the Perron polynomial $L(\lambda)$ in terms of $S_L(1)$ [PT04].

Proposition 3.1.4. *Let* $L(\lambda)$ *be Perron polynomial given in (3.1) and* $S_L(\lambda)$ *be given in (3.7). Then, the following hold.*

(i) $\rho(S_L(1)) \le \rho \le \rho(S_L(1))^{1/m}$ *if* $\rho(S_L(1)) \le 1$*;*

(ii) $\rho(S_L(1))^{1/m} \le \rho \le \rho(S_L(1))$ *if* $\rho(S_L(1)) \ge 1$*;*

(iii) $\rho < 1$ *if and only if* $\rho(S_L(1)) < 1$*;*

(iv) $\rho = 1$ *if and only if* $\rho(S_L(1)) = 1$.

Note that functions of this type were studied for non-negative compact operators in [FN91, Rau92]. The following is one of the results in these papers that can directly be applied to Perron polynomials as follows.

Theorem 3.1.5. *Let the spectral radius of* $L(\lambda)$ *in (3.1) be positive. Further, let* C_L *and* $S_L(\lambda)$ *be given in (3.4) and (3.7). Then,* $\rho = \rho(C_L) = \rho(S_L(\rho))$.

3.2 Matrix Polynomials in the Max Algebra

In the spirit of Psarrakos and Tsatsomeros [PT04], we now consider max-algebraic matrix polynomials and their associated companion matrices. We argue that there is a perfect correspondence between the max-algebraic eigenvalues and eigenvectors of the polynomial and those of the companion matrix. This allows us to apply the Perron-Frobenius theorem for the max algebra introduced in Section 2.2.2 to obtain a corresponding result for max matrix polynomials.

Formally, we consider polynomials given by

$$P(\lambda) = A_0 \oplus \lambda A_1 \oplus \cdots \oplus \lambda^{m-1} A_{m-1} \tag{3.8}$$

where $A_0, A_1, \ldots, A_{m-1}$ are in $\mathbb{R}_+^{n\times n}$. We refer to $P(\lambda)$ as a *max matrix polynomial of degree* $m-1$. In analogy with the definitions for the conventional algebra presented in Section 3.1.2, we say that

(i) $\kappa \geq 0$ is a *right max eigenvalue* of $P(\lambda)$ with corresponding *right max eigenvector* $x \geq 0$ if

$$P(\kappa) \otimes x = \kappa^m x. \tag{3.9}$$

(κ, x) is then a *right max eigenpair* of $P(\lambda)$.

(ii) $\tau \geq 0$ is a *left max eigenvalue* of $P(\lambda)$ with corresponding *left max eigenvector* $y \geq 0$ if

$$y^T \otimes P(\tau) = \tau^m y^T. \tag{3.10}$$

(τ, y) is then a *left max eigenpair* of $P(\lambda)$.

The key result of this section, which allows us to directly apply the Perron-Frobenius theorem for the max algebra to obtain corresponding statements for max matrix polynomials, is Proposition 3.2.1 below. Essentially, as was explained in Section 3.1.2 for the conventional algebra, this establishes a one to one correspondence between the max eigenpairs of the polynomial $P(\lambda)$ in

(3.8) and the max eigenpairs of the *companion matrix* of the form

$$C_P = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & I \\ A_0 & A_1 & \dots & A_{m-2} & A_{m-1} \end{bmatrix} \in \mathbb{R}_+^{mn \times mn}. \tag{3.11}$$

Proposition 3.2.1. *Consider the max matrix polynomial $P(\lambda)$ given by (3.8) and the corresponding companion matrix given by C_P in (3.11). Then $(\kappa, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is a right max eigenpair of $P(\lambda)$ if and only if $(\kappa, \hat{x}) \in \mathbb{R}_+ \times \mathbb{R}_+^{mn}$ is a right max eigenpair of C_P, where*

$$\hat{x} = \begin{bmatrix} x \\ \kappa x \\ \vdots \\ \kappa^{m-1} x \end{bmatrix}. \tag{3.12}$$

Moreover, $(\tau, y) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is a left max eigenpair of $P(\lambda)$ if and only if $(\tau, \hat{y}) \in \mathbb{R}_+ \times \mathbb{R}_+^{mn}$ is a left max eigenpair of C_P, where

$$\hat{y} = \begin{bmatrix} \frac{1}{\tau} A_0^T \otimes y \\ (\frac{1}{\tau^2} A_0^T \oplus \frac{1}{\tau} A_1^T) \otimes y \\ \vdots \\ (\frac{1}{\tau^{m-1}} A_0^T \oplus \frac{1}{\tau^{m-2}} A_1^T \oplus \cdots \oplus \frac{1}{\tau} A_{m-2}^T) \otimes y \\ y \end{bmatrix}. \tag{3.13}$$

Proof: It is a straightforward calculation to verify that $C_P \otimes \hat{x}$ is given by

$$\begin{bmatrix} \kappa x \\ \kappa^2 x \\ \vdots \\ P(\kappa) \otimes x \end{bmatrix}.$$

Hence, if (κ, x) is a right max eigenpair of $P(\lambda)$, it is immediate that $C_P \otimes \hat{x} = \kappa \hat{x}$.

For the converse, it is clear that any right eigenvector of C_P must be of the form (3.12). Then equating the last rows of $C_P \otimes \hat{x} = \kappa \hat{x}$, we have

$$P(\kappa) \otimes x = A_0 \otimes x \oplus \kappa A_1 \otimes x \oplus \cdots \oplus \kappa^{m-1} A_{m-1} \otimes x = \kappa^m x.$$

For the left eigenpair statement, we have $\hat{y}^T \otimes C_P$ given by

$$\begin{bmatrix} A_0^T \otimes y \\ (\frac{1}{\tau}A_0^T \oplus A_1^T) \otimes y \\ \vdots \\ \frac{P(\tau)^T}{\tau^{m-1}} \otimes y \end{bmatrix}^T.$$

Since (τ, y) is a left max eigenpair of $P(\lambda)$, it follows that $\hat{y}^T \otimes C_P = \tau\hat{y}^T$. For the converse, equating the last columns of $\hat{y}^T \otimes C_P$ and $\tau\hat{y}^T$, we see that

$$y^T \otimes \frac{P(\tau)}{\tau^{m-1}} = \tau y^T \Rightarrow y^T \otimes P(\tau) = \tau^m y^T.$$

□

It follows from Proposition 3.2.1 that the largest right and left max eigenvalues of the polynomial $P(\lambda)$ coincide. We now define

$$\mu = \mu(P(\lambda)) \tag{3.14}$$

to be the *largest right (or left) max eigenvalue* of $P(\lambda)$. μ will be used for $\mu(P(\lambda))$ throughout this chapter. Then, $\mu = \mu(C_P)$, the maximal cycle geometric mean of $D(C_P)$. The following results, which extend the Perron-Frobenius theorems to matrix polynomials over the max algebra now follow easily from combining Proposition 3.2.1 with Theorems 2.2.1 and 2.2.2.

Theorem 3.2.2. *Consider the max matrix polynomial $P(\lambda)$ given by (3.8) and let C_P be the corresponding companion matrix in (3.11). Further, let μ be the largest max eigenvalue of $P(\lambda)$. The following are true.*

(i) $\mu \geq 0$;

(ii) There exist non-negative and nonzero right and left eigenvectors $x, y \in \mathbb{R}_+^n$ such that $P(\mu) \otimes x = \mu^m x$ and $y^T \otimes P(\mu) = \mu^m y^T$.

It has been pointed out in [PT04] that C_P will be irreducible if A_0 is irreducible. Note that the irreducibility of C_P only implies that the eigenvalue μ is unique. However, in contrast with the conventional algebra, there may be multiple eigenvectors corresponding to μ. The following result describes the extension of Theorem 3.1.2 to max matrix polynomials for the situation in which the eigenvector is also unique.

Theorem 3.2.3. *Consider the max matrix polynomial $P(\lambda)$ given by (3.8) and let C_P be the corresponding companion matrix (3.11). Further, let μ be the largest max eigenvalue of $P(\lambda)$. Suppose that C_P is irreducible. The following are true.*

(i) $\mu > 0$;

(ii) μ *is the unique max eigenvalue of* $P(\lambda)$;

(iii) *There exists positive right and left eigenvectors* $x, y > 0$ *such that* $P(\mu) \otimes x = \mu^m x$ *and* $y^T \otimes P(\mu) = \mu^m y^T$;

(iv) x *and* y *are unique (up to a scalar multiple) if and only if* $D^C(C_P)$ *is strongly connected.*

3.3 Multi-step Difference Equations in the Max Algebra

In this section, we investigate the implications of the results of Section 3.2 for the convergence of multi-step difference equations in the max algebra. We pursue the association of matrix polynomials with multi-step difference equations over the max algebra. As noted in Section 3.1.1, equations of this type have been previously studied in the max-plus setting with the view to applications to time scheduling.

Consider the multi-step difference equation over the max algebra:

$$u_{k+m} = A_{m-1} \otimes u_{k+m-1} \oplus \cdots \oplus A_1 \otimes u_{k+1} \oplus A_0 \otimes u_k \quad (k = 0, 1, ...) \tag{3.15}$$

where $A_0, A_1 \cdots, A_{m-1} \in \mathbb{R}_+^{n \times n}$ are coefficient matrices and $u_0, u_1, ...u_{m-1} \in \mathbb{R}_+^n$ are initial values. As with multi-step difference equations for the conventional algebra [PT04], the system (3.15) is equivalent to the single-step difference equation given by

$$v(k+1) = C_P \otimes v(k) \quad (k = 0, 1, ...). \tag{3.16}$$

This is seen by setting $v(k) = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+m-1} \end{bmatrix} \in \mathbb{R}_+^{mn}$.

(3.16) is a result of the following matrix equation:

$$\underbrace{\begin{bmatrix} u_{k+1} \\ u_{k+2} \\ \vdots \\ u_{k+m} \end{bmatrix}}_{v(k+1)} = \underbrace{\begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & I \\ A_0 & \dots & \dots & \dots & A_{m-2} & A_{m-1} \end{bmatrix}}_{C_P} \otimes \underbrace{\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+m-1} \end{bmatrix}}_{v(k)}$$

For a given initial condition $v(0) = \begin{bmatrix} u_0 & u_1 & \cdots & u_{m-1} \end{bmatrix}^T \in \mathbb{R}_+^{mn}$, the solution of (3.16) is

$$v(k) = C_{P_\otimes}^k \otimes v(0) \quad k \geq 0. \tag{3.17}$$

Hence, the solution of (3.15) can be written in the following form:

$$u_k = \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}}_{\in \mathbb{R}_+^{n \times mn}} \otimes C_{P_\otimes}^k \otimes v(0) \quad k \geq 0. \tag{3.18}$$

It can be seen from (3.18) that the the behaviour of the solution depends on the asymptotic properties of the max powers of C_P.

Throughout this section, we will assume that C_P is irreducible and that the critical matrix C_P^C is primitive. Therefore, it follows from Theorem 3.2.3 that $P(\lambda)$ has unique left and right max eigenpairs.

Under the above assumptions, it follows from Theorem 2.2.10 that the max powers of the normalized companion matrix $\frac{1}{\mu^k} C_{P_\otimes}^k$ converge in finitely many steps to a matrix $\bar{C}$. In fact, Theorem 2.2.10 establishes that for any irreducible $A \in \mathbb{R}_+^{n \times n}$ with A^C primitive, there is some $K \in \mathbb{R}_+$ and some $\bar{A} \in \mathbb{R}_+^{n \times n}$ such that for $\forall k \geq K$,

$$\frac{A_\otimes^k}{\mu(A)^k} = \bar{A}. \tag{3.19}$$

The following lemma restates the above convergence result in terms of the normalised max eigenvectors of A.

Lemma 3.3.1. *Let $A \in \mathbb{R}_+^{n \times n}$ be irreducible and A^C be primitive. Then there exists some $K > 0$ such that*

$$\frac{A_\otimes^k}{\mu(A)^k} = x \otimes y^T, \quad \textit{for } k \geq K \tag{3.20}$$

where $x > 0$ and $y > 0$ are the unique right and left max eigenvectors of A satisfying $A \otimes x = \mu(A)x$, $A^T \otimes y = \mu(A)y$ and $x^T \otimes y = 1$.

Proof: It follows from (3.19) that there is some $K > 0$ and a matrix $\bar{A}$ such that $\frac{A_\otimes^k}{\mu(A)^k} = \bar{A}$ for all $k \geq K$. Now calculate $A \otimes \bar{A}$:

$$A \otimes \bar{A} = A \otimes \frac{A_\otimes^K}{\mu(A)^K} = \mu(A)\frac{A_\otimes^{K+1}}{\mu(A)^{K+1}} = \mu(A)\bar{A}.$$

It follows immediately that the columns of $\bar{A}$ are right max eigenvectors of A and hence that $\bar{A} = x \otimes v^T$ for some $v \in \mathbb{R}_+^n$.

On the other hand,

$$A^T \otimes \bar{A}^T = A^T \otimes \frac{(A_\otimes^K)^T}{\mu(A)^K} = A^T \otimes \frac{(A_\otimes^T)^K}{\mu(A)^K} = \mu(A)\frac{(A_\otimes^T)^{K+1}}{\mu(A)^{K+1}} = \mu(A)\bar{A}^T.$$

But, $\bar{A} = x \otimes v^T$ and hence,

$$A^T \otimes v \otimes x^T = \mu(A)v \otimes x^T \Rightarrow A^T \otimes v = \mu(A)v.$$

Thus, $v = \lambda y$ for some $\lambda \in \mathbb{R}_+$. It is explicit that $\bar{A} \otimes \bar{A} = \bar{A}$. Since $\bar{A} = \lambda x \otimes y^T$, we have

$$\lambda^2 x \otimes \underbrace{y^T \otimes x}_{1} \otimes y^T = \lambda x \otimes y^T \Rightarrow \lambda = 1.$$

Therefore, we conclude that $v = y$ and $\bar{A} = x \otimes y^T$. $\square$

A generalisation of the so-called Fundamental Theorem of Demography was presented in Section 3.1.3 in the conventional algebra. In the following result, we present a max-algebraic version of this fact.

Theorem 3.3.2. *Let $P(\lambda)$ be the max matrix polynomial given by (3.8) and C_P be the corresponding companion matrix in (3.11); let C_P be irreducible and C_P^C be primitive. Let x and y be the unique right and left max eigenvectors of $P(\lambda)$ corresponding to μ, normalised so that*

$$\begin{bmatrix} y^T \otimes \frac{A_0}{\mu} & y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) & \cdots & y^T \end{bmatrix} \otimes \begin{bmatrix} x \\ \mu x \\ \cdots \\ \mu^{m-1}x \end{bmatrix} = 1. \tag{3.21}$$

Write $u_k, k = 0, 1, \ldots$ for the solution of the multi-step difference equation ([3.15](#)) corresponding to a nonzero initial vector $v(0) = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \in \mathbb{R}_+^{mn}$.

Then there is some positive integer K such that for all $k \geq K$

$$\frac{u_k}{\mu^k} = \left[y^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{i=0}^{j-1} \frac{A_i}{\mu^{j-i}} \right) \otimes u_{j-1} \right) \right] \otimes x. \tag{3.22}$$

Proof: Let $\hat{x}$ and $\hat{y}$ be the right and left max eigenvectors of C_P given by (3.12), (3.13) respectively. Lemma 3.3.1 implies that there is some integer $K > 0$ such that for all $k \geq K$,

$$\begin{aligned}
\frac{C_{P_\otimes}^k}{\mu^k} &= \hat{x} \otimes \hat{y}^T \\
&= \begin{bmatrix} x \\ \mu x \\ \cdots \\ \mu^{m-1} x \end{bmatrix} \otimes \begin{bmatrix} y^T \otimes \frac{A_0}{\mu} & y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) & \cdots & y^T \end{bmatrix} \\
&= \begin{bmatrix} x \otimes y^T \otimes \frac{A_0}{\mu} & x \otimes y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) & \cdots & x \otimes y^T \\ \mu x \otimes y^T \otimes \frac{A_0}{\mu} & \mu x \otimes y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) & \cdots & \mu x \otimes y^T \\ \vdots & \vdots & & \vdots \\ \mu^{m-1} x \otimes y^T \otimes \frac{A_0}{\mu} & \mu^{m-1} x \otimes y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) & \cdots & \mu^{m-1} x \otimes y^T \end{bmatrix}.
\end{aligned}$$

The solution of (3.15) is given by $u_k = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \otimes C_{P_\otimes}^k \otimes v(0)$. It immediately follows from the above calculation that for all $k \geq K$,

$$\begin{aligned}
\frac{u_k}{\mu^k} &= \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \otimes \frac{C_{P_\otimes}^k}{\mu^k} \otimes v(0) \\
&= \begin{bmatrix} x \otimes y^T \otimes \frac{A_0}{\mu} & x \otimes y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) & \cdots & x \otimes y^T \end{bmatrix} \otimes \begin{bmatrix} u_0 \\ u_1 \\ \cdots \\ u_{m-1} \end{bmatrix} \\
&= x \otimes (y^T \otimes \frac{A_0}{\mu} \otimes u_0 \oplus y^T \otimes (\frac{A_0}{\mu^2} \oplus \frac{A_1}{\mu}) \otimes u_1 \oplus \cdots \oplus y^T \otimes u_{m-1}).
\end{aligned}$$

Using the fact that $y^T = y^T \otimes \frac{P(\mu)}{\mu^m} = y^T \otimes \left(\frac{A_0}{\mu^m} \oplus \frac{A_1}{\mu^{m-1}} \oplus \cdots \oplus \frac{A_{m-1}}{\mu}\right)$, we find that for all $k \geq K$,

$$\frac{u_k}{\mu^k} = \left[y^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{i=0}^{j-1} \frac{A_i}{\mu^{j-i}}\right) \otimes u_{j-1}\right)\right] \otimes x$$

as claimed. □

Note that the above result implies that

$$\lim_{k\to\infty} \frac{u_k}{\mu^k} = \left[y^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{i=0}^{j-1} \frac{A_i}{\mu^{j-i}}\right) \otimes u_{j-1}\right)\right] \otimes x. \tag{3.23}$$

which is a direct generalisation of Theorem 4.2 of [PT04].

In population dynamics, the spectral radius of matrix polynomials determines the growth or decay rate of the considered population model [LS02]. In particular: if it is less than one, the population converges to zero; if it is greater than one, the population grows to infinity; otherwise, the population stays finite. Similar facts can be interpreted for the max-algebraic models based on the value of μ. Essentially, the solution u_k is given by μ^k times a constant vector once k is large enough. This means that in the max algebra, μ completely characterises the rate of convergence or divergence of the solution for any initial condition. The following is a direct implication of Theorem 3.3.2.

$$\lim_{k\to\infty} ||u_k|| = \begin{cases} 0 & \text{if } \mu < 1, \\ \left|\left|\left[y^T \otimes \left(\bigoplus_{j=1}^{m} \left(\bigoplus_{i=0}^{j-1} \frac{A_i}{\mu^{j-i}}\right) \otimes u_{j-1}\right)\right] \otimes x\right|\right| & \text{if } \mu = 1, \\ \infty & \text{if } \mu > 1. \end{cases}$$

where $||.||$ denotes the infinity norm of a vector on $\mathbb{R}^n_+$, i.e, $||x|| = \max_{1\leq i\leq n} x_i$ for some $x \in \mathbb{R}^n_+$ [PT04].

3.4 Some Bounds on $\mu(P(\lambda))$

In this section, we derive a number of inequalities for the largest max eigenvalue of a max matrix polynomial in terms of the largest max eigenvalue of a fixed matrix naturally associated with the polynomial. First, consider a max version of the rational matrix function in (3.7) given by

$$S_P(\lambda) = A_{m-1} \oplus \frac{1}{\lambda} A_{m-2} \oplus \cdots \oplus \frac{1}{\lambda^{m-1}} A_0. \tag{3.24}$$

Following similar notation for the max eigenpairs of $P(\lambda)$, we say that

(i) $\kappa \geq 0$ is said to be a *right max eigenvalue* of $S_P(\lambda)$ with corresponding *right max eigenvector* $x \geq 0$ if

$$S_P(\kappa) \otimes x = \kappa x. \tag{3.25}$$

(κ, x) is then a *right max eigenpair* of $S_P(\lambda)$.

(ii) $\tau \geq 0$ is said to be a *left max eigenvalue* of $S_P(\lambda)$ with corresponding *left max eigenvector* $y \geq 0$ if

$$y^T \otimes S_P(\tau) = \tau y^T. \tag{3.26}$$

(τ, y) is then a *left max eigenpair* of $S_P(\lambda)$.

Next, we show the relation between the spectral radius of a matrix polynomial and the corresponding rational function over the max algebra.

Proposition 3.4.1. *Consider the max matrix polynomial $P(\lambda)$ given by (3.8) and the corresponding max rational matrix function given by $S_P(\lambda)$ (3.24). Then $(\kappa, x) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is a right max eigenpair of $P(\lambda)$ if and only if (κ, x) is a right max eigenpair of $S_P(\lambda)$. Moreover, $(\tau, y) \in \mathbb{R}_+ \times \mathbb{R}_+^n$ is a left max eigenpair of $P(\lambda)$ if and only if (τ, y) is a left max eigenpair of $S_P(\lambda)$.*

Proof: It is a straightforward calculation to verify that

$$S_P(\lambda) = \frac{1}{\lambda^{m-1}} P(\lambda).$$

For the right max eigenpair, take $\lambda = \kappa$ and multiply the above equation by x from right. The equality is immediate from the definition of right max eigenpair of $P(\lambda)$ and $S_P(\lambda)$ respectively. Similarly for the left max eigenpair, take $\lambda = \tau$ and multiply the above equation by y^T from the left. Using the definition of left max eigenpair of $P(\lambda)$ and $S_P(\lambda)$ respectively, we can verify the equality. □

Using the equality of max eigenpairs of $P(\lambda)$ and $S_P(\lambda)$, Theorem 3.1.5 can directly be extended to the max algebra as follows.

Corollary 3.4.2. *Let $P(\lambda)$ given in (3.8) be a max matrix polynomial. Assume that μ is positive. Further, let C_P be given in (3.11) and $S_P(\lambda)$ be given in (3.24). Then, $\mu = \mu(C_P) = \mu(S_P(\mu))$.*

In particular, we denote $S_P(1)$ by S where $S \in \mathbb{R}_+^{n\times n}$. We explore the relationship between the largest max eigenvalues of the max matrix polynomial (3.8) and

$$S = A_0 \oplus A_1 \oplus \cdots \oplus A_{m-1}. \tag{3.27}$$

We next present a number of results relating μ and $\mu(S)$ that are similar to those given in Proposition 3.1.4 for matrix polynomials over the conventional algebra. For the remainder of this section, given a set of coefficient matrices $A_0, \ldots, A_{m-1}$ associated with $P(\lambda)$, we write

$$\Psi = \{A_0, A_1, ..., A_{m-1}\}. \tag{3.28}$$

Let $M(\Psi)$ denote the *multigraph* associated with the set Ψ. The maximal cycle geometric mean in $M(\Psi)$ is denoted by $\mu(M(\Psi))$ analogously to the case of a simple graph. Critical cycles are defined for $M(\Psi)$ in the obvious manner.

Lemma 3.4.3. *Let $\mu(M(\Psi))$ denote the maximal cycle geometric mean of the multigraph associated with Ψ in (3.28) and let $\mu(S)$ be the maximal cycle geometric mean of $D(S)$ associated with S in (3.27). Then $\mu(M(\Psi)) = \mu(S)$.*

Proof: First it is immediate that any cycle in $D(S)$ is also a cycle in $M(\Psi)$. This implies that $\mu(S) \leq \mu(M(\Psi))$. On the other hand, if Γ_M is a critical cycle in $M(\Psi)$ with product $a_{i_1i_2}^{p_1}a_{i_2i_3}^{p_2}\ldots a_{i_ki_1}^{p_k}$, it is clear that $(i_1, i_2, \ldots, i_k = i_1)$ is also a cycle in $D(S)$. Moreover, from the definition of S,

$$s_{i_1i_2}s_{i_2i_3}\cdots s_{i_ki_1} \geq a_{i_1i_2}^{p_1}a_{i_2i_3}^{p_2}\ldots a_{i_ki_1}^{p_k}.$$

This implies that $\mu(S) \geq \mu(M(\Psi))$. Hence $\mu(S) = \mu(M(\Psi))$ as claimed. $\square$

Before proceeding, note that the argument used above also shows that $\mu(M(\Psi)) = 0$ if and only if $\mu(S) = 0$.

The following result plays a central role in the proof of the main result of this section. It shows that there is a one to one correspondence between cycles in the multigraph $M(\Psi)$ and cycles in the directed graph $D(C_P)$. In the proof of this result we write $c_{i,j}$ for the $(i,j)^{\text{th}}$ entry of C_P.

Lemma 3.4.4. *Let Γ_M be a cycle in the multigraph $M(\Psi)$ with cycle product $\pi(\Gamma_M)$ and length j. Then there exists a cycle Γ_C in $D(C_P)$ of length $k \geq j$ such that $\pi(\Gamma_C) = \pi(\Gamma_M)$. Conversely, for every cycle Γ_C in $D(C_P)$ of length k, there exists a cycle Γ_M of length j in $M(\Psi)$ with cycle product $\pi(\Gamma_M) = \pi(\Gamma_C)$ and length $j \leq k$.*

Proof: Let Γ_M be a cycle in $M(\Psi)$ with product

$$\pi(\Gamma_M) = a^{p_1}_{i_1 i_2} a^{p_2}_{i_2 i_3} \dots a^{p_j}_{i_j i_1}.$$

Note that for $1 \leq s \leq j$, the entry $a^{p_s}_{i_s i_{s+1}}$ corresponds to the entry in the companion matrix C_P given by $c_{(m-1)n+i_s, p_s n+i_{s+1}}$. Now note that the form of C_P means that for any p with $0 \leq p < m-1$, and any i with $1 \leq i \leq n$, there exists a simple path in $D(C_P)$ from the vertex $pn+i$ to $(m-1)n+i$. Further, all the entries of C_P used to construct this path are equal to one. It follows immediately from this that there exists a cycle Γ_C in $D(C_P)$ whose product is equal to $\pi(\Gamma_M)$ but whose length k may be greater than j (as extra edges of weight 1 may have been added to define the cycle in $D(C_P)$).

For the converse, note that any cycle Γ_C of length k in $D(C_P)$ must contain at least one vertex corresponding to an index i with $(m-1)n+1 \leq i \leq mn$ (an index from the bottom n rows of C_P). Suppose the product $\pi(\Gamma_C)$ contains j terms from the bottom n rows of C_P and is given by

$$c_{(m-1)n+i_1, p_1 n+i_2} c_{(m-1)n+i_2, p_2 n+i_3} \cdots c_{(m-1)n+i_j, p_j n+i_1}$$

(where we have omitted terms equal to one from the product).

Then the cycle Γ_M in $M(\Psi)$ consisting of the vertices $i_1, \dots, i_j, i_{j+1} = i_1$ and the edges with weights

$$a^{p_1}_{i_1 i_2}, a^{p_2}_{i_2 i_3}, \dots, a^{p_j}_{i_j i_1}$$

has length j with $j \leq k$ and moreover, it is immediate that $\pi(\Gamma_M) = \pi(\Gamma_C)$. □

Again, note that the above argument shows that $\mu(M(\Psi)) = 0$ if and only if $\mu = 0$. Hence, from Lemma 3.4.3, $\mu(S) = 0$ if and only if $\mu = 0$. As all of the following results are trivial in the case where $\mu = \mu(S) = 0$, we henceforth assume that $\mu \neq 0$.

Corollary 3.4.5. *Let μ denote the largest max eigenvalue of the max matrix polynomial given by (3.8) and let $\mu(S)$ denote the largest max eigenvalue of the matrix S given by (3.27). Then, there exist integers j_1, j_2, k_1, k_2 with $0 < j_1 \leq k_1$, $0 < j_2 \leq k_2$ such that*

$$\mu(S)^{j_1/k_1} \leq \mu \leq \mu(S)^{j_2/k_2}. \tag{3.29}$$

Proof: First let Γ_M be a critical cycle in $M(\Psi)$ of length j_1. Then the product of Γ_M is given by $\mu(M(\Psi))^{j_1}$. From Lemma 3.4.4 there is a corresponding cycle, not necessarily critical, Γ_C in $D(C_P)$ of length $k_1 \geq j_1$ with the same cycle product. It follows from the definition of μ that $\mu(M(\Psi))^{j_1/k_1} \leq \mu$.

