Input-Output Linearisation by Velocity-Based Gain-Scheduling

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Abstract

The velocity-based analysis framework provides direct support for divide and conquer design approaches, such as the gain-scheduling control design methodology, whereby the design of a nonlinear system is decomposed into the design of an associated family of linear systems. The velocity-based gain-scheduling approach is quite general and directly supports the design of feedback configurations for which the closed-loop dynamics are nonlinear. However, the present paper concentrates on the velocity-based design of controllers which, when combined with a nonlinear plant, attain linear closed-loop dynamics. The resulting approach is a direct generalisation to nonlinear systems of classical frequency-domain pole-zero inversion which is, in many ways, complementary to the Input-Output Linearisation approach. In particular the former is dynamic and reduces to open-loop inversion in the linear case whilst the latter is essentially static and utilises full state feedback.

1. Introduction

The velocity-based analysis framework recently developed in Leith & Leithead (1998b,c,d) establishes an alternative description of a nonlinear system in terms of a family of linear systems; namely, the velocity-based linearisation family. This approach rigorously generalises and extends the conventional series expansion linearisation at an equilibrium operating point and associates a linearisation with every operating point, not just equilibrium operating points. It is emphasised that the velocity-based formulation involves no loss of information and, in particular, is not confined to equilibrium operating points alone but instead encompasses every operating point, including those far from equilibrium.

The velocity-based analysis framework provides direct support for divide and conquer design approaches, such as the gain-scheduling control design methodology, whereby the design of a nonlinear system is decomposed into the design of an associated family of linear systems (Leith & Leithead 1998c). Specifically, since a velocity-based linearisation family is associated with the nonlinear plant, a corresponding linear controller family can be obtained by designing a linear controller for each member of the plant family. A nonlinear controller may then be determined for which the velocity-based linearisation family is the designed linear controller family. It should be noted that the velocity-based gain-scheduling approach resolves many of the deficiencies of the conventional gain-scheduling approach including the restriction to a possibly excessively small neighbourhood of the equilibrium operating points (Leith & Leithead 1998c). By allowing information about the plant dynamics at non-equilibrium operating points to be directly incorporated into the controller design, both sustained non-equilibrium operation and dynamic transitions between equilibrium operating points, including those which take the system far from equilibrium, can be accommodated. In addition, conventional gain-scheduling design approaches employ a number of different, and quite distinct, linearisations (Leith & Leithead 1998c). Consequently, since the analysis and design frameworks employ different linearisations it is difficult to incorporate insight provided by analysis into the controller design. In contrast, the velocity-based approach provides a consistent, unified design and analysis framework.

The velocity-based gain-scheduling approach is quite general and directly supports the design of feedback configurations for which the closed-loop dynamics are nonlinear. However, the purpose of this paper is to study the velocity-based design of controllers which, when combined with a nonlinear plant, attain linear closed-loop dynamics. Whilst of interest in its own right, an additional motivation for this study is to investigate the relationship, if any, between velocity-based gain-scheduling and dynamic linearisation approaches such as Input-Output Linearisation (see, for example, Isidori 1995).

The paper is organised as follows. In section 2, the velocity-based analysis framework is briefly reviewed. The dynamic linearisation of a class of square MIMO nonlinear systems using the velocity-based approach is investigated in section 3 and, in section 4, the assumptions in section 3 are relaxed and the analysis is extended to include a very broad class of square MIMO nonlinear systems. The dynamic linearisation of a nonlinear system in the absence of plant measurements is briefly considered in section 5. In section 6, the velocity-based approach is compared with the Input-Output Linearisation approach and the conclusions are summarised in section 7.

2. Review of velocity-based analysis

In this section, the approach of Leith & Leithead (1998b) to the analysis of a nonlinear system, by relating its dynamic characteristics to those of an associated family of linear systems, is briefly reviewed. Consider nonlinear plants with dynamics,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{1}$$

where $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable nonlinear functions with Lipschitz continuous first derivatives and $\mathbf{r}\in\Re^m$ denotes the input to the plant, $\mathbf{y} \in \Re^p$ the output and $\mathbf{x} \in \Re^n$ the states. The set of equilibrium operating points of the nonlinear plant, (1), consists of those points, $(\mathbf{x_o}, \mathbf{r_o})$, for which

$$\mathbf{F}(\mathbf{x_o}, \mathbf{r_o}) = 0 \tag{2}$$

and the corresponding equilibrium output is

$$\mathbf{y_o} = \mathbf{G}(\mathbf{x_o}, \mathbf{r_o}) \tag{3}$$

Let $\Phi:\Re^n\times\Re^m$ denote the space consisting of the union of the state, $\mathbf{x}$, with the input, $\mathbf{r}$. The set of equilibrium operating points of the nonlinear plant, (1), forms a locus of points, $(\mathbf{x_o}, \mathbf{r_o})$, in Φ and the response of the plant to a general time-varying input, $\mathbf{r}(t)$, is depicted by a trajectory in Φ.

