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Brief paper

# A result on second order nonlinear operators arising in high-speed networking applications☆

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ABSTRACT

We consider the behaviour of a class of nonlinear difference equations that arise in networks employing congestion control protocols. For a class of such systems we show that trajectories, experiencing the same set of congestion notifications, but starting at different initial conditions, converge exponentially to each other.



## 1. Introduction

Recent years have witnessed a huge interest in both the analysis and design of congestion control protocols for deployment in the internet (Low, Paganini, & Doyle, 2002). In particular, protocols based on the *Additive-Increase Multiplicative-Decrease* (AIMD) paradigm, of which TCP (Jacobson, 1988) is one, have been the subject of much scrutiny. The study of such protocols is interesting for a number of reasons. Clearly, given the importance of the internet, the study of network congestion control is of immense practical value. Secondly, network congestion control brings together many concepts and ideas from diverse areas of Applied Mathematics. These include (to name but a few): Probability Theory (Barnsley, Demko, Elton, & Geronimo, 1988; Stenflo, 2002); the theory of Stochastic Matrices (Shorten, Wirth, & Leith, 2006); Positive Systems and Hybrid Systems (Baccelli & Hong, 2002). Many practical problems that arise in the study of congestion control therefore motivate mathematical questions that are not only of interest in the context of the application, but also merit study in a more general and abstract setting. Our problem is motivated by such generalisations of the *Transmission Control Protocol* (TCP). Standard TCP is based on the AIMD algorithm of Chiu and Jain. In their original paper (Chiu & Jain, 1989), Chiu and Jain consider a system in which $n$ users compete for a resource. The users' actions consist of (continuously) probing the availability of the resource by submitting requests for its use. A key assumption in the model is that the users do not communicate directly and that the only information they have about the availability of the resource is when the collective utilization of the resource reaches some capacity constraint. At such time instances, referred to as *congestion events*, some or all users are informed through feedback and respond instantly by decentralized down-scaling of their individual utilization rates. Given this basic setting, the problem is then to develop an algorithm that produces probing strategies for the users so that each user will infer its "fair" share of the shared resource in a decentralized manner, while at the same time preventing congestion collapse (Low et al., 2002).

The AIMD algorithm of Chiu and Jain describes probing strategies that evolve in cycles, each cycle having two phases (Rothblum & Shorten, 2007). The first phase is instantaneous. It occurs when capacity is reached, a subset of users are notified, and notified users respond by down-scaling their utilization rates (abruptly) by a multiplicative factor. During the second phase of a cycle, each user increases its utilization rate linearly until congestion is reached again, at which point the first phase of the next cycle is entered. Let $w_i(k) \geq 0$ be the $i$th user's share of the resource at the beginning of the $k$th cycle. Then,

$$w_i(k+1) = b_i(k)w_i(k) + \alpha_i T(k), \quad (1)$$

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where $T(k)$ is the time taken by the $k$th cycle; we call this the *inter-congestion time*. The positive scalar $\alpha_i$ is the rate of increase in the second phase. The scalar $b_i(k)$ depends on whether or not the $i$th user is informed of congestion at the $k$th event; if it is informed then $b_i(k)$ equals some constant $\beta_i$ (call it the drop parameter) which satisfies $0 \leq \beta_i < 1$; if not, $b_i(k) = 1$. If $C$ is the total capacity of the resource available then, at congestion,

$$w_1(k) + \cdots + w_n(k) = w_1(k+1) + \cdots + w_n(k+1) = C. \quad (2)$$

The inter-congestion time $T$ is determined by (1) and (2).

### 1.1. Previous results and contribution

A primary objective of network congestion control is to prevent network congestion collapse, or more generally to maintain a reasonable throughput of useful information for users (as opposed to resending the same information that is continually being dropped). A second, equally important, objective is to ensure a fair and equitable allocation of network resources (bandwidth) in a distributed manner without communication between network users.

For this dynamical system it is essential to settle questions such as the existence and uniqueness of a trajectory which is independent of initial conditions, and the rate at which the system approaches this steady state trajectory. These convergence issues have been studied extensively for the standard AIMD-TCP model described above, and a complete picture for this model is presented in Shorten, King, and Wirth (2007). Even under very general assumptions, questions relating to the existence and uniqueness of network equilibria have been settled and are reported in Shorten et al. (2007).

Our objective in the present paper is to extend these results to generalisations of standard TCP. Such generalisations have been motivated by the view that standard TCP is deficient in most modern network scenarios (Floyd, 2003; Jin, Wei, & Low, 2003; Kelly, 2002; Xu, Harfoush, & Rhee, 2004). Recently nonlinear AIMD variants (NAIMD) have been proposed for deployment in networks. Remarkably, despite the fact that nonlinear TCPs are widely deployed in the internet (cubic is the default in Linux), few mathematical results exist that characterise their behaviour (one way or the other).

Generally speaking, NAIMD protocols differ from the standard (linear) AIMD in the second (AI) phase, whereby the linear increase in the utilization rate of each user $i$ is replaced by a nonlinear function of time that we denote by $a_i$. Thus a user that is informed of congestion at the $k$th congestion event evolves as follows:

$$w_i(k+1) = \beta_i w_i(k) + a_i(T(k)). \quad (3)$$

However for a user that is not informed of congestion, the future evolution of its utilization rate depends in addition on how long it has been growing in the second phase. Specifically let $t_i(k)$ be the time elapsed since user $i$ was last informed of a congestion event. Then the evolution equation for its utilization rate is

$$w_i(k+1) = w_i(k) + a_i(t_i(k) + T(k)) - a_i(t_i(k)). \quad (4)$$

Thus a complete description of the state at the $k$th congestion event must include the times $t_i$ as well as the utilization rates $w_i$. So the state $X$ of the system at a congestion event is a vector in $\mathbb{R}^{2n}$, namely

$$X = (w, t) \quad \text{where } w = (w_1, \ldots, w_n) \text{ and } t = (t_1, \ldots, t_n).$$

Also Eqs. (3) and (4) must be supplemented by the evolution equations for $t_i$. If user $i$ is informed of congestion then

$$t_i(k+1) = T(k) \quad (5)$$

while if user $i$ is not informed of congestion then

$$t_i(k+1) = t_i(k) + T(k). \quad (6)$$

Together (3)–(6) determine the evolution of the state vector $X(k) \mapsto X(k+1)$ between successive congestion events.

Initial attempts to study general nonlinear AIMD models are presented in Rothblum and Shorten (2007) and King, Shorten, Wirth, and Akar (2008); albeit for the case of synchronised communication networks. Motivated by the desire to ultimately extend this work to a stochastic setting mimicking that in Shorten et al. (2007), we present here in this present paper an initial analysis of nonlinear AIMD protocols which we call *eventually linear*. These models are characterised by growth functions $a_i$ which become (affine) linear after some time $T_{m_i}$. Specifically, such a growth function $a_i$ satisfies

$$a_i(\tau') = a_i(T_{m_i}) + \alpha_{H_i}(\tau' - T_{m_i}) \quad \text{for } \tau' \geq T_{m_i} \quad (7)$$

where $\alpha_{H_i}$ is constant, and where $T_{m_i}$ is the 'elbow' in the function. So after time $T_{m_i}$ in the second phase the model reverts to standard linear TCP with utilization rate growing at the rate $\alpha_{H_i}$. For the synchronised case it is known that all solutions converge to a unique fixed point at a geometrical rate (Rothblum & Shorten, 2007). Here we begin the analysis of the unsynchronised case of the eventually linear model, by addressing the question of how the solution depends on the initial value $w(0)$ for the user shares.