On the other hand, let Γ_C be a critical cycle in $D(C_P)$ of length k_2. Then as above the cycle product of Γ_C is μ^{k_2} and there exists a (not necessarily critical) cycle in $M(\Psi)$ of length $j_2 \leq k_2$ with the same cycle product. This implies that

$$\mu^{k_2/j_2} \leq \mu(M(\Psi)).$$

Rearranging this, we see that

$$\mu \leq \mu(M(\Psi))^{j_2/k_2}.$$

As $\mu(M(\Psi)) = \mu(S)$ from Lemma 3.4.3 the result follows. $\square$

Next, we present a numerical example to illustrate the result in Lemma 3.4.4.

Example 3.4.1. Let $P(\lambda)$ be given by

$$P(\lambda) = \begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 2 \\ 4 & 2 \end{bmatrix} \lambda \oplus \begin{bmatrix} 0 & 1 \\ 3 & 5 \end{bmatrix} \lambda^2$$

Then, the corresponding companion matrix and S are as follows.

$$C_P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 3 & 0 & 2 & 0 & 1 \\ 0 & 1 & 4 & 2 & 3 & 5 \end{bmatrix}, S = \begin{bmatrix} 0 & 3 \\ 4 & 5 \end{bmatrix}$$

Consider the cycle Γ_M in $M(\Psi)$ whose product is $\pi(\Gamma_M) = a_{12}^0 a_{21}^1 = s_{12}s_{21} = 12$ with $l(\Gamma_M) = 2$. We get the following cycle in $D(C_P)$:

- $\Gamma_C = (5, 2, 4, 6, 3, 5)$ with $\pi(\Gamma_C) = c_{52}c_{24}c_{46}c_{63}c_{35} = 12$ and $l(\Gamma_1) = 5$

Next, consider the cycle Γ_C in $D(C_P)$ whose product is $\pi(\Gamma_C) = c_{54}c_{46}c_{65} = 6$ with $l(\Gamma_C) = 3$. Then, we get the following cycle in $M(\Psi)$:

- $\Gamma_M = (1, 2, 1)$ with $\pi(\Gamma_C) = a_{12}^1 a_{21}^2 = 6$ and $l(\Gamma_1) = 2$

We are now able to state the main result of this section, which provides a max algebra version of Proposition 3.1.4.

Theorem 3.4.6. *Let $P(\lambda)$ be the max matrix polynomial in (3.8) and S be given in (3.27). Further, μ is the largest max eigenvalue of $P(\lambda)$ and $\mu(S)$ is the largest max eigenvalue of S. Then, the following hold.*

(i) $\mu(S) < 1$ *if and only if* $\mu < 1$*;*

(ii) $\mu(S) > 1$ *if and only if* $\mu > 1$*;*

(iii) $\mu(S) = 1$ *if and only if* $\mu = 1$*.*

Proof: This result follows immediately from the identity

$$\mu(S)^{j_1/k_1} \leq \mu \leq \mu(S)^{j_2/k_2}$$

established in Corollary 3.4.5. $\square$

The following Corollary is obtained immediately from Corollary 3.4.5.

Corollary 3.4.7. *Let μ denote the largest max eigenvalue of the max matrix polynomial given by (3.8) and let $\mu(S)$ denote the largest max eigenvalue of the matrix S given by (3.27).*

$$\begin{aligned} &\mu(S) > 1 \Rightarrow \mu \leq \mu(S) \\ &\mu(S) < 1 \Rightarrow \mu \geq \mu(S) \\ &\mu(S) = 1 \Rightarrow \mu = \mu(S) \end{aligned} \tag{3.30}$$

Theorem 3.4.6 shows that $\mu = \mu(S)$ when $\mu(S) = 1$. In the next result, we give a necessary condition for $\mu = \mu(S)$ when $\mu(S) \neq 1$.

Corollary 3.4.8. *Let μ denote the largest max eigenvalue of the max matrix polynomial given by (3.8) and let $\mu(S)$ denote the largest max eigenvalue of the matrix S given by (3.27). If $\mu(S) \neq 1$ and $\mu = \mu(S)$, then $\mu = \mu(A_{m-1})$.*

Proof: Consider $\mu = \mu(S)$.

Case 1: Let $\mu(S) > 1$. Using Corollary 3.4.5, we have $\mu \leq \mu^{j_2/k_2} \Rightarrow \mu^{1-j_2/k_2} \leq 1$. This is only possible when $j_2 = k_2$. This means that there is some critical cycle in $D(C_P)$ whose product only contains terms from the last n rows of C_P. This immediately implies that all the terms in this product are in A_{m-1}, so in this case $\mu = \mu(A_{m-1})$.

Case 2: Let $\mu(S) < 1$. As above, using Corollary 3.4.5, we have $\mu^{j_1/k_1} \leq \mu \Rightarrow 1 \leq \mu^{1-j_1/k_1}$. This is only possible when $j_1 = k_1$. As in Case 1 this implies that $\mu = \mu(A_{m-1})$ $\square$

As a final point, we note that the converse of the previous result does not hold. Specifically, the example below has $\mu = \mu(A_{m-1})$ with $m = 2$ but $\mu \neq \mu(S)$.

Example 3.4.2.

$$C_P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0.2 & 1 & 0.1 & 1 & 0.5 & 3 \\ 2 & 1 & 0.2 & 1.5 & 0.1 & 1 \\ 0.3 & 2 & 2 & 2 & 5 & 0.6 \end{bmatrix} \Rightarrow \mu = 2.8231$$

$$A_1 = \begin{bmatrix} 1 & 0.5 & 3 \\ 1.5 & 0.1 & 1 \\ 2 & 5 & 0.6 \end{bmatrix} \Rightarrow \mu(A_1) = 2.8231$$

$$S = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 2 & 5 & 2 \end{bmatrix} \Rightarrow \mu(S) = 3.1072$$

3.5 Concluding Remarks

Our main goal in this chapter was to extend the spectral theory of non-negative matrix polynomials to the max algebra. In this context,

- we extended results on Perron polynomials to the max algebra;
- we derived convergence results for the solution of multi-step difference equations over the max algebra;
- we proved a number of results giving inequalities for the largest max eigenvalue of a max matrix polynomial.

CHAPTER 4

Asymptotic Stability in the Max Algebra

In this chapter, we introduce the class of P_{max}-matrices for the max algebra and derive some properties of these that echo similar results for P-matrices in the conventional algebra. We obtain results elucidating the relationship between the P_{max}-property, S_{max}-property and the stability of delayed difference equations. Moreover, we define P_{max}-matrix sets, the row-P_{max}-property and the S_{max}-property of a finite set of non-negative matrices. We describe a number of equivalent results for P_{max}-matrix sets and relate these concepts to stability questions for sets of matrices and discrete inclusions with delay.

4.1 Motivation and Mathematical Background

In this section, we are concerned with P-matrices over the conventional algebra. We define the row-P-property and S-property of sets. We briefly recall some results on the stability of positive switched dynamical systems and discrete linear inclusions.

4.1.1 Introduction

A principal submatrix of order i of an $n \times n$ matrix A is a matrix formed by deleting $n - i$ rows and the corresponding $n - i$ columns (with the same indices) of A ($i \in \{1, 2, ..., n\}$). *A principal minor of order i* of A is the determinant of a principal submatrix of order i of A. $A \in \mathbb{R}^{n \times n}$ is said to be a *P-matrix* (*P_0-matrix*) if the principal minors of order i are positive (non-negative) for $1 \leq i \leq n$. In other words, A is a P-matrix (P_0-matrix) if all of its principal submatrices have positive (non-negative) determinant. P-matrices were introduced by Fiedler and Pták in [FP62].

For a complex number λ, we adopt the notation $Re(\lambda)$ for the real part of λ. Then, we say that A is *positive-stable* if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ [HJ90]. The results to be presented in this section relate most directly to characterisations of P-matrices in terms of matrix stability within the class of so-called Z-matrices. A is a *Z-matrix* if $a_{ij} \leq 0$ for all $i, j (i \neq j)$. It is well-known that for a Z-matrix A, the following conditions are equivalent [FP62, HJ90, BP94].

Theorem 4.1.1. *Let $A \in \mathbb{R}^{n \times n}$ be a Z-matrix. Then, the following are equivalent.*

(i) A is a P-matrix;

(ii) A is positive-stable;

(iii) For every non-zero $x \in \mathbb{R}^n$ there is some i with $x_i(Ax)_i > 0$;

(iv) Every principal submatrix of A is positive-stable;

(v) There exists some $v > 0$ with $Av > 0$.

Property (i) above is usually referred to as the *P-property* while Property (v) is referred to as the *S-property* of the matrix [FP62, FP66]. A Z-matrix with Property (ii) is said to be an *M-matrix* [HJ90].

The class of P-matrices has been extensively studied due to its importance in fields such as statistics, optimisation and dynamical systems [Par83, HS98, Sou06, CPS09]. The relevance of such matrices to the linear complementarity problem is well documented and details can be found in [CPS09]. P-matrices are also intimately connected with the stability theory of positive linear systems, with the long-term behaviour of Lotka-Volterra systems in ecological modelling [HS98] and with chemical reaction systems [BDB07]. Yet another context in which P-matrices play a role is in the study of globally univalent functions, motivated by applications in Economics and Biology [Par83, Sou06].

Results on asymptotic stability of a non-negative matrix in the max algebra were discussed in [Lur05]. The analyses in this work was then extended to sets of non-negative matrices in [Lur06, Pep08]. Here, a max algebra version of the generalised spectral radius for sets of matrices was defined and results were presented in the context of stability and convergence properties of discrete linear inclusions. More recently, the class of Z and M-matrices were considered in connection with the solution of matrix equations over the max algebra [BSS12].

Inspired by Song, Gowda and Ravindran [SGR99], we shall be concerned with extending results concerning P-properties of single matrices and sets of matrices to the setting of the max algebra. In this direction, the layout of this chapter is as follows. First, we briefly give the main results over the conventional algebra. In Section 4.2, we show that equivalences analogous to (i) - (v) in Theorem 4.1.1 also hold in the max algebra. Moreover, we explore the connection between matrix stability in the max algebra and max-algebraic dynamical systems. In Section 4.4, we extend the results for sets of matrices in [SGR99] to the max algebra. The work contained in this chapter has resulted in the publication: [BGM11a].

4.1.2 P-matrix Sets, Row-P-property and S-property

Song, Gowda and Ravindran extended the P-property and S-property of a single matrix to sets of matrices in [SGR99]. Specifically, they introduced the *row-P*-property and showed the relation between the *row-P*-property and S-property of a set.

Throughout this chapter, Ψ denotes a set of non-negative matrices in $\mathbb{R}_+^{n\times n}$. We recall the notation from (3.28) that

$$\Psi = \{A_1, A_2, ..., A_m\} \tag{4.1}$$

where each $A_i \geq 0$ and $A_i \neq 0$ for some $i \in \{1, 2, ..., m\}$.

Following the notation in [SGR99] we define the *row representative set* of Ψ as follows

$$\hat{\Psi} = \{M \in \mathbb{R}_+^{n\times n} \mid \text{for } 1 \leq j \leq n \text{ there exists } A_{i_j} \in \Psi \text{ with } M_{j.} = (A_{i_j})_{j.}\} \tag{4.2}$$

where $M_{j.}$ denotes the j^{th} row of M. Briefly, the matrices $M \in \hat{\Psi}$ are formed by choosing corresponding rows from some $A_{i_j} \in \Psi$ where $1 \leq i_j \leq m$. This is illustrated by the following simple example.

Example 4.1.1. Consider the matrices in $\mathbb{R}_+^{2\times 2}$ given by

$$A_1 = \begin{bmatrix} 1/2 & 1/3 \\ 4/5 & 1/4 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2/3 & 0 \\ 1 & 5/8 \end{bmatrix}.$$

For $\Psi = \{A_1, A_2\}$, it is easy to see that $\hat{\Psi} = \{A_1, A_2, M_1, M_2\}$ where

$$M_1 = \begin{bmatrix} 1/2 & 1/3 \\ 1 & 5/8 \end{bmatrix} \quad M_2 = \begin{bmatrix} 2/3 & 0 \\ 4/5 & 1/4 \end{bmatrix}.$$

Note that the results of Song, Gowda and Ravindran [SGR99] are for a set of general $n \times n$ matrices. However, we recall the following definitions for Ψ given in (4.1). The row-P-property and S-property of Ψ are described as follows [SGR99].

(i) Ψ has the *row-P-property* (*row-P_0-property*) if every matrix in $\hat{\Psi}$ is a P-matrix (P_0-matrix);

(ii) Ψ has the *S-property* if there is $v > 0$ such that $Av > 0$ for all $A \in \Psi$.

Note that $\Psi \subset \hat{\Psi}$. Thus, if Ψ has the row-P-property (row-P_0-property), each matrix in Ψ is automatically a P-matrix (P_0-matrix). In this direction Ψ is called a *P-matrix set* (*P_0-matrix set*) [SGR99].

The following result demonstrates that Theorem 4.1.1 (iii) holds uniformly for all matrices in Ψ [SGR99].

Theorem 4.1.2. *Let $\Psi \subset \mathbb{R}_+^{n\times n}$ be given in (4.1). Then, Ψ has the row-P-property if and only if for every non-zero $x \in \mathbb{R}_+^n$ there is some $i \in \{1, 2, ..., n\}$ with $x_i(Ax)_i > 0$ for all $A \in \Psi$.*

The following result states the equivalence of the row-P-property and S-property for a compact set of Z-matrices [SGR99].

Theorem 4.1.3. *Let $\Psi \subset \mathbb{R}_+^{n\times n}$ be given in (4.1) and be a set of Z-matrices. Then, the following are equivalent.*

(i) *Ψ has the row-P-property;*

(ii) *There exists some $v > 0$ such that $Av > 0$ for all $A \in \Psi$.*

4.1.3 On the Stability of Positive Switched Linear Systems

The results on P-matrix sets in [SGR99] echo similar results on the existence of a common linear copositive Lyapunov function in the context of switched dynamical systems.

Let a continuous-time linear system be given by

$$\dot{x}(t) = Ax(t), x(0) = x_0, 0 \leq t < \infty \tag{4.3}$$

where $x \in \mathbb{R}^n$ is called the state vector and $A \in \mathbb{R}^{n\times n}$ is referred as the system matrix. (4.3) is said to be a *positive system* if $x(t) \in \mathbb{R}_+^n$ for all $t \geq 0$ for any $x_0 \in \mathbb{R}_+^n$. It follows from [FR00] that (4.3) is positive if and only if A is a *Metzler matrix* , i.e., $a_{ij} \geq 0$ for all i, j when $i \neq j$. It is well known that the asymptotic stability of the positive system (4.3) is characterised by whether or not the matrix A is *Hurwitz* (meaning that all of its eigenvalues lie in the open left half plane) [GSM07, MS07, KMS09].

A positive switched linear system is formed by a set of continuous-time positive linear systems and a switching mechanism that arbitrarily switches between them. More generally, one can consider positive differential inclusions of the form

$$\dot{x}(t) \in \{A(t)x(t), A(t) \in \{A_1, A_2, ..., A_m\}\}, x(0) = x_0, 0 \leq t < \infty \tag{4.4}$$

where A_i is a Metzler matrix for $1 \leq i \leq m$. It is well known that Lyapunov theory is a powerful tool for stability analysis of the systems of this type. In particular, the authors of [MS07] studied the existence of common linear copositive Lyapunov functions for a pair of continuous-time positive linear systems. Their results were then extended to a finite set of continuous-time positive linear systems in [KMS09]. Following [MS07], the function $V(x) = v^T x$ is a *common linear copositive Lyapunov function* for (4.4) if and only if $v \in \mathbb{R}^n$ satisfies the following

(i) $v > 0$;

(ii) $A_i^T v < 0$ for all $1 \leq i \leq m$.

The existence of such a Lyapunov function is a sufficient condition for the stability of (4.4). The following result, which is essentially a special case of Theorem 4.1.3, defines an equivalent condition for its existence [KMS09].

Theorem 4.1.4. *Let $A_1, A_2, ..., A_m \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz. Then, the following are equivalent.*

(i) Any matrix in the row representative set of $\{A_1^T, A_2^T, ..., A_m^T\}$ is Hurwitz;

(ii) There exists some $v > 0$ such that $A_i^T v < 0$ for all $1 \leq i \leq m$.

4.1.4 Generalised Spectral Radius

The generalised spectral radius plays a key role when extending the results in [SGR99] to the max algebra.

Two different concepts have been proposed to generalise the spectral radius to sets of matrices: *the generalised spectral radius* [DL92] and *the joint spectral radius* [RS60]. They are respectively defined for a bounded set of $n\times n$ complex matrices (there exists an upper bound on the norms of the matrices in the set) denoted by $\sum$ as follows.

$$\rho(\textstyle\sum) = \lim_{k\to\infty} \sup \left(\sup_{A\in\sum^k} \rho(A) \right)^{\frac{1}{k}} \tag{4.5}$$

$$\hat{\rho}(\textstyle\sum) = \lim_{k\to\infty} \left(\sup_{A\in\sum^k} ||A|| \right)^{\frac{1}{k}} \tag{4.6}$$

Here, $\sum^k$ denotes the set of all products of matrices from $\sum$ of length $k \geq 1$ and $||.||$ is any matrix norm on $\mathbb{C}^{n\times n}$. Note that (4.5) and (4.6) are generalisation of the fundamental formulae relating the spectral radius of an $n \times n$ matrix in (2.8). It has been proven by [BW92] that $\rho(\sum) = \hat{\rho}(\sum)$ for any bounded set $\sum$. These concepts have been studied by several authors in [BW92, HS95, LW95, Gur95, Wir02]. They have various applications in wavelet theory [HS95] and discrete inclusions [Wir02]. In particular, the role of the generalised spectral radius in the stability analysis of discrete inclusions was investigated in [Gur95, Wir02]. Following the notation in these papers, consider the discrete linear inclusion of the form

$$x(k+1) \in \{A(k)x(k), A(k) \in \sum\}, x(0) = x_0, k = 0, 1, ... \tag{4.7}$$

A sequence $\{x(k)\}_{k\geq 0}$ is said to be a solution of (4.7) starting with an initial condition $x(0) = x_0$ if for all $k \geq 0$ there exists some $A(k) \in \sum$ such that $x(k+1) = A(k)x(k)$. Then, we obtain $x(k) = A(k-1)A(k-2)...A(0)x_0$ for some k. The convergence of the solution to the origin can be characterised as follows: $\lim\limits_{k\to\infty} A(0)A(1)...A(k-1) = 0$ for all $k \geq 0$ if and only if $\rho(\sum) < 1$ [Gur95, Wir02].

A max algebra version of the generalised spectral radius, which plays a central role in determining stability and convergence properties of discrete linear inclusions and nonhomogeneous matrix products over the max algebra was introduced in [Lur06]. Subsequent work showing the connection between the max version of the generalised spectral radius and the conventional spectral radius of Hadamard powers was presented in [Pep08].

Let Ψ be given by (4.1). Then, the max version of the generalised and joint spectral radius of Ψ are respectively given by

$$\mu(\Psi) = \lim_{k\to\infty} \sup \left(\max_{A\in\Psi^k_\otimes} \mu(A) \right)^{\frac{1}{k}} \tag{4.8}$$

$$\hat{\mu}(\Psi) = \lim_{k\to\infty} \left(\max_{A\in\Psi^k_\otimes} \eta_{||.||}(A) \right)^{\frac{1}{k}} \tag{4.9}$$

where $\Psi^k_\otimes$ denotes the set of all products of matrices from Ψ of length $k \geq 1$ in the max algebra. Formally,

$$\Psi^k_\otimes := \{A_{j_1} \otimes \cdots \otimes A_{j_k} : j_i \in \{1, 2, ..., m\} \text{ for } 1 \leq i \leq k\}. \tag{4.10}$$

Since all vector norms are equivalent on a finite dimensional space, $\hat{\mu}(\Psi) = \lim_{k\to\infty}\left(\max_{A\in\Psi_{\otimes}^{k}} ||A||\right)^{\frac{1}{k}}$ for any matrix norm $||.||$ [Lur05]. Note that (4.8) and (4.9) are generalisations of the formulae relating the largest max eigenvalue of an $n\times n$ non-negative matrix in (2.15). It has been shown in [Lur06] that $\mu(\Psi) = \hat{\mu}(\Psi)$ is true for the finite set Ψ. Moreover, some of the inequalities for the largest max eigenvalue discussed in Section 2.2.2 can be extended to the max generalised spectral radius. For instance, $\mu(\Psi) \le \rho(\Psi) \le n\mu(\Psi)$ [Lur06]. The following result shows the connection between the max generalised spectral radius and the asymptotic behaviour of products of matrices from Ψ [Lur06].

Theorem 4.1.5. *Let Ψ be given by (4.1). Then, the following are equivalent.*

(i) $\lim_{k\to\infty} A_{j_1}\otimes\cdots\otimes A_{j_k} = 0$ *where* $j_i \in \{1, 2, ..., m\}$ *for* $1 \le i \le k$*;*

(ii) $\mu(\Psi) < 1$.

4.2 The class of P_{max}-matrices

In this section, we define the class of P_{max}-matrices. Further, we demonstrate the relationship between these matrices and the stability properties of matrices and difference equations in the max algebra. The results presented here echo similar facts presented in Theorem 4.1.1 for the conventional algebra.

We deal with the permanent of a matrix in order to form the analogue of P-matrices since the notion of a determinant does not directly extend to the max algebra because of the minus sign [Bap95, But03]. Note that the max version of the permanent plays an important role in the linear assignment problem [BB03].

Let S_n denote the set of all permutations of the numbers $1, 2, ..., n$ and σ be a permutation in S_n. Formally, the max permanent is given by

$$per_{\max}(A) = \max_{\sigma\in S_n} \bigotimes_{i=1}^{n} a_{i,\sigma(i)}. \tag{4.11}$$

Simply, for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}_{+}^{2\times 2}$, $per_{\max}(A) = \max(a_{11}a_{22}, a_{12}a_{21})$.

$A \in \mathbb{R}_+^{n\times n}$ is said to be a P_{max}-*matrix* if $per_{\max}(B) < 1$ for all principal submatrices B of A. We next relate the P_{max}-matrices with the matrix stability over the max algebra. The specific notion of matrix stability considered here is that explored in [Lur05] and corresponds to asymptotic stability of the discrete-time system

$$x(k+1) = A \otimes x(k), x(0) = x_0, k = 0, 1, ... \tag{4.12}$$

We say that (4.12) is asymptotically stable if all solutions $x(k)$ converge to zero as k tends to ∞. As with discrete-time systems in the conventional algebra, the largest max eigenvalue is intimately related to the asymptotic stability of (4.12). As stated in Theorem 2.2.4, asymptotic stability is equivalent to $\mu(A) < 1$.

The following theorem presents some equivalent conditions for $A \in \mathbb{R}_+^{n\times n}$ to be a P_{max}-matrix.

Theorem 4.2.1. *Let $A \in \mathbb{R}_+^{n\times n}$. Then, the following are equivalent:*

(i) A is a P_{max}-matrix;

(ii) A is asymptotically stable, that is, $\mu(A) < 1$;

(iii) For each $x \neq 0$ in $\mathbb{R}_+^n$, there exists an $i \in \{1, 2, ..., n\}$ such that $(A\otimes x)_i < x_i$;

(iv) For all principal submatrices B of A, $\mu(B) < 1$;

(v) There exists a vector $v > 0$ such that $A \otimes v < v$.

Proof:

(i) $\iff$ (ii) Assume that we have $per_{\max}(B) < 1$ for all principal submatrices B of A. Let $(i_1, i_2, ..., i_k, i_1)$ be a critical cycle in $D(A)$. (If there is no cycle in $D(A)$ then $\mu(A) = 0$ and we are done.) Further, let $B \in \mathbb{R}_+^{k\times k}$ be the principal submatrix of A corresponding to $i_1, i_2, ..., i_k$. Then we have

$$a_{i_1i_2}a_{i_2i_3}...a_{i_ki_1} = b_{1,\sigma(1)}b_{2,\sigma(2)}...b_{k,\sigma(k)} \leq per_{\max}(B)$$

for some permutation $\sigma \in S_k$. It follows immediately that $\mu(A) < 1$.

For the converse, assume $\mu(A) < 1$. So, all cycle products of any length in $D(A)$ are less than 1. Let a principal submatrix $B \in \mathbb{R}_+^{k\times k}$ of A be given with $per_{\max}(B)$ equal to $b_{i_1,\sigma(i_1)}b_{i_2,\sigma(i_2)}...b_{i_k,\sigma(i_k)}$ for some $1 \leq i_1, i_2, ..., i_k \leq n$. Since $\sigma \in S_k$ is a permutation and can be written as a product of cyclic permutations, it follows that $per_{\max}(B)$ can be decomposed into cycle products. It is immediate that $per_{\max}(B) < 1$.

(ii) $\Longleftrightarrow$ (iii) Let $\mu(A) < 1$. Suppose that there exists $x \neq 0$ in $\mathbb{R}_+^n$ such that $(A \otimes x)_i \geq x_i$ for each $i \in \{1, 2, ..., n\}$. Then $A \otimes x \geq x$. This implies that $A_\otimes^k \otimes x \geq x$ for some $x \neq 0$ in $\mathbb{R}_+^n$. Thus, as $k \to \infty$, the k^{th} power of A doesn't converge to zero which contradicts $\mu(A) < 1$ (Theorem 2.2.4).

Conversely, assume (iii) and let $i_1, i_2, ..., i_k, i_{k+1} = i_1$ be a cycle of length k with the cycle product $a_{i_1i_2}a_{i_2i_3}...a_{i_ki_1}$ in $D(A)$ for $i_1, i_2, ..., i_k \in \{1, 2, ..., n\}$. (If $D(A)$ contains no cycles, then $\mu(A) = 0$ and we are done.) Define $x \in \mathbb{R}_+^n$ as follows:

$$\begin{aligned} x_{i_2} &= 1 \\ x_{i_j} &= \frac{x_{i_{j-1}}}{a_{i_{j-1}i_j}}, j = 3, ..., k \\ x_{i_1} &= \frac{x_{i_k}}{a_{i_ki_1}} \\ x_p &= 0, p \neq \{i_1, i_2, ..., i_k\}. \end{aligned}$$

By assumption there exists some index i with $(A \otimes x)_i < x_i$. Clearly i must be in $\{i_1, i_2, ..., i_k\}$. Consider the following two cases.

- $i = i_1 \Rightarrow a_{i_1i_1}x_{i_1} \oplus a_{i_1i_2}x_{i_2} \oplus ... \oplus a_{i_1i_k}x_{i_k} < x_{i_1}$. Since $x_{i_1} = \frac{x_{i_k}}{a_{i_ki_1}} \neq 0$, it easily follows from the second term in the left side that $a_{i_1i_2}x_{i_2} < x_{i_1}$. Hence,

$$a_{i_1i_2}x_{i_2} < \frac{x_{i_k}}{a_{i_ki_1}} = \frac{x_{i_2}}{a_{i_2i_3}a_{i_3i_4}...a_{i_ki_1}} \Rightarrow a_{i_1i_2}a_{i_2i_3}...a_{i_ki_1} < 1.$$

- $i = i_j (1 < j \leq k) \Rightarrow a_{i_ji_1}x_{i_1} \oplus a_{i_ji_2}x_{i_2} \oplus ... \oplus a_{i_ji_k}x_{i_k} < x_{i_j}$. Similarly, it follows from the $(j+1)^{\text{th}}$ term that

$$a_{i_ji_{j+1}}x_{i_{j+1}} < x_{i_j} \Rightarrow a_{i_ji_{j+1}}\frac{x_{i_j}}{a_{i_ji_{j+1}}} < x_{i_j} \Rightarrow 1 < 1.$$

The second condition is not possible. As a result, we have $a_{i_1i_2}a_{i_2i_3}...a_{i_ki_1} < 1$. As this is true for any cycle in $D(A)$, it follows that $\mu(A) < 1$.