The nonlinear system, (1), may be reformulated, equivalently, as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\rho), \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\rho) \tag{4}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are appropriately dimensioned constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ are nonlinear functions and $\rho(\mathbf{x},\mathbf{r})\in\Re^q$, $q\leq m+n$, embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_\mathbf{x}\rho$, $\nabla_\mathbf{r}\rho$ functions of ρ alone. Trivially, this reformulation can always be achieved by letting $\rho = [\mathbf{x}^T \ \mathbf{r}^T]^T$, in which case q=m+n. However, the nonlinearity of the system is frequently dependent on only a subset of the elements of the state and input, in which case the dimension, q, of ρ is less than m+n. Since $\nabla_\mathbf{x}\rho$, $\nabla_\mathbf{r}\rho$ are functions of ρ alone, the variable, $\rho(\mathbf{x},\mathbf{r})$, equals the constant value, ρ_1, upon a surface of co-dimension q in Φ and $\nabla_\mathbf{x}\rho$ and $\nabla_\mathbf{r}\rho$ are constant over each surface. Hence, the normal to each surface is identical at every point on the surface and each surface is, therefore, affine. Moreover, to ensure that ρ is a unique function of $\mathbf{x}$ and $\mathbf{r}$, these surfaces must be parallel for all ρ. Consequently, it may in fact be assumed, without loss of generality, that $\nabla_\mathbf{x}\rho$ and $\nabla_\mathbf{r}\rho$ are constant.

Suppose that the nonlinear system, (4) , is evolving along a trajectory, $(\mathbf{x}(t), \mathbf{r}(t))$, in Φ and at time, t_1, the trajectory has reached the point, $(\mathbf{x_1}, \mathbf{r_1})$. It is emphasised that the point, $(\mathbf{x_1}, \mathbf{r_1})$, need not be an equilibrium operating point and, indeed, may lie far from the locus of equilibrium operating points. From Taylor series expansion theory, the subsequent behaviour of the nonlinear system can be approximated, locally to the point, $(\mathbf{x_1}, \mathbf{r_1})$, by the first order representation,

$$\delta\dot{\hat{\mathbf{x}}} = (\mathbf{A}\mathbf{x_1}+\mathbf{B}\mathbf{r_1}+\mathbf{f}(\rho_1)) + (\mathbf{A}+\nabla\mathbf{f}(\rho_1)\nabla_\mathbf{x}\rho)\delta\hat{\mathbf{x}} + (\mathbf{B}+\nabla\mathbf{f}(\rho_1)\nabla_\mathbf{r}\rho)\delta\mathbf{r} \tag{5}$$

$$\delta\hat{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\rho_1)\nabla_\mathbf{x}\rho)\delta\hat{\mathbf{x}} + (\mathbf{D}+\nabla\mathbf{g}(\rho_1)\nabla_\mathbf{r}\rho)\delta\mathbf{r} \tag{6}$$

$$\rho_1=\rho(\mathbf{x_1},\mathbf{r_1}), \quad \delta\mathbf{r} = \mathbf{r} - \mathbf{r_1}, \quad \hat{\mathbf{y}} = \mathbf{C}\mathbf{x_1} + \mathbf{D}\mathbf{r_1} + \mathbf{g}(\rho_1) + \delta\hat{\mathbf{y}}, \quad \hat{\mathbf{x}} = \delta\hat{\mathbf{x}} + \mathbf{x_1}, \ \dot{\hat{\mathbf{x}}} = \delta\dot{\hat{\mathbf{x}}} \tag{7}$$

provided $\mathbf{x_1}+\delta\hat{\mathbf{x}} \subseteq N_\mathbf{x}$ $\mathbf{r_1}+\delta\mathbf{r}\subseteq N_\mathbf{r}$, where the neighbourhoods, $N_\mathbf{x}$ and $N_\mathbf{r}$, of, respectively, $\mathbf{x_1}$ and $\mathbf{r_1}$ are sufficiently small. When $\delta\hat{\mathbf{x}}(t_1)$ is zero,

$$\hat{\mathbf{x}}(t_1) = \mathbf{x_1} = \mathbf{x}(t_1) \tag{8}$$

$$\dot{\hat{\mathbf{x}}}(t_1) = \mathbf{A}\mathbf{x_1}+\mathbf{B}\mathbf{r_1}+\mathbf{f}(\rho_1) = \dot{\mathbf{x}}(t_1) \tag{9}$$

$$\ddot{\hat{\mathbf{x}}}(t_1) = (\mathbf{A}+\nabla\mathbf{f}(\rho_1)\nabla_\mathbf{x}\rho \ \dot{\mathbf{x}}(t_1))\dot{\hat{\mathbf{x}}}(t_1) + (\mathbf{B}+\nabla\mathbf{f}(\rho_1)\nabla_\mathbf{r}\rho)\dot{\mathbf{r}}(t_1) = \ddot{\mathbf{x}}(t_1) \tag{10}$$

$$\hat{\mathbf{y}}(t_1) = \mathbf{C}\mathbf{x_1} + \mathbf{D}\mathbf{r_1} + \mathbf{g}(\rho_1) = \mathbf{y}(t_1) \tag{11}$$

$$\dot{\hat{\mathbf{y}}}(t_1) = (\mathbf{C}+\nabla\mathbf{g}(\rho_1)\nabla_\mathbf{x}\rho)\dot{\hat{\mathbf{x}}}(t_1)+(\mathbf{D}+\nabla\mathbf{g}(\rho_1)\nabla_\mathbf{r}\rho)\dot{\mathbf{r}}(t_1) = \dot{\mathbf{y}}(t_1) \tag{12}$$