## 2. Eventually linear congestion control protocols

The main motivation for moving away from linear AIMD is to realise protocols whose properties scale with increasing network bandwidths. This requirement suggests developing algorithms whose probing phase is nonlinear or piecewise linear in nature, where each linear phase is designed with a typical network (bandwidth) scenario in mind. In this latter setting, it is entirely reasonable to assume that it only rarely occurs that network flows do not achieve the phase of the probing action that is suitable for current network conditions. From a mathematical perspective, this is a very appealing assumption. In essence, this means that while eventually linear protocols are nonlinear and thus include all the complexities of NAIMD, including long-term memory effects, this latter assumption means that the nonlinear features (which dominate only when the inter-congestion times are short) can be treated fully without approximation in this regime. In particular, if all inter-congestion times are greater than a certain minimum value then the growth functions of every flow move past the 'elbow' before the next congestion event occurs. This latter fact means that the equations describing the difference dynamics are the same as in standard TCP and all our previous results apply. In what follows we explore this intuition in a more formal setting. We begin by defining the notion of a *regular* congestion event.

**Definition 2.1.** A congestion event $k$ is *regular* if $T(k) \geq T_{m_i}$ for every user $i$.

Thus a regular congestion event occurs whenever the subsequent inter-congestion time exceeds all the times $T_{m_i}$, meaning that every flow enters its linear phase before the next congestion event. The essential feature of regular events is that they ensure that all memory of the past is lost and the network dynamics are in effect linear. To explore the consequences of regular events we will compare two trajectories, both experiencing the same selection of dropped flows at each congestion event, but evolving from two different initial states. Our principal assumption is that the congestion events are all regular for both trajectories. We denote the set of states of the first trajectory at successive congestion events by $\{w(0), w(1), \ldots, w(k)\}$, where $w(0)$ is the initial condition, and the set of states associated with the second trajectory by $\{v(0), v(1), \ldots, v(k)\}$. Denote further the difference vectors by $E(k) = w(k) - v(k)$. Then, the assumption that all events are regular implies that the difference vectors evolve linearly between the $t_i(k)$, that is,

$$\begin{aligned} E_i(k+1) &= w_i(k+1) - v_i(k+1) \\ &= b_i(k)(w_i(k) - v_i(k)) + \alpha_{H_i}(T(k) - S(k)) \end{aligned} \tag{8}$$

where $b_i(k) = \beta$ or $b_i(k) = 1$ depending on whether or not this flow experiences a drop at the event, and where $T(k), S(k)$ are the inter-congestion times for the $w, v$ flows respectively. Since both $E(k)$ and $E(k+1)$ are perpendicular to the vector $e = [1, 1, \ldots, 1]^{\mathrm{T}}$ this set of equations determines the value of $T(k) - S(k)$, and leads to the linear relation

$$E(k+1) = A(k)E(k), \quad A(k) \in \mathcal{A} \tag{9}$$

where $\mathcal{A}$ is precisely a set of matrices obtained from the dynamics of standard unsynchronised TCP. In particular, each $A(k)$ is entrywise non-negative and column stochastic and hence non-expansive in the one-norm; thus $\|E(k+1)\| \leq \|E(k)\|$ for all $k$. Here and throughout the paper $\|\cdot\|$ denotes the one-norm of a vector, that is, the sum of the absolute values of its components.

**Definition 2.2.** A drop sequence is *typical* if there is an integer $N$ such that every flow experiences at least one drop during $N$ successive congestion events.

We then have the following theorem.

**Theorem 2.1.** *Let $w(0), v(0)$ be any initial conditions, and consider a typical drop sequence for which both flow sequences $\{w(k)\}$ and $\{v(k)\}$ experience only regular congestion events. Then $\|E(k)\| = \|w(k) - v(k)\|$ converges to zero at an exponential rate as $k \to \infty$, and the rate is uniform in $w(0), v(0)$.*

**Proof.** It follows from (9) that

$$E(k) = A(k)\,A(k-1)\,\ldots A(0)\,E(0). \tag{10}$$

Since the drop sequence is typical there is an integer $N$ such that every flow experiences a drop at least once in every $N$ events. Let $k = \kappa N + l$ where $0 \leq l \leq N-1$ is the remainder, and define for $j = 0, \ldots, \kappa - 1$

$$B(j) = A(jN+N-1)\,A(jN+N-2)\,\ldots\,A(jN). \tag{11}$$

Then

$$E(k) = A(\kappa N + l)\,\ldots\,A(\kappa N)\,B(\kappa-1)\,\ldots B(1)\,B(0)\,E(0).$$

Each matrix $A(k)$ is non-expansive on the hyperplane in $\mathbb{R}^n$ orthogonal to $[1, 1, \ldots, 1]^{\mathrm{T}}$, that is $\|A(k)E\| \leq \|E\|$, for all $E$ in this hyperplane. Each matrix $B(j)$ is column stochastic and entrywise positive. This follows from the assumption that the drop sequence is regular, and from the fact that the columns of $A(k)$ corresponding to its dropped flows are entrywise positive. It therefore follows that each matrix $B(j)$ is a contraction on the hyperplane in $\mathbb{R}^n$ orthogonal to $[1, 1, \ldots, 1]^{\mathrm{T}}$. Since the set $\mathcal{A}$ is finite (it has at most $2^n - 1$ elements), it follows that there is there is a finite number of different $B(j)$ matrices (at most $(2^n-1)^N$). Hence there is $r < 1$ such that for all $j$ and for all $E$ orthogonal to $[1, 1, \ldots, 1]^{\mathrm{T}}$,

$$\|B(j)E\| \leq r\,\|E\|. \tag{12}$$

It follows that

$$\|E(k)\| \leq r^{\kappa}\,\|E(0)\| \leq r^{k/N-1}\,\|E(0)\|. \tag{13}$$

which yields exponential convergence at rate $r^{1/N}$, uniform in $w(0), v(0)$. □

Before proceeding to the statement of our main result, we present the following lemma (which is proven in the Appendix).

**Lemma 2.1.** *If $t_i(k^*) \geq T_{m_i}$ for every user $i$ which is not dropped at some stage $k^*$ then, $w(k)$ is independent of $t(k^*)$ for all $k \geq k^*$.*

As we shall see in the proof of our main result, this lemma is central to all that follows and with its help we shall prove that in spite of nonlinear and long-term memory effects, trajectories of two-flow systems starting from different initial conditions but experiencing the same drop sequence converge to each other exponentially, and that this is true for every possible drop sequence.

## 3. A convergence result for two users

We now present the main mathematical results of this paper. Our main result says that for a two-user network, trajectories starting from different initial conditions, but experiencing the same set of congestion events, converge exponentially to each other.