(ii) $\Longleftrightarrow$ (iv) First, let $\mu(A) < 1$. Then, all cycle products in $D(A)$ are less than one. Let a principal submatrix B^* of A be given and let Γ be a critical cycle in $D(B^*)$. Since Γ also defines a cycle in $D(A)$, $\pi(\Gamma) < 1$. As Γ was arbitrary, $\mu(B^*) < 1$.

For the converse, let $\mu(B) < 1$ for all principal submatrices B of A. Since A is a principal submatrix of order n, it is immediate that $\mu(A) < 1$.

(ii) $\Longleftrightarrow$ (v) First, suppose $\mu(A) < 1$. Let $\mathbf{1}_n \in \mathbb{R}^n_+$ denote the vector of all ones. We can choose $\epsilon > 0$ so that $\mu(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T) < 1$. Since $A + \epsilon\mathbf{1}_n\mathbf{1}_n^T$ is an irreducible matrix, it follows from Theorem 2.2.2 on the max version of the Perron-Frobenius theorem that there is some $v > 0$ with $(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T) \otimes v = \mu(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T)v < v$. It follows immediately that

$$A \otimes v \leq (A + \epsilon\mathbf{1}_n\mathbf{1}_n^T) \otimes v < v.$$

For the converse, assume that there exists $v > 0$ satisfying $A \otimes v < v$. As above, choose $\epsilon > 0$ so that

$$(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T) \otimes v < v.$$

As $(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T)$ is irreducible, $A + \epsilon\mathbf{1}_n\mathbf{1}_n^T$ has a positive left max eigenvector $w > 0$. Multiplying both sides of the above equation with w^T from the left, we see that

$$w^T \otimes (A + \epsilon\mathbf{1}_n\mathbf{1}_n^T) \otimes v < w^T \otimes v.$$

Since w is the left max eigenvector of $A + \epsilon\mathbf{1}_n\mathbf{1}_n^T$, it follows that $\mu(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T)w^T \otimes v < w^T \otimes v$.

But $w^T \otimes v > 0$ which implies directly that

$$\mu(A) \leq \mu(A + \epsilon\mathbf{1}_n\mathbf{1}_n^T) < 1.$$

This completes the proof. $\square$

We call Property (i) in Theorem 4.2.1 the *P_{max}-property* and Property (v) the *S_{max}-property* of a matrix in the max algebra. The following example illustrates Theorem 4.2.1.

Example 4.2.1. Let a 3×3 matrix be given by

$$A = \begin{bmatrix} 1/2 & 1/3 & 0 \\ 0 & 3/5 & 1/4 \\ 1/5 & 1/6 & 2/3 \end{bmatrix}.$$

There are three first order principal submatrices: $a_{11} = 1/2$, $a_{22} = 3/5$ and $a_{33} = 2/3$ all of which are less than one. Thus, automatically all have max permanent less than one.

There are three second order principal submatrices:

$$A_{12} = \begin{bmatrix} 1/2 & 1/3 \\ 0 & 3/5 \end{bmatrix}, per_{\max}(A_{12}) = 3/10 < 1;$$
$$A_{13} = \begin{bmatrix} 1/2 & 0 \\ 1/5 & 2/3 \end{bmatrix}, per_{\max}(A_{13}) = 1/3 < 1;$$
$$A_{23} = \begin{bmatrix} 3/5 & 1/4 \\ 1/6 & 2/3 \end{bmatrix}, per_{\max}(A_{23}) = 2/5 < 1.$$

The third order principal submatrix is A where $per_{\max}(A) = \max(a_{11}a_{22}a_{33}, a_{11}a_{23}a_{32}, a_{12}a_{21}a_{33}, a_{12}a_{23}a_{31}, a_{13}a_{21}a_{32}, a_{13}a_{22}a_{31}) = 1/5 < 1$.

Hence, A is a P_{max}-matrix. For A, $\mu(A) = 0.667$ and $\mu(B) < 1$ where $B \in \{a_{11}, a_{22}, a_{33}, A_{12}, A_{13}, A_{23}, A\}$. Moreover, for $v = \begin{bmatrix} 1/5 & 1/3 & 1/4 \end{bmatrix}^T$, $A \otimes v < v$.

We are next concerned with the relation of the P_{max}-property to the stability of delayed difference equations over the max algebra. In [HS00], it was shown for the conventional algebra that off-diagonal delays had no effect on the stability of a differential equation if and only if $-A$ is a P-matrix where A is the system matrix. We shall prove a corresponding fact for difference equations in the max algebra without restricting diagonal delays to be zero.

Consider the delayed system of difference equations given by

$$x_i(k+1) = \bigoplus_{j=1}^{n} a_{ij} x_j(k - \tau_{ij}), k \geq 0, i = 1, 2, ..., n \tag{4.13}$$

where $A \in \mathbb{R}_+^{n \times n}$ and $\tau_{ij} \geq 0$ are non-negative integers for all $1 \leq i, j \leq n$.

Theorem 4.2.2. *Consider the system of delayed difference equations (4.13) where $\tau_{ij} \geq 0$ for all i, j. The following are equivalent:*

(i) A is a P_{max}-matrix;

(ii) (4.13) is asymptotically stable for all $\tau_{ij} \geq 0$;

(iii) (4.13) is asymptotically stable for some $\tau_{ij} \geq 0$.

Proof: We shall prove that (i) implies (ii) and that (iii) implies (i). The implication (ii) $\Rightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (ii): Assume that A is a P_{max}-matrix. Define the state vector by $x(k) = \begin{bmatrix} x_1(k) & x_2(k) & \cdots & x_n(k) \end{bmatrix}^T \in \mathbb{R}_+^n$. Let $\tau_{ij} \geq 0$ be any set of non-negative integer delays and suppose that the delays τ_{ij} take values in the set $\{0, 1, ..., \tau_{\max}\}$ for all $1 \leq i, j \leq n$, where $\tau_{\max} = \max\limits_{i,j} \tau_{ij}$.

As all delays are non-negative integers less than or equal to $\tau_{\max}$, we can write the delayed system in (4.13) in the following form

$$x(k+1) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1) \oplus ... \oplus A_{\tau_{\max}} \otimes x(k - \tau_{\max}) \qquad (4.14)$$

where the matrices $A_p (p = 0, 1, ..., \tau_{\max})$ in $\mathbb{R}_+^{n \times n}$ are defined as follows. The $(i, j)^{\text{th}}$ entry of A_p is equal to a_{ij} if $\tau_{ij} = p$ and all other entries of A_p are zero. Note that

$$A = A_0 \oplus A_1 \oplus ... \oplus A_{\tau_{\max}}.$$

By setting $\hat{x}(k) = \begin{bmatrix} x(k - \tau_{\max}) & x(k - \tau_{\max} + 1) & \cdots & x(k) \end{bmatrix}^T \in \mathbb{R}_+^{n(\tau_{\max}+1)}$, we see that the stability of (4.13) is equivalent to the stability of

$$\underbrace{\begin{bmatrix} x(k - \tau_{\max} + 1) \\ x(k - \tau_{\max} + 2) \\ \vdots \\ x(k) \\ x(k+1) \end{bmatrix}}_{\hat{x}(k+1)} = \underbrace{\begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & I \\ A_{\tau_{\max}} & \dots & \dots & \dots & A_1 & A_0 \end{bmatrix}}_{C} \otimes \underbrace{\begin{bmatrix} x(k - \tau_{\max}) \\ x(k - \tau_{\max} + 1) \\ \vdots \\ x(k-1) \\ x(k) \end{bmatrix}}_{\hat{x}(k)}$$

where $C \in \mathbb{R}_+^{n(\tau_{\max}+1) \times n(\tau_{\max}+1)}$ is the companion matrix associated to (4.13).

As A is a P_{max}-matrix it follows from Theorem 4.2.1 that $\mu(A) < 1$. Since $A = A_0 \oplus A_1 \oplus ... \oplus A_{\tau_{\max}}$, it follows from Theorem 3.4.6 that $\mu(C) < 1$. Thus, the system (4.13) is asymptotically stable. As this is true for any delays $\tau_{ij} \geq 0$, we conclude that (ii) holds.

(iii) $\Rightarrow$ (i): Now assume that for some integer values of $\tau_{ij} \geq 0$, the system (4.13) is asymptotically stable. Then we can proceed as above to write the system in the form (4.14). By assumption the companion matrix C associated with the system will have $\mu(C) < 1$. It then follows from Theorem 3.4.6 that $\mu(A) < 1$ and hence that A is a P_{max}-matrix by Theorem 4.2.1.

This completes the proof. $\square$

4.3 The class of P^0_{max}-matrices

In this section, we define the class of P^0_{max}-matrices. Recall that P_0-matrices are introduced in [FP62] in the conventional algebra. They are also described in [HJ90, BP94] in the context of matrix stability.

We say that $A \in \mathbb{R}^{n\times n}_+$ is a P^0_{max}-*matrix* if $per_{\max}(B) \leq 1$ for all principal submatrices B of A. Equivalent conditions for P_{max}-matrices stated in Theorem 4.2.1 can also be stated for P^0_{max}-matrices that echo similar results over the conventional algebra. We next present these conditions in Theorem 4.3.1 below. Notice that Property (iii) in this theorem shows that a P^0_{max}-matrix can be converted into a P_{max}-matrix by a small perturbation.

Theorem 4.3.1. *Let $A \in \mathbb{R}^{n\times n}_+$. Then, the following are equivalent:*

(i) A is a P^0_{max}-matrix;

(ii) $\mu(A) \leq 1$;

(iii) αA is a P_{max}-matrix for all $0 < \alpha < 1$;

(iv) For each $x \neq 0$ in $\mathbb{R}^n_+$, there exists an $i \in \{1, 2, ..., n\}$ such that $(A \otimes x)_i \leq x_i$;

(v) For all principal submatrices B of A, $\mu(B) \leq 1$;

(vi) There exists a vector $v > 0$ such that $A \otimes v \leq v$.

Proof:

(i) $\iff$ (ii) Assume that we have $per_{\max}(B) \leq 1$ for all principal submatrices B of A. Following the same approach in the proof of Theorem 4.2.1, let $(i_1, i_2, ..., i_k, i_1)$ be a critical cycle in $D(A)$ and $B \in \mathbb{R}_+^{k\times k}$ be the principal submatrix of A corresponding to $i_1, i_2, ..., i_k$. Then, $a_{i_1 i_2} a_{i_2 i_3} ... a_{i_k i_1} = b_{1,\sigma(1)} b_{2,\sigma(2)} ... b_{k,\sigma(k)} \leq per_{\max}(B)$ for some permutation $\sigma \in S_k$. We automatically get $\mu(A) \leq 1$.

Next, assume that $\mu(A) \leq 1$. Let $B \in \mathbb{R}_+^{k\times k}$ be a principal submatrix of A where $per_{\max}(B) = b_{i_1,\sigma(i_1)} b_{i_2,\sigma(i_2)} ... b_{i_k,\sigma(i_k)}$ for some $1 \leq i_1, i_2, ..., i_k \leq n$. Similar to the proof of Theorem 4.2.1, $per_{\max}(B)$ can be written as a union of several cycles products in $D(A)$. Since all cycle products of any length in $D(A)$ are less than 1, it follows immediately that $per_{\max}(B) \leq 1$.

(i) $\iff$ (iii) Assume that $A \in \mathbb{R}_+^{n\times n}$ is a P_{max}^0-matrix. Then, $\mu(A) \leq 1$ from (ii). Let α be in $(0, 1)$. Then, $\mu(\alpha A) = \alpha\mu(A) < \mu(A) \leq 1$. Thus, αA is a P_{max}-matrix from (ii) in Theorem 4.2.1.

For the converse, assume that $\alpha A \in \mathbb{R}_+^{n\times n}$ is a P_{max}-matrix for any $\alpha \in (0, 1)$. Similarly, it is immediate from (ii) in Theorem 4.2.1 that $\alpha\mu(A) < 1$. Let α converge to one. Then, we automatically have $\mu(A) \leq 1$. Thus, A is a P_{max}^0-matrix from (ii).

(i) $\iff$ (iv) Assume that $A \in \mathbb{R}_+^{n\times n}$ is a P_{max}^0-matrix. Then, αA is a P_{max}-matrix for all $0 < \alpha < 1$ from (iii). It follows from Theorem 4.2.1 (iii) that for each $x \neq 0$ in $\mathbb{R}_+^n$, there exists an $i \in \{1, 2, ..., n\}$ such that $(\alpha A \otimes x)_i < x_i$. By letting α converge to one, we get $(A \otimes x)_i \leq x_i$.

Next, let for each $x \neq 0$ in $\mathbb{R}_+^n$, there exists an $i \in \{1, 2, ..., n\}$ such that $(A \otimes x)_i \leq x_i$. Let α be in $(0, 1)$. It is straightforward that $(\alpha A \otimes x)_i < (A \otimes x)_i \leq x_i$ implies $(\alpha A \otimes x)_i < x_i$. Then, $\alpha A \in \mathbb{R}_+^{n\times n}$ is a P_{max}-matrix for any $\alpha \in (0, 1)$ from Theorem 4.2.1 (iii). Thus, $A \in \mathbb{R}_+^{n\times n}$ is a P_{max}^0-matrix from (iii).

(i) $\iff$ (v) First, let $A \in \mathbb{R}_+^{n\times n}$ is a P_{max}^0-matrix and $B \in \mathbb{R}_+^{k\times k}$ be a principal submatrix of A for $1 \leq k \leq n$. Then, $\alpha A \in \mathbb{R}_+^{n\times n}$ is a P_{max}-matrix for all $\alpha \in (0, 1)$ from (iii). Moreover, $\alpha B \in \mathbb{R}_+^{k\times k}$ a principal submatrix of αA for $1 \leq k \leq n$ since α is a scalar. It follows from Theorem 4.2.1 (iv) that $\mu(\alpha B) < 1$ for all $k \in \{1, 2, ..., n\}$. By letting α converge to one, we get $\mu(B) \leq 1$ for all principal submatrices B of A.

The converse is immediate.

(i) $\iff$ (vi) Assume that $A \in \mathbb{R}_+^{n\times n}$ is a P_{max}^0-matrix. Then $\mu(A) \leq 1$ from (ii). We choose $\epsilon > 0$ and define a positive matrix $B \in \mathbb{R}_+^{n\times n}$ as follows

$$b_{ij} = \begin{cases} \epsilon & \text{if } a_{ij} = 0, \\ a_{ij} & \text{otherwise.} \end{cases}$$

for all i, j so that $\mu(B) \leq 1$. Since B is irreducible, it follows from Theorem 2.2.2 on the max version of the Perron-Frobenius theorem that there is some $v > 0$ with $B \otimes v = \mu(B)v \leq v$. As $A \leq B$, it follows immediately that $A \otimes v \leq v$.

Now assume that there exists $v > 0$ satisfying $A \otimes v \leq v$. Let α be in $(0, 1)$. Then, $\alpha A \otimes v < A \otimes v \leq v$. Since $\alpha A \otimes v < v$ for some $v > 0$, αA is a P_{max}-matrix from (v) in Theorem 4.2.1 and A is a P_{max}^0-matrix from (iii).
$\square$

4.4 The Row-P_{max}-property and S_{max}-property of Sets of Matrices

In the spirit of the results recalled in Section 4.1.2, we extend the P_{max}-property of a matrix to sets of non-negative matrices. We derive analogous results to the equivalence of (i), (ii), (iii) and (v) established in Theorem 4.2.1. Further, we are concerned with the relation between the row-P_{max}-property for sets of matrices, the S_{max}-property and the stability of discrete inclusions in the max algebra.

Let $\Psi \subset \mathbb{R}_+^{n\times n}$ denote the finite set of $n \times n$ non-negative matrices defined in (4.1) and $\hat{\Psi} \subset \mathbb{R}_+^{n\times n}$ denote the row representative set of Ψ given by (4.2). The following two definitions play a central role in what follows.

(i) Ψ has the *row-P_{max}-property* (*row-P_{max}^0-property*) if every matrix $M \in \hat{\Psi}$ is a P_{max}-matrix (P_{max}^0-matrix).

(ii) Ψ has the *S_{max}-property* if there is $v > 0$ such that $A_i \otimes v < v$ for all $i \in \{1, 2, ..., m\}$.

Following the notation in Section 4.1.2, if Ψ has the row-P_{max}-property (row-P^0_{max}-property), then each $A_i \in \Psi$ is also a P_{max}-matrix (P^0_{max}-matrix). In this case, Ψ is said to be a P_{max}-matrix set (P^0_{max}-matrix set).

In our main result, Theorem 4.4.1 below, we shall present some facts relating P_{max}-matrix sets and the stability of discrete inclusions in the max algebra. The generalised spectral radius for the max algebra will play a key role in what follows. Formally, we consider a max version of the inclusion given in (4.7) as follows:

$$x(k+1) \in \{A_p \otimes x(k), p = 1, 2, ..., m\}, x(0) = x_0, k = 0, 1, ... \tag{4.15}$$

associated with the set of matrices Ψ given in (4.1). We say that (4.15) is asymptotically stable if all solutions $x(k)$ converge to zero as k tends to ∞. As with discrete linear inclusions in the conventional algebra, the max generalised spectral radius is intimately related to the asymptotic stability of (4.15). As stated in Theorem 4.1.5, asymptotic stability is equivalent to $\mu(\Psi) < 1$.

Before stating our main result, we make the following definitions. Given Ψ in (4.1), we recall the notation from (3.27) that

$$S = A_1 \oplus A_2 \oplus ... \oplus A_m. \tag{4.16}$$

Moreover, we consider the *max convex hull* of Ψ given by

$$CO_{\max}(\Psi) = \{\bigoplus_{i=1}^{m} \alpha_i A_i \mid \alpha_i \geq 0, 1 \leq i \leq m \text{ and } \bigoplus_{i=1}^{m} \alpha_i = 1\}. \tag{4.17}$$

We say that $CO_{\max}(\Psi)$ is asymptotically stable if $\mu(A) < 1$ for all $A \in CO_{\max}(\Psi)$. In particular, we are interested in relating the asymptotic stability of $CO_{\max}(\Psi)$ with the S_{max}-property of Ψ.

The next result shows the relationship between the row-P_{max}-property, the S_{max}-property and the stability of discrete inclusions with delay for the max algebra.

Theorem 4.4.1. *Let Ψ be a set of $n \times n$ non-negative matrices given by (4.1) and $\hat{\Psi}$ be the row representative set of Ψ given by (4.2). Then the following are all equivalent:*

(i) Ψ *has the row-P_{max}-property;*

(ii) The max generalised spectral radius $\mu(\Psi) < 1$*;*

(iii) The max generalised spectral radius $\mu(\hat{\Psi}) < 1$*;*

(iv) Ψ *has the S_{max}-property;*

(v) $CO_{\max}(\Psi)$ *is asymptotically stable;*

(vi) The delayed difference inclusion given by

$$x_i(k+1) \in \{\bigoplus_{j=1}^{n} a_{ij}^{p} x_j(k-\tau_{ij}), p = 1, 2, ..., m\}, k \geq 0, i = 1, 2, ..., n \quad (4.18)$$

is asymptotically stable for all $\tau_{ij} \geq 0$ *for* $1 \leq i, j \leq n$*.*

Before proving this result, we shall state two key propositions. First, we relate the stability of the matrix S given by (4.16) to the S_{max}-property of the set $\hat{\Psi}$.

Proposition 4.4.2. *Let S be the matrix given by (4.16) and $v > 0$ be given. Then, $S \otimes v < v$ is equivalent to $M \otimes v < v$ for all $M \in \hat{\Psi}$.*

Proof: Let $v > 0$ be given and let M be a matrix in $\hat{\Psi}$. From the definition of $\hat{\Psi}$, for each $j \in \{1, 2, ..., n\}$ there exists some $A_{i_j} \in \Psi$ with $1 \leq i_j \leq m$ such that $M_{j.} = (A_{i_j})_{j.}$. It is explicit that for all j, if $S \otimes v < v$, then

$$M_{j.} \otimes v = (A_{i_j})_{j.} \otimes v \leq S_{j.} \otimes v < v_j.$$

Hence, $M \otimes v < v$ for all $M \in \hat{\Psi}$.

For the converse, if $M \otimes v < v$ for all $M \in \hat{\Psi}$, $A_i \otimes v < v$ for all $A_i \in \Psi$ since every matrix is also a row representative of itself. Thus, we observe that

$$\bigoplus_{i=1}^{m} A_i \otimes v < \bigoplus_{i=1}^{m} v \Rightarrow S \otimes v < v.$$

$\square$

The next proposition is a restatement of a result of [Gau95a] for the max-plus algebra, which was phrased in the language of discrete event systems. In the interests of clarity and completeness we have provided a direct max-algebraic proof below. It is an important result stating that the max generalised spectral radius of a finite set in $\mathbb{R}_+^{n\times n}$ can be calculated using the largest max eigenvalue of an $n \times n$ non-negative matrix.

Proposition 4.4.3. *Let Ψ be a set of $n \times n$ non-negative matrices given by (4.1). Let S be the matrix given by (4.16). Then, $\mu(\Psi) = \mu(S)$.*

Proof: We shall first show that $\mu(\Psi) \leq \mu(S)$. Consider some $\psi \in \Psi_\otimes^k$. It is explicit that $\psi \leq S_\otimes^k$. Then, we have $\mu(\psi) \leq \mu(S_\otimes^k)$. Since this is true for any ψ, we can write

$$\max_{\psi \in \Psi_\otimes^k} \mu(\psi) \leq \mu(S_\otimes^k).$$

Taking k^{th} root and $\limsup_{k\to\infty}$ of both sides, we obtain

$$\limsup_{k\to\infty} (\max_{\psi \in \Psi_\otimes^k} \mu(\psi))^{\frac{1}{k}} \leq \limsup_{k\to\infty} \mu(S_\otimes^k)^{\frac{1}{k}} = \mu(S),$$

where the final equality follows from (2.15). Thus, we have $\mu(\Psi) \leq \mu(S)$.

To complete the proof, we show that $\mu(S) \leq \mu(\Psi)$. Let Γ be a critical cycle of length p in $D(S)$ with product $\pi(\Gamma) = s_{i_1 i_2} s_{i_2 i_3} ... s_{i_p i_1}$ $(i_1, i_2, ..., i_p \in \{1, 2, ..., n\})$. Since $S = A_1 \oplus A_2 \oplus ... \oplus A_m$, it follows that there are indices $j_1, j_2, ..., j_p \in \{1, 2, ..., m\}$ such that

$$\mu(S)^p = \pi(\Gamma) = a_{i_1 i_2}^{j_1} a_{i_2 i_3}^{j_2} ... a_{i_p i_1}^{j_p} \leq (A_{j_1} \otimes A_{j_2} \otimes ... \otimes A_{j_p})_{i_1 i_1}.$$

Write $M = A_{j_1} \otimes A_{j_2} \otimes ... \otimes A_{j_p}$. Then, $M \in \Psi_\otimes^p$. For all $r \geq 1$,

$$(M_\otimes^r)_{i_1 i_1} \geq \mu(S)^{pr}.$$

Note that $M_\otimes^r \in \Psi_\otimes^{pr}$ and the above relation implies that $\max_{\psi \in \Psi_\otimes^{pr}} \mu(\psi)^{\frac{1}{pr}} \geq \mu(S)$. Let $k = pr$. If we take $\limsup_{k\to\infty}$ of both sides, we obtain

$$\limsup_{k\to\infty} (\max_{\psi \in \Psi_\otimes^k} \mu(\psi))^{\frac{1}{k}} \geq \mu(S).$$

Thus, we have $\mu(S) \leq \mu(\Psi)$.

So, $\mu(S) = \mu(\Psi)$ as claimed.

□

Proof:(Theorem 4.4.1) We will show that each of the conditions from (i) to (vi) is equivalent to $\mu(S) < 1$.

(i) : First, denote the multigraph associated with the set Ψ by $M(\Psi)$. Recall that it consists of the vertices $\{1, 2, ..., n\}$ with an edge of weight a_{ij}^{p} from i to j for every $A_p \in \Psi$ with $1 \leq p \leq m$ for which $a_{ij}^{p} > 0$. With analogous definitions to the case of a simple graph, $\mu(M(\Psi))$ denotes the maximal cycle geometric mean of $M(\Psi)$.

Now, assume that Ψ has the row-P_{max}-property. Then, $\mu(M) < 1$ for all $M \in \hat{\Psi}$. This implies that all cycle products in $M(\Psi)$ are less than one. It follows from Lemma 3.4.3 that $\mu(M(\Psi)) = \mu(S)$. So, we obtain that $\mu(S) < 1$.

For the converse, assume that $\mu(S) < 1$. Then, from Theorem 4.2.1 there exists a vector $v > 0$ such that $S \otimes v < v$. It automatically follows from Proposition 4.4.2 that $\mu(M) < 1$ for all $M \in \hat{\Psi}$. So, every $M \in \hat{\Psi}$ is a P_{max}-matrix. Thus, Ψ has the row-P_{max}-property.

(ii) : It is immediate from Proposition 4.4.3 that $\mu(\Psi) < 1$ if and only if $\mu(S) < 1$.

(iii) : Let $\mu(\hat{\Psi}) < 1$. Then, for all $M \in \hat{\Psi}$, we get $\mu(M) < 1$ since $\mu(\hat{\Psi}) = \mu(\bigoplus_{M\in\hat{\Psi}} M)$ from Proposition 4.4.3. Thus, every $M \in \hat{\Psi}$ is a P_{max}-matrix from Theorem 4.2.1. Hence, Ψ has the row-P_{max}-property. From (i), we automatically have $\mu(S) < 1$.

For the converse, let $\mu(S) < 1$. From the definition of $\hat{\Psi}$, for each $j \in \{1, 2, ..., n\}$ there exists some $A_{i_j} \in \Psi$ with $1 \leq i_j \leq m$ such that $M_{j.} = (A_{i_j})_{j.}$. Then, for every $M \in \hat{\Psi}$, we have $M \leq S$. By taking max sum of both sides so that $\mu(\hat{\Psi}) = \mu(\bigoplus_{M\in\hat{\Psi}} M)$, we obtain $\mu(\hat{\Psi}) \leq \mu(S) < 1$.

(iv) : First, assume that Ψ has the S_{max}-property. Then, there exists a vector $v > 0$ such that $A_i \otimes v < v$ for $1 \leq i \leq m$. As in the proof of Proposition 4.4.2 if we add both sides from 1 to m such that $\bigoplus_{i=1}^{m} A_i \otimes v < \bigoplus_{i=1}^{m} v$, we obtain $S \otimes v < v$. Thus, $\mu(S) < 1$.

For the converse, assume that $\mu(S) < 1$. Then, there exists a vector $v > 0$ such that $S \otimes v < v$ from Theorem 4.2.1. It implies that $A_i \otimes v \leq S \otimes v < v$ for $1 \leq i \leq m$. Thus, Ψ has the S_{max}-property.

(v) : Let $CO_{\max}(\Psi)$ be asymptotically stable. Notice that $S \in CO_{\max}(\Psi)$. We immediately see that $\mu(S) < 1$.

Now, let $\mu(S) < 1$. Since $A \leq S$ for all $A \in CO_{\max}(\Psi)$, $CO_{\max}(\Psi)$ is asymptotically stable.

(vi) : Following the same procedure as in Theorem 4.2.2, we can define $\tau_{\max} = \max_{i,j} \tau_{ij}$ and rewrite (4.18) in the following form

$$x(k+1) \in \{B_q^p \otimes x(k-q)\}, k \geq 0, p \in \{1,2,...,m\}, q \in \{0,1,...,\tau_{\max}\} \quad (4.19)$$

where the matrices B_q^p in $\mathbb{R}_+^{n\times n}$ are defined as follows. The $(i,j)^{\text{th}}$ entry of B_q^p is equal to a_{ij}^p if $\tau_{ij} = q$ and all other entries of B_q^p are zero. Note that

$$A_p = B_0^p \oplus B_1^p \oplus ... \oplus B_{\tau_{\max}}^p$$

for $1 \leq p \leq m$. By setting $\hat{x}(k) = (x(k-\tau_{\max}), x(k-\tau_{\max}+1), ..., x(k))^T \in \mathbb{R}_+^{n(\tau_{\max}+1)}$, we see that the inclusion (4.19) is equivalent to the inclusion

$$\hat{x}(k+1) \in \{C_p \otimes \hat{x}(k)\}, k \geq 0, p \in \{1,2,...,m\} \quad (4.20)$$

where

$$C_p = \begin{bmatrix} 0 & I & 0 & ... & 0 & 0 \\ 0 & 0 & I & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & ... & ... & ... & 0 & I \\ B_{\tau_{\max}}^p & ... & ... & ... & B_1^p & B_0^p \end{bmatrix}$$

for $1 \leq p \leq m$.