Hence, the solution, $\hat{\mathbf{x}}(t)$, to (5)-(7), initially at time t_1, is tangential to the solution, $\mathbf{x}(t)$, of (4). Indeed, locally to time t_1, $\dot{\hat{\mathbf{x}}}(t)$ provides a first-order approximation to $\dot{\mathbf{x}}(t)$ and a second-order approximation to $\mathbf{x}(t)$ and $\hat{\mathbf{y}}(t)$ provides a first-order approximation to $\mathbf{y}(t)$.

The solution, $\hat{\mathbf{x}}(t)$, to the first-order series expansion, (5)-(7), provides a valid approximation only while the solution, $\mathbf{x}(t)$, to the nonlinear system remains in the vicinity of the operating point, $(\mathbf{x_1}, \mathbf{r_1})$. However, the solution, $\mathbf{x}(t)$, to the nonlinear system need not stay in the vicinity of a single operating point. Consider the time interval, [0,T]; the initial time can, without loss of generality, always be taken as zero. An approximation to $\mathbf{x}(t)$ is obtained by partitioning the interval into a number of short sub-intervals. Over each sub-interval, the approximate solution is the solution to the first-order series expansion relative to the operating point reached at the initial time for the sub-interval (with the initial conditions for each sub-interval chosen to ensure continuity of the approximate solution). The number of local solutions employed is dependent on the duration of the sub-intervals, but the local solutions are accurate to second order; that is, the approximation error is proportional to the duration of the sub-interval cubed. Hence, as the number of sub-intervals increases, the approximation error associated with each rapidly decreases and the overall approximation error also decreases. Indeed, the overall approximation error tends to zero as the maximum size of the sub-intervals tends to zero (Leith & Leithead 1998b). Hence, the family of first-order series expansions, with members defined by (5)-(7), can provide an arbitrarily accurate approximation to the solution of the nonlinear system. Moreover, this approximation property holds throughout Φ and is not confined to the vicinity of a single equilibrium operating point or even of the locus of equilibrium operating points.

The foregoing analysis shows that the solution to the nonlinear system, (4), is approximated by the piecewise combination of the solutions to the members, (5)-(7), of the family of first-order series expansions. It should be noted that the state, input and output transformations, (7), depend on the operating point relative to which the series expansion is carried out. When the solution to the nonlinear system is confined to a neighbourhood about a single operating point, the transformations, (7), are static and the dynamic behaviour is described by the system, (5)-(6), alone. However, when the solution to the nonlinear system traces a trajectory which is not confined to a neighbourhood about a single operating point, the transformations, (7), are no longer static and the dynamic behaviour is no longer described solely by the system, (5)-(6). Instead, the dynamic behaviour is described by (5)-(7). Combining (5) and (6) with the local input, output and state transformations, (7), each member, (5)-(7), of the family of first-order representations may be reformulated as,

$$\dot{\hat{\mathbf{x}}} = \{ \mathbf{f}(\rho_1)-\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{x}}\rho\, \mathbf{x_1}-\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{r}}\rho\, \mathbf{r_1} \} + (\mathbf{A}+\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{x}}\rho)\hat{\mathbf{x}} + (\mathbf{B}+\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{r}}\rho)\mathbf{r} \quad (13)$$

$$\hat{\mathbf{y}} = \{ \mathbf{g}(\rho_1)-\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{x}}\rho\, \mathbf{x_1}-\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{r}}\rho\, \mathbf{r}_1 \} + (\mathbf{C}+\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{x}}\rho)\hat{\mathbf{x}} + (\mathbf{D}+\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{r}}\rho)\mathbf{r} \quad (14)$$

In contrast to the representation, (5)-(6), the state, input and output are now the same for all members of the reformulated family. The dynamics, (13)-(14), of an individual member of the family are affine rather than linear even when $(\mathbf{x_1}, \mathbf{r_1})$ is an equilibrium operating point. The inhomogeneous terms in (13)-(14) may, in general, be extremely large and can dominate the solution.