**Setting:** In what follows $w = (w_1, w_2)$ and $t = (t_1, t_2)$, and the evolution of the our dynamic system is described by the four difference equations

$$\begin{aligned} w_i(k+1) &= b_i w_i + a_i(d_i t_i + T) - a_i(d_i t_i) \\ t_i(k+1) &= d_i t_i + T \end{aligned} \qquad i = 1, 2 \tag{14}$$

where, for simplicity of presentation, we sometimes omit the dependence on $k$ of variables on the right-hand side of a difference equation. The scalars $b_i(k)$ and $d_i(k)$ are defined as follows: if the $i$th user is dropped at event $k$ then, $(b_i(k), d_i(k)) = (\beta_i, 0)$; if the $i$th user is not dropped at stage $k$ then, $(b_i(k), d_i(k)) = (1, 1)$. As usual, the time between congestion events $T$, is determined by the congestion condition that

$$w_1(k) + w_2(k) = w_1(k+1) + w_2(k+1) = C$$

and system (14) evolves on the extended state space

$$\mathcal{S}^{\mathrm{ext}} = \{(w, t) : w \in \mathcal{S}, t_1, t_2 \geq 0\} \tag{15}$$

where

$$\mathcal{S} = \{w \in \mathbb{R}^2 : w_1, w_2 \geq 0 \text{ and } w_1 + w_2 \leq C\} \tag{16}$$

is the rate space.

We assume that both users have growth rate functions which are eventually affine as described in (7). We also assume that each $a_i$ is *increasing and convex* on $[0, \infty)$, $a_i(0) = 0$ and there exists $\alpha_{L_i} > 0$ such that $a_i(\tau_2) - a_i(\tau_1) \geq \alpha_{L_i}(\tau_2 - \tau_1)$ for all $\tau_2 \geq \tau_1$.

**Statement of the main result:** For the special case of two users we show that for a given drop sequence, all trajectories converge exponentially to a unique trajectory.

The solution of system (14) is determined by the initial state $(w(0), t(0))$ and by the sequence of drops at the congestion events. Our main result is that when the system capacity $C$ is sufficiently large, the long-term behaviour of $w$ is determined solely by the sequence of drops and is independent of the initial state.

Consider first the cases in which one of the users is never dropped. If the first user is never dropped, one can show that, regardless of initial conditions, $\lim_{k\to\infty} w(k) = (1, 0)$. For the second user, the consequence is $\lim_{k\to\infty} w(k) = (0, 1)$. We call the drop sequences corresponding to these cases trivial.

Our main result requires that $C, T_{m_1}, T_{m_2}$ satisfy the following condition.

$$C \geq \max\{C_0, C_1, C_2\} \tag{17}$$

where

$$C_0 = \underline{w}_1 + \underline{w}_2 \tag{18}$$

$$C_1 = \frac{(\alpha_{H_1} + \alpha_{H_2})\underline{w}_1 + \alpha_{H_1} a_2(T_m) - \alpha_{H_2} a_1(T_m)}{(1-\beta_2)\alpha_{H_1}} \tag{19}$$

$$C_2 = \frac{(\alpha_{H_1} + \alpha_{H_2})\underline{w}_2 + \alpha_{H_2} a_1(T_m) - \alpha_{H_1} a_2(T_m)}{(1-\beta_1)\alpha_{H_2}} \tag{20}$$

with

$$\underline{w}_1 = \frac{a_1(T_m) + \alpha_{H_2} T_m}{1-\beta_1}, \qquad \underline{w}_2 = \frac{a_2(T_m) + \alpha_{H_1} T_m}{1-\beta_2} \tag{21}$$

and

$$T_m = \max\{T_{m_1}, T_{m_2}\}. \tag{22}$$

Note that $C_0, C_1, C_2$ go to zero as $T_{m_1}, T_{m_2}$ go to zero. Hence, for a given system capacity $C$, condition (17) can be satisfied by choosing $T_{m_1}$ and $T_{m_2}$ sufficiently small.

We now present the main result of the paper.

**Theorem 3.1.** *When condition (17) is satisfied, there is a constant $r < 1$ such that the following holds for every non-trivial drop sequence. There are constants $c > 0$ and $k^*$ such that if $(w, t)$ and $(v, s)$ are any two solutions of (14) then,*

$$\|w(k) - v(k)\| \le c r^{k/2}(\|w(0) - v(0)\| + \|t(0) - s(0)\|)$$

*for all $k \ge k^*$.*

Theorem 3.1 shows that rate vector histories corresponding to different initial conditions exponentially converge onto a trajectory which is determined solely by the sequence of dropped users at the congestion events. Hence the long-run behaviour of the rate vector does not depend on the initial conditions.

**Comment:** Our main theorem does not exclude the possibility that the long-run behaviour could depend on how the dropped users are selected at the congestion events, since different sequences of drops will lead to different trajectories.

## 4. Proof of the main result

We now present the proof of our main result. An essential component of our proof is the existence of a reduced order dynamic system that captures the convergence properties of the original system.

### 4.1. Preamble to proof: A reduced order system

At a congestion event, there are only three possibilities: only user one is dropped, only user two is dropped or both users are dropped. We use the integer $q_k$ to indicate the set of users dropped at an event $k$. We let $q_k = 1, 2$, or 3 corresponding to the three possibilities as listed in the above order. With $X = (w, t)$, the evolution of system (14) can then be described by

$$X(k+1) = \Psi_{q_k}(X(k)) \tag{23}$$

where $q_k = 1, 2$ or 3 and the maps $\Psi_1, \Psi_2, \Psi_3$ are determined from (14) by choosing appropriate values of $b_i$ and $d_i$. For example, $\Psi_1$ is obtained by letting $(b_1, d_1) = (\beta_1, 0)$ and $(b_2, d_2) = (1, 1)$ in (14). We also introduce the following two-step maps

$$\Psi_4 = \Psi_2 \circ \Psi_1, \qquad \Psi_5 = \Psi_1 \circ \Psi_2. \tag{24}$$

Consider the following five subsets of the extended state space:

$$\mathcal{M}_1 = \{X \in \mathcal{S}^{\text{ext}} : t_2 \ge T_{m_2}\} \tag{25}$$

$$\mathcal{M}_2 = \{X \in \mathcal{S}^{\text{ext}} : t_1 \ge T_{m_1}\} \tag{26}$$

$$\mathcal{M}_3 = \mathcal{S}^{\text{ext}}, \qquad \mathcal{M}_4 = \mathcal{M}_1, \qquad \mathcal{M}_5 = \mathcal{M}_2. \tag{27}$$

For $p = 1, 2, 3$, the condition that $X \in \mathcal{M}_p$ is equivalent to the requirement that $t_i \ge T_{m_i}$ if the $i$th user is not dropped for the map $\Psi_p$. For $p = 4$ or 5, $X \in \mathcal{M}_p$ means that $t_i \ge T_{m_i}$ if the $i$th user is not dropped for the first stage of the map $\Psi_p$.