Then By Proposition 4.4.3, (4.20) is asymptotically stable if and only if $\mu(C_1 \oplus C_2 \oplus ... \oplus C_m) < 1$.

Define $\bar{C} = C_1 \oplus C_2 \oplus ... \oplus C_m$ and write

$$\bar{C} = \begin{bmatrix} 0 & I & 0 & ... & 0 & 0 \\ 0 & 0 & I & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & ... & ... & ... & 0 & I \\ \bar{B}_{\tau_{\max}} & ... & ... & ... & \bar{B}_1 & \bar{B}_0 \end{bmatrix}.$$

Then for $i = 0, \ldots, \tau_{\max}$, $\bar{B}_i = \bigoplus_{p=1}^{m} B_i^p$. It follows from Theorem 3.4.6 that $\mu(\bar{C}) < 1$ if and only if

$$\mu\left(\bigoplus_{i=0}^{\tau_{\max}} \bar{B}_i\right) < 1.$$

However

$$\begin{aligned}\bigoplus_{i=0}^{\tau_{\max}} \bar{B}_i &= \bigoplus_{i=0}^{\tau_{\max}} \bigoplus_{p=1}^{m} B_i^p \\ &= \bigoplus_{p=1}^{m} \bigoplus_{i=0}^{\tau_{\max}} B_i^p \\ &= \bigoplus_{p=1}^{m} A_p = S.\end{aligned}$$

Thus we have shown that (4.20) is asymptotically stable if and only if $\mu(S) < 1$. This completes the proof.

□

The above result establishes that Ψ has the S_{max}-property if and only if $\mu(M) < 1$ for all M in $\hat{\Psi}$, thus furnishing a max-algebraic version of Theorem 4.1.3 on the relation of row-P-property and S-property in the conventional algebra and Theorem 4.1.4 on linear copositive Lyapunov functions. The equivalence of (ii) and (v) shows that asymptotic stability of the inclusion (4.15) is equivalent to the asymptotic stability of the max-convex hull of the set Ψ. Note that as in Theorem 4.2.2, point (vi) above is also equivalent to the asymptotic stability of (4.20) for some $\tau_{ij} \geq 0$.

We illustrate Theorem 4.4.1 below.

Example 4.4.1. Consider the set $\Psi = \{A_1, A_2\} \subset \mathbb{R}_+^{2\times 2}$ and the row representative set $\hat{\Psi} = \{A_1, A_2, M_1, M_2\} \subset \mathbb{R}_+^{2\times 2}$ given in Example (4.1.1). Since $\mu(A_i) < 1$ and $\mu(M_i) < 1$ for $i = 1, 2$, Ψ has the row-P_{max}-property.

For $v = \begin{bmatrix} 1/2 & 2/3 \end{bmatrix}^T$, $A_i \otimes v < v$ for $i = 1, 2$. Thus, Ψ has the S_{max}-property.

Moreover, $S = \begin{bmatrix} 2/3 & 1/3 \\ 1 & 5/8 \end{bmatrix}$ where $\mu(S) = 2/3$. Since $\mu(\Psi) = \mu(\hat{\Psi}) = \mu(S)$, (ii) and (iii) hold.

Example 4.4.2. Consider the set $\Psi = \{A_1, A_2\} \subset \mathbb{R}_+^{2\times 2}$ where

$$A_1 = \begin{bmatrix} 1/2 & 4/3 \\ 0 & 1/4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2/3 & 0 \\ 1 & 5/8 \end{bmatrix}$$

with $\mu(A_i) < 1$ for $i = 1, 2$. For $M_1 = \begin{bmatrix} 1/2 & 4/3 \\ 1 & 5/8 \end{bmatrix} \in \hat{\Psi}$, $\mu(M_1) \nless 1$. Thus, the row-P_{max}-property doesn't hold. (v) implies that the max convex hull of Ψ is not asymptotically stable. Indeed, $\mu(A_1 \oplus 0.8A_2) \nless 1$.

Finally in this section, we present a max-algebra version of Theorem 4.1.2.

Proposition 4.4.4. *Let Ψ be a set of $n \times n$ non-negative matrices given by (4.1). Ψ has the row-P_{max}-property if and only if for any $x \neq 0$ in $\mathbb{R}^n_+$, there exists an index $k (1 \leq k \leq n)$ such that $(A_i \otimes x)_k < x_k$ for every matrix $A_i \in \Psi$ $(1 \leq i \leq m)$.*

Proof: Let Ψ have the row-P_{max}-property. Assume that there exists an $x^* \neq 0$ in $\mathbb{R}^n_+$ such that for every index j with $1 \leq j \leq n$ there is $A_{i_j} \in \Psi$ satisfying $(A_{i_j} \otimes x^*)_j \geq x^*_j$. It is obvious that $(S \otimes x^*)_j \geq x^*_j$. For each j, there exists an index $k \in \{1, 2, ..., n\}$ such that $s_{jk} x^*_k \geq x^*_j$. Since $s_{jk} = a^{i_j}_{jk}$ for some $i_j \in \{1, 2, ..., m\}$, we have $(A_{i_j})_{j.} \otimes x^* \geq x^*_j$. We can then construct $M \in \hat{\Psi}$ by setting $M_{j.} = (A_{i_j})_{j.}$ and it is clear that $M \otimes x^* \geq x^*$. This contradicts the assumption that every matrix in $\hat{\Psi}$ is a P_{max}-matrix.

Conversely, let $M \in \hat{\Psi}$ be given and let $x \neq 0$ be in $\mathbb{R}^n_+$. Then, there is some k such that $(A_i \otimes x)_k < x_k, \forall i \in \{1, 2, ..., m\}$. Since it is true for all $A_i \in \Psi$, we also have $(S \otimes x)_k < x_k$. It implies that $(M \otimes x)_k < x_k$. Hence, M is a P_{max}-matrix. Thus, Ψ has the row-P_{max}-property. This completes the proof. □

4.5 Concluding Remarks

Our main goal in this chapter was to extend the class of P-matrices and P-matrix sets to the max algebra and investigate the relationship between these concepts and the asymptotic stability of delayed difference equations and inclusions over the max algebra. In this context,

- we defined P_{max}-matrices (P^0_{max}-matrices) over the max algebra;
- we extended a number of properties of P-matrix sets to the max algebra;
- we derived some stability results for sets and discrete inclusions over the max algebra.

CHAPTER 5

The AHP, Max Algebra and Multi-objective Optimisation

In this chapter, we are interested in the application of the max algebra to the Analytic Hierarchy Process (AHP). We consider a novel approach to derive a single ranking scheme for alternatives in the multi-criteria AHP. In particular, we extend the single objective optimisation problem based on the max algebra to the general multi-criteria AHP. In this context, we consider three optimisation problems associated with a set of error functions corresponding to a set of symmetrically reciprocal matrices (SR-matrices). Respectively: we relate the existence of globally optimal solutions to the commutativity properties of the matrices; we show that min-max optimal solutions are intimately related to the generalised spectral radius; and we prove that Pareto optimal solutions are guaranteed to exist. We present a practical example to compare the rankings generated by these Pareto solutions to those of the classical AHP.

5.1 Motivation and Mathematical Background

In this section, we explain the classical Analytic Hierarchy Process (AHP) and illustrate Saaty's Eigenvalue method (EM) [Saa77a]. Moreover, we recall the max algebra approach suggested by Elsner and van den Driessche [EvdD04, EvdD10].

5.1.1 Introduction

The AHP is a method widely used for decision making problems involving more than one criterion. It was originally developed by Thomas L. Saaty in the early 1970s [Saa77a]. It is used to rank the alternatives in a decision problem. It consists of a three layer hierarchical structure: the overall goal is at the top; the criteria are in the next level; and the alternatives are in the bottom level. See Figure 5.1.

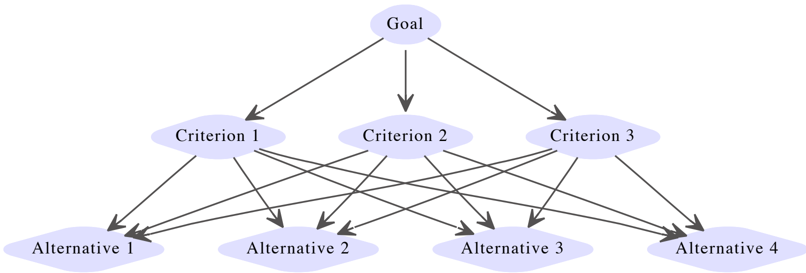


Figure 5.1: Hierarchical structure of the AHP

The AHP has a quite large number of interesting applications. One of the earliest applications, which was developed by Saaty, involved deciding a strategic plan for the future of Sudan's transportation system [Saa77b]. Since then, the AHP has been applied to various resource allocation problems. See for instance [RG95] for its consideration in the allocation of energy resources. Another application of the AHP describes how it can be used in the process of selecting questions from a database for an examination system in education [MNO95]. One application in which the AHP was used for selecting the best web site for online advertising is described in [Nga03]. Some of the recent applications of the AHP appear in the banking sector. For instance, it was used to evaluate the performance of Turkish banks with respect to both financial and non-financial factors [SBK09]. For more applications, see [Zah86, SV01, FG01, VK06, IL11] and references therein.

The essence of the AHP can be described as follows. Given n alternatives we construct a *pairwise comparison matrix* (PC-matrix), $A > 0$ for each criterion, in which a_{ij} indicates the strength of alternative i relative to alternative j for that criterion. In the AHP, a_{ij} is assigned from the fundamental $1-9$ scale [Saa77a]. See Table 5.1.

Table 5.1: $1-9$ Scale

Strength	Definition
1	Equal Importance
3	Moderate Importance
5	Strong Importance
7	Demonstrated Importance
9	Extreme Importance
$2,4,6,8$	Intermediate values

Furthermore, ratios arising from the $1-9$ scale can be assigned to a_{ij}.

A PC-matrix with the property that

$$a_{ij}a_{ji} = 1 \text{ for all } i, j(i \neq j) \text{ and } a_{ii} = 1 \text{ for all } i \tag{5.1}$$

is called a *symmetrically reciprocal matrix* (SR-matrix) [Far07]. Given n alternatives, $n(n-1)/2$ pairwise comparisons are required to construct such a matrix by a decision maker. An example of an SR-matrix is given in (5.2). Here, the first alternative is slightly more important than the second, extremely important compared to the third and the third alternative is strongly more important than the second.

$$A = \begin{bmatrix} 1 & 2 & 9 \\ 1/2 & 1 & 1/5 \\ 1/9 & 5 & 1 \end{bmatrix} \tag{5.2}$$

Note that the SR notation was used for strongly regular matrices in the max algebra by Butkovič [But94]. Here, we follow Farkas' notation in the AHP context [Far07]. We would also like to note that there is an additive version of an SR-matrix which is *anti-symmetric* in the sense that $a_{ij} = -a_{ji}$ for all $i, j(i \neq j)$ and $a_{ii} = 1$ for all i [Tra11]. However, we are only interested in the multiplicative version (5.1) throughout this chapter.

Once an SR-matrix is constructed, the next step in the AHP is to derive a vector $(w_1, \ldots, w_n)$ of positive weights, which can be used to rank the alternatives, with w_i quantifying the weight of alternative i. For two alternatives $i, j \in \{1, 2, ..., n\}$,

- if $w_i > w_j$, then i is preferred to j. This is denoted by $i > j$;
- if $w_i < w_j$, then j is preferred to i. This is denoted by $j > i$;
- if $w_i = w_j$, then i and j are equally preferred. This is denoted by $i = j$.

Example 5.1.1. Let w be given by $[0.8 \quad 1 \quad 0.5]^T$. Since $w_2 > w_1 > w_3$, we get the ranking: $2 > 1 > 3$.

The ideal situation is where the *SR*-matrix describing pairwise comparisons is of the form $a_{ij} = w_i/w_j$ for all i, j. In this case A is said to be a *transitive matrix*. Formally, this means

$$a_{ij}a_{jk} = a_{ik} \text{ for all } i, j, k (i \neq j, i \neq k, j \neq k) \text{ and } a_{ii} = 1 \text{ for all } i. \tag{5.3}$$

In the matrix form,

$$\begin{bmatrix} 1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & 1 & \cdots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & 1 \end{bmatrix}. \tag{5.4}$$

For a general *SR*-matrix A, $a_{ij} > 1$ and $a_{jk} > 1$ does not imply $a_{ik} > 1$. This creates a problem in ranking the alternatives. It is then necessary to approximate A with a transitive matrix T, where $t_{ij} = w_i/w_j$ for some positive *weight vector* $w = (w_1, \ldots, w_n)$.

The problem in the AHP is then how to derive w or T given A. Several approaches have been proposed including Saaty's suggestion to take w to be the Perron vector of A [Saa77a] or the approach of Farkas et al. [FLR03], which chooses w to minimise the Euclidean error $\sum\limits_{i,j}(a_{ij} - x_i/x_j)^2$. Other variant of this idea is to take the entrywise logarithmic transformation of the Euclidean error and minimise $\sum\limits_{i,j}(\log a_{ij} - \log x_i + \log x_j)^2$ with respect to x [Cra87]. A-nother well-known approach which was proposed by Dahl in [Dah05] is to select w as a solution of the following optimisation problem: $\inf\limits_{x} \max\limits_{i,j} \frac{a_{ij}x_j}{x_i}$. A recent approach introduced by Elsner and van den Driessche [EvdD04, EvdD10] is to select w to be the max-algebraic eigenvector of A. This is similar in spirit to Saaty's approach [Saa77a] and also uses the same error measure as [Dah05].

Our motivation in this chapter comes from the max algebra approach for the single criterion AHP described in [EvdD04, EvdD10]. Instead of mimicking Saaty's Eigenvalue Method (EM), we consider a different approach cast in the framework of multi-objective optimisation and max-algebraic spectral theory. In this direction, the layout of this chapter is as follows. First, we briefly review the EM with an illustrative example and discuss the max algebra approach for the one criterion case. In Section 5.2, we address the ranking problem by considering the multi-criteria AHP as a multi-objective optimisation problem. In Section 5.3, we investigate the existence of a single transitive matrix with a minimum distance to all matrices in the set of SR-matrices simultaneously. We remark that this amounts to finding a common subeigenvector of the given matrices. As this will not in general be possible, we consider two other questions. The first in Section 5.4 is concerned with obtaining a transitive matrix that minimises the maximal distance to any of the given SR-matrices. In this section, we also extend the results in [EvdD04, EvdD10] for an SR-matrix to the set of SR-matrices. The second question in Section 5.5 concerns the existence of a transitive matrix that is Pareto optimal for the given set of matrices. The work contained in this chapter has resulted in the publication: [BGMS13].

5.1.2 Saaty's Eigenvalue Method

For a transitive matrix T given in the form of (5.4), the following eigenequation is satisfied by w:

$$Tw = nw. \tag{5.5}$$

All eigenvalues of T are zero except n. Moreover, it follows from Theorem 2.1.3 (The Perron Theorem) that there exists a unique eigenvector (up to a scalar multiple) associated with n. In this direction, $\rho(T) = n$ and w is the Perron vector of T.

Saaty's rationale was as follows [Saa86b, Saa90, Saa99]. A small perturbation of a positive matrix generates a small perturbation of the eigenvector corresponding to an unrepeated eigenvalue. He suggested therefore that the Perron vector of an *SR*-matrix corresponding to the Perron root can be used as a weight vector:

$$Aw = \rho(A)w. \tag{5.6}$$

As A is positive, w in (5.6) is unique (up to a scalar multiple) (Theorem 2.1.3). He also proposed a way to measure the inconsistency of the pairwise comparisons in the *SR*-matrix. The difference between $\rho(A)$ and n is used to calculate a ratio called the *consistency ratio* [Saa08, Saa90]. If it is bigger than a desired value, then the pairwise comparisons are reconsidered .

For the classical AHP involving multiple criteria, the *Eigenvalue Method (EM)* is as follows. A set of *SR*-matrices is constructed: one for each criterion. One additional *SR*-matrix is constructed based on comparisons of the different criteria. Once weight vectors are obtained for each individual criterion, these are then combined using the entries of the weight vector for the *criteria-comparison matrix*. As an illustration, we take the following numerical example from [Saa77a] and show how the Perron vectors of the *SR*-matrices are used to construct an overall weight vector.

Example 5.1.2. The problem is to decide where to go for a one week vacation among the alternatives:

1. Short trips;
2. Quebec;
3. Denver;
4. California.

Five criteria are considered:

1. cost of the trip;
2. sight-seeing opportunities;
3. entertainment;
4. means of travel;

5. dining.

The SR-matrix for the criteria and its Perron vector are given by

$$C = \begin{bmatrix} 1 & 1/5 & 1/5 & 1 & 1/3 \\ 5 & 1 & 1/5 & 1/5 & 1 \\ 5 & 5 & 1 & 1/5 & 1 \\ 1 & 5 & 5 & 1 & 5 \\ 3 & 1 & 1 & 1/5 & 1 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 0.179 \\ 0.239 \\ 0.431 \\ 0.818 \\ 0.237 \end{bmatrix}.$$

The above matrix C describes the pairwise comparisons between the different criteria. For instance, as $c_{21} = 5$, criterion 2 is rated more important than criterion 1; $c_{32} = 5$ indicates that criterion 3 is rated more important than criterion 2 and so on. The vector w contains the weights of the criteria; in this method, criterion 4 is given most weight, followed by criterion 3 and so on. Thus, we get the ranking: $4 > 3 > 2 > 5 > 1$ for the criteria.

The SR-matrices, $A_1, ..., A_5$, for each of the 5 criteria, their Perron vectors and corresponding ranking schemes are given below. For instance, for criterion 1, the first alternative is preferred to the second as the $(1,2)$ entry of A_1 is 3. Similarly, for criterion 3, the 4th alternative is preferred to the 1st as the $(4,1)$ entry of A_3 is 2.

For the cost of the trip:

$$A_1 = \begin{bmatrix} 1 & 3 & 7 & 9 \\ 1/3 & 1 & 6 & 7 \\ 1/7 & 1/6 & 1 & 3 \\ 1/9 & 1/7 & 1/3 & 1 \end{bmatrix}, \quad w^{(1)} = \begin{bmatrix} 0.877 \\ 0.46 \\ 0.123 \\ 0.064 \end{bmatrix}, \quad 1 > 2 > 3 > 4$$

For the sight-seeing opportunities:

$$A_2 = \begin{bmatrix} 1 & 1/5 & 1/6 & 1/4 \\ 5 & 1 & 2 & 4 \\ 6 & 1/2 & 1 & 6 \\ 4 & 1/4 & 1/6 & 1 \end{bmatrix}, \quad w^{(2)} = \begin{bmatrix} 0.091 \\ 0.748 \\ 0.628 \\ 0.196 \end{bmatrix}, \quad 2 > 3 > 4 > 1$$

For the entertainment:

$$A_3 = \begin{bmatrix} 1 & 7 & 7 & 1/2 \\ 1/7 & 1 & 1 & 1/7 \\ 1/7 & 1 & 1 & 1/7 \\ 2 & 7 & 7 & 1 \end{bmatrix}, \quad w^{(3)} = \begin{bmatrix} 0.57 \\ 0.096 \\ 0.096 \\ 0.81 \end{bmatrix}, \quad 4 > 1 > 2 = 3$$

For the means of travel:

$$A_4 = \begin{bmatrix} 1 & 4 & 1/4 & 1/3 \\ 1/4 & 1 & 1/2 & 3 \\ 4 & 2 & 1 & 3 \\ 3 & 1/3 & 1/3 & 1 \end{bmatrix}, \quad w^{(4)} = \begin{bmatrix} 0.396 \\ 0.355 \\ 0.768 \\ 0.357 \end{bmatrix}, \quad 3 > 1 > 4 > 2$$

For the dining:

$$A_5 = \begin{bmatrix} 1 & 1 & 7 & 4 \\ 1 & 1 & 6 & 3 \\ 1/7 & 1/6 & 1 & 1/4 \\ 1/4 & 1/3 & 4 & 1 \end{bmatrix}, \quad w^{(5)} = \begin{bmatrix} 0.722 \\ 0.642 \\ 0.088 \\ 0.242 \end{bmatrix}, \quad 1 > 2 > 4 > 3$$

Next, a matrix is constructed such that the i^{th} column correspond to the weight vector $w^{(i)}$ associated with the SR-matrix A_i $(i = 1, 2, ..., 5)$. It is then multiplied by the Perron vector of C.

$$\begin{bmatrix} 0.877 & 0.091 & 0.57 & 0.396 & 0.722 \\ 0.46 & 0.748 & 0.096 & 0.355 & 0.642 \\ 0.123 & 0.628 & 0.096 & 0.768 & 0.088 \\ 0.064 & 0.196 & 0.81 & 0.357 & 0.242 \end{bmatrix} \begin{bmatrix} 0.179 \\ 0.239 \\ 0.431 \\ 0.818 \\ 0.237 \end{bmatrix} = \begin{bmatrix} 0.919 \\ 0.745 \\ 0.862 \\ 0.757 \end{bmatrix}$$

Then, the overall weight vector gives the ranking: $1 > 3 > 4 > 2$.

5.1.3 Max Algebra Approach

The max algebra approach to the AHP was first suggested by Elsner and van den Driessche in 2004 [EvdD04]. Specifically, w is taken to be a max eigenvector of the SR-matrix A corresponding to $\mu(A)$:

$$A \otimes w = \mu(A)w. \tag{5.7}$$

This is similar in spirit to Saaty's approach and also generates a transitive matrix that minimises the maximal relative error $\max_{i,j} |a_{ij} - x_i/x_j|/a_{ij}$. As noted in [EvdD10], minimising this functional is equivalent to minimising

$$e_A(x) = \max_{1\leq i,j\leq n} a_{ij}x_j/x_i. \tag{5.8}$$

We next make a number of observations on the max-algebraic spectral properties of *SR*-matrices. Notice that an *SR*-matrix A is irreducible. Thus, $\mu(A)$ is positive and unique from Theorem 2.2.2. Moreover, there exists a positive max eigenvector (not necessarily unique) corresponding to it. Since $a_{ii} = 1$ for all $i \in \{1, 2, ..., n\}$, we get $\mu(A) \geq 1$. Further if all cycle products of length 3 equal to 1 ($a_{ij}a_{jk}a_{ki} = 1$ for all $i, j, k \in \{1, 2, ..., n\}$), then all cycle products of any length are 1. In this case, $\mu(A) = 1$ and A is a transitive matrix [EvdD04]. Note that for a transitive matrix, its max eigenvector and Perron vector are the same.

While our interest is in *SR*-matrices and the AHP, we shall describe results for general irreducible matrices where appropriate. Some of these results can be generalized to reducible matrices. Results of this type, obtained by Sergeĭ Sergeev, are in [BGMS13].

For notation, we denote the set of all n-tuples of positive real numbers by

$$\mathrm{int}(\mathbb{R}^n_+) = \{x \in \mathbb{R}^n \mid x_i > 0, 1 \leq i \leq n\}. \tag{5.9}$$

For an irreducible matrix $A \in \mathbb{R}^{n\times n}_+$ and a real number $r > 0$, we will consider the following set, which was introduced in [EvdD10]

$$\mathcal{C}_{A,r} = \{x \in \mathrm{int}(\mathbb{R}^n_+) \mid A \otimes x \leq rx\}. \tag{5.10}$$

For the special case of $r = \mu(A)$, $\mathcal{C}_{A,\mu(A)}$ is denoted by $\mathcal{C}_A$ [EvdD10]

$$\mathcal{C}_A = \{x \in \mathrm{int}(\mathbb{R}^n_+) \mid A \otimes x \leq \mu(A)x\}. \tag{5.11}$$

Not only does a max eigenvector w minimises (5.8) but also any vector from the set $\mathcal{C}_A$ minimises it. Thus, (5.11) is said to be a *minimum error requirement set* for A and the following holds [EvdD04, EvdD10].

$$\mu(A) = \min_{x\in\mathrm{int}(\mathbb{R}^n_+)} e_A(x) = e_A(w) \text{ for any } w \in \mathcal{C}_A. \tag{5.12}$$

Note that (5.12) was introduced in [CG79] in the context of linear programming. It was recently studied in [Kri05] for idempotent linear algebra.

We next recall the definition of the normalised set of $\mathcal{C}_{A,r}$ from [EvdD10]

$$\mathcal{D}_{A,r} = \{x \in \text{int}(\mathbb{R}^n_+) \mid x \in \mathcal{C}_{A,r}, x_1 = 1\}. \tag{5.13}$$

As above, $\mathcal{D}_A$ is used to denote the special case where $r = \mu(A)$.

The relations between the sets $\mathcal{C}_{A,r}$, the error function (5.8) and $\mu(A)$ were clarified in [EvdD10] and are recalled in the following propositions.

Proposition 5.1.1. *Let $A \in \mathbb{R}^{n\times n}_+$ be an irreducible matrix. Then:*

(i) $\mathcal{C}_{A,r} \neq \emptyset \iff r \geq \mu(A)$*;*

(ii) $x \in \mathcal{C}_{A,r} \iff e_A(x) \leq r$*.*

In particular, $\mathcal{C}_A \neq \emptyset$ for an irreducible matrix A. However, it may not in general be possible to obtain a unique ranking scheme from the set $\mathcal{C}_A$. The following result defines a necessary and sufficient condition for the uniqueness. Recall that A^C denotes the critical matrix and $N^C(A)$ denotes the set of critical vertices in the critical digraph $D^C(A)$.

Proposition 5.1.2. *Let $A \in \mathbb{R}^{n\times n}_+$ be an irreducible matrix. $x \in \mathcal{C}_A$ is unique (up to a scalar multiple) if and only if A^C is irreducible and $N^C(A) = N(A)$.*

In particular, $\mathcal{C}_A$ is the same as $V^*(A)$ defined in (2.27) since A is irreducible. Recall from Section 2.2.3 that $V^*(A)$ is a subeigencone of A associated with $\mu(A)$. In addition, it follows from Proposition 2.2.7 that $V^*(A) = \text{span}_\oplus(\hat{A}^*)$ where $\hat{A} = \frac{A}{\mu(A)}$ and $\hat{A}^*$ is the Kleene star of $\hat{A}$. Throughout this section, we use the notation C_A of [EvdD10]. It follows from [EvdD10, BS05] that x is in $\mathcal{C}_A$ if and only if there exists a vector $x(0) \in \mathbb{R}^n_+$, $x(0) \neq 0$ such that $x = \hat{A}^* \otimes x(0)$. We can calculate such a vector x by using Algorithm 3 below.

We next revisit Example 5.1.2 and highlight some issues that arise in the max-algebraic setting; in particular, the issue of non-unique ranking is raised. In the next Chapter, we discuss some ideas on how to deal with this issue.

Algorithm 3 Calculate $x \in \mathcal{C}_A$

A : SR-matrix, $x(0) \in \mathbb{R}^n_+$, $x(0) \neq 0$
for $i = 1, ..., n-1$ **do**
  $x(i) = \frac{A}{\mu(A)} \otimes x(i-1)$
end for
$x = x(0) \oplus x(1) \oplus ... \oplus x(n-1)$

Example 5.1.3. Let $C, A_1, \ldots, A_5$ be as in Example 5.1.2. By following the same way in Example 2.2.1, we find $D^C(C)$. We see that C^C is irreducible and $N^C(C) = \{1, 2, 3, 4\}$. Since all vertices are not critical, there exist multiple vectors in $\mathcal{C}_C$ from Proposition 5.1.2. It follows from Theorem 2.2.5 that there exists a unique max eigenvector. It is given by

$$\begin{bmatrix} 1 & 1.495 & 2.236 & 3.344 & 0.897 \end{bmatrix} \text{ with the ranking } 4 > 3 > 2 > 1 > 5.$$

By using Algorithm 3, we can find others vectors in $\mathcal{C}_C$ giving at least three more rankings: $4 > 5 > 3 > 2 > 1$, $4 > 3 > 2 > 5 > 1$ and $4 > 3 > 5 > 2 > 1$.