On differentiating (13)-(14)

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \quad (15)$$

$$\dot{\hat{\mathbf{w}}} = (\mathbf{A}+\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{x}}\rho)\hat{\mathbf{w}} + (\mathbf{B}+\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{r}}\rho)\dot{\mathbf{r}} \quad (16)$$

$$\dot{\hat{\mathbf{y}}} = (\mathbf{C}+\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{x}}\rho)\hat{\mathbf{w}} + (\mathbf{D}+\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{r}}\rho)\dot{\mathbf{r}} \quad (17)$$

It should be noted that this differentiation operation is purely formal and differentiation of noisy measurements is not required. The system, (15)-(17), is dynamically equivalent to the system, (13)-(14), in the sense that with appropriate initial conditions, namely,

$$\hat{\mathbf{x}}(t_1) = \mathbf{x_1}, \quad \hat{\mathbf{w}}(t_1) = \mathbf{A}\,\mathbf{x_1}+\mathbf{B}\mathbf{r_1}+\mathbf{f}(\rho_1), \quad \hat{\mathbf{y}}(t_1) = \mathbf{C}\mathbf{x_1}+\mathbf{D}\mathbf{r_1}+\mathbf{g}(\rho_1) \quad (18)$$

the solution, $\hat{\mathbf{x}}$, to (15)-(17), is the same as the solution, $\hat{\mathbf{x}}$, to (13)-(14). However, in contrast to (13)-(14) , the transformed system, (15)-(17), is linear. The relationship between the nonlinear system and its velocity-based linearisation, (15)-(17), is direct. Differentiating (4), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w} \quad (19)$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\rho)\nabla_{\mathbf{x}}\rho)\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\rho)\nabla_{\mathbf{r}}\rho)\dot{\mathbf{r}} \quad (20)$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\rho)\nabla_{\mathbf{x}}\rho)\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\rho)\nabla_{\mathbf{r}}\rho)\dot{\mathbf{r}} \quad (21)$$

Dynamically, (19)-(21), with appropriate initial conditions corresponding to (18), and (4) are equivalent (have the same solution, $\mathbf{x}$). (When $\mathbf{w} = \mathbf{F}(\mathbf{x}, \mathbf{r})$, $\mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r})$ is invertible for every $(\mathbf{x}, \mathbf{r})$, so that $\mathbf{x}$ may be expressed as a function of $\mathbf{w}$, $\mathbf{r}$ and $\mathbf{y}$, then the transformation relating (19)-(21) to (4) is, in fact, algebraic).

Clearly, the velocity-based linearisation, (15)-(17), is simply the frozen form of (19)-(21) at the operating point, ($\mathbf{x_1}$, $\mathbf{r_1}$). There exists a velocity-based linearisation, (15)-(17), for every point in Φ. Hence, a velocity-based linearisation family, with members defined by (15)-(17), can be associated with the nonlinear system, (4). Similarly to the family of first-order expansions, the solutions to the members of the family of velocity-based linearisations, (15)-(17), can be pieced together (with the initial conditions for each sub-interval chosen to ensure continuity of $\hat{\mathbf{x}}$, $\hat{\mathbf{w}}$ and $\hat{\mathbf{y}}$) to approximate the solution to the nonlinear system, (19)-(21) to an arbitrary degree of accuracy (Leith & Leithead 1998b).

There exists a rigorous, and direct, relationship between the dynamic characteristics of a nonlinear system and those of a related family of linear systems, namely, the velocity-based linearisation family, and a related family of affine systems, namely the first-order series expansion family. Since the solutions to the members of the families can be combined to approximate the solution to the nonlinear system arbitrarily accurately, the families embody the entire dynamics of the nonlinear system, (4), with no loss of information and therefore provide alternative representations of the nonlinear system. The situation is depicted in figure 1. Whilst these representations are equivalent in the sense that they each embody the entire dynamics of the nonlinear system, they are not necessarily equivalent with respect to other considerations. In particular, the *direct* relationship between the velocity-form of the nonlinear system and the velocity-based linearisation family and the linearity of the members of the latter family provides continuity with established linear theory. Furthermore, the velocity-based linearisation of the cascade/feedback interconnection of a plant and controller is simply the cascade/feedback connection of the velocity-based linearisations of the plant and controller (Leith & Leithead 1998c). Hence, both analysis (Leith & Leithead 1998b) and controller design (Leith & Leithead 1998c) are facilitated by adopting the velocity-based linearisation representation.

3. Dynamic inversion of square nonlinear systems with well-defined point-wise relative degree of zero

The requirement is to design a system which, when combined with the nonlinear system, (4), is equivalent to a linear input-output map. (The former is, hereafter, often referred to as the "controller" whilst the latter is denoted the "plant"). Moreover, in accordance with the divide and conquer philosophy, and to maintain continuity with well established linear methods, it is desired that the nonlinear design task is decomposed into a number of linear sub-problems.