As we have stated our objective here is to show that our original system is equivalent in some sense to a reduced order system. We establish this equivalence using Lemmas 4.1–4.4 and Corollaries 4.1, 4.2. When not explicitly given, proofs are presented in the Appendix. The first of these results is an immediate consequence of Lemma 2.1.

**Lemma 4.1.** *For $p = 1, \ldots, 5$, $\Psi_p(X)$ is independent of $t$ when $X \in \mathcal{M}_p$.*

Now let $P$ be the projection of $\mathcal{S}^{\text{ext}}$ onto $\mathcal{S}$ defined by

$$P(w, t) = w. \tag{28}$$

For $p = 1, \ldots, 5$, it follows from the last result that if we define the map $\Phi_p$ on the rate space $\mathcal{S}$ by

$$\Phi_p(w) = P\Psi(w, T_m, T_m) \tag{29}$$

where $T_m$ is defined in (22) then,

$$P\Psi_p(X) = \Phi_p(PX) \quad \text{for all } X \in \mathcal{M}_p. \tag{30}$$

Recalling (21) we have the following result and corollary.

**Lemma 4.2.** *If a user $i^*$ is dropped at an event $k$ where $w_{i^*}(k) \ge \underline{w}_{i^*}$ then, $T(k) \ge T_m$ and $t_i(k+1) \ge T_m$ for $i = 1, 2$.*

In particular, the following is an immediate consequence of the previous discussion.

**Corollary 4.1.** *If both users are dropped at an event $k$ then, $T(k) \ge T_m$ and $t_i(k+1) \ge T_m$ for $i = 1, 2$.*

**Proof.** Assumption (17) on $C$ tells us that $C \ge C_0 = \underline{w}_1 + \underline{w}_2$. Since $w_1 + w_2 = C$ we must have $w_1 \ge \underline{w}_1$ or $w_2 \ge \underline{w}_2$. Since both users are dropped, it now follows from Lemma 4.2 that $T(k) \ge T_m$ and $t_i(k+1) \ge T_m$ for $i = 1, 2$. □

Furthermore, the following lemma and corollary also follow.

**Lemma 4.3.** *If user $i^*$ is not dropped at an event $k$ then, $w_{i^*}(k+1) \ge \underline{w}_{i^*}$.*

**Corollary 4.2.** *If for some $i^*$, user $i^*$ is not dropped at an event $k$ but is dropped at event $k+1$ then, $T(k+1) \ge T_m$ and $t_i(k+2) \ge T_m$ for $i = 1, 2$.*

**Proof.** Since user $i^*$ is not dropped at event $k$, Lemma 4.3 tells us that $w_{i^*}(k+1) \ge \underline{w}_{i^*}$. It now follows from Lemma 4.2 that $T(k+1) \ge T_m$ and $t_i(k+2) \ge T_m$ for $i = 1, 2$. □

The following result motivates the introduction of $\Psi_4$ and $\Psi_5$; it follows from Corollaries 4.1 and 4.2. In this result

$$\mathcal{M} = \{X \in \mathcal{S}^{\text{ext}} : t_1, t_2 \ge T_m\}. \tag{31}$$

Note that $\mathcal{M} \subset \mathcal{M}_p$ for all $p$.

**Lemma 4.4.** *If $p = 1, \ldots, 5$ then, $\mathcal{M}_p$ is invariant for $\Psi_p$, that is, $\Psi_p(X) \in \mathcal{M}_p$ for $X \in \mathcal{M}_p$. If $p = 3, 4, 5$ then, $\Psi_p(X) \in \mathcal{M}$ for all $X \in \mathcal{S}^{\text{ext}}$.*

We now have the main result of this section.

**Lemma 4.5** (*Reduction Lemma*). *Suppose $(w, t)$ is any solution of (14) and $t_i(k^*) \geq T_m$ for $i = 1, 2$ at some event $k^*$. Then for any event $k^f \geq k^*$, there is a subsequence $\{k_0, k_1, \ldots, k_l\}$ of $\{k^*, k^* + 1, \ldots, k^f\}$ with $k_0 = k^*$, $k_l = k^f$, $l \geq (k^f - k^*)/2$ and a sequence $\{p_0, p_1, \ldots, p_{l-1}\}$ such that*

$$w(k_{j+1}) = \Phi_{p_j}(w(k_j)) \tag{32}$$

*for $j = 0, \ldots, l - 1$.*

**Remark 4.1.** The above lemma tells us that if $t_1(k^*), t_2(k^*) \geq T_m$ at some event $k^*$ then, the resulting behaviour of the rate vector $w$ for the original full order system (14) is completely determined by the reduced order system (32).

### 4.2. Contractions and the proof of the main result

We now proceed to prove the main result of the paper. As before, proofs not explicitly given, are presented in the Appendix. Before we proceed let us now define the contraction parameter

$$r = \max\{r_1, r_2, r_4, r_5\}, \tag{33}$$

where

$$r_1 = \frac{\alpha_{H_1} + \alpha_{L_2}\beta_1}{\alpha_{H_1} + \alpha_{L_2}}, \qquad r_2 = \frac{\alpha_{H_2} + \alpha_{L_1}\beta_2}{\alpha_{H_2} + \alpha_{L_1}}$$

$$r_4 = \frac{\alpha_{L_1}(\alpha_{H_1}\beta_2 + \alpha_{H_2}\beta_1) + \alpha_{H_2}(\alpha_{H_1}(1 + \beta_1\beta_2 - \beta_1) + \alpha_{H_2}\beta_1)}{(\alpha_{L_1} + \alpha_{H_2})(\alpha_{H_1} + \alpha_{H_2})}$$

$$r_4 = \frac{\alpha_{L_2}(\alpha_{H_2}\beta_1 + \alpha_{H_1}\beta_2) + \alpha_{H_1}(\alpha_{H_2}(1 + \beta_1\beta_2 - \beta_2) + \alpha_{H_1}\beta_2)}{(\alpha_{L_2} + \alpha_{H_1})(\alpha_{H_1} + \alpha_{H_2})}.$$

Careful examination of the above expressions reveals that $r < 1$.

Our first result in this section states that $\Psi_1, \Psi_2, \Psi_3$ are globally Lipschitz and $\Phi_1, \Phi_2, \Phi_3$ are contractive.

**Lemma 4.6.** (i) *There is a constant $c_1$ such that, for all $X, Y \in \mathcal{S}^{\text{ext}}$,*

$$\|\Psi_q(X) - \Psi_q(Y)\| \leq c_1\|X - Y\| \quad \text{for } q = 1, 2, 3. \tag{34}$$

(ii) *For all $w, v \in \mathcal{S}$,*

$$\|\Phi_q(w) - \Phi_q(v)\| \leq r\|w - v\| \quad \text{for } q = 1, 2, 3, \tag{35}$$

*where $r < 1$ is given by (33).*

The next result states that the maps $\Phi_4$ and $\Phi_5$ are contractive.

**Lemma 4.7.** *For all $w, v \in \mathcal{S}$,*

$$\|\Phi_p(w) - \Phi_p(v)\| \leq r\,\|w - v\| \quad \text{for } p = 4, 5, \tag{36}$$

*where $r < 1$ is given by (33).*

Now, armed with these preliminary results we give the proof of the main result of the paper.