For $1 \leq i \leq 3$, $N^C(A_i) = N(A_i)$ and A_i^C is irreducible. Thus, there exists a unique vector in $\mathcal{C}_{A_i}$ which is the max eigenvector of A_i. We list each $w^{(i)} \in \mathcal{C}_{A_i}$ and the corresponding ranking below.

$$w^{(1)} = \begin{bmatrix} 1 & 0.522 & 0.136 & 0.071 \end{bmatrix} \text{ with the ranking 1>2>3>4.}$$

$$w^{(2)} = \begin{bmatrix} 1 & 8.801 & 7.746 & 2.272 \end{bmatrix} \text{ with the ranking 2>3>4>1.}$$

$$w^{(3)} = \begin{bmatrix} 1 & 0.181 & 0.181 & 1.587 \end{bmatrix} \text{ with the ranking 4>1>2=3.}$$

For A_4, $N^C(A_4) = \{1, 2, 4\}$ and A_4^C is irreducible. For A_5, $N^C(A_5) = \{1, 3, 4\}$ and A_5^C is irreducible. There exist multiple vectors in $\mathcal{C}_{A_4}$ and $\mathcal{C}_{A_5}$. A_4 and A_5 have unique max eigenvectors. We list these eigenvectors as follows.

$$w^{(4)} = \begin{bmatrix} 1 & 0.825 & 1.211 & 0.909 \end{bmatrix} \text{ with the ranking 3>1>4>2.}$$

$$w^{(5)} = \begin{bmatrix} 1 & 0.760 & 0.108 & 0.329 \end{bmatrix} \text{ with the ranking 1>2>4>3.}$$

Then, we construct the following coefficient matrix in the same way as in Example 5.1.2. As a weight vector for A, we choose the vector that produces the same ranking with the Perron vector of C from the set $\mathcal{C}_C$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.522 & 8.801 & 0.181 & 0.825 & 0.76 \\ 0.136 & 7.746 & 0.181 & 1.211 & 0.108 \\ 0.071 & 2.272 & 1.587 & 0.909 & 0.329 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1.495 \\ 2.236 \\ 3.344 \\ 1.114 \end{bmatrix} = \begin{bmatrix} 3.344 \\ 13.161 \\ 11.583 \\ 3.549 \end{bmatrix}$$

The overall weight vector gives the ranking: $2 > 3 > 4 > 1$. It was $1 > 3 > 4 > 2$ for the EM. Although both approaches can produce same schemes for local rankings, the overall ranking scheme can be different.

5.2 Multi-criteria AHP and Multi-objective Optimisation

The application of the max algebra to the AHP is motivated in [EvdD04, EvdD10] by the following considerations. First, it is observed that, for an *SR*-matrix A, vectors in the set $\mathcal{C}_A$ minimise the function (5.8) and hence the relative error. Based on this observation, these vectors are used to construct transitive matrices to obtain an overall ranking of the alternatives in the decision problem. In light of the properties of $\mathcal{C}_A$, this is justified by the fact that the transitive matrices constructed in this way are closest to the original *SR*-matrix A in the sense of the relative error. Thus, the approach to construct a ranking vector for a single *SR*-matrix taken in [EvdD04, EvdD10] amounts to solving the following optimisation problem.

$$\min_{x \in \mathrm{int}(\mathbb{R}^n_+)} \{e_A(x)\}. \tag{5.14}$$

Throughout this chapter, we are concerned with extending the max algebra approach to the general AHP with n alternatives and m criteria within the framework of multi-objective optimisation. Formally, we are given m *SR*-matrices; one for each criterion. Let Ψ in (5.15) denote the set of these matrices by (Recall the notation from (4.1))

$$\Psi = \{A_1, A_2, ..., A_m\}. \tag{5.15}$$

For each $A_i \in \Psi$, there is an error function $e_{A_i} : \mathrm{int}(\mathbb{R}^n_+) \to \mathbb{R}_+$ defined as in (5.8). In contrast to the approach taken in the classical AHP, we view the construction of a ranking vector for the m criteria as a multi-objective optimisation problem for the error functions e_{A_i}, $1 \leq i \leq m$.

5.3 Globally Optimal Solutions

To begin with, we seek a vector that simultaneously minimises all of the functions e_{A_i} for $A_i \in \Psi$. Such a vector is said to be a *globally optimal solution* for the multi-objective optimisation problem.

$$\min_{x \in \mathrm{int}(\mathbb{R}^n_+)} \{e_{A_i}(x)\}, i = 1, 2, ..., m. \tag{5.16}$$

For each $A_i \in \Psi$, the set of vectors that minimise $e_{A_i} : \mathrm{int}(\mathbb{R}^n_+) \to \mathbb{R}_+$ is precisely $\mathcal{C}_{A_i}$ in (5.11). Formally,

$$\mathcal{C}_{A_i} = \{x \in \mathrm{int}(\mathbb{R}^n_+) \mid A_i x \leq \mu(A_i)x\}, \quad i = 1, 2, ..., m. \tag{5.17}$$

Hence, the problem of finding a vector $x \in \mathrm{int}(\mathbb{R}^n_+)$ that simultaneously minimises all the error functions e_{A_i} amounts to determining when

$$\bigcap_{i=1}^{m} \mathcal{C}_{A_i} \neq \emptyset.$$

Equivalently, x simultaneously minimises all the error functions if and only if it is a *common subeigenvector* of A_i for all $i \in \{1, 2, ..., m\}$. The remainder of this section is divided into two parts: we first consider the existence of common subeigenvectors for arbitrary non-negative irreducible matrices in the next subsection; we then specialise to sets of SR-matrices and globally optimal solutions of the optimisation problem (5.16).

5.3.1 Common Subeigenvectors of Irreducible Matrices

First of all, we consider the general problem of finding a common subeigenvector for a set of non-negative irreducible matrices (not necessarily SR-matrices). Our results are clearly related to the work in [KSS12] concerning the intersection of eigencones of commuting matrices over the max and non-negative algebra.

We adopt the following notation (Recall the notation from (4.16)).

$$S = A_1 \oplus A_2 \oplus ... \oplus A_m. \tag{5.18}$$

Note that S is irreducible if at least one A_i is irreducible and $\mu(S) > 0$ if at least one $\mu(A_i) > 0$. In an abuse of notation, we denote the max sum of $\hat{A}_i = \frac{A_i}{\mu(A_i)}$ $(1 \leq i \leq m)$ by

$$\hat{S} = \bigoplus_{i=1}^{m} \hat{A}_i. \tag{5.19}$$

Theorem 5.3.1. *Consider the set $\Psi \subset \mathbb{R}_+^{n\times n}$ in (5.15). Assume that each $A_i \in \Psi$ is irreducible. Then, the following are equivalent.*

(i) $\mu(\hat{S}) = 1$;

(ii) There exists some $x \in \mathrm{int}(\mathbb{R}_+^n)$ with $A_i \otimes x \leq \mu(A_i)x$ for all $A_i \in \Psi$;

(iii) $\mu(A_{j_1} \otimes \cdots \otimes A_{j_k}) \leq \mu(A_{j_1}) \cdots \mu(A_{j_k})$ *where $1 \leq j_i \leq m$ for $1 \leq i \leq k$. (We say that μ is submultiplicative on Ψ).*

Proof: (i)⇒(ii): First, assume that $\mu(\hat{S}) = 1$. Then, there exists $x \in \mathrm{int}(\mathbb{R}_+^n)$ such that $\hat{S} \otimes x = x$ since $\hat{S}$ is irreducible. Thus, $\hat{A}_i \otimes x \leq \hat{S} \otimes x = x$ for all $A_i \in \Psi$. Hence, $A_i \otimes x \leq \mu(A_i)x$ for all i.

(ii)⇒(iii): Suppose that there exists some $x \in \mathrm{int}(\mathbb{R}_+^n)$ with $A_i \otimes x \leq \mu(A_i)x$ for all $A_i \in \Psi$. Pick some $\psi \in \Psi_\otimes^k$ such that $\psi = A_{j_1} \otimes A_{j_2} \otimes \cdots \otimes A_{j_k}$ where $1 \leq j_i \leq m$ for $1 \leq i \leq k$. Then,

$$\begin{aligned}
\psi \otimes x &= A_{j_1} \otimes A_{j_2} \otimes ... \otimes A_{j_k} \otimes x \\
&\leq \mu(A_{j_k}) A_{j_1} \otimes A_{j_2} \otimes ... \otimes A_{j_{k-1}} \otimes x \\
&\vdots \\
&\leq \mu(A_{j_k})\mu(A_{j_{k-1}})...\mu(A_{j_2}) A_{j_1} \otimes x \\
&\leq \mu(A_{j_k})\mu(A_{j_{k-1}})...\mu(A_{j_2})\mu(A_{j_1})x.
\end{aligned}$$

Writing $r = \mu(A_{j_1})\mu(A_{j_2})...\mu(A_{j_k})$, we see that $x \in \mathcal{C}_{\psi,r}$ from the definition (5.10). Hence, $\mathcal{C}_{\psi,r} \neq \emptyset$. Point (i) in Proposition 5.1.1 implies that $r \geq \mu(\psi)$. Thus, $\mu(A_{j_1})\mu(A_{j_2})...\mu(A_{j_k}) \geq \mu(A_{j_1} \otimes A_{j_2} \otimes ... \otimes A_{j_k})$.

(iii)⇒(i): Consider the set of normalised matrices

$$\Psi_n = \{\hat{A}_1, \hat{A}_2, ..., \hat{A}_m\} \tag{5.20}$$

where $\mu(\hat{A}_i) = 1$ for all $i \in \{1, 2, ..., m\}$. Pick some $\psi \in \Psi_{n\otimes}^{k}$. As μ is submultiplicative on Ψ, it is automatically submultiplicative on Ψ_n. Thus, we have $\mu(\psi) \leq 1$. As this is true for any $\psi \in \Psi_{n\otimes}^{k}$, it follows that

$$\max_{\psi \in \Psi_{n\otimes}^{k}} \mu(\psi) \leq 1.$$

Taking the k^{th} root and $\limsup_{k\to\infty}$ of both sides, we see that the generalised spectral radius

$$\mu(\Psi_n) \leq 1.$$

Proposition 4.4.3 then implies that $\mu(\hat{S}) \leq 1$. Furthermore, since $\mu(\hat{S}) \geq \mu(\hat{A}_i) = 1$ for all $i \in \{1, 2, ..., m\}$ we obtain $\mu(\hat{S}) = 1$. □

Following Theorem 5.3.1, we simply list steps for deriving a common subeigenvector.

Algorithm 4 Find a common subeigenvector of $A_1, A_2, ..., A_m$

A_i : SR-matrix, $i \in \{1, 2, ..., m\}$
Calculate $\mu(A_i)$ for all $A_i \in \Psi$ by using Karp's algorithm.
Normalise each matrix and obtain $\hat{A}_i$ for all i.
Find $\hat{S} = \bigoplus_{i=1}^{m} \hat{A}_i$ and calculate $\mu(\hat{S})$.
if $\mu(\hat{S}) = 1$ **then**
  $x \in \mathcal{C}_{\hat{S}}$ is a common subeigenvector. (use Algorithm 3)
else
  There is no subeigenvector.
end if

Example 5.3.1. For two irreducible matrices (SR-matrices) given by

$$A = \begin{bmatrix} 1 & 8 & 1/4 & 7 \\ 1/8 & 1 & 6 & 1/4 \\ 4 & 1/6 & 1 & 4 \\ 1/7 & 4 & 1/4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 4 & 5 & 9 \\ 1/4 & 1 & 1/8 & 9 \\ 1/5 & 8 & 1 & 1/8 \\ 1/9 & 1/9 & 8 & 1 \end{bmatrix}.$$

By using Algorithm 4, we get $x = \begin{bmatrix} 1 & 0.721 & 0.693 & 0.667 \end{bmatrix}$ is common subeigenvector.

Note that the equivalence (i) and (ii) in Theorem 5.3.1 can be regarded as a special case of [HS03] Theorem 2.5 in the context of simultaneous non-negative matrix scaling. See also [BS05] Theorem 3.5 for an extension. The problem in these works is to find a diagonal matrix $X > 0$ such that $XA_iX^{-1} \leq B_i$ for $i = 1, \ldots, m$. In this context, our problem amounts to finding $X = \text{diag}\ (x)$ where $x \in \text{int}(\mathbb{R}^n_+)$ such that

$$X^{-1}A_iX \leq \mu(A_i)\mathbf{1}_n\mathbf{1}_n^T \text{ for all } i \tag{5.21}$$
$$(X^{-1}A_iX)_{kl} = \mu(A_i) \text{ for all } i \text{ and } (k,l) \in E^C(A_i).$$

In the terminology of [SSB09, But10], this yields a *simultaneous visualisation* of all matrices in Ψ (See Section 2.2.4 for the definition of visualisation scaling.).

The following result represents a relation of the critical graphs of two non-negative irreducible matrices in the context of simultaneous visualisation.

Proposition 5.3.2. *Let $A, B \in \mathbb{R}^{n\times n}_+$ be irreducible with $\mu(A) = \mu(B) = 1$. If $\mu(A \oplus B) = 1$, then*

(i) $a_{ij} = b_{ij}$ for all edges $(i,j) \in E^C(A) \cap E^C(B)$;

(ii) $a_{ij}b_{ji} = 1$ for $(i,j) \in E^C(A)$ and $(j,i) \in E^C(B)$.

Proof: As, $\mu(A) = \mu(B) = 1$, it follows that $\hat{A} = A$ and $\hat{B} = B$. Thus, $\hat{S} = A \oplus B$. From the assumption, we obtain $\mu(\hat{S}) = 1$. It now follows from Theorem 5.3.1 that there exists some $x \in \text{int}(\mathbb{R}^n_+)$ with $A \otimes x \leq x$, $B \otimes x \leq x$. Let $X = \text{diag}\ (x)$ and consider the diagonally scaled matrices

$$X^{-1}AX, \quad X^{-1}BX.$$

From the choice of X it is immediate that

$$X^{-1}AX \leq \mathbf{1}_n\mathbf{1}_n^T, \quad X^{-1}BX \leq \mathbf{1}_n\mathbf{1}_n^T. \tag{5.22}$$

(i): Let $(i,j) \in E^C(A) \cap E^C(B)$ be given. It follows from Proposition 2.2.8 that $a_{ij}x_j = x_i$ and $b_{ij}x_j = x_i$.

$$\frac{a_{ij}x_j}{x_i} = \frac{b_{ij}x_j}{x_i} = 1.$$

Hence
$$a_{ij} = b_{ij} = \frac{x_i}{x_j}$$
and this completes the proof.

(ii): Let $(i,j) \in E^C(A)$ and $(j,i) \in E^C(B)$. From Proposition 2.2.8, we get $a_{ij}x_j = x_i$ and $b_{ji}x_i = x_j$.
$$\frac{a_{ij}x_j}{x_i} = \frac{b_{ji}x_i}{x_j} = 1,$$
and hence
$$a_{ij}b_{ji} = \frac{a_{ij}x_j}{x_i} \cdot \frac{b_{ji}x_i}{x_j} = 1.$$
□

We next recall the following result, which was established in [KSS12] and shows that commutativity is a sufficient condition for the existence of a common eigenvector for irreducible matrices.

Proposition 5.3.3. *Consider the set $\Psi \subset \mathbb{R}_+^{n\times n}$ in (5.15). Assume that each $A_i \in \Psi$ is irreducible and moreover that*
$$A_i \otimes A_j = A_j \otimes A_i \quad \textit{for } 1 \le i,j \le m. \tag{5.23}$$
Then there exists some $x \in int(\mathbb{R}_+^n)$ with $A_i \otimes x = \mu(A_i)x$ for $1 \le i \le m$.

The next corollary is an immediate consequence of Proposition 5.3.3 and the fact that for an irreducible matrix A, the set $\mathcal{C}_A$ is the subeigencone $V^*(A)$, which coincides with the eigencone $V(\hat{A}^*)$.

Corollary 5.3.4. *Consider the set $\Psi \subset \mathbb{R}_+^{n\times n}$ in (5.15). Assume that each $A_i \in \Psi$ is irreducible and moreover that*
$$\hat{A}_i^* \otimes \hat{A}_j^* = \hat{A}_j^* \otimes \hat{A}_i^* \quad \textit{for } 1 \le i,j \le m. \tag{5.24}$$
Then there exists some $x \in int(\mathbb{R}_+^n)$ with $A_i \otimes x \le \mu(A_i)x$ for $1 \le i \le m$.

5.3.2 *SR*-matrices and Globally Optimal Solutions

In the remainder of this section, we will only focus on *SR*-matrices. We first present the following corollary of Theorem 5.3.1 which develops the concept of simultaneous visualisation for SR-matrices. Before stating it, we define the *anticritical graph* of an SR-matrix to consist of the edges $E^{\overline{C}}(A)$ given by:

$$(i,j) \in E^{\overline{C}}(A) \Leftrightarrow (j,i) \in E^{C}(A). \tag{5.25}$$

Corollary 5.3.5. *Consider the set $\Psi \subset \mathbb{R}_+^{n\times n}$ in (5.15). Assume that each $A_i \in \Psi$ is an SR-matrix. If any of the equivalent statements of Theorem 5.3.1 holds, then there exists some $x \in int(\mathbb{R}_+^n)$ such that for $X = \mathrm{diag}\ (x)$ and $1 \le i \le m$, we have*

$$\mu^{-1}(A_i) \cdot \mathbf{1}_n \mathbf{1}_n^T \le X^{-1} A_i X \le \mu(A_i) \cdot \mathbf{1}_n \mathbf{1}_n^T. \tag{5.26}$$

In particular,

$$\begin{aligned} (X^{-1}A_iX)_{kl} &= \mu(A_i) \text{ for all } (k,l) \in E^C(A_i) \\ (X^{-1}A_iX)_{kl} &= \mu^{-1}(A_i) \text{ for all } (k,l) \in E^{\overline{C}}(A_i). \end{aligned} \tag{5.27}$$

Proof: (We drop the subscript for the ease of use.) Let Theorem 5.3.1 (ii) hold for all $A \in \Psi$. Then, there exists some $x \in \mathrm{int}(\mathbb{R}_+^n)$ such that $A \otimes x \le \mu(A)x$ for all $A \in \Psi$. The right-hand side inequality of (5.26) follows by setting $X = \mathrm{diag}\ (x)$. For the remaining left-hand side inequality of (5.26), we observe that $x_i^{-1} a_{ij} x_j \le \mu(A)$ is equivalent to $x_j^{-1} a_{ij}^{-1} x_i \ge \mu^{-1}(A)$ for all $A \in \Psi$. Then, we apply $a_{ij}^{-1} = a_{ji}$ since A is an SR-matrix.

Let $(k,l) \in E^C(A)$ for some $A \in \Psi$. Then, it directly follows from Proposition 2.2.8 that $a_{kl}x_l = \mu(A)x_k$. Since $(k,l) \in E^C(A)$, $(l,k) \in E^{\overline{C}}(A)$ from (5.25). Similarly, using the SR-matrix property, we get $a_{lk}x_k = \mu(A_i)^{-1}x_l$. Then, (5.27) follows by setting $X = \mathrm{diag}\ (x)$. □

Remark 5.3.1. We note that two distinct SR-matrices A, B in $\mathbb{R}_+^{2\times 2}$ cannot have a common subeigenvector. Let

$$A = \begin{bmatrix} 1 & a \\ 1/a & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & b \\ 1/b & 1 \end{bmatrix}$$

and assume that $A \neq B$. Clearly, $\mu(A) = \mu(B) = 1$ and $\hat{S} = A \oplus B$. If $a > b$, then $1/a < 1/b$ and $\mu(\hat{S}) = a/b > 1$. If $b > a$, then $1/b < 1/a$ and $\mu(\hat{S}) = b/a > 1$. In both cases, $\mu(\hat{S}) \neq 1$. Hence by Theorem 5.3.1, A and B do not have a common subeigenvector.

Remark 5.3.2. Consider a 3×3 *SR*-matrix A. It is immediate that all cycle products of length one and two in $D(A)$ are equal to 1. Further, there are two possible cycle products of length 3 in $D(A)$: $a_{12}a_{23}a_{31}$ and $a_{13}a_{32}a_{21}$. As A is an *SR*-matrix, it follows that

$$a_{12}a_{23}a_{31} = \frac{1}{a_{13}a_{32}a_{21}}.$$

Since $\mu(A) \geq 1$, at least one of the above products must be greater than or equal to 1 and $D^C(A)$ must contain either $(1, 2, 3, 1)$ or $(1, 3, 2, 1)$. In this case, one of the cycles of length three is critical, and the other cycle is anticritical. We conclude that $N^C(A) = N(A)$ and A^C is irreducible. It follows from Proposition 5.1.2 that A has a unique subeigenvector (up to a scalar multiple) in $\mathcal{C}_A$ which is its max eigenvector.

Proposition 5.3.3 shows that commuting irreducible matrices possess a common max eigenvector. We now show that for 3×3 *SR*-matrices, commutativity is both necessary and sufficient for the existence of a common subeigenvector.

Theorem 5.3.6. *Let a set $\{A, B\} \subset \mathbb{R}_+^{3\times 3}$ of SR-matrices be given. Write $\hat{S} = \hat{A} \oplus \hat{B}$. The following are equivalent.*

(i) $\mu(\hat{S}) = 1$*;*

(ii) A and B commute;

(iii) There exists a vector $x \in int(\mathbb{R}_+^n)$ with $A \otimes x \leq \mu(A)x$, $B \otimes x \leq \mu(B)x$.

Proof: The equivalence of (i) and (iii) follows immediately from Theorem 5.3.1 so we will show that (ii) and (iii) are also equivalent.

(ii) $\Rightarrow$ (iii) follows immediately from Proposition 5.3.3. We prove (iii)$\Rightarrow$(ii). First note that it follows from Remark 5.3.2 that for distinct i, j, k, the edges (i, j), (j, k) are either both critical or both anti-critical for A. The same is true for B. Calculating $X^{-1}AX$ and $X^{-1}BX$ where $X = \text{diag}\,(x)$, it follows from Theorem 5.3.1 and Corollary 5.3.5 that $a_{ij}b_{jk} = b_{ij}a_{jk} = \mu(A)\mu(B)x_i/x_k$ or $a_{ij}b_{jk} = b_{ij}a_{jk} = \mu(A)\mu^{-1}(B)x_i/x_k$ for any distinct i, j, k. Thus,

$$a_{ij}b_{jk} = b_{ij}a_{jk} \tag{5.28}$$

for any $i, j, k \in \{1, 2, 3\}, k \neq i,\ k \neq j,\ i \neq j$. It now follows from (5.28) that for $i \neq j$

$$\begin{aligned}(A \otimes B)_{ij} &= a_{ii}b_{ij} \oplus a_{ij}b_{jj} \oplus a_{ik}b_{kj} \\ &= b_{ij}a_{jj} \oplus b_{ii}a_{ij} \oplus b_{ik}a_{kj} \\ &= (B \otimes A)_{ij}\end{aligned}$$

where $k \neq i,\ k \neq j$. Rewriting (5.28) as $a_{ji}b_{ik} = b_{ji}a_{ik}$, it follows readily that $b_{ik}a_{ki} = a_{ij}b_{ji}$ and $a_{ik}b_{ki} = b_{ij}a_{ji}$. It now follows that for $1 \leq i \leq 3$,

$$\begin{aligned}(A \otimes B)_{ii} &= a_{ii}b_{ii} \oplus a_{ij}b_{ji} \oplus a_{ik}b_{ki} \\ &= b_{ii}a_{ii} \oplus b_{ij}a_{ji} \oplus b_{ik}a_{ki} \\ &= (B \otimes A)_{ii}\end{aligned}$$

Thus, $A \otimes B = B \otimes A$ as claimed. $\square$

It is straightforward to extend the above result to an arbitrary finite set of SR-matrices in $\mathbb{R}_+^{3\times3}$.

Theorem 5.3.7. *Let a set $\{A_1, \ldots, A_m\} \subset \mathbb{R}_+^{3\times3}$ of SR-matrices be given. Write $\hat{S} = \hat{A}_1 \oplus \cdots \oplus \hat{A}_m$. The following are equivalent.*

(i) $\mu(\hat{S}) = 1$;

(ii) $A_i \otimes A_j = A_j \otimes A_i$ *for all* i, j;

(iii) There exists a vector $x \in int(\mathbb{R}_+^n)$ *with* $A_i \otimes x \leq \mu(A_i)x$ *for all* i.

Proof: As above, the equivalence of (i) and (iii) follows immediately from Theorem 5.3.1 so we will show that (i) and (ii) are also equivalent.

(ii) $\Rightarrow$ (i) follows immediately from Proposition 5.3.3. To show that (i) $\Rightarrow$ (ii), suppose $\mu(\hat{S}) = 1$. Then it follows that for all i, j,

$$\hat{A}_i \oplus \hat{A}_j \leq \hat{S}$$

and hence that $\mu(\hat{A}_i \oplus \hat{A}_j) \leq 1$. As $\mu(\hat{A}_i \oplus \hat{A}_j) \geq 1$, it is immediate that

$$\mu(\hat{A}_i \oplus \hat{A}_j) = 1$$

for all i, j in $\{1, \ldots, m\}$. It follows immediately from Theorem 5.3.6 that

$$A_i \otimes A_j = A_j \otimes A_i$$

for $1 \leq i, j \leq m$ as claimed. $\square$

We note with the following example that commutativity is not a necessary condition for 4×4 SR-matrices to possess a common subeigenvector.

Example 5.3.2. Consider the SR-matrices given in Example 5.3.1. They have a common subeigenvector. However, it can be readily verified that $A \otimes B \neq B \otimes A$.

5.4 Min-max Optimal Solutions

In general, given a set of SR-matrices, it will not always be possible to find a single vector x that is globally optimal in the sense considered in the previous section. With this in mind, we next consider a different notion of optimal solution for the multiple objective functions $e_{A_i} : int(\mathbb{R}^n_+) \to \mathbb{R}_+$, $1 \leq i \leq m$. In fact, we consider the following optimisation problem.

$$\min_{x \in \mathrm{int}(\mathbb{R}^n_+)} \left\{ \max_{1 \leq i \leq m} e_{A_i}(x) \right\}. \tag{5.29}$$

We are seeking a weight vector that minimises the maximal relative error where the maximum is taken over the m criteria. Note that since we later show that there always exists such a minimum point, we use min notation instead of inf. In the following, we first consider extensions of the results in Section 5.1.3 for an irreducible matrix to a set of irreducible matrices. We then relate min-max optimal solutions to the max generalised spectral radius.

5.4.1 Max Generalised Spectral Radius

Inspired by the approach to the single criterion problem, we associate a set $\mathcal{C}_{\Psi,r}$ with a set Ψ of irreducible matrices and show how the properties of $\mathcal{C}_{A,r}$ in (5.10) for a single irreducible matrix extend to this new setting.

For $r > 0$, define

$$\mathcal{C}_{\Psi,r} = \{x \in \mathrm{int}(\mathbb{R}^n_+) \mid e_{A_i}(x) \leq r \text{ for all } A_i \in \Psi\} = \bigcap_{i=1}^{m} \mathcal{C}_{A_i,r}. \tag{5.30}$$

We also consider the set of normalised vectors:

$$\mathcal{D}_{\Psi,r} = \{x \in \mathcal{C}_{\Psi,r} \mid x_1 = 1\}. \tag{5.31}$$

We will use the notations $\mathcal{C}_\Psi$ for $\mathcal{C}_{\Psi,\mu(\Psi)}$ and $\mathcal{D}_\Psi$ for $\mathcal{D}_{\Psi,\mu(\Psi)}$ where $\mu(\Psi)$ is the max generalised spectral radius of Ψ. The following result shows that the set $\mathcal{C}_{\Psi,r}$ and the set $\mathcal{C}_{S,r}$ are equal. This will allow us to readily extend algebraic and geometric properties of $\mathcal{C}_{A,r}$ established in [EvdD10] to sets of matrices.