The velocity-based analysis framework establishes a rigorous, and direct, relationship between the dynamic characteristics of a nonlinear system and those of a related family of linear systems; namely, the velocity-based linearisation family. Since the velocity-based linearisation of two cascaded nonlinear systems is simply the cascade of the velocity-based linearisations of the individual nonlinear systems (similarly for feedback interconnections, Leith & Leithead 1998c), consider designing a linear controller for each member of the velocity-based linearisation family. In view of the present requirements, select each linear controller such that the input-output dynamics (of the combined plant linearisation and controller) are identical in each case. Specifically, consider target input-output dynamics where the output is simply equal to the input in each case; that is, to the identity map. This choice corresponds to dynamic inversion of each plant linearisation. Because each member of the plant family is linear, conventional *linear* design methods can be utilised to design the members of the controller family. The solution, to the combination of the nonlinear plant with a nonlinear controller which has velocity-based linearisation family corresponding to the designed controller family, is approximated arbitrarily accurately by the appropriate piecewise combination of the solutions to the members of the family of combined plant linearisations/linear controllers. Owing to the uniformity of the input-output dynamics of the latter, it might be expected that the input/output dynamics of the nonlinear plant/controller system are linear and equal to the identity mapping; that is, dynamic inversion of the original nonlinear plant is achieved. (It should be noted that whilst consideration is hereafter confined to identity input/output dynamics, any other linear input/output dynamics can, of course, always be obtained by combining a suitable linear pre-compensator in cascade with the proposed dynamic inversion controller).

Dynamic inversion of linear systems can be achieved for state feedback (Silverman 1969). (The approach of Silverman (1969) is subsumed by the more recent nonlinear state feedback linearisation approach (see, for example, Isidori 1995)). However, since this involves repeated differentiation of the output, the direct relationship between the direct formulations and the velocity formulations (figure 1) is obscured. Moreover, dynamic inversion of linear systems is more usually approached from a frequency-domain pole-zero cancellation perspective which does not involve repeated differentiation and has the advantage that full state information is not required. In this approach, which is adopted hereafter, the controller zeroes are placed at the poles of the plant and the controller poles are placed at the zeroes of the plant. The resulting controller is a realisation of the plant inverse and, provided the plant and controller are stable, the cascade connection of the plant and controller is stable with unity transfer function. (When the relative degree of the plant is greater than zero, the pole-zero inverse is improper and, therefore, not realisable. This may be resolved by augmenting the inverse with additional poles such that the augmented system is proper. By placing the additional poles sufficiently far left in the complex plane, the augmented inverse system approximates the exact inverse system arbitrarily accurately).

3.1 Pole-zero inversion of linear systems with relative degree of zero

Consider the SISO linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}r, \quad y = \mathbf{c}\mathbf{x} + dr \tag{22}$$

for which the transfer function, G(s), relating y to r is

$$G(s) = Y(s)/R(s) = \frac{b_1 s^n + b_2 s^{n-1} + \ldots + b_n s + b_{n+1}}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} \tag{23}$$

where **A**, **b**, **c**, d are appropriately dimensioned, r, $y \in \Re$, $\mathbf{x} \in \Re^n$ and Y(s), R(s) are, respectively, the Laplace transforms of y(t) and r(t). Assume that b_1 is non-zero so that the transfer function has relative degree zero. In addition, assume that the transfer function poles and zeroes are stable; that is, lie in the left half complex plane. The requirement is to determine a minimal-order inverse system such that the cascade combination of this system with the original system, (22), has unity transfer function. Working in the frequency domain, it follows from standard linear theory that every realisation of the minimal inverse system has transfer function

$$G^{-1}(s) = \frac{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}{b_1 s^n + b_2 s^{n-1} + \ldots + b_n s + b_{n+1}} \tag{24}$$

Although encountered less frequently, this result can, of course, be reformulated in the state-space time-domain. Assume, without loss of generality, that (22) is of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & & 0 & 1 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} -a_1 b_1 + b_2 \\ -a_2 b_1 + b_3 \\ \vdots \\ -a_{n-1} b_1 + b_n \\ -a_n b_1 + b_{n+1} \end{bmatrix} r, \quad y = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_1 r \tag{25}$$

It is straightforward to show that (25) has transfer function (23). Every linear system with transfer function, (23), is related to (25) by a linear state transformation. It can be seen that the coefficients of the state-space representation, (25), are directly related to the coefficients of its transfer function, (23). Adopting a realisation of this form for the inverse system, it is immediately clear that a system with transfer function (24) is

$$\dot{\mathbf{x}}^i = \mathbf{A}^i \mathbf{x}^i + \mathbf{b}^i v, \quad r = \mathbf{c}^i \mathbf{x}^i + d^i v \tag{26}$$

where $\mathbf{x}^i = [x^i_1 \,..\, x^i_n]^T$ and

$$\mathbf{A}^i = \begin{bmatrix} -\frac{b_2}{b_1} & 1 & 0 & \cdots & 0 & 0 \\ -\frac{b_3}{b_1} & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\frac{b_n}{b_1} & 0 & 0 & & 0 & 1 \\ -\frac{b_{n+1}}{b_1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \mathbf{b}^i = \begin{bmatrix} -\frac{b_2}{b_1} + a_1 \\ -\frac{b_3}{b_1} + a_2 \\ \vdots \\ -\frac{b_n}{b_1} + a_{n-1} \\ -\frac{b_{n+1}}{b_1} + a_n \end{bmatrix}, \mathbf{c}^i = \frac{1}{b_1}[1 \quad 0 \quad \cdots \quad 0 \quad 0], \; d^i = \frac{1}{b_1} \tag{27}$$