**Proof of Theorem 3.1.** Let $X = (w, t)$ and $Y = (v, s)$ be any two solutions of (14) corresponding to some non-trivial drop sequence $\{q_1, q_2, \ldots\}$. Then there exists an event $k^*$ such that either $q_{k^*-1} = 3$ or $\{q_{k^*-2}, q_{k^*-1}\}$ equals $\{1, 2\}$ or $\{2, 1\}$. It follows from Corollaries 4.1 and 4.2 that for any of the above occurrences,

$$t_i(k^*), s_i(k^*) \geq T_m$$

for $i = 1, 2$.

Consider any event $k = k^f > k^*$. It now follows from the Reduction Lemma (Lemma 4.5) that there is a subsequence $\{k_0, k_1, \ldots, k_l\}$ of $\{k^*, k^* + 1, \ldots, k^f\}$ with $k_0 = k^*$, $k_l = k^f$, $l \geq (k^f - k^*)/2$ and a sequence $\{p_0, \ldots, p_{l-1}\}$ such that

$$w(k_{j+1}) = \Phi_{p_j}(w(k_j)) \quad \text{and} \quad v(k_{j+1}) = \Phi_{p_j}(v(k_j)) \tag{37}$$

for $j = 0, \ldots, l-1$. Recalling the contraction results in Lemmas 4.6 and 4.7 we now obtain that

$$\|w(k_{j+1}) - v(k_{j+1})\| \leq r\|w(k_j) - v(k_j)\| \tag{38}$$

for $j = 0, \ldots, l - 1$. It now follows from (38), $k^f = k_l$, $k_0 = k^*$ and $l \geq (k^f - k^*)/2$ that

$$\begin{aligned}\|w(k^f) - v(k^f)\| &= \|w(k_l) - v(k_l)\| \leq r^l\|w(k_0) - v(k_0)\| \\ &\leq r^{(k^f - k^*)/2}\|w(k^*) - v(k^*)\|.\end{aligned}$$

Hence,

$$\|w(k^f) - v(k^f)\| \leq r^{(k^f - k^*)/2}\|w(k^*) - v(k^*)\|. \tag{39}$$

From Lemma 4.6 there exists a constant $c_1$ independent of $w(0), v(0), t(0), s(0)$ such that (34) holds. Since $X(k + 1) = \Psi_{q_k}(X(k))$ and $Y(k + 1) = \Psi_{q_k}(Y(k))$ where $q_k = 1, 2$ or 3, we have

$$\|X(k + 1) - Y(k + 1)\| \leq c_1\|X(k) - Y(k)\|$$

for all $k$. This implies that

$$\begin{aligned}\|w(k^*) - v(k^*)\| &\leq \|X(k^*) - Y(k^*)\| \leq c_1^{k^*}\|X(0) - Y(0)\| \\ &= c_1^{k^*}(\|w(0) - v(0)\| + \|t(0) - s(0)\|).\end{aligned}$$

This, combined with (39) yields

$$\|w(k^f) - v(k^f)\| \leq c\, r^{k^f/2}(\|w(0) - v(0)\| + \|t(0) - s(0)\|)$$

where $c = (c_1/r^{1/2})^{k^*}$. □

## 5. Example

As we have already mentioned, our ultimate objective is to demonstrate the existence of a unique stationary measure for a wide class of nonlinear congestion control protocols. Roughly speaking, the existence of such a measure means that the long-term average behaviour of the network does not depend on initial conditions. In situations, where simulation is used as the principal tool for analysis, knowledge that the average behaviour of the network is independent of starting condition is a fundamental requirement. In what follows we give an example to illustrate our basic result.

Our basic setup is as follows. Two identical NAIMD flows compete for ten units of capacity. Upon notification of congestion a flow reduces its rate to half the value it had achieved before congestion. After congestion notified flows probe for available bandwidth with piecewise linear growth rate

$$a_i(\tau') = \begin{cases} \alpha_L\tau' & \text{if } 0 \leq \tau' \leq 1 \\ \alpha_L + \alpha_H(\tau' - 1) & \text{if } 1 \leq \tau' \end{cases}$$

where $\alpha_L = 1$ and $\alpha_H = 5$. In these examples, we abuse notation and let $w(\tau) = (w_1(\tau), w_2(\tau))$ be the flow rate vector at time $\tau \in [0, \infty)$.

**Example 1** (*A Random Drop Sequence*). Here we generate a sequence of fifty congestion notifications using the random number generator in Matlab. We follow the evolution of two trajectories, each experiencing these identical congestion notifications, but starting from different initial conditions. Figs. 1 and 2 depict the evolution of $w(\tau)$ from $w(0) = (2, 8)$ and $w(0) = (5, 5)$, respectively; in both cases $t(0) = (0, 0)$. Finally, we show in Fig. 3 the distance between these trajectories as measured by the 1-norm as a function of congestion event.