Theorem 5.4.1. *Consider the set $\Psi \subset \mathbb{R}_+^{n\times n}$ given by (5.15). Assume that A_i is irreducible for $1 \leq i \leq m$ and let S be given by (5.18). Then:*

$$\mathcal{C}_{\Psi,r} = \mathcal{C}_{S,r}.$$

Proof: Let x be in $\mathcal{C}_{\Psi,r}$ be given. Then, $e_{A_i}(x) \leq r$ which implies $A_i \otimes x \leq rx$ for each $A_i \in \Psi$ from (ii) in Proposition 5.1.1. Taking the maximum of both sides from 1 to m, we see that $S \otimes x \leq rx$. It follows that $x \in \mathcal{C}_{S,r}$. Thus, $\mathcal{C}_{\Psi,r} \subset \mathcal{C}_{S,r}$.

Now choose some $x \in \mathcal{C}_{S,r}$. Then $e_S(x) \leq r$ from (ii) in Proposition 5.1.1. Since $e_{A_i}(x) \leq e_S(x) \leq r$ for all $1 \leq i \leq m$, we obtain $x \in \mathcal{C}_{\Psi,r}$. Thus, $\mathcal{C}_{S,r} \subset \mathcal{C}_{\Psi,r}$. Hence $\mathcal{C}_{\Psi,r} = \mathcal{C}_{S,r}$. $\square$

The following corollary extends Proposition 5.1.1 to a set of irreducible matrices Ψ. Noting that $\mu(\Psi) = \mu(S)$ from Theorem 4.4.3, it is an immediate consequence of Theorem 5.4.1.

Corollary 5.4.2. *Consider the set $\Psi \subset \mathbb{R}_+^{n\times n}$ given by (5.15). Assume that A_i is irreducible for $1 \leq i \leq m$ and let S be given by (5.18). Then:*

(i) $\mathcal{C}_{\Psi,r} \neq \emptyset \iff r \geq \mu(\Psi)$;

(ii) $x \in \mathcal{C}_{\Psi,r} \iff e_S(x) \leq r$.

Since Theorem 5.4.1 establishes that $\mathcal{C}_{\Psi,r} = \mathcal{C}_{S,r}$, it is immediate that we also have $\mathcal{D}_{\Psi,r} = \mathcal{D}_{S,r}, \mathcal{C}_\Psi = \mathcal{C}_S$ and $\mathcal{D}_\Psi = \mathcal{D}_S$. Therefore, studying these sets for a collection of irreducible matrices reduces to studying the sets associated with the single matrix S. This fact means that the properties of $\mathcal{C}_{A,r}$ discussed in

Theorem 1 of [EvdD10] can be directly extended to $\mathcal{C}_{\Psi,r}$. We use the following notation.

$$x^j = (x_1^j, ..., x_n^j)^T \text{ where } x \in \text{int}(\mathbb{R}_+^n), j \in \mathbb{R}.$$

$$x \circ y = (x_1 y_1, ..., x_n y_n)^T \text{ where } x \in \text{int}(\mathbb{R}_+^n), y \in \text{int}(\mathbb{R}_+^n).$$

$$\Psi^T = \{A_1^T, A_2^T, ..., A_m^T\}.$$

Theorem 5.4.3. *Assume that all matrices in the set $\Psi \subset \mathbb{R}_+^{n\times n}$ given in (5.15) are irreducible. Let S be given by (5.18). Then:*

(i) $x \in \mathcal{C}_{\Psi,r} \Leftrightarrow x^{-1} \in \mathcal{C}_{\Psi^T,r}$;

(ii) *If* $x, y \in \mathcal{C}_{\Psi,r}$ *then*

- *a.* $\gamma x + (1-\gamma)y \in \mathcal{C}_{\Psi,r} \forall \gamma \in [0,1]$. *($\mathcal{C}_{\Psi,r}$ is a convex cone.)*
- *b.* $x^\gamma \circ y^{1-\gamma} \in \mathcal{C}_{\Psi,r} \forall \gamma \in [0,1]$.
- *c.* $(x^p + y^p)^{1/p} \in \mathcal{C}_{\Psi,r} \forall p \in \mathbb{R}, p \neq 0$.
- *d.* $x \oplus y \in \mathcal{C}_{\Psi,r}$. *($\mathcal{C}_{\Psi,r}$ is a max convex set.)*
- *e.* $x^{-1} \oplus y^{-1} = (\min(x_i, y_i)) \in \mathcal{C}_{\Psi,r}$.

(iii) $\mathcal{D}_{\Psi,r}$ *is compact;* $\mathcal{D}_\Psi$ *is a convex polytope.*

(iv) *The vector* $x = (x_i) = (\min(y_i, y \in \mathcal{D}_\Psi))$ *is in* $\mathcal{D}_\Psi$. *If in addition* $1 \in N^C(S)$, *then* x *is a max eigenvector of* S.

(v) *Fot* $y = (y_i) = (\max(z_i, z \in \mathcal{D}_\Psi))$, y^{-1} *is in* $\mathcal{D}_{\Psi^T}$. *If in addition* $1 \in N^C(S)$, *then* y^{-1} *is a max eigenvector of* S^T.

(vi) $\mathcal{D}_\Psi$ *consists of only one vector if and only if* S^C *is irreducible and* $N^C(S) = N(S)$.

Proof: We only prove statement (i). As $\mathcal{C}_{\Psi,r} = \mathcal{C}_{S,r}$ from Theorem 5.4.1, the others are immediate from Theorem 1 of [EvdD10].

(i): From Proposition 5.1.1 (ii), $x \in \mathcal{C}_{\Psi,r}$ if and only if $\max_{i,j} s_{ij} x_j / x_i = \max_{i,j} s_{ji}^T (1/x_i)/(1/x_j) \leq r$. Since $S^T = A_1^T \oplus A_2^T \oplus ... \oplus A_m^T$, $\mathcal{C}_{\Psi^T,r} = \mathcal{C}_{S^T,r}$ from Theorem 5.4.1. Thus, it is equivalent to $x^{-1} \in \mathcal{C}_{\Psi^T,r}$. $\square$

We remark that the sets $\mathcal{C}_{\Psi,r}$ and $\mathcal{D}_{\Psi,r}$ have appeared under various names like Kleene cones [SSB09] (See Section 2.2.3 for the definition of the Kleene cone.), tropical polytropes [JK10], or zones [Min04].

5.4.2 Minimum Error Requirement Set for $\boldsymbol{SR}$-matrices

First, we state the following immediate result on the relation of $e_S(x)$ and $e_{A_i}(x)$ for $i = 1, 2, ..., m$.

Lemma 5.4.4. *Let $\Psi \subset \mathbb{R}_+^{n\times n}$ be a set of irreducible matrices given by (5.15). For $x \in int(\mathbb{R}_+^n)$, $e_S(x) = \max_{1\le i\le m} e_{A_i}(x)$.*

Proof: Using the definition of S in (5.18),

$$\begin{aligned}
e_S(x) &= \max_{i,j}(a_{ij}^1 \oplus a_{ij}^2 \oplus ... \oplus a_{ij}^m)x_j/x_i \\
&= \max_{i,j}(\max(a_{ij}^1, a_{ij}^2, ..., a_{ij}^m))x_j/x_i \\
&= \max(\max_{i,j} a_{ij}^1 x_j/x_i, \max_{i,j} a_{ij}^2 x_j/x_i, ..., \max_{i,j} a_{ij}^m x_j/x_i) \\
&= \max_{i,j} a_{ij}^1 x_j/x_i \oplus \max_{i,j} a_{ij}^2 x_j/x_i \oplus ... \oplus \max_{i,j} a_{ij}^m x_j/x_i \\
&= e_{A_1}(x) \oplus e_{A_2}(x) \oplus ... \oplus e_{A_m}(x)
\end{aligned}$$

□

Corollary 5.4.2 has the following interpretation together with Lemma 5.4.4 in terms of the optimisation problem given in (5.29).

Proposition 5.4.5. *Consider the set Ψ given by (5.15). Assume that A_i is irreducible for $1 \le i \le m$. Then:*

(i) $\mu(\Psi) = \min_{x\in int(\mathbb{R}_+^n)} \left(\max_{1\le i\le m} e_{A_i}(x)\right)$;

(ii) x solves (5.29) if and only if $x \in \mathcal{C}_\Psi$.

Proof: Corollary 5.4.2 shows that there exists some $x \in \mathrm{int}(\mathbb{R}_+^n)$ with

$$\max_{1\le i\le m} e_{A_i}(x) \le r$$

if and only if $r \ge \mu(\Psi)$. (i) follows from this observation. The result of (ii) is then immediate from the definition of $\mathcal{C}_\Psi$. □

Inspired by Proposition 5.4.5, we call $\mathcal{C}_\Psi$ as *a minimum error requirement set for* Ψ. We illustrate Proposition 5.4.5 below.

Example 5.4.1. Consider the *SR*-matrices given in Example 5.3.1.

$$S = A \oplus B = \begin{bmatrix} 1 & 8 & 5 & 9 \\ 1/4 & 1 & 6 & 9 \\ 4 & 8 & 1 & 4 \\ 1/7 & 4 & 8 & 1 \end{bmatrix}$$

where $\mu(S) = 8.3203$. The vector $x = \begin{bmatrix} 1 & 1 & 0.962 & 0.924 \end{bmatrix}^T$ is in $\mathcal{C}_\Psi$. We see that $e_S(x) = 8.3203$.

5.5 Pareto Optimal Solutions

Thus far, we have considered two different approaches to the multi-objective optimisation problem associated with the AHP. In this section we turn our attention to what is arguably the most common framework adopted in multi-objective optimisation: Pareto Optimality [Mie99, Bew07].

As above, we are concerned with the existence of optimal points for the set of objective functions e_{A_i}, for $1 \leq i \leq m$ associated with the set Ψ of *SR*-matrices. A Pareto optimal point for $e_{A_1}, \ldots, e_{A_m}$ has the property that any other point that decreases one of the functions must increase the value of another function.

5.5.1 Existence of Pareto Optimal Solutions

We first recall the notion of weak Pareto optimality.

Definition 5.5.1. $w \in int(\mathbb{R}^n_+)$ is said to be a *weak Pareto optimal point* for the functions $e_{A_i} : int(\mathbb{R}^n_+) \to \mathbb{R}_+ (1 \leq i \leq m)$ if there does not exist $x \in \text{int}(\mathbb{R}^n_+)$, $x \neq w$ such that

$$e_{A_i}(x) < e_{A_i}(w)$$

for all $i = 1, 2, ..., m$.

The next lemma shows that every point in the set $\mathcal{C}_\Psi$ is a weak Pareto optimal point for $e_{A_1}, \ldots, e_{A_m}$.

Lemma 5.5.1. *Let $\Psi \subset \mathbb{R}_+^{n\times n}$ given by (5.15) consist of irreducible matrices. Any $w \in \mathcal{C}_\Psi$ is a weak Pareto optimal point for $e_{A_1}, \dots, e_{A_m}$.*

Proof: Let $w \in \mathcal{C}_\Psi$ be given. Then $e_{A_i}(w) \leq \mu(\Psi)$ for $1 \leq i \leq m$. If there exists some $x \in \text{int}(\mathbb{R}_+^n)$ such that $e_{A_i}(x) < e_{A_i}(w)$ for $1 \leq i \leq m$, then for this x

$$e_{A_i}(x) < \mu(\Psi)$$

for $1 \leq i \leq m$. This contradicts Proposition 5.4.5. □

We next recall the usual definition of a Pareto optimal point.

Definition 5.5.2. $w \in int(\mathbb{R}_+^n)$ is said to be a *Pareto optimal point* for the functions $e_{A_i} : int(\mathbb{R}_+^n) \to \mathbb{R}_+ (1 \leq i \leq m)$ if $x \neq w$, $e_{A_i}(x) \leq e_{A_i}(w)$ for $1 \leq i \leq m$ implies $e_{A_i}(x) = e_{A_i}(w)$ for all $1 \leq i \leq m$.

We later show that the multi-objective optimisation problem associated with the AHP always admits a Pareto optimal point. We first present some simple facts concerning such points.

Theorem 5.5.2. *Let $\Psi \subset \mathbb{R}_+^{n\times n}$ given by (5.15) consist of irreducible matrices. Then:*

(i) If $w \in \mathcal{C}_\Psi$ is unique (up to a scalar multiple), then it is a Pareto optimal point for $e_{A_1}, \dots, e_{A_m}$;

(ii) If $w \in \mathcal{C}_{A_i}$ is unique (up to a scalar multiple) for at least one $i \in \{1, 2, ..., m\}$, then it is a Pareto optimal point for $e_{A_1}, \dots, e_{A_m}$.

Proof: (i) Assume that $w \in \mathcal{C}_\Psi$ is unique (up to a scalar multiple). Pick some $x \in \text{int}(\mathbb{R}_+^n)$ such that $e_{A_i}(x) \leq e_{A_i}(w)$ for all i. Then, $e_{A_i}(x) \leq \mu(\Psi)$ for all i which implies that $x \in \mathcal{C}_\Psi$. Thus, $x = \alpha w$ for some $\alpha \in \mathbb{R}_+$. Hence, $e_{A_i}(x) = e_{A_i}(w)$ for all i and w is a Pareto optimal point.

(ii) Assume that for some $A_i \in \Psi$, $w \in \mathcal{C}_{A_i}$ is unique up to scalar multiple. Suppose $x \in \text{int}(\mathbb{R}_+^n)$ is such that $e_{A_j}(x) \leq e_{A_j}(w)$ for all $1 \leq j \leq m$. In particular, $x \in \mathcal{C}_{A_i}$, and this implies that $x = \alpha w$ for some $\alpha \in \mathbb{R}$. Further, it is immediate that for any other $A_j \in \Psi$ $(i \neq j)$, we have $e_{A_j}(x) = e_{A_j}(w)$. Thus, w is a Pareto optimal point. □

By Theorem 5.4.3 (vi), condition (i) implies that $N^C(S) = N(S)$ and S^C is irreducible. By Proposition 5.1.2, condition (ii) implies that $N^C(A_i) = N(A_i)$ and A_i^C is irreducible for some $A_i \in \Psi$.

Corollary 5.5.3. *Let the set $\Psi \subset \mathbb{R}_+^{n\times n}$ given by (5.15) consist of SR-matrices. For $n \in \{2,3\}$, any $w \in \mathcal{C}_{A_i}(1 \leq i \leq m)$ is a Pareto optimal point for $e_{A_1}, \ldots, e_{A_m}$.*

Proof: Notice that from Remark 5.3.2 for 3×3 case, there exists a unique subeigenvector (up to a scalar multiple) in each $\mathcal{C}_{A_i}$ for $1 \leq i \leq m$. This is also true for the 2×2 case because $N^C(A) = N(A)$ and A^C is irreducible. The result directly follows from (ii) in Theorem 5.5.2. $\square$

The following example demonstrates point (i) in Theorem 5.5.2.

Example 5.5.1. Consider the following matrices given by

$$A = \begin{bmatrix} 1 & 9 & 1/4 & 2 \\ 1/9 & 1 & 6 & 3 \\ 4 & 1/6 & 1 & 1/4 \\ 1/2 & 1/3 & 4 & 1 \end{bmatrix} B = \begin{bmatrix} 1 & 1/2 & 4 & 1/8 \\ 2 & 1 & 3 & 2 \\ 1/4 & 1/3 & 1 & 5 \\ 8 & 1/2 & 1/5 & 1 \end{bmatrix}.$$

The S matrix is given by

$$S = \begin{bmatrix} 1 & 9 & 4 & 2 \\ 2 & 1 & 6 & 3 \\ 4 & 1/3 & 1 & 5 \\ 8 & 1/2 & 4 & 1 \end{bmatrix}.$$

By following the same way in Example 2.2.1, we find $D^C(S)$. We see that $N^C(S) = N(S)$ and S^C is irreducible. By Theorem 5.4.3, there is a unique vector, w, (up to a scalar multiple) in $\mathcal{C}_\Psi$. We obtain this vector by using Algorithm 3. Then, $w = \begin{bmatrix} 1 & 0.758 & 0.861 & 1.174 \end{bmatrix}^T$.

Figure 5.2 shows the values of $e_A(x)$ and $e_B(x)$ at w and at several points from the sets $\mathcal{C}_A$, $\mathcal{C}_B$ and from $\mathrm{int}(\mathbb{R}_+^n)$. Pareto optimality is observed at w where $e_A(w) = 6.817$ and $e_B(w) = 6.817$. In general, $e_A(w)$ and $e_B(w)$ are not necessarily equal. We specifically choose this example to illustrate the role of the Pareto optimal point. For instance, it can be seen from the graphic that for any point $x \in \mathcal{C}_B$, $e_B(x) < e_B(w)$ while $e_A(x) > e_A(w)$.

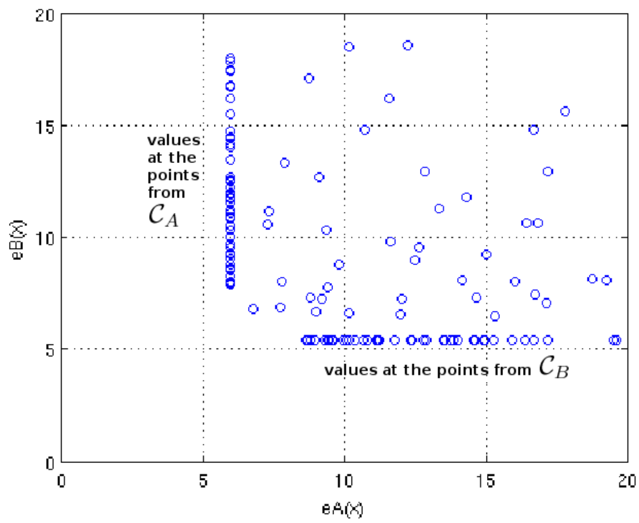


Figure 5.2: Refer to Example 5.5.1

Our objective in the remainder of this section is to show that the multi-objective optimisation problem associated with the AHP always admits a Pareto optimal solution. We first recall the following general result giving a sufficient condition for the existence of a Pareto optimal point with respect to a set $E \subset \mathbb{R}^n$ [Bew07]. Essentially, this is a direct application of the fact that a continuous function on a compact set always attains its minimum.

Theorem 5.5.4. *Let $E \subseteq \mathbb{R}^n$ be nonempty and compact. Let a set $\{f_1, \ldots, f_m\}$ of continuous functions be given where*

$$f_i : E \to \mathbb{R}_+$$

for $1 \leq i \leq m$. There exists $w \in E$ such that $x \in E$, $f_i(x) \leq f_i(w)$ for $1 \leq i \leq m$ implies $f_i(x) = f_i(w)$ for $1 \leq i \leq m$.

This result follows from elementary real analysis and the observation that if w minimises the (continuous) weighted sum $\sum_{i=1}^{m} \alpha_i f_i(x)$ where $\alpha_i > 0$ for $1 \leq i \leq m$, then w must be Pareto optimal for the functions $f_1, \ldots, f_m$. A point w satisfying the conclusion of the theorem is said to be *Pareto optimal for $f_1, \ldots, f_m$ with respect to E.*

In essence the above result shows that for any multi-objective optimisation problem with continuous objective functions defined on a compact set, there exists a point that is Pareto optimal with respect to the given set. To apply the above result to the AHP, we first note that for a set Ψ of *SR*-matrices, $\mathcal{D}_\Psi$ is compact from Theorem 5.4.3 (iii). Moreover, each function e_{A_i} is continuous on D_Ψ as it is a composition of continuous functions. We will also prove this by using the definition of continuity.

Lemma 5.5.5. *The error functions* $e_{A_i} : int(\mathbb{R}^n_+) \to \mathbb{R}_+ (1 \leq i \leq m)$ *are continuous on* D_Ψ.

Proof: Take some $A \in \Psi$. (we drop the subscript to make the subsequent notation clearer.) Let the sequence $x^k \in D_\Psi$ satisfy $\lim\limits_{k\to\infty} x^k = x^0$ where $x^0 \in D_\Psi$. We must show that $e_A(x^k) \to e_A(x^0)$ as $k \to \infty$.

As $D_\Psi \subset \text{int}(\mathbb{R}^n_+)$, for all i, j, $\frac{a_{ij}x_j^k}{x_i^k} \to \frac{a_{ij}x_j^0}{x_i^0}$ as $k \to \infty$. Suppose that $e_A(x^0) = \max\limits_{i,j} \frac{a_{ij}x_j^0}{x_i^0} = \frac{a_{pq}x_q^0}{x_p^0}$ for some $p, q \in \{1, 2, ..., n\}$. Then $\frac{a_{pq}x_q^k}{x_p^k} \to \frac{a_{pq}x_q^0}{x_p^0}$ as $k \to \infty$.

Let $\epsilon > 0$ be given. Then there exists some N_1 such that for $k \geq N_1$,

$$\frac{a_{pq}x_q^k}{x_p^k} \geq \frac{a_{pq}x_q^0}{x_p^0} - \epsilon.$$

Thus, for $k \geq N_1$,

$$e_A(x^k) \geq \frac{a_{pq}x_q^k}{x_p^k} \geq e_A(x^0) - \epsilon.$$

On the other hand, for all i, j, we know that $\frac{a_{ij}x_j^0}{x_i^0} \leq \frac{a_{pq}x_q^0}{x_p^0}$. As $\frac{a_{ij}x_j^k}{x_i^k} \to \frac{a_{ij}x_j^0}{x_i^0}$ as $k \to \infty$, there exists some N_2 such that for $k \geq N_2$,

$$\frac{a_{ij}x_j^k}{x_i^k} \leq \frac{a_{pq}x_q^0}{x_p^0} + \epsilon$$

for all i, j. This implies that $e_A(x^k) \leq e_A(x^0) + \epsilon$ for all $k \geq N_2$. Choosing, $N = \max\{N_1, N_2\}$, we see that for $k \geq N$,

$$e_A(x^0) - \epsilon \leq e_A(x^k) \leq e_A(x^0) + \epsilon.$$

As $\epsilon > 0$ was arbitrary, we immediately see that $\lim\limits_{k\to\infty} e_A(x^k) = e_A(x^0)$. $\square$

We now show that any point in $\mathcal{D}_\Psi$ that is Pareto optimal with respect to $\mathcal{D}_\Psi$ is also Pareto optimal with respect to the set $\text{int}(\mathbb{R}^n_+)$. Although we don't specifically use the *SR* property in the following, we assume Ψ to consist of *SR*-matrices as we are primarily interested in the AHP application.

Lemma 5.5.6. *Consider the set Ψ in (5.15) and assume that A_i is an SR-matrix for $1 \leq i \leq m$. Let w be a Pareto optimal point for $e_{A_1}, \ldots, e_{A_m}$ with respect to $\mathcal{D}_\Psi$. Then, w is also Pareto optimal for $e_{A_1}, \ldots, e_{A_m}$ with respect to int$(\mathbb{R}^n_+)$.*

Proof: Assume that $w \in \mathcal{D}_\Psi$ is a Pareto optimal point with respect to $\mathcal{D}_\Psi$. Suppose $x \in \text{int}(\mathbb{R}^n_+)\backslash\mathcal{C}_\Psi$. Then from the definition of $\mathcal{C}_\Psi$ (5.30), it follows that

$$e_{A_{i_0}}(x) > \mu(\Psi) \text{ for some } i_0. \tag{5.32}$$

As $w \in \mathcal{D}_\Psi$, $e_{A_i}(w) \leq \mu(\Psi)$ for $1 \leq i \leq m$. It follows immediately from (5.32) that for any $x \notin \mathcal{C}_\Psi$, it cannot happen that $e_{A_i}(x) \leq e_{A_i}(w)$ for $1 \leq i \leq m$.

Let x in $\mathcal{C}_\Psi$ be such that

$$e_{A_i}(x) \leq e_{A_i}(w) \quad \text{for } 1 \leq i \leq m.$$

As $e_{A_i}(\lambda x) = e_{A_i}(x)$ for all $\lambda > 0$, $1 \leq i \leq m$, and w is Pareto optimal with respect to $\mathcal{D}_\Psi$, it follows that $e_{A_i}(x) = e_{A_i}(w)$ for $1 \leq i \leq m$. $\square$

Our next step is to show that there exists a point $x \in \mathcal{D}_\Psi$ that is Pareto optimal with respect to $\mathcal{D}_\Psi$.

Proposition 5.5.7. *Consider the set Ψ in (5.15) and assume that A_i is an SR-matrix for $1 \leq i \leq m$. There exists $x \in \mathcal{D}_\Psi$ that is Pareto optimal for $e_{A_1}, \ldots, e_{A_m}$ with respect to $\mathcal{D}_\Psi$.*

Proof: First note that $\mathcal{D}_\Psi \neq \emptyset$ since $\mu(\Psi) > 0$. Theorem 5.4.3 shows that $\mathcal{D}_\Psi$ is compact. Furthermore, the functions $e_{A_i} : \mathcal{D}_\Psi \to \mathbb{R}_+ (1 \leq i \leq m)$ are continuous as shown in Lemma 5.5.5. Theorem 5.5.4 implies that there exists w in $\mathcal{D}_\Psi$ that is Pareto optimal with respect to $\mathcal{D}_\Psi$. $\square$

Combining Proposition 5.5.7 with Lemma 5.5.6, we immediately obtain the following result.

Corollary 5.5.8. *Consider the set Ψ in (5.15) and assume that A_i is an SR-matrix for $1 \leq i \leq m$. There exists $x \in \mathcal{D}_\Psi$ that is Pareto optimal for $e_{A_1}, \ldots, e_{A_m}$ with respect to int$(\mathbb{R}^n_+)$.*

Corollary 5.5.8 means that there exists a vector x of positive weights that is simultaneously Pareto optimal and also optimal in the min-max sense of Section 5.4.2 for the error functions e_{A_i}, $1 \leq i \leq m$.

5.5.2 Pareto Optimisation Problem

As has been discussed above, our approach to construct an overall ranking vector for m given SR-matrices amounts to solving the following optimisation problem.

$$\min_{x \in \mathcal{D}_\Psi} \{e_{A_i}(x)\}, i = 1, 2, ..., m. \tag{5.33}$$

Recall that the solution of the following weighted sum

$$\min_{x \in \mathcal{D}_\Psi} \sum_{i=1}^{m} \alpha_i e_{A_i}(x). \tag{5.34}$$

where $\alpha_i > 0$ for $1 \leq i \leq m$ is a Pareto optimal solution for (5.33). This approach is called the *weighted sum method*, which is one of the most commonly used approaches for multi-objective optimisation problems [Mie99]. The idea in the weighted sum method is to take a linear combination of the objective functions with positive weights. This leads to a constrained optimisation problem with one objective function that can be solved using standard methods and software such as MATLAB.

In (5.34), we choose α as a max eigenvector of the criteria-comparison matrix. Recall from Section 5.1.3 that the max eigenvector of this matrix can be used as a weight vector for the criteria with respect to the main goal. Thus, α_i is the weight of the criterion i for the error function $e_{A_i}(x)$ associated with the SR-matrix A_i for $1 \leq i \leq m$.

To illustrate, we revisit Example 5.1.2.

Example 5.5.2. Let $C, A_1, \ldots, A_5$ be as in Example 5.1.2. We first observe that

$$\mu(\hat{S}) = 4.985.$$

So, there is no common subeigenvector for A_i, $i = 1, 2, ..., 5$.

For the solution of $\min\limits_{x \in \mathcal{D}_\Psi} \sum\limits_{i=1}^{5} \alpha_i e_{A_i}(x)$, we first find α. We know from Example 5.1.3 that there is a unique max eigenvector of C given by

$$\alpha = \begin{bmatrix} 1 \\ 1.495 \\ 2.236 \\ 3.344 \\ 0.897 \end{bmatrix}$$

with the ranking $4 > 3 > 2 > 1 > 5$.

Next, we find the S matrix.

$$S = \begin{bmatrix} 1 & 7 & 7 & 9 \\ 5 & 1 & 6 & 7 \\ 6 & 2 & 1 & 6 \\ 4 & 7 & 7 & 1 \end{bmatrix}$$

Note that $N^C(S) = \{1, 3, 4\}$ and S^C is irreducible. It follows from Theorem 5.4.3 (vi) that there is not a unique vector in D_Ψ. In this case, we find several Pareto optimal points giving at least two possible distinct rankings: $1 > 3 > 4 > 2$ and $1 > 3 > 2 > 4$.

Notice that the first ranking scheme is the same as the one obtained from the classical EM used in Example 5.1.2. We see that α_3 and α_4 are the two highest weight factors that are applied to $e_{A_3}(x)$ and $e_{A_4}(x)$. Thus, they increase the effect of these error functions which correspond to the SR-matrices A_3 and A_4. If we analyse the local rankings associated with these matrices in Example 5.1.2, we see that $4 > 2$ for both matrices.