Every linear realisation of the minimal inverse system, with transfer function, (24), is related to (26) by a linear state transformation. To confirm that (26) is, indeed, a realisation of the inverse of (22), let

$$\mathbf{z} = \mathbf{x} + \mathbf{x}^i \tag{28}$$

The cascade combination of (22) and (26) is

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{b}\mathbf{c}^i + \mathbf{A}^i - \mathbf{A} \\ 0 & \mathbf{A}^i \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{b}d^i + \mathbf{b}^i \\ \mathbf{b}^i \end{bmatrix} v, \quad y = [\mathbf{c} \quad d\mathbf{c}^i - \mathbf{c}] \begin{bmatrix} \mathbf{z} \\ \mathbf{x}^i \end{bmatrix} + dd^i v \tag{29}$$

and it follows from (25) and (27) that

$$\mathbf{b}\mathbf{c}^i + \mathbf{A}^i - \mathbf{A} = 0, \quad \mathbf{b}d^i + \mathbf{b}^i = 0, \quad d\mathbf{c}^i - \mathbf{c} = 0, \quad dd^i = 1 \tag{30}$$

Hence,

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A}^i \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{b}^i \end{bmatrix} v, \quad y = [\mathbf{c} \quad 0] \begin{bmatrix} \mathbf{z} \\ \mathbf{x}^i \end{bmatrix} + v \tag{31}$$

The dynamics of the cascade system comprise an unforced, observable component with state, **z**, and a forced, unobservable component with state $\mathbf{x}^i$. Since the poles and zeroes of (23) lie in the left half complex plane, **A** and $\mathbf{A}^i$ are Hurwitz and (31) is stable. It can be seen that the sub-system comprising the state, **z**, is unforced and so **z** decays to zero exponentially. Consequently, the error, y-v, decays exponentially to zero and the output, y, asymptotically tracks the input, v, as required. When the initial condition of $\mathbf{x}^i$ is such that **z**(0) is zero, the tracking error is identically zero.

3.2 Point-wise relative degree

The relative degree of the nonlinear system, (4), is $\{r_1(\mathbf{x},\mathbf{r}),\ldots,r_p(\mathbf{x},\mathbf{r})\}$, where the characteristic number, $r_j(\mathbf{x},\mathbf{r})$, is defined as the minimum number of times that, at the operating point $(\mathbf{x},\mathbf{r})$, the j^{th} output must be differentiated such that it is directly coupled to the input (see, for example, Isidori 1995). The relative degree is uniform when it is the same at every operating point, $(\mathbf{x},\mathbf{r})$. Associated with the nonlinear system, (4), is the velocity-based linearisation family, (15)-(17). The point-wise relative degree of the nonlinear system, (4), is defined as $\{r_1(\rho),\ldots,r_p(\rho)\}$, where $\{r_1(\rho_1),\ldots,r_p(\rho_1)\}$ is the relative degree of the member of the velocity-based linearisation family associated with the operating points at which ρ equals ρ_1. Letting

$$\mathbf{A}(\rho) = \mathbf{A}+\nabla\mathbf{f}(\rho)\,\nabla_{\mathbf{x}}\rho\,, \quad \mathbf{B}(\rho) = \mathbf{B}+\nabla\mathbf{f}(\rho)\,\nabla_{\mathbf{r}}\rho\,, \quad \mathbf{C}(\rho) = \mathbf{C}+\nabla\mathbf{g}(\rho)\,\nabla_{\mathbf{x}}\rho\,, \quad \mathbf{D}(\rho) = \mathbf{D}+\nabla\mathbf{g}(\rho)\,\nabla_{\mathbf{r}}\rho \tag{32}$$

it follows that when $\mathbf{D}_j(\rho_1) \neq 0$, where $\mathbf{D}_j(\rho_1)$ denotes the j^{th} row of $\mathbf{D}(\rho_1)$, the corresponding characteristic number, $r_j(\rho_1)$, is 0. When $\mathbf{D}_j(\rho_1) = 0$, $r_j(\rho_1)$ is defined (see, for example, Rugh 1996 Definition 14.10) by

$$\mathbf{C}_j(\rho_1)\mathbf{A}^k(\rho_1)\mathbf{B}(\rho_1) = 0 \quad \forall k < r_j(\rho_1) - 1 \tag{33}$$

$$\mathbf{C}_j(\rho_1)\mathbf{A}^{r_j(\rho_1)-1}(\rho_1)\mathbf{B}(\rho_1) \neq 0 \tag{34}$$

where $\mathbf{C}_j(\rho_1)$ denotes the j^{th} row of $\mathbf{C}(\rho_1)$. The nonlinear system has a point-wise relative degree which is uniformly zero when the characteristic numbers, $r_j(\rho_1)$, j=1,..p, are zero for all values of ρ_1; that is, when every row of $\mathbf{D}(\rho_1)$ is non-zero for all values of ρ_1. The nonlinear system has a well-defined point-wise relative degree of zero when, in addition, the rank of $\mathbf{D}(\rho_1)$ is p for all values of ρ_1.