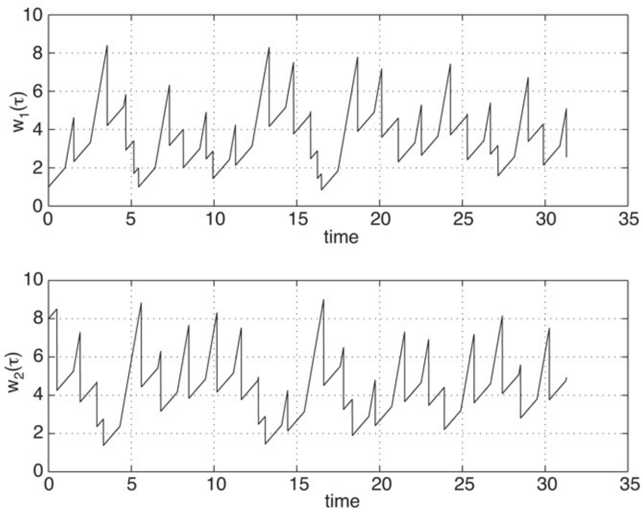


**Fig. 1.** Example: Evolution of $w(\tau)$ from $w(0) = (2, 8)$.

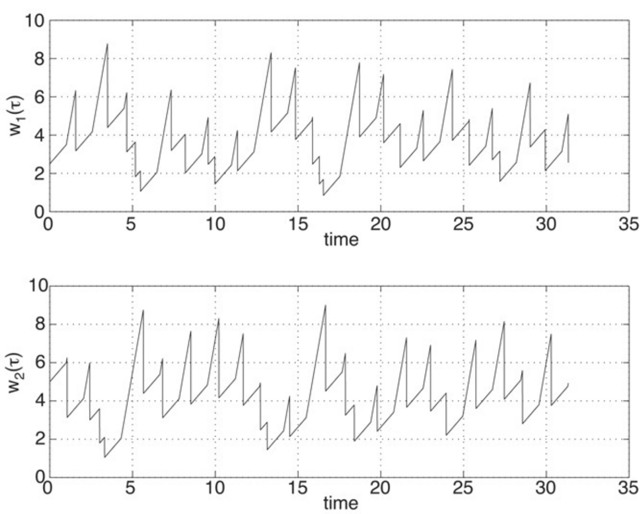


**Fig. 2.** Example 1: Evolution of $w(\tau)$ from $w(0) = (5, 5)$.

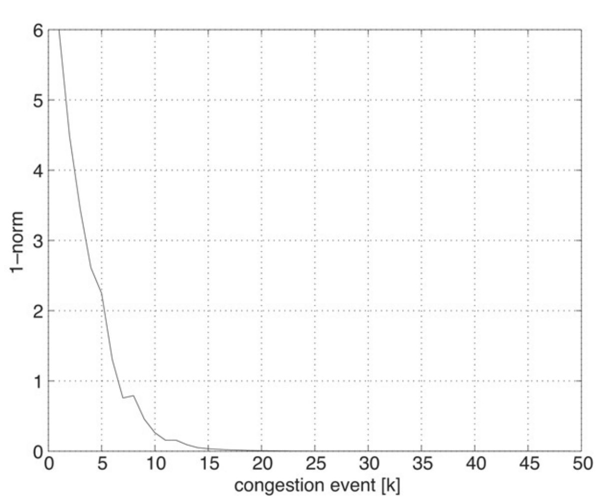


**Fig. 3.** Example 1: Distance between flow trajectories. Note that the 1-norm does not decrease monotonically; but the convergence is exponential.

## 6. Conclusions

In this brief paper we have presented initial results for nonlinear AIMD protocols operating in stochastic environments. Future work will involve extending our results to more realistic network scenarios.

## Acknowledgement

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## Appendix

**Proof of Lemma 2.1.** For $k \geq k^*$, let $\mathcal{D}(k)$ denote the set of users that have been dropped at least once during the events $k^*, \ldots, k$. We will show by induction that for all $k > k^*$ and $j \in \mathcal{D}(k-1)$, the quantities $w(k)$ and $t_j(k)$ are independent of $t(k^*)$.

So suppose that this statement holds for some event $k > k^*$. If $i \notin \mathcal{D}(k-1)$ then, user $i$ has not been dropped during events $k^*, \ldots, k-1$ and (6) implies that $t_i(k) \geq t_i(k^*) \geq T_{m_i}$. It follows from the fact that $a_i$ is affine after $T_{m_i}$ (recall (7)) that

$$a_i(t_i(k) + T(k)) - a_i(t_i(k)) = \alpha_{H_i} T(k). \tag{40}$$

Hence Eqs. (2)–(4) are independent of $t_i(k)$. Thus, $T(k)$ and $w(k+1)$ are independent of $t_i(k)$ and depend only on $w(k)$ and $t_j(k)$ where $j \in \mathcal{D}(k-1)$. Since, by assumption, $w(k)$ and $t_j(k)$ do not depend on $t(k^*)$ for all $j \in \mathcal{D}(k-1)$ it follows that $T(k)$ and $w(k+1)$ are independent of $t(k^*)$. Consider now any user $j \in \mathcal{D}(k)$. It follows from (5) and (6) that $t_j(k+1)$ depends only on $w(k)$ and $t_j(k)$ where $j \in \mathcal{D}(k-1)$; hence $t_j(k+1)$ is independent of $t(k^*)$.

We now show that the statement holds for $k = k^* + 1$. If user $i$ is not dropped at $k^*$, $t_i(k^*) \geq T_{m_i}$ and $a_i(t_i(k^*) + T(k^*)) - a_i(t_i(k^*)) = \alpha_{H_i} T(k^*)$. Eqs. (2)–(4) imply that $T(k^*)$ and $w(k^*+1)$ are independent of $t(k^*)$. If $j \in \mathcal{D}(k^*)$ then (5) now implies that $t_j(k^*+1)$ is independent of $t(k^*)$. □

**Proof of Lemma 4.2.** Suppose user one is dropped at an event $k$. Then

$$w_1(k+1) = \beta_1 w_1 + a_1(T) \tag{41}$$

$$w_2(k+1) = b_2 w_2 + a_2(d_2 t_2 + T) - a_2(d_2 t_2) \tag{42}$$

$$t_1(k+1) = T \tag{43}$$

$$t_2(k+1) = d_2 t_2 + T. \tag{44}$$

Here $(b_2, d_2) = (\beta_2, 0)$ if user two is dropped while $(b_2, d_2) = (1, 1)$ if user two is not dropped. Considering $\tau = \max\{T_{m_2},\ d_2 t_2\}$ and using the convexity of $a$ yield

$$a_2(d_2 t_2 + T) - a_2(d_2 t_2) \leq a_2(\tau + T) - a_2(\tau) = \alpha_{H_2} T. \tag{45}$$

Since $b_2 \leq 1$, relationships (42) and (45) result in

$$w_2(k+1) \leq w_2 + \alpha_{H_2} T. \tag{46}$$

Recalling that $w_1(k+1) + w_2(k+1) = w_1 + w_2 = C$, relationships (41) and (46) imply that

$$a_1(T) + \alpha_{H_2} T \geq (1-\beta_1) w_1 \geq (1-\beta_1) \underline{w}_1 = a_1(T_m) + \alpha_{H_2} T_m,$$

that is, $a_1(T) + \alpha_{H_2} T \geq a_1(T_m) + \alpha_{H_2} T_m$. Since $a_1$ is increasing and $\alpha_{H_2} > 0$, we must have $T \geq T_m$. Hence $t_i(k+1) \geq T_m$ for $i = 1, 2$. If user two is dropped, the proof proceeds in the same fashion. □

**Proof of Lemma 4.3.** Suppose user two is not dropped. Then user one must be dropped; hence

$$w_1(k+1) = \beta_1 w_1 + a_1(T) \tag{47}$$

$$w_2(k+1) = w_2 + a_2(t_2 + T) - a_2(t_2). \tag{48}$$

Consider first the situation in which $w_1 \le \underline{w}_1$. Since $w_1 + w_2 = C \ge \underline{w}_1 + \underline{w}_2$, we must have $w_2 \ge \underline{w}_2$. Clearly $w_2(k+1) \ge w_2$; hence $w_2(k+1) \ge \underline{w}_2$.

Consider now the situation in which $w_1 \ge \underline{w}_1$. Since user one is dropped, it follows from Lemma 4.2 that $T \ge T_m$; hence (47) implies that

$$w_1(k+1) = \beta_1 w_1 + a_1(T_m) + \alpha_{H_1}(T - T_m). \quad (49)$$

The convexity of $a$ and $t_2 \ge 0$ implies that $a_2(t_2+T) - a_2(t_2) \ge a_2(T) - a_2(0) = a_2(T) = \alpha_{H_2}(T - T_m) + a_2(T_m)$; hence (48) implies that

$$w_2(k+1) \ge w_2 + a_2(T_m) + \alpha_{H_2}(T - T_m). \quad (50)$$

Combining (49) and (50) to eliminate $T$ and using the identities $w_1(k+1) + w_2(k+1) = w_1 + w_2 = C$ result in

$$(\alpha_{H_1} + \alpha_{H_2})w_2(k+1) \ge (\alpha_{H_1} + \alpha_{H_2})C - (\alpha_{H_1} + \beta_1\alpha_{H_2})w_1 + \alpha_{H_1}a_2(T_m) - \alpha_{H_2}a_1(T_m).$$

Now use the inequalities $w_1 \le C$ and $C \ge C_2$, where $C_2$ is given by (20), to obtain that $w_2(k+1) \ge \underline{w}_2$.