The second ranking scheme is also reasonable. Similarly, from the local ranking schemes, we see that $2 > 4$ for A_1, A_2 and A_5. In particular, 2 is preferred to all other alternatives for A_2.

5.6 Concluding Remarks

Our main goal in this chapter was to develop a new approach for ranking alternatives in the multi-criteria AHP based on the max algebra, thereby combining the max-algebraic spectral theory and multi-objective optimisation. In this context,

- we presented results on the existence of a single transitive matrix with a minimum distance to all SR-matrices in the set simultaneously and related these with the commutativity properties of matrices;
- we proved that the generalised spectral radius provides a lower bound on the maximal error in approximating a set of SR-matrices with a transitive matrix;
- we showed that it is always possible to find a Pareto optimal point for which the maximal error is minimal.

CHAPTER 6

Max Eigenvectors, Directed Spanning Trees and Ranking

In this chapter, we are concerned with the matrix tree theorem for Markov chains. We consider its max-algebraic extension. In particular, we relate the max-algebraic spectral properties of an irreducible max-stochastic matrix to the maximal weights of rooted directed spanning trees (RSTs) in its associated digraph. We show that the vector of maximal RST weights is always a max eigenvector of the matrix. We also present some results relating this vector to the rows of the Kleene star. Finally, we discuss possible applications of our results to ranking problems.

6.1 Motivation and Mathematical Background

In this section, we provide a brief introduction to the so-called Markov Chain Tree Theorem and introduce a number of definitions and results concerning rooted directed spanning trees (RSTs) of irreducible matrices.

6.1.1 Introduction

A *Markov chain* is a stochastic process over a set of states denoted by $\mathcal{S} = \{1, 2, ..., n\}$ [Sen06]. It is considered as a process without memory so that the probability of being in a state at a given time depends only on the previous state. If we let a_{ij} denote the probability of moving from state i to state j, then the chain can be represented by a matrix $A \in \mathbb{R}_+^{n\times n}$ where n is the number of the states in $\mathcal{S}$. Given a vector x in which x_i represents the probability of being in state i initially, $x(t) = A^t x$ represents the probability of being in state i at time t for $1 \leq i \leq n$. The behaviour of the Markov chain as t tends to infinity is of particular interest.

The transition probability matrix A is generally a non-negative and row stochastic matrix. If A is irreducible, the chain possesses a unique *stationary distribution* vector (Theorem 2.1.5) which is the solution of

$$x^T A = x^T, \sum_{1\leq i\leq n} x_i = 1. \tag{6.1}$$

Recall from Section 2.2.6 that x can be found by the power method. The classical matrix tree theorem for Markov chains provides an alternative way to calculate the stationary distribution vector.

The original Matrix Tree Theorem goes back to the work of Kirchhoff [Kir47]. It provides a characterisation for the number of spanning trees in a graph based on the determinant of an $n \times n$ matrix. Since then, variants of it have been studied in the context of electrical networks. A version of this theorem for irreducible Markov chains (referred to as the Markov Chain Tree Theorem) relates the stationary distribution of an irreducible Markov chain with the weights of directed spanning trees of its associated digraph. This core result has appeared in a variety of different contexts. See [Shu75, KV80, FW84, AT89, Bro89, Ald90, Son99, GP01, Wic09]. One of the earliest appearances of this result is provided by Freĭdlin and Wentzell [FW84] (Translation of Russian ed., Nauka, Moscow, 1979). It was independently discovered by [Shu75] in connection with flow-graph methods. It was also stated in [KV80] in the context of biological modelling. For another reference which discusses its extension to general, not necessarily irreducible Markov chains, see Leighton and Rivest [LR82].

The motivation of our study of the matrix tree theorem for Markov chains comes from its connection with the spectral theory of non-negative matrices and graph theory so that it provides a different characterisation of the Perron vector of an irreducible matrix. In this direction, the layout of this chapter is as follows. First, we describe necessary notation and preliminary results related to RSTs of irreducible matrices. We also state the Markov Chain Tree Theorem over the conventional algebra. In Section 6.2, we derive its max-algebraic version. In Section 6.3, we discuss how to associate our main result with the max-algebraic spectral theory. In particular, we consider its connection with the Kleene star of an irreducible max-stochastic matrix. In Section 6.4, we discuss the possibility of applying of these results to decision making problems. The work contained in this chapter has resulted in the publication: [BGKMS13].

6.1.2 Directed Spanning Trees

We denote the weighted directed graph of $A \in \mathbb{R}_+^{n\times n}$ by $D(A) = (N(A), E(A))$. We say that the edge $e = (i, j)$ is *outgoing* from i and write $t(e) = i$. A spanning subgraph $T = (N(A), E_T)$ of $D(A)$ is a *directed spanning tree rooted at* $i \in \{1, \dots, n\}$ if the following conditions are satisfied:

(i) for every $j \neq i$ in $\{1, \dots, n\}$, there is exactly one outgoing edge $e \in E_T$ with $t(e) = j$;

(ii) there is no edge $e \in E_T$ with $t(e) = i$;

(iii) the subgraph $(N(A), E_T)$ contains no directed cycle.

We recall that directed spanning trees are also referred to as rooted branchings or arborescences by some authors [BR91].

Given a directed spanning tree $T = (N(A), E_T)$, the weight of T is given by the product of the weights of the edges in T and is denoted by $\pi(T)$. We adopt the notation $\mathcal{T}_i$ for the set of all directed spanning trees of $D(A)$ rooted at i for $1 \le i \le n$. We demonstrate these concepts with the following example.

Example 6.1.1. Consider the following 3×3 matrix

$$A = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1/4 & 0 & 1 \\ 1 & 1/3 & 1/9 \end{bmatrix}. \tag{6.2}$$

There are three directed spanning trees rooted at 1 as shown in Figure 6.1. Let $\mathcal{T}_1 = \{T_1^{(1)}, T_1^{(2)}, T_1^{(3)}\}$.

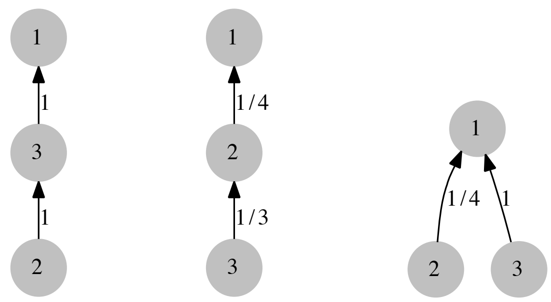


Figure 6.1: $\mathcal{T}_1 = \{T_1^{(1)}, T_1^{(2)}, T_1^{(3)}\}$

- $T_1^{(1)}$: $(2,3),(3,1)$ with $\pi(T_1^{(1)}) = a_{23}a_{31} = 1$
- $T_1^{(2)}$: $(3,2),(2,1)$ with $\pi(T_1^{(2)}) = a_{32}a_{21} = 1/12$
- $T_1^{(3)}$: $(2,1),(3,1)$ with $\pi(T_1^{(3)}) = a_{21}a_{31} = 1/4$

Directed spanning trees rooted at 2 and 3 are constructed in the same way.

For future use, we record the following simple facts concerning directed spanning trees.

Lemma 6.1.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible. Then for every $i \in \{1,\ldots,n\}$, there exists a directed spanning tree rooted at i.*

Proof: To begin with, let P_1 be a directed path of maximal length ending at i. If $N(P_1) = \{1,\ldots,n\}$, then P_1 is a directed spanning tree rooted at i. Otherwise, choose $j \notin N(P_1)$ and let Q_1 be the shortest directed path in $D(A)$ from j to P_1. Let P_2 be the union of P_1 and Q_1. Again, if $N(P_2) = \{1,\ldots,n\}$ then P_2 will be a directed spanning tree rooted at i. Otherwise, we repeat the above procedure and as the number of vertices in $D(A)$ is finite, we must eventually arrive at a directed spanning tree rooted at i as claimed. $\square$

Lemma 6.1.2. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and let $T = (N(A), E_T)$ be a directed spanning tree of $D(A)$ rooted at $i \in \{1, \ldots, n\}$. Then for each $j \neq i$ in $\{1, \ldots, n\}$, there exists a unique directed path from j to i in T.*

Proof: Let $j \neq i$ be given. There exists a unique edge (j, i_1) in E_T outgoing from j. If $i_1 \neq i$, there exists a unique outgoing edge from i_1, (i_1, i_2). Continuing in this fashion, we construct a sequence of vertices $j = i_0, i_1, i_2, \ldots$ such that $(i_j, i_{j+1}) \in E_T$ for $j = 0, 1, \ldots$. As T is acyclic, the vertices $i_0, i_1, i_2, \ldots$ are distinct. Hence, the process must terminate at some vertex i_P, which has no outgoing edge in E_T. Hence, $i_P = i$ and $j = i_0, i_1, \ldots, i_P = i$ is a directed path from j to i. To see uniqueness, suppose there is a distinct path

$$j = j_0, j_1, \ldots, j_Q = i$$

from j to i. Let k be the smallest integer such that $j_{k+1} \neq i_{k+1}$. As the paths are distinct, such a k exists. But then $i_k = j_k$ has two outgoing edges (i_k, i_{k+1}), (i_k, j_{k+1}) in E_T, which is a contradiction.

6.1.3 Matrix Tree Theorem for Markov Chains

In this section, we state the matrix tree theorem for Markov chains, also known as the Freĭdlin and Wentzell formula [FW84, Son99]. This classical result links the weights of directed spanning trees in the graph associated with an irreducible chain to its stationary distribution. The stationary distribution of the chain is essentially a normalised left eigenvector of the corresponding transition probability matrix. With this theorem, we see that it can be expressed in terms of the weights of RSTs of its associated digraph .

Recall that a matrix $A \in \mathbb{R}_+^{n\times n}$ is said to be a *row stochastic matrix* if all of its row sums equal to one. Formally, $\sum\limits_{1\leq j\leq n} a_{ij} = 1$ for $1 \leq i \leq n$.

The classical matrix tree theorem for Markov chains can be stated as follows.

Theorem 6.1.3. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible (row) stochastic matrix. Define $w \in \mathbb{R}_+^n$ by*

$$w_i = \sum_{T\in\mathcal{T}_i} \pi(T). \tag{6.3}$$

Then $A^T w = w$. In particular, $\frac{w}{\sum_{i=1}^{n} w_i}$ is the unique stationary distribution of the Markov chain with transition matrix A.

6.2 Matrix Tree Theorem for Markov Chains in the Max Algebra

In the main result below, we show via a direct combinatorial argument that Theorem 6.1.3 extends to the max algebra.

Motivated by Theorem 6.1.3, we first consider an irreducible matrix A in $\mathbb{R}_+^{n\times n}$ which is row stochastic in a max-algebraic sense. Formally, we assume that for $1 \leq i \leq n$, $\max_{1\leq j\leq n} a_{ij} = 1$ or using the max-algebraic notation

$$A \otimes \mathbf{1}_n = \mathbf{1}_n.$$

In a convenient abuse of notation, we refer to matrices satisfying the above condition as *max-stochastic*. Our main result shows that Theorem 6.1.3 extends in a natural way to the max-algebra.

Theorem 6.2.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible max-stochastic matrix. Define the vector w by*

$$w_i = \bigoplus_{T\in\mathcal{T}_i} \pi(T), \quad 1 \leq i \leq n. \tag{6.4}$$

Then

$$A^T \otimes w = w.$$

Proof: We first show that $A^T \otimes w \leq w$. To this end, let $i \in \{1, \dots, n\}$ be given. Then as $a_{ii} \leq 1$, it is immediate that

$$a_{ii}w_i \leq w_i. \tag{6.5}$$

Now consider $j \neq i$. We can write

$$w_j = a_{i_1j_1}a_{i_2j_2}\cdots a_{i_{n-1}j_{n-1}} \tag{6.6}$$

where the edges $(i_1, j_1), \dots, (i_{n-1}, j_{n-1})$ correspond to a maximal directed spanning tree T rooted at j. Then

$$a_{ji}w_j = a_{ji}a_{i_1j_1}a_{i_2j_2}\cdots a_{i_{n-1}j_{n-1}} \tag{6.7}$$

As $j \neq i$, it follows that there is an outgoing edge from i in T. Hence, $i = i_p$ for some $p \in \{1, \ldots, n-1\}$. Moreover $a_{i_p j_p} \leq 1$ which implies that

$$a_{ji} w_j \leq a_{ji} a_{i_1 j_1} \cdots a_{i_{p-1} j_{p-1}} a_{i_{p+1} j_{p+1}} \cdots a_{i_{n-1} j_{n-1}}. \tag{6.8}$$

If $a_{ji} = 0$, then it is immediate that $a_{ji} w_j \leq w_i$. On the other hand, if $a_{ji} > 0$, the right hand side of (6.8) corresponds to the set of edges

$$E' = \{(j, i), (i_1, j_1), \ldots (i_{p-1}, j_{p-1}), (i_{p+1}, j_{p+1}), \ldots, (i_{n-1}, j_{n-1})\}.$$

Consider the spanning subgraph $T' = (N(A), E')$. There is exactly one outgoing edge from every $k \neq i$ in $N(A)$ and no outgoing edge from i. Assume that there is a cycle in T'. It must contain the edge (j, i) otherwise it would define a cycle in the original spanning tree T. However there is no outgoing edge from i in T'. We get a contradiction. So, T' is acyclic. Therefore, the right hand side of (6.8) is the weight of a directed spanning tree rooted at i. From the definition of w, this immediately implies that $a_{ji} w_j \leq w_i$ for every $j \neq i$. Combining this with (6.5) yields immediately that $(A^T \otimes w)_i \leq w_i$. As this is true for any i, we see that $A^T \otimes w \leq w$.

To complete the proof, we show that $A^T \otimes w \geq w$. Let $i \in \{1, \ldots, n\}$ be given. Then as w_i is the weight of a maximal directed spanning tree, T_i, rooted at i, we can write

$$w_i = a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_{n-1} j_{n-1}}.$$

The edges in the spanning tree corresponding to the above expression are

$$E_i = \{(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})\}.$$

As A is a max-stochastic matrix by assumption, we know that $a_{ij} = 1$ for some j. If $j = i$, then

$$(A^T \otimes w)_i \geq a_{ii} w_i = w_i.$$

Now suppose $j \neq i$. Consider the set S of indices

$$S = \{i_p \mid j_p = i\}.$$

As E_i defines a directed spanning tree rooted at i, S is non-empty by Lemma 6.1.1. We need to consider two cases.

First, suppose that $j \in S$ (i.e. there is an outgoing edge from j to i). Then consider the set of edges

$$E_i' := \{(i_1, j_1), \ldots, (i_{p-1}, j_{p-1}), (i, j), (i_{p+1}, j_{p+1}), \ldots, (i_{n-1}, j_{n-1})\}$$

where we replace (j, i) by (i, j) in E_i. We claim that E_i' defines a directed spanning tree rooted at j. It is clear that there is exactly one outgoing edge from each vertex other than j and that j has no outgoing edge. Moreover, the digraph determined by E_i' is acyclic. To see this, first note that any cycle in T_i' must contain the edge (i, j) as otherwise it would define a cycle in the spanning tree T_i. As there is no outgoing edge in E_i' from j, it follows that T_i' must be acyclic as claimed. As $a_{ij} = 1$, it follows that

$$\pi(T_i') = a_{i_1 j_1} \cdots a_{i_{p-1} j_{p-1}} a_{i_{p+1} j_{p+1}} \cdots a_{i_{n-1} j_{n-1}}. \tag{6.9}$$

Moreover, as T_i' is rooted at j, it follows that

$$w_j \geq \pi(T_i')$$

and hence that

$$(A^T \otimes w)_i \geq a_{ji} w_j = a_{i_p j_p} w_j \geq a_{i_p j_p} \pi(T_i') = w_i.$$

We still need to consider the case where $j \notin S$ (i.e. the distance from j to i is greater than one). As T_i is a directed spanning tree rooted at i, it follows from Lemma 6.1.2 that there exists a unique $i_p \in S$ for which there is a directed path in T_i from j to i_p. Consider the set of edges E_i' where, as before, we replace the edge (i_p, j_p) (where $j_p = i$) with the edge (i, j). Let T' denote the associated spanning subgraph. Then there is exactly one outgoing edge from each vertex other than i_p in T' and there is no outgoing edge from i_p. Furthermore, if there exists a cycle in T', it must contain the edge (i, j) as otherwise it would define a cycle in T. This would then imply that there exists a directed path in T' from j to i, all of whose edges are also edges in T. This is impossible however, as the only such path in T contains the edge (i_p, j_p) which is not an edge in T'. Therefore T' is indeed a directed spanning tree rooted at i_p whose weight $\pi(T')$ is given by (6.9). As argued for the previous case, $w_{i_p} \geq \pi(T')$ and

$$(A^T \otimes w)_i \geq a_{i_p i} w_{i_p} = a_{i_p j_p} w_{i_p} \geq a_{i_p j_p} \pi(T') = w_i.$$

We have shown that

$$(A^T \otimes w)_i \geq w_i$$

for any $i \in \{1, \ldots, n\}$. Hence $A^T \otimes w \geq w$. This completes the proof. $\square$

In [BGKMS13], we present an alternative proof for Theorem 6.2.1 using a procedure that can be seen as an instance of the Maslov dequantization [LM98]. It was suggested and contributed by Sergeĭ Sergeev. See [BGKMS13] for the details.

The vector w defined in Theorem 6.2.1 will be called the *max-spanning-tree eigenvector* of A. We next present some numerical examples to illustrate Theorem 6.2.1 each will be referred later.

Example 6.2.1. Consider the following max-stochastic matrix

$$A = \begin{bmatrix} 1 & 3/4 & 5/6 & 0 \\ 1/2 & 1 & 1/4 & 9/10 \\ 0 & 0 & 1 & 7/8 \\ 1/3 & 0 & 1 & 4/5 \end{bmatrix}. \tag{6.10}$$

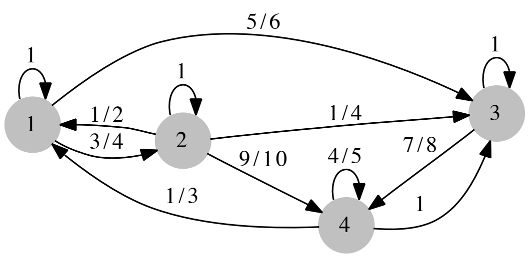


Figure 6.2: $D(A)$ for (6.10)

Let T_i be the max-spanning-tree rooted at i for $i = 1, 2, 3, 4$. Then,

- $T_1 : (3,4), (2,4), (4,1) \quad w_1 = \pi(T_1) = a_{34}a_{24}a_{41} = 21/80$
- $T_2 : (3,4), (4,1), (1,2) \quad w_2 = \pi(T_2) = a_{34}a_{41}a_{12} = 7/32$
- $T_3 : (1,3), (2,4), (4,3) \quad w_3 = \pi(T_3) = a_{13}a_{24}a_{43} = 3/4$
- $T_4 : (2,4), (1,3), (3,4) \quad w_4 = \pi(T_4) = a_{24}a_{13}a_{34} = 21/32$

Hence, $w = \begin{bmatrix} 21/80 \\ 7/32 \\ 3/4 \\ 21/32 \end{bmatrix}$ and $A^T \otimes w = w$.

Example 6.2.2. Consider the following max-stochastic matrix

$$A = \begin{bmatrix} 2/5 & 1 & 2/5 & 0 \\ 1 & 1/4 & 3/4 & 5/6 \\ 0 & 0 & 9/10 & 1 \\ 1/2 & 0 & 1 & 3/5 \end{bmatrix}. \tag{6.11}$$

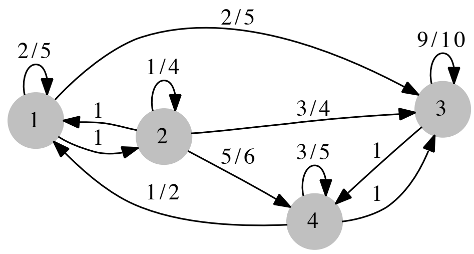


Figure 6.3: $D(A)$ for (6.11)

Let T_i be the max-spanning-tree rooted at i for $i = 1, 2, 3, 4$. Then,

- $T_1 : (3,4), (4,1), (2,1) \quad w_1 = \pi(T_1) = a_{34}a_{41}a_{21} = 1/2$
- $T_2 : (3,4), (4,1), (1,2) \quad w_2 = \pi(T_2) = a_{34}a_{41}a_{12} = 1/2$
- $T_3 : (1,2), (2,4), (4,3) \quad w_3 = \pi(T_3) = a_{12}a_{24}a_{43} = 5/6$
- $T_4 : (1,2), (2,4), (3,4) \quad w_4 = \pi(T_4) = a_{12}a_{24}a_{34} = 5/6$

Hence, $w = \begin{bmatrix} 1/2 \\ 1/2 \\ 5/6 \\ 5/6 \end{bmatrix}$ and $A^T \otimes w = w$.

6.3 Max-spanning-Tree Eigenvector and Kleene Star

We have shown that the vector of maximal rooted spanning trees is always a left max eigenvector of a max-stochastic matrix. However, in contrast to the conventional algebra, the irreducibility of A is not sufficient to guarantee uniqueness (up to scalar multiple) of the max eigenvector. This naturally leads to the question of how to identify the max-spanning-tree eigenvector using tools of the max-algebraic spectral theory such as the Kleene star. We next consider this question.

Recall that we denote the critical digraph of $A \in \mathbb{R}_+^{n\times n}$ by $D^C(A) = (N^C(A), E^C(A))$ where $N^C(A)$ is the set of critical vertices and $E^C(A)$ is the set of critical edges. Moreover, recall that Theorem 2.2.5 shows the connection of the max eigenvectors of A with columns of the Kleene star A^*. By applying it to A^T, the same relation also holds between the left max eigenvectors of A and rows of A^*. Notice that $N^C(A) = N^C(A^T)$, $\mu(A) = \mu(A^T)$ and $(A^T)^* = (A^*)^T$. We adopt the notation $A^*_{i.}$ for the i^{th} row, and the notation $A^*_{.i}$ for the i^{th} column of the matrix A^*.

We will now make a number of observations on the Kleene star of max-stochastic matrices. Notice that a max-stochastic matrix has max eigenvalue 1, and $a_{ij} \leq 1$ for all i, j. This implies that $\mu(A) = 1$, and that $a_{ij} = 1$ for $(i, j) \in E^C(A)$. Recall from Section 2.2.4 that such matrices are called visualized. Note that the max-stochastic matrices have an additional property: each vertex in $D(A)$ has an outgoing edge with weight 1.

The Kleene star of a definite visualised matrix (and hence of a max-stochastic matrix) has a very specific structure, as described, for example, in Proposition 4.1 of [SSB09], which we now recall. Before stating it, we recall a number of necessary definitions from [SSB09]. Define $D^{C*}(A)$ to be the directed graph formed by adding trivial graphs each consisting of just one non-critical vertex to $D^C(A)$ (we add one such graph for each non-critical vertex). Let $D^{C*}(A)$ have r' strongly connected components with vertex sets $N_1, \ldots, N_{r'}$ where N_α denotes the set of vertices corresponding to the component α of $D^{C^*}(A)$ for $1 \leq \alpha \leq r'$.

For $1 \leq \alpha, \beta \leq r'$, denote by $A_{\alpha\beta}$ the submatrix of A formed from the rows with indices in N_α and from the columns with indices in N_β. Let A^{red} be the $r' \times r'$ matrix with entries $a^{\text{red}}_{\alpha\beta} = \max\{a_{ij} \mid i \in N_\alpha,\, j \in N_\beta\}$. We now state Proposition 4.1 of [SSB09] below.

Proposition 6.3.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be a definite visualised matrix and r' be* *the number of strongly connected components of $D^{C*}(A)$. Then,*

1. $a^{red}_{\alpha\alpha} = 1$ *for all* $1 \leq \alpha \leq r'$ *and* $a^{red}_{\alpha\beta} \leq 1$ *(resp.* $a^{red}_{\alpha\beta} < 1$ *for* $\alpha \neq \beta$*), where* $\alpha, \beta \in \{1, 2, ..., r'\}$*;*

2. *For* $1 \leq \alpha, \beta \leq r'$*, the corresponding* (α, β)*-submatrix of* A^* *is equal to* $A^*_{\alpha\beta} = a^{red*}_{\alpha\beta}\mathbf{1}_\alpha \mathbf{1}_\beta{}^T$*, where* $a^{red*}_{\alpha\beta}$ *is the* $(\alpha, \beta)^{th}$ *entry of* $(A^{red})^*$*, and* $\mathbf{1}_\alpha \mathbf{1}_\beta{}^T$ *is the* (α, β)*-submatrix of* $\mathbf{1}_n \mathbf{1}_n^T$*.*

From Proposition 6.3.1, we get the following form for A^* by applying suitable permutations to rows and columns [SSB09].

$$A^* = \begin{bmatrix} \mathbf{1}_1 \mathbf{1}_1{}^T & a^{\mathrm{red}*}_{12} \mathbf{1}_1 \mathbf{1}_2{}^T & \ldots & a^{\mathrm{red}*}_{1r'} \mathbf{1}_1 \mathbf{1}_{r'}{}^T \\ a^{\mathrm{red}*}_{21} \mathbf{1}_2 \mathbf{1}_1{}^T & \mathbf{1}_2 \mathbf{1}_2{}^T & \ldots & a^{\mathrm{red}*}_{2r'} \mathbf{1}_2 \mathbf{1}_{r'}{}^T \\ \vdots & \vdots & \ddots & \vdots \\ a^{\mathrm{red}*}_{r'1} \mathbf{1}_{r'} \mathbf{1}_1{}^T & a^{\mathrm{red}*}_{r'2} \mathbf{1}_{r'} \mathbf{1}_2{}^T & \ldots & \mathbf{1}_{r'} \mathbf{1}_{r'}{}^T \end{bmatrix}. \tag{6.12}$$

The following is an immediate consequence of Proposition 6.3.1.

Corollary 6.3.2. *Let* $A \in \mathbb{R}^{n\times n}_+$ *be an irreducible max-stochastic matrix. Then for all* $i, j \in N^C(A)$ *belonging to the same strongly connected component in* $D^C(A)$*,*

(i) $a^*_{ij} = 1$*;*

(ii) $A^*_{.i} = A^*_{.j}$*;*

(iii) $A^*_{i.} = A^*_{j.}$*.*

Before deriving the main result of this section, which is Theorem 6.3.5, we introduce a number of preliminary results.

Lemma 6.3.3. *Let* $A \in \mathbb{R}^{n\times n}_+$ *be an irreducible max-stochastic matrix. Then, for* $1 \leq i \leq n$*,* $\min\limits_{1\leq j\leq n} a^*_{ji} = \min\limits_{q\in N^C(A)} a^*_{qi}$*.*

Proof: Let $i \in \{1, \ldots, n\}$ be given. It is immediate that

$$\min_{1\leq j\leq n} a^*_{ji} \leq \min_{q\in N^C(A)} a^*_{qi}. \tag{6.13}$$

To show the reverse inequality, consider some $l \notin N^C(A)$. We claim that there exists a path from l to some $k \in N^C(A)$ of weight 1. As A is max-stochastic, there exists at least one outgoing edge from l of weight 1, $a_{lk_1} = 1$. Moreover, as l is not critical, $k_1 \neq l$. If k_1 is critical, we are done. If not, then there exists $k_2 \notin \{l, k_1\}$ with $a_{k_1k_2} = 1$. Continuing in this fashion, we must eventually arrive at some vertex $k = k_p$ which is in $N^C(A)$. By construction, $l, k_1, \ldots, k_p = k$ is a path of weight 1.

a^*_{ki} is the maximal weight of a path between k and i. Using the path constructed above from l to k of weight 1, and remembering that all the entries of A are less than or equal to 1, it is easy to see that we can construct a path from l to i whose weight is at least a^*_{ki}. It follows that

$$a^*_{li} \geq a^*_{ki} \geq \min_{q \in N^C(A)} a^*_{qi}.$$

As this must hold for any $l \notin N^C(A)$, we have that

$$\min_{1 \leq j \leq n} a^*_{ji} \geq \min_{q \in N^C(A)} a^*_{qi}. \tag{6.14}$$

Combining (6.14) and (6.13) yields the result. $\square$

Lemma 6.3.4. *Let $A \in \mathbb{R}_+^{n \times n}$ be an irreducible max-stochastic matrix. Assume that $D^C(A)$ is strongly connected. Let w be the max-spanning-tree eigenvector of A. Then for all $i \in N^C(A)$, $w_i = 1$.*

Proof: Let $i \in N^C(A)$ be given. Choose any $k_1 \in N^C(A) \backslash \{i\}$. As $N^C(A)$ is strongly connected, $\mu(A) = 1$ and A is max-stochastic, there exists a path P_1, from k_1 to i of weight 1, consisting of vertices in $N^C(A)$. If $N(P_1) \neq N^C(A)$, choose $k_2 \in N^C(A) \backslash N(P_1)$, and let Q_1 be the shortest path in $D^C(A)$ from k_2 to P_1. Set P_2 to be the union of P_1 and Q_1. Continuing in this way, we can construct a directed tree T_1 of weight one, rooted at i, with $N(T_1) = N^C(A)$. If $N^C(A) = N(A)$, we are done.