The relative degree and point-wise relative degree are not equivalent. For example, consider the nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}x_1\\ 0\end{bmatrix}+\begin{bmatrix}-\dfrac{2x_1x_2}{1+x_2^2}\\ 1\end{bmatrix}r,\quad y=x_1+x_1x_2^2 \tag{35}$$

for which

$$\mathbf{C}(\rho_1)\mathbf{B}(\rho_1)=\begin{bmatrix}1+x_2^2 & 2x_1x_2\end{bmatrix}\begin{bmatrix}-\dfrac{2x_1x_2}{1+x_2^2}\\ 1\end{bmatrix}=0 \tag{36}$$

$$\mathbf{C}(\rho_1)\mathbf{A}(\rho_1)\mathbf{B}(\rho_1)=2x_1x_2+\frac{4x_1x_2^2r}{1+x_2^2} \tag{37}$$

Hence, the point-wise relative degree is two at the operating points for which $\mathbf{C}(\rho_1)\mathbf{A}(\rho_1)\mathbf{B}(\rho_1)$ is non-zero. However, since

$$\nabla\mathbf{h}(\mathbf{x})\mathbf{g}(\mathbf{x})=\mathbf{C}(\rho_1)\mathbf{B}(\rho_1)=\mathbf{0} \tag{38}$$

$$\nabla\mathbf{h}(\mathbf{x})\nabla\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})+\nabla^2\mathbf{h}(\mathbf{x})\mathbf{g}(\mathbf{x})\mathbf{f}(\mathbf{x})=\begin{bmatrix}1+x_2^2 & 2x_1x_2\end{bmatrix}\begin{bmatrix}-\dfrac{2x_1x_2}{1+x_2^2}\\ 1\end{bmatrix}=\mathbf{0} \tag{39}$$

it follows that the relative degree is greater than two. Indeed, it is straightforward to show that the relative degree of the nonlinear system is uniformly infinite; that is, the input never appears at the output regardless of the number of times that the output is differentiated.

■

3.3 Nonlinear MIMO systems with well-defined point-wise relative degree of zero

Consider the MIMO nonlinear system

$$\dot{\mathbf{x}}=\mathbf{Ax}+\mathbf{Br}+\mathbf{f}(\rho),\quad \mathbf{y}=\mathbf{Cx}+\mathbf{Dr}+\mathbf{g}(\rho) \tag{40}$$

for which the velocity-based linearisation associated with the operating points at which ρ equals ρ_1 is

$$\dot{\mathbf{x}}=\mathbf{w} \tag{41}$$

$$\dot{\mathbf{w}}=\mathbf{A}(\rho_1)\,\mathbf{w}+\mathbf{B}(\rho_1)\,\dot{\mathbf{r}} \tag{42}$$

$$\dot{\mathbf{y}}=\mathbf{C}(\rho_1)\,\mathbf{w}+\mathbf{D}(\rho_1)\,\dot{\mathbf{r}} \tag{43}$$

where $\mathbf{r}\in\Re^m$, $\mathbf{y}\in\Re^p$, $\mathbf{x}\in\Re^n$ and

$$\mathbf{A}(\rho_1)=\mathbf{A}+\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{x}}\rho,\quad \mathbf{B}(\rho_1)=\mathbf{B}+\nabla\mathbf{f}(\rho_1)\nabla_{\mathbf{r}}\rho,\quad \mathbf{C}(\rho_1)=\mathbf{C}+\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{x}}\rho,\quad \mathbf{D}(\rho_1)=\mathbf{D}+\nabla\mathbf{g}(\rho_1)\nabla_{\mathbf{r}}\rho \tag{44}$$

This defines a velocity-based linearisation family as ρ_1 varies throughout its range. Since the velocity-based linearisation family defined by (41)-(43) is directly related to the velocity-form of the nonlinear system, see figure 1, a slight abuse of notation is adopted and the same state, input and output are used for both the nonlinear system, (40), and the members of its velocity-based linearisation family. Assume, for the moment, that the nonlinear system is square and has well-defined point-wise relative degree zero; that is, the number of inputs, m, equals the number of outputs, p, and $\mathbf{D}(\rho)$ is non-singular for all values of ρ. This assumption is relaxed later in section 4.