The proof proceeds in the same fashion if user one is not dropped. □

**Proof of Lemma 4.5.** Consider the sequence $\mathcal{Q} = \{q_{k^*}, \ldots, q_{k^f-1}\}$ in $\{1, 2, 3\}$. Using the following algorithm, we obtain a new sequence $\mathcal{P} = \{p_0, p_1, \ldots p_{l-1}\}$ in $\{1, 2, 3, 4, 5\}$ by replacing selected subsequences of the form $\{1, 2\}$ or $\{2, 1\}$ in $\mathcal{Q}$ with 4 and 5, respectively. The new sequence $\mathcal{P}$ does not contain the subsequences $\{1, 2\}$, $\{2, 1\}$, $\{1, 5\}$ or $\{2, 4\}$. This algorithm is illustrated in (51).

Algorithm. Starting with index $K = k^*$, find the first $k \ge K$ with $\{q_k, q_{k+1}\}$ equal to $\{1, 2\}$ or $\{2, 1\}$. If $\{q_k, q_{k+1}\} = \{1, 2\}$, replace it with 4; otherwise replace it with 5. Set the index $K$ to $k+2$.

$$\begin{array}{lll} \mathcal{Q} & 11322221121 & 2232113 \\ \mathcal{P} & 113222541 & 223513 \end{array} \quad (51)$$

From the above algorithm, it should be clear that there is a subsequence $\{k_0, k_1 \cdots, k_l\}$ of $\{k^*, k^*+1, \ldots, k^f\}$ such that $k^* = k_0$, $k^f = k_l$ and

$$X(k_{j+1}) = \Psi_{p_j}(X(k_j)) \quad (52)$$

for $j = 0, \ldots, l-1$. We now show by induction that

$$X(k_j) \in \mathcal{M}_{p_j} \quad \text{for } j = 0, \ldots, l-1. \quad (53)$$

So suppose that $X(k_j) \in \mathcal{M}_{p_j}$ for some $j$. If $p_j = 3$, 4 or 5, Lemma 4.4 tells us that $X(k_{j+1}) = \Psi_{p_j}(X(k_j))$ belongs to $\mathcal{M}$; hence it is in $\mathcal{M}_{p_{j+1}}$. If $p_j = 1$ then, $X(k_j) \in \mathcal{M}_1$ and Lemma 4.4 tells us that $X(k_{j+1}) = \Psi_1(X(k_j))$ belongs to $\mathcal{M}_1$. Since $\mathcal{P}$ does not contain the subsequences $\{1, 2\}$ and $\{1, 5\}$, we must have $p_{j+1} = 1$, 3 or 4, and therefore $X(k_{j+1}) \in \mathcal{M}_{p_{j+1}}$. A similar reasoning applies if $p_j = 2$. Thus, $X(k_j) \in \mathcal{M}_{p_j}$ implies that $X(k_{j+1}) \in \mathcal{M}_{p_{j+1}}$. Since $X(k_0) = X(k^*) \in \mathcal{M} \subset \mathcal{M}_{p_0}$, we obtain (53) by induction.

It now follows from (52), (53) and (30) that, for $j = 0, \ldots, l-1$,

$$PX(k_{j+1}) = P\Psi_{p_j}(X(k_j)) = \Phi_{p_j}(PX(k_j)),$$

that is,

$$w(k_{j+1}) = \Phi_{p_j}(w(k_j)). \quad \square$$

**Proof of Lemma 4.6.** Part (i). Consider $q = 1$ or 3. Since user one is dropped in either case, the map $\Psi_q : X \mapsto X'$ is given by

$$\begin{aligned} w_1' &= \beta_1 w_1 + a_1(T), \qquad t_1' = T \\ w_2' &= b_2 w_2 + a_2(d_2 t_2 + T) - a_2(d_2 t_2), \qquad t_2' = d_2 t_2 + T \end{aligned} \quad (54)$$

where $T$, the inter-congestion time associated with $X$, is given by $w_1' + w_2' = C$. Here $(b_2, d_2) = (1, 1)$ if $q = 1$ while $(b_2, d_2) = (\beta_2, 0)$ if $q = 3$.

Consider any $X, Y \in \mathcal{S}^{\text{ext}}$. With $(w, t) = X$ and $(v, s) = Y$, let $\eta = w - v$ and $\tau = t - s$. Since $\eta_1 + \eta_2 = 0$, we have $\|\eta\| = 2|\eta_1|$; hence

$$\|X - Y\| = \|\eta\| + \|\tau\| = 2|\eta_1| + \|\tau\|. \quad (55)$$

Define $(\eta', \tau') = \Psi_q(X) - \Psi_q(Y)$. Then, $\eta_1' + \eta_2' = 0$ implies that

$$\|\Phi_q(X) - \Phi_q(Y)\| = 2|\eta_1'| + \|\tau'\|. \quad (56)$$

Let $\delta T = T - S$ where $T$ and $S$ are the inter-congestion times associated with $X$ and $Y$, respectively. As a consequence of the properties of $a_1$ and $a_2$, there exist scalars $\alpha_1, \alpha_2, \alpha_3$ such that

$$\begin{aligned} & a_1(T) - a_1(S) = \alpha_1 \delta T \\ & a_2(d_2 t_2 + T) - a_2(d_2 s_2 + S) = \alpha_2(d_2\tau_2 + \delta T) \\ & a_2(d_2 t_2) - a_2(d_2 s_2) = \alpha_3 d_2 \tau_2 \end{aligned} \quad (57)$$

with

$$\alpha_{L_1} \le \alpha_1 \le \alpha_{H_1} \quad \text{and} \quad \alpha_{L_2} \le \alpha_2, \alpha_3 \le \alpha_{H_2}.$$

To see this consider, for example,

$$\alpha_1 = \begin{cases} \dfrac{a_1(T) - a_1(S)}{T - S} & \text{if } T \ne S \\ \alpha_{H_1} & \text{if } T = S. \end{cases} \quad (58)$$

The convexity of $a_2$ implies that $\alpha_3 \le \alpha_2$. It now follows from description (54) of $\Psi_q$ that

$$\begin{aligned} \eta_1' &= \beta_1\eta_1 + \alpha_1\delta T, \qquad \tau_1' = \delta T \\ \eta_2' &= b_2\eta_2 + \alpha_2\delta T + \gamma d_2\tau_2, \qquad \tau_2' = d_2\tau_2 + \delta T \end{aligned} \quad (59)$$

where $\gamma = \alpha_2 - \alpha_3 \ge 0$. Since $\eta_1' + \eta_2' = \eta_1 + \eta_2 = 0$, $\delta T$ is given by

$$(\beta_1 - b_2)\eta_1 + (\alpha_1 + \alpha_2)\delta T + \gamma d_2 \tau_2 = 0.$$