Otherwise, to complete this to a spanning tree of weight 1, let $j_1 \notin N^C(A)$ be given. As A is max-stochastic there exists some edge (j_1, p_1) in $D(A)$ of weight 1. In fact, it is easy to see that there exists a directed path P'_1 from j_1 to $N^C(A)$ of weight 1, consisting of non-critical vertices. Let T_2 be the union of T_1 and P'_1. If $N(T_2) = \{1, \ldots, n\}$, then T_2 is a directed spanning tree rooted at i of weight 1. Otherwise, we can choose $j_2 \notin N(T_2)$ and repeat the previous step, choosing a path from j_2 to $N(T_2)$ consisting of edges of weight one and vertices not in $N(T_2)$. Continuing in this fashion we will arrive at a directed spanning tree T rooted at i of weight 1. As A is max-stochastic, it follows that T is maximal and hence that $w_i = 1$ as claimed. $\square$

The next result, which is the main result of this section, shows the relationship between the max-spanning-tree eigenvector and rows of the Kleene star of A. Note that minimum is taken entry-wise.

Theorem 6.3.5. *Let $A \in \mathbb{R}_+^{n\times n}$ be an irreducible max-stochastic matrix. Let w be the max-spanning-tree eigenvector of A. Then, the following are true.*

(i) $w \leq \min_{q\in N^C(A)} A^*_{q.}$;

(ii) If $D^C(A)$ is strongly connected then $w = A^*_{q.}$ for all $q \in N^C(A)$;

(iii) If $D^C(A)$ has no more than two components and $N^C(A) = N(A)$, then $w = \min_{q\in N^C(A)} A^*_{q.}$.

Proof:

(i) Consider a maximal spanning tree T rooted at i $(1 \leq i \leq n)$ with weight w_i. There exists a path in T from j to i for $1 \leq j \leq n$, $i \neq j$. Let $\pi(j, i)$ be the weight of this path. Then,

$$w_i \leq \pi(j, i) \leq a^*_{ji} \text{ for all } j \in \{1, 2, ..., n\}.$$

Thus, $w_i \leq \min_{1\leq j\leq n} a^*_{ji}$. Using Lemma 6.3.3, we have $w_i \leq \min_{q\in N^C(A)} a^*_{qi}$.

(ii) If there exists one strongly connected component in $D^C(A)$, then it follows from Theorem 2.2.5 (i) that there exists a positive vector $x > 0$ such that $A^*_{i.} = x$ for all $i \in N^C(A)$. We claim $w = x$. It follows from Theorem 2.2.5 (ii) that w is a scalar multiple of x. Combining point (iii) of Corollary 6.3.2 with Lemma 6.3.4, we see that $w = x$.

(iii) Assume that $D^C(A)$ has two strongly connected components and $N^C(A) = N(A)$. It follows from Corollary 6.3.2 that there exist linearly independent (in a max-algebraic sense) vectors v, u such that every row of A^* is equal to v^T

or u^T. Let $D_v^C(A)$, $D_u^C(A)$ denote the corresponding strongly connected components of $D^C(A)$. Using Corollary 6.3.2 again, (after possibly relabelling the vertices in $D^C(A)$), we can assume that

$$v = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ v_{p+1} \\ \vdots \\ v_n \end{bmatrix}, u = \begin{bmatrix} u_1 \\ \vdots \\ u_p \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

In this case, $D_v^C(A)$ consists of vertices from 1 to p while $D_u^C(A)$ consists of vertices from $p+1$ to n. Denote $N_v^C(A) = \{1, 2, ..., p\}$ and $N_u^C(A) = \{p+1, p+2, ..., n\}$.

We need to show that

$$w = \min(u, v) = \begin{bmatrix} u_1 \\ \vdots \\ u_p \\ v_{p+1} \\ \vdots \\ v_n \end{bmatrix}. \tag{6.15}$$

From (i), we know that $w \leq \min(u, v)$. Hence, it is enough to show that for each $i \in \{1, \ldots, n\}$, there exists a spanning tree rooted at i of weight u_i if $i \leq p$ and of weight v_i if $i \geq p+1$.

First, let $i \in \{1, \ldots, p\}$ be given. For any $j_0 \in \{p+1, \ldots, n\}$, there exists a path P_0 in $D(A)$ from j_0 to i of weight $a_{ji}^* = u_i$. Next, note that for any $k \neq i$ in $\{1, \ldots, p\}$, as P_0 terminates in $D_v^C(A)$, there exists a shortest path Q_0 in $D_v^C(A)$ (necessarily of weight 1) from k to P_0. Set P_1 to be the union of P_0 and Q_0. Repeat this process until we have included all the vertices in $\{1, \ldots, p\}$. This will yield a directed tree T_1 rooted at i of weight u_i, which includes all the vertices in $D_v^C(A)$.

Next note that as P_0 and hence T_1 contains j_0, which lies in $D_u^C(A)$, we can mimic the previous steps in $D_u^C(A)$. First pick some k_1 in $D_u^C(A)$ that is not in $N(T_1)$ and choose a shortest path R_1 in $D_u^C(A)$ from k_1 to T_1. If the union of T_1 and R_1 does not contain all vertices in $\{p+1,\dots,n\}$, repeat this process; eventually, we arrive at a directed spanning tree, rooted at i, of weight u_i.

The argument to establish that there exists a directed spanning tree rooted at each $i \in \{p+1,\dots,n\}$ of weight v_i is identical. $\square$

We next illustrate Theorem 6.3.5 (i) and (iii).

Example 6.3.1. Consider the matrix given in Example 6.2.1. There exist three strongly connected components in $D^C(A)$ and $N^C(A) = \{1,2,3\}$ where $N_1^C(A) = \{1\}$, $N_2^C(A) = \{2\}$ and $N_3^C(A) = \{3\}$. The Kleene star of A is given by

$$A^* = \begin{bmatrix} 1 & 3/4 & 5/6 & 35/48 \\ 1/2 & 1 & 9/10 & 9/10 \\ 7/24 & 7/32 & 1 & 7/8 \\ 1/3 & 1/4 & 1 & 1 \end{bmatrix}.$$

The left max eigenvectors are $A^*_{1.}$, $A^*_{2.}$ and $A^*_{3.}$.

Recall that $w = \begin{bmatrix} 21/80 \\ 7/32 \\ 3/4 \\ 21/32 \end{bmatrix}$. Then, $w \leq \min_{q \in N^C(A)} A^*_{q.} = \begin{bmatrix} 7/24 \\ 7/32 \\ 5/6 \\ 35/48 \end{bmatrix}$.

Example 6.3.2. Consider the matrix given in Example 6.2.2. There exist two strongly connected components in $D^C(A)$ and $N^C(A) = \{1,2,3,4\}$ where $N_1^C(A) = \{1,2\}$ and $N_2^C(A) = \{3,4\}$. The Kleene star of A is given by

$$A^* = \begin{bmatrix} 1 & 1 & 5/6 & 5/6 \\ 1 & 1 & 5/6 & 5/6 \\ 1/2 & 1/2 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1 \end{bmatrix}.$$

The left max eigenvectors are $A^*_{1.}$ and $A^*_{3.}$.

Recall that $w = \begin{bmatrix} 1/2 \\ 1/2 \\ 5/6 \\ 5/6 \end{bmatrix}$. Note that $w = \min_{q \in N^C(A)} A^*_{q.}$.

As a final point, we note that Theorem 6.3.5 (iii) does not hold when $N^C(A) \neq \{1, 2, ..., n\}$ or when there exists more than two strongly connected components in $D^C(A)$. Specifically, the example below for the former case shows that $w \neq \min_{q \in N^C(A)} A^*_{q.}$.

Example 6.3.3. Consider the following 3×3 matrix

$$A = \begin{bmatrix} 1 & 1/4 & 1/3 \\ 3/4 & 1 & 2/3 \\ 1 & 0 & 0 \end{bmatrix}. \tag{6.16}$$

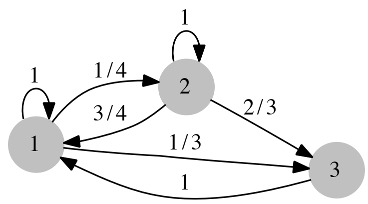


Figure 6.4: $D(A)$ for (6.16)

Let T_i be the max-spanning-tree rooted at i for $i = 1, 2, 3$. Then,

- $T_1 : (2,1), (3,1) \quad w_1 = \pi(T_1) = a_{21}a_{31} = 3/4$
- $T_2 : (3,1), (1,2) \quad w_2 = \pi(T_2) = a_{31}a_{12} = 1/4$
- $T_3 : (2,1), (1,3) \quad w_3 = \pi(T_3) = a_{21}a_{13} = 1/4$

There exist two strongly connected components in $D^C(A)$ and $N^C(A) \neq N(A)$. $N_1^C(A) = \{1\}$ and $N_2^C(A) = \{2\}$. The Kleene star of A is given by

$$A^* = \begin{bmatrix} 1 & 1/4 & 1/3 \\ 3/4 & 1 & 2/3 \\ 1 & 1/4 & 1 \end{bmatrix}.$$

The left max eigenvectors are $A^*_{1.}$ and $A^*_{2.}$.

Note that $w = \begin{bmatrix} 3/4 \\ 1/4 \\ 1/4 \end{bmatrix}$ and $w \neq \min_{q \in N^C(A)} A^*_{q.} = \begin{bmatrix} 3/4 \\ 1/4 \\ 1/3 \end{bmatrix}$.

6.4 Applications to the AHP and Voting

In this section, we briefly discuss the application of Theorem 6.2.1 to ranking schemes in the single criterion Analytic Hierarchy Process (AHP) and voting problems.

6.4.1 Single Criterion AHP

We first show how the results of the previous sections can be used to provide a max-algebraic characterisation of maximal RST vectors for general irreducible non-negative matrices.

Proposition 6.4.1. *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible. Let the diagonal matrix D be given by*

$$d_i = \max_{1\leq j\leq n} a_{ij}.$$

Further let w be max-spanning-tree eigenvector of A. Then

$$A^T \otimes w = Dw \tag{6.17}$$

where $D = \text{diag}\,(d_1, d_2, ..., d_n)$.

Proof: Let $\hat{A} = D^{-1}A$ where $A \in \mathbb{R}_+^{n\times n}$ is irreducible. Then $\hat{A}$ is irreducible and max-stochastic. Consider a spanning tree $\hat{T}_i = (N(A), E')$ in $D(\hat{A})$ rooted at $i \in \{1, 2, ..., n\}$ where

$$E' = \{(i_1, j_1), (i_2, j_2), ..., (i_{n-1}, j_{n-1})\}.$$

It is clear that the weight of $\hat{T}$ takes the form

$$\hat{a}_{i_1 j_1} \cdots \hat{a}_{i_{n-1} j_{n-1}} = \frac{1}{d_{i_1} d_{i_2} \cdots d_{i_{n-1}}} a_{i_1 j_1} \cdots a_{i_{n-1} j_{n-1}}$$

where $\{i_1, \ldots, i_{n-1}\} = \{1, \ldots, n\} \backslash \{i\}$. In fact, it is clear that there is a bijective correspondence between spanning trees T_i in $D(A)$ rooted at i and spanning trees $\hat{T}_i$ rooted at i in $D(\hat{A})$ for $1 \leq i \leq n$ with

$$\pi(\hat{T}_i) = \frac{d_i}{\det(D)} \pi(T_i). \tag{6.18}$$

It follows that if we write $\hat{w}$ for the max-spanning-tree eigenvector of $\hat{A}$, then

$$\hat{w} = \frac{D}{\det(D)} w. \tag{6.19}$$

As $\hat{A}$ is max-stochastic, we know from Theorem 6.2.1 that $\hat{A}^T \otimes \hat{w} = \hat{w}$. Noting that $\hat{A}^T = A^T D^{-1}$, we can use (6.19) to rewrite this as

$$\frac{1}{\det(D)} A^T \otimes w = \frac{D}{\det(D)} w.$$

The result follows immediately. $\square$

Now recall the AHP from Chapter 5. Suppose that A is an SR-matrix where a_{ij} indicates the relative strength of alternative i to alternative j. A central question in the AHP was to determine a weight vector w in which w_i represents the weight given to alternative i. Saaty [Saa77a] suggested to take w to be the Perron vector of A. Elsner and van den Driessche [EvdD04, EvdD10] suggested selecting w from the set of vectors, including the max-algebraic eigenvector, that minimises the functional (5.8) given by

$$e_A(x) = \max_{1 \leq i,j \leq n} a_{ij} x_j / x_i.$$

Recall that the set of vectors that minimise (5.8) is denoted by $\mathcal{C}_A$ in (5.11) (the subeigencone of A with respect to $\mu(A)$). See Section 5.1.3 for the details.

In this context, a spanning tree in $D(A^T)$ rooted at i represents an accumulation of relative scores with respect to all other alternatives in $\{1, \ldots, n\}$. With this in mind, a reasonable choice of weight vector w is the max-spanning-tree eigenvector of A^T. From Proposition 6.4.1, we know that w must solve the generalised max-eigenvector equation

$$A \otimes w = Dw \tag{6.20}$$

where D is diagonal and satisfies $d_i = \max\limits_{1 \leq j \leq n} a_{ji}$. It is worth noting that such a w does not minimise the maximal relative error functional given in (5.8) and may give different rankings to the schemes considered there. On the other hand, it has the advantage that the max-spanning-tree eigenvector is unique, while the optimisation problem explained in Section 5.1.3 may give rise to multiple rankings.

Next, we present an illustrative example and compare rankings from the Perron vector [Saa77a], the max algebra approach [EvdD04, EvdD10] and the max-spanning-tree eigenvector of A^T which satisfies (6.20).

Example 6.4.1. Consider the following *SR*-matrix

$$A = \begin{bmatrix} 1 & 1/2 & 3 & 2 \\ 2 & 1 & 1/2 & 4 \\ 1/3 & 2 & 1 & 1/5 \\ 1/2 & 1/4 & 5 & 1 \end{bmatrix}. \tag{6.21}$$

Let x be the Perron vector of A, v be in $\mathcal{C}_A$ and w be the max-spanning-tree eigenvector of A^T. All are normalised.

$x = \begin{bmatrix} 1 & 1.322 & 0.689 & 0.931 \end{bmatrix}^T$ with the ranking $2 > 1 > 4 > 3$.

Since $N^C(A) = \{2, 3, 4\}$ and A^C is irreducible, $\hat{A}^*_{.2}$, $\hat{A}^*_{.3}$ and $\hat{A}^*_{.4}$ are multiples of each other and correspond to the max eigenvector of A from Theorem 2.2.5 (ii). For $v = \begin{bmatrix} 1 & 1.949 & 1.14 & 1.667 \end{bmatrix}^T$, we get the ranking: $2 > 4 > 3 > 1$. By using Algorithm 3, we find subeigenvectors producing at least three more possible distinct rankings: $1 > 2 > 4 > 3$, $2 > 1 > 4 > 3$ and $2 > 4 > 1 > 3$.

$w = \begin{bmatrix} 1 & 1.667 & 0.667 & 0.833 \end{bmatrix}^T$ with the same ranking with the Perron vector.

6.4.2 Judges and Competitors

Next, we consider a particular type of voting problem. Suppose that we have a set of m *judges* and n *competitors*. Each judge is asked to give the competitors a score between 0 and 1 with the highest ranked competitor scoring a 1 and the others scored accordingly. Moreover, each competitor is asked to score the judges in the same way. The judges scores will generate a matrix $J \in \mathbb{R}_+^{m \times n}$ with a row for each judge, while the competitors' scores will generate a matrix $C \in \mathbb{R}_+^{n \times m}$ with a row for each competitor's scores.

Consider now the matrix $\hat{C} = C \otimes J$ in $\mathbb{R}_+^{n \times n}$. For $1 \leq p, q \leq n$, consider the entry

$$\hat{c}_{pq} = \max_{1 \leq r \leq m} c_{pr} j_{rq}.$$

Each product $c_{pr} j_{rq}$ can be viewed as an indirect score given by competitor p to competitor q via judge r. Thus the entry $\hat{c}_{pq}$ is the maximal such score over all judges. It is easy to see that the matrix $\hat{C}$ will be max-stochastic. The max-spanning-tree eigenvector w associated with $D(\hat{C})$ can be used to rank the competitors and Theorem 6.2.1 shows that w is a max eigenvector of $\hat{C}^T$. Similar remarks apply to the matrix $\hat{J} = J \otimes C$ in $\mathbb{R}_+^{m\times m}$. In this case, the max-spanning-tree eigenvector of $D(\hat{J})$ can be used to rank the judges and Theorem 6.2.1 shows that it is a max eigenvector of $\hat{J}^T$.

Example 6.4.2. Assume that we have three judges and four competitors. The following matrices represent judges scores to competitors and competitors scores to judges.

$$\text{J}= \begin{array}{c|cccc} & C_1 & C_2 & C_3 & C_4 \\ \hline J_1 & 1 & 1/2 & 1/3 & 0 \\ J_2 & 1/2 & 1 & 0 & 1/5 \\ J_3 & 1 & 1/2 & 1/2 & 1/4 \end{array} \qquad \text{C}= \begin{array}{c|ccc} & J_1 & J_2 & J_3 \\ \hline C_1 & 1/6 & 1 & 1/4 \\ C_2 & 0 & 1 & 1/2 \\ C_3 & 1/3 & 2/3 & 1 \\ C_4 & 1/5 & 1 & 1/2 \end{array}$$

$$\hat{C} = \begin{bmatrix} 1/2 & 1 & 1/8 & 1/5 \\ 1/2 & 1 & 1/4 & 1/5 \\ 1 & 2/3 & 1/2 & 1/4 \\ 1/2 & 1 & 1/4 & 1/5 \end{bmatrix}$$

For $\hat{C}$, we have $w = \begin{bmatrix} 1/2 \\ 1 \\ 1/4 \\ 1/5 \end{bmatrix}$ with the ranking $2 > 1 > 3 > 4$ for competitors.

$$\hat{J} = \begin{bmatrix} 1/6 & 1 & 1/3 \\ 1/12 & 1 & 1/2 \\ 1/6 & 1 & 1/2 \end{bmatrix}$$

For $\hat{J}$, we have $w = \begin{bmatrix} 1/12 \\ 1 \\ 1/2 \end{bmatrix}$ with the ranking $2 > 3 > 1$ for judges.

6.5 Concluding Remarks

Our main goal in this chapter was to extend the so-called matrix tree theorem for Markov chains to the max algebra and investigate the relationship between the max-spanning-tree eigenvector and rows of the Kleene star of the corresponding max-stochastic matrix. In this context,

- we defined the class of max-stochastic matrices over the max algebra;
- we presented a max version of the matrix tree theorem for max-stochastic matrices;
- we discussed possible applications of this theorem to ranking problems and showed that it is always possible to find a unique ranking scheme.

CHAPTER 7

Conclusions and Future Work

In this final chapter, we summarise the results presented throughout the thesis and suggest directions for future research related to the work described here.

7.1 Summary

Recently, there has been great interest in the correspondence between the max algebra and non-negative linear algebra which was the motivation of the work presented in this thesis. We can divide the presented work into two parts. In the first part, we introduced theoretical results on the spectral and stability properties of non-negative matrices in the max algebra. Our main concern here was matrix polynomials and sets of matrices. In the second part, we discussed applications to ranking problems. Our primary interest here was in the association of the max-algebraic spectral theory with constructing a ranking scheme for the alternatives in the Analytic Hierarchy Process (AHP). We will now review the work in the preceding chapters, highlighting the motivation and main contributions accomplished in the thesis.

We started with a brief introduction to the class of non-negative matrices and the max algebra in Chapter 1. We recalled several examples to motivate the work carried out in this thesis. In particular, we focused on the AHP to emphasize the role of the Perron-Frobenius theory and the max-algebraic spectral theory in applications to ranking problems.

In Chapter 2, we reviewed relevant definitions and preliminary results on which our work in this thesis rests. We recalled the celebrated Perron-Frobenius theory. We highlighted the relation of non-negative matrices with graph theory. We introduced essential tools in the max algebra that played important roles in proving the results developed in the subsequent chapters. Particularly, we stated a max version of the Perron-Frobenius theory and discussed numerical algorithms for the solution of the max eigenvalue problem.

In Chapter 3, we were motivated by the work of Psarrakos and Tsatsomeros [PT04]. We extended their main results on the Perron-Frobenius theory for matrix polynomials to matrix polynomials defined over the max algebra. We derived convergence results for multi-step difference equations over the max algebra analogous to those for the conventional algebra. We presented a number of results relating the largest max eigenvalue of a max matrix polynomial $P(\lambda)$. In particular, we showed that it can be characterised by using the largest max eigenvalue of an $n \times n$ matrix. We emphasized the role of the multigraph of the coefficient matrices of the polynomial in this analysis.

In Chapter 4, motivated by the work of Song, Gowda and Ravindran [SGR99], we defined the class of P_{max}-matrices and showed how some basic properties of P-matrices in the conventional algebra extend to the max algebra. Further, the relation between the P_{max}-property, the S_{max}-property and the stability of delayed difference equations over the max algebra was described. We extended the P_{max}-property of a matrix to sets of non-negative matrices in the spirit of [SGR99]. We showed that the relation between P-matrix sets and the S-property for Z-matrices in the conventional algebra extends to this new setting. We highlighted the role of the max generalised spectral radius in this extension. We explored the implications of the row-P_{max}-property for the stability of max-convex hulls, as well as delayed and undelayed difference inclusions defined over the max algebra.

In Chapter 5, building on the work of Elsner and van den Driessche [EvdD04, EvdD10], we considered a max-algebraic approach to the multi-criteria AHP within the framework of multi-objective optimisation. [EvdD04, EvdD10] characterise the max eigenvectors and subeigenvectors of a single SR-matrix as solving an optimisation problem with a single objective. We extended this work to the multi-criteria AHP by directly considering several natural extensions of this basic optimisation problem to the multiple objective case. Specifically, we presented results on three types of optimal solutions: globally optimal solutions; min-max optimal solutions; Pareto optimal solutions. In each case, the optimal solution was used to construct a ranking scheme for the alternatives in the process. We related the existence of globally optimal solutions to the existence of common subeigenvectors and highlighted connections between this question and commutativity. We established the connection between the max generalised spectral radius and min-max optimal solutions. We proved the existence of Pareto optimal solutions and showed that it is possible to simultaneously solve the Pareto and min-max optimisation problems.

Finally, in Chapter 6, we derived a max-algebraic version of the Markov Chain Tree Theorem (the Freĭdlin and Wentzell formula) for a max stochastic matrix [FW84]. We observed that the maximal weights of rooted directed spanning trees in its associated digraph is a left max eigenvector. We were concerned with relating the max-spanning-tree eigenvector to the max-algebraic spectral theory. In particular, we showed its connection with rows of the Kleene star of the matrix. We discussed possible applications of our results to decision making problems to derive a unique ranking scheme.

7.2 Directions for Future Research

In this final section, we discuss a number of questions that arise from the work of this thesis.

One of the main results in Chapter 3 describes how the behaviour of the solution of multi-step difference equations depends on the max powers of the associated companion matrices (Theorem 3.3.2). In this result, we assumed that the conditions in Theorem 2.2.10 hold in order to conclude convergence. Recall that Theorem 2.2.10 and Theorem 2.2.11 define conditions for the convergence of the max-algebraic powers of irreducible matrices in a finite number of steps and Corollary 2.2.12 describes an upper bound for the required number of steps. The idea of applying these results to companion matrices $C_P \in \mathbb{R}_+^{mn\times mn}$ of type (3.11) gives rise to a number of interesting questions due to their special structure:

1. *When is C_P^C primitive?*

2. *When do the following convergence conditions defined in Theorem 2.2.11 hold?*

   (i) $N^C(C_P) = \{1, 2, ..., mn\}$

   (ii) C_P^C *is a direct sum of primitive matrices*

3. *Is there a better bound for the number of finite steps required for convergence than $(mn)^2 - 1$ in Corollary 2.2.12?*

Another possible direction for future work is to investigate the potential for extending the results of Chapter 3 to sets of matrix polynomials over the max algebra. In particular, it would be interesting to investigate whether the results of Section 3.4 could be extended to the generalized max spectral radius studied in Section 4.1.4. Recall that the generalised spectral radius of a set can be calculated by the largest max eigenvalue of an $n \times n$ matrix in the max algebra (Proposition 4.4.3). In this direction, we ask the following questions:

1. *What is $\mu(\hat{P})$ where $\hat{\mathcal{P}} = \{P_1(\lambda), P_2(\lambda), ..., P_l(\lambda)\}$?*

2. *Can the relation defined in Proposition 4.4.3 for the max version of the generalized spectral radius for sets of matrices be defined for the set of max matrix polynomials?*

3. *Do similar relations between $\mu(P(\lambda))$ and $\mu(S)$ in Section 3.4 hold for $\mu(\hat{P})$ and $\mu(\{S_1, S_2, ..., S_l\})$?*

In Section 5.1.3, we explained the max algebra approach of Elsner and van den Driessche [EvdD04, EvdD10] for the single criterion AHP. In their approach, a max eigenvector or a subeigenvector from the minimum error requirement set C_A of an *SR*-matrix A is used as a weight vector for the alternatives. In contrast to the classical Eigenvalue Method (EM) for the AHP, this approach does not in general give rise to a unique ranking vector (Proposition 5.1.2). If $x \in \mathcal{C}_A$ is not unique (up to a scalar multiple), we may get more than one ranking scheme. Thus, we have the following obvious question:

> *Given multiple vectors in C_A of an SR-matrix A, which one should be chosen as a weight vector for the alternatives?*

Sergeĭ Sergeev suggested applying the max-balancing scaling [SS90, SS91, RSS92] mentioned in Section 2.2.4 as a means of ranking alternatives. The max-balancing vector always exists and is unique for an irreducible matrix. Furthermore, it strictly visualises the matrix so that it is in the minimum error requirement set of A. Thus, it can be used as a weight vector for the alternatives. This leads us to the following question:

> *Is there an interpretation of the max-balancing scaling in terms of pairwise comparisons in the AHP?*

In Chapter 5, we suggested a novel approach for the multi-criteria AHP based on combining the max algebra approach of Elsner and van den Driessche [EvdD04, EvdD10] and multi-objective optimisation. In Section 5.5, we showed that Pareto optimal solutions play a central role in determining an overall ranking scheme for the alternatives. These solutions in general will not be unique (Example 5.5.2). In view of this, it is natural to ask the following question:

> *Given a set of SR-matrices $A_1, A_2, ..., A_m$, is it possible to characterise the Pareto set for the max-algebraic errors $e_{A_i} : int(\mathbb{R}^n_+) \to \mathbb{R}_+ (1 \leq i \leq m)$ and decide which vectors in it to choose as a weight vector for the alternatives?*

In Chapter 6, we were concerned with max-stochastic matrices and discussed possible applications of these matrices in decision making problems. It would be interesting to investigate the potential of these ideas to be applied in the multi-criteria AHP and other ranking problems such as the Judges and Competitors scenario discussed above.

In finishing, this thesis has hopefully demonstrated that while the max algebra is now a well-developed research field with a rich body of results, there is no shortage of challenging open questions in the area.

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