Consider the candidate inverse system, in velocity-form,

$$\dot{\mathbf{x}}^i=\mathbf{w}^i \tag{45}$$

$$\dot{\mathbf{w}}^i=\mathbf{A}^i(\rho^i)\,\mathbf{w}^i+\mathbf{B}^i(\rho^i)\,\dot{\mathbf{v}} \tag{46}$$

$$\dot{\mathbf{r}}=\mathbf{C}^i(\rho^i)\,\mathbf{w}^i+\mathbf{D}^i(\rho^i)\,\dot{\mathbf{v}} \tag{47}$$

for which the velocity-based linearisation associated with the operating points at which ρ^i equals ρ^i_1 (that is, the corresponding frozen form of (45)-(47)) is

$$\dot{\mathbf{x}}^i=\mathbf{w}^i \tag{48}$$

$$\dot{\mathbf{w}}^i=\mathbf{A}^i(\rho^i_1)\,\mathbf{w}^i+\mathbf{B}^i(\rho^i_1)\,\dot{\mathbf{v}} \tag{49}$$

$$\dot{\mathbf{r}}=\mathbf{C}^i(\rho^i_1)\,\mathbf{w}^i+\mathbf{D}^i(\rho^i_1)\,\dot{\mathbf{v}} \tag{50}$$

where $\mathbf{v}\in\Re^m$, $\mathbf{x}^i\in\Re^n$. Following the linear analysis of section 3.1, let $\mathbf{A}^i(\rho^i_1)$, $\mathbf{B}^i(\rho^i_1)$, $\mathbf{C}^i(\rho^i_1)$, $\mathbf{D}^i(\rho^i_1)$ be solutions of

$$\mathbf{B}(\rho_1)\mathbf{C}^i(\rho^i_1)+\mathbf{A}^i(\rho^i_1)-\mathbf{A}(\rho_1)=0,\quad \mathbf{B}(\rho_1)\mathbf{D}^i(\rho^i_1)+\mathbf{B}^i(\rho^i_1)=0,\quad \mathbf{D}(\rho_1)\mathbf{C}^i(\rho^i_1)-\mathbf{C}(\rho_1)=0,\quad \mathbf{D}(\rho_1)\mathbf{D}^i(\rho^i_1)=\mathbf{I} \tag{51}$$

Since $\mathbf{D}(\rho_1)$ is uniformly non-singular, the solution to (51) is well-defined and unique: namely,
$\rho^i_1=\rho_1$, $\mathbf{A}^i(\rho^i_1)=\mathbf{A}(\rho_1)-\mathbf{B}(\rho_1)\mathbf{D}^{-1}(\rho_1)\mathbf{C}(\rho_1)$, $\mathbf{B}^i(\rho^i_1) = -\mathbf{B}(\rho_1)\mathbf{D}^{-1}(\rho_1)$, $\mathbf{C}^i(\rho^i_1)=\mathbf{D}^{-1}(\rho_1)\mathbf{C}(\rho_1)$, $\mathbf{D}^i(\rho^i_1)=\mathbf{D}^{-1}(\rho_1)$ (52)
Let

$$\mathbf{z} = \mathbf{w} + \mathbf{w}^i \tag{53}$$

(It should be noted that the choice of transformation, (53), is not unique. Other state transformations lead to alternative inverse system realisations, provided the same transformation is employed with every member of the velocity-based linearisation family; for example, $\mathbf{z} = \mathbf{w} + \mathbf{T}\mathbf{w}^i$ where $\mathbf{T}$ is a non-singular constant matrix). It is straightforward to show (Leith & Leithead 1998c) that the velocity-based linearisation family of the cascade connection of (40) and (45)-(47) consists simply of the cascade connection of each member of the velocity-based linearisation family, (41)-(43), with the corresponding member of the velocity-based linearisation family of the inverse system, (48)-(50). The cascade combination of (41)-(43) and (48)-(50) is

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} \tag{54}$$

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\rho_1) & \mathbf{B}(\rho_1)\mathbf{C}^i(\rho_1) + \mathbf{A}^i(\rho_1) - \mathbf{A}(\rho_1) \\ 0 & \mathbf{A}^i(\rho_1) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \begin{bmatrix} \mathbf{B}(\rho_1)\mathbf{D}^i(\rho_1) + \mathbf{B}^i(\rho_1) \\ \mathbf{B}^i(\rho_1) \end{bmatrix} \dot{\mathbf{v}} \tag{55}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{C}(\rho_1) & \mathbf{D}(\rho_1)\mathbf{C}^i(\rho_1) - \mathbf{C}(\rho_1) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \mathbf{D}(\rho_1)\mathbf{D}^i(\rho_1)\dot{\mathbf{v}} \tag{56}$$

Hence, it follows from (52) that

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} \tag{57}$$

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\rho_1) & 0 \\ 0 & \mathbf{A}^i(\rho_1) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}^i(\rho_1) \end{bmatrix} \dot{\mathbf{v}} \tag{58}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{C}(\rho_1) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \dot{\mathbf{v}} \tag{59}$$

The velocity-based linearisation family of the cascade connection of (40) and (45)-(47) is (57)-(59).

The velocity-form of the cascaded system is obtained when ρ is permitted to vary in (57)-(59). The structure of the cascade system is depicted, in block diagram form, in figure 2. Assume that the inverse system is bounded-input bounded-output stable. This ensures that in the cascade connection the unobservable $\mathbf{