Solving for $\delta T$ and substituting into (59) yield

$$\eta_1' = \left(\frac{\alpha_2\beta_1 + \alpha_1 b_2}{\alpha_1 + \alpha_2}\right)\eta_1 - d_2\left(\frac{\gamma\alpha_1}{\alpha_1 + \alpha_2}\right)\tau_2 \quad (60)$$

$$\tau_1' = \left(\frac{b_2 - \beta_1}{\alpha_1 + \alpha_2}\right)\eta_1 \quad (61)$$

$$\tau_2' = \left(\frac{b_2 - \beta_1}{\alpha_1 + \alpha_2}\right)\eta_1 + d_2\left(1 - \frac{\gamma}{\alpha_1 + \alpha_2}\right)\tau_2. \quad (62)$$

As a consequence of the upper and lower bounds on $\alpha_1, \alpha_2, \alpha_3$ there exists a constant $c_{13}$ independent of $X$ and $Y$ such that

$$2|\eta_1'| + \|\tau'\| \le c_{13}(2|\eta_1| + \|\tau\|). \quad (63)$$

Hence, $\|\Psi_q(X) - \Psi_q(X)\| \le c_{13}\|X - Y\|$. One can follow the same procedure to obtain a corresponding constant $c_{23}$ for $q = 2, 3$. Finally, let $c_1 = \max\{c_{13}, c_{23}\}$ to obtain the desired result.

Part (ii). Consider first $q = 1$ or 3 and recall that $\Phi_q(w) = P\Psi_q(w, T_m, T_m)$. Hence $\|\Phi_q(w) - \Phi_q(v)\| = \|\eta'\| = 2|\eta_1'|$. When $q = 3$ we have $d_2 = 0$. When $q = 1$, $d_2 = 1$. Recall that $a_2$ is affine with slope $\alpha_{H_2}$ after $T_{m_2} \le T_m$. Hence, when $q = 1$ and $t_2 = s_2 = T_m$, Eq. (57) tells us that $\alpha_2 = \alpha_3 = \alpha_{H_2}$; hence $\gamma = 0$. In either case, it follows from (60) that

$$\eta_1' = \left(\frac{\alpha_2\beta_1 + \alpha_1 b_2}{\alpha_1 + \alpha_2}\right)\eta_1. \quad (64)$$

Since $b_2 \leq 1, \beta_1 < 1, 0 < \alpha_1 \leq \alpha_{H_1}$ and $0 < \alpha_{L_2} \leq \alpha_2$, we must have

$$\frac{\alpha_2\beta_1 + \alpha_1 b_2}{\alpha_1 + \alpha_2} \leq \frac{\alpha_2\beta_1 + \alpha_1}{\alpha_1 + \alpha_2} \leq \frac{\alpha_{L_2}\beta_1 + \alpha_{H_1}}{\alpha_{H_1} + \alpha_{L_2}} \leq r$$

and it follows from (64) that $|\eta_1'| \leq r|\eta_1|$. Since $\|\Phi_q(w)-\Phi_q(v)\| = 2|\eta_1'|$ and $\|w - v\| = 2|\eta_1|$, we obtain the desired result that $\|\Phi_q(w) - \Phi_q(v)\| \leq r\|w - v\|$. The proof for $q = 2$ or 3 is similar. □

**Proof of Lemma 4.7.** The map $\Phi_4 : w \mapsto w''$ is given by

$$\begin{aligned} w_1' &= \beta_1 w_1 + a_1(T), \qquad w_1'' = w_1' + a_1(T' + T) - a_1(T) \\ w_2' &= w_2 + \alpha_{H_2} T, \qquad w_2'' = \beta_2 w_2' + a_2(T') \end{aligned} \tag{65}$$

where the inter-congestion times $T$ and $T'$ are calculated by imposing the conditions $w_1' + w_2' = w_1'' + w_2'' = C$. Also, Corollary 4.2 tells us that $T' \geq T_m$. For any $w, v \in \mathcal{S}$, let $\eta = w - v$ and $\eta'' = \Phi_4(w) - \Phi_4(v)$. The conditions $\eta_1 + \eta_2 = 0$ and $\eta_1'' + \eta_2'' = 0$ yield

$$\|w - v\| = 2|\eta_1| \quad \text{and} \quad \|\Phi_4(w) - \Phi_4(v)\| = 2|\eta_1''|. \tag{66}$$

Let $\alpha_1$ be given by (58) where $T, T'$ and $S, S'$ are the inter-congestion times associated with $w$ and $v$, respectively. Convexity of $a(\cdot)$ implies that $\alpha_1 \leq \alpha_H$. The description of $\Phi_4$ in (65) now yields

$$\eta_1' = \beta_1\eta_1 + \alpha_1\delta T \tag{67}$$

$$\eta_2' = \eta_2 + \alpha_{H_2}\delta T \tag{68}$$

$$\eta_1'' = \eta_1' + \alpha_{H_1}(\delta T' + \delta T) - \alpha_1\delta T \tag{69}$$

$$\eta_2'' = \beta_2\eta_2' + \alpha_{H_2}\delta T' \tag{70}$$

where $\delta T := T - S$ and $\delta T' = T' - S'$. Since $\eta_1 + \eta_2 = \eta_1' + \eta_2' = 0$, Eqs. (67) and (68) imply that

$$\delta T = \left(\frac{1 - \beta_1}{\alpha_1 + \alpha_{H_2}}\right)\eta_1, \qquad \eta_1' = \left(\frac{\alpha_1 + \alpha_{H_2}\beta_1}{\alpha_1 + \alpha_{H_2}}\right)\eta_1. \tag{71}$$

Using $\eta_1' + \eta_2' = \eta_1'' + \eta_2'' = 0$, Eqs. (69) and (70) result in

$$\eta_1'' = \left(\frac{\alpha_{H_2} + \alpha_{H_1}\beta_2}{\alpha_{H_1} + \alpha_{H_2}}\right)\eta_1' + \left(\frac{\alpha_{H_2}(\alpha_{H_1} - \alpha_1)}{\alpha_{H_1} + \alpha_{H_2}}\right)\delta T.$$

Substituting the previously obtained expressions (71) for $\eta_1'$ and $\delta T$ results in $\eta_1'' = \rho\eta_1$ where

$$\rho = \frac{\alpha_1(\alpha_{H_1}\beta_2 + \alpha_{H_2}\beta_1) + \alpha_{H_2}(\alpha_{H_1}(1 + \beta_1\beta_2 - \beta_1) + \alpha_{H_2}\beta_1)}{(\alpha_1 + \alpha_{H_2})(\alpha_{H_1} + \alpha_{H_2})}.$$

Since $1 + \beta_1\beta_2 - \beta_1 \geq \beta_2$, the above expression for $\rho$ achieves it maximum value when $\alpha_1$ is at its minimum value of $\alpha_{L_1}$; the corresponding maximum value of $\rho$ is $r_4$. Hence $|\eta_1''| \leq r_4|\eta_1| \leq r|\eta_1|$. Thus $\|\Phi_4(w) - \Phi_4(v)\| \leq r\|w - v\|$. The map $\Phi_5$ is handled similarly. □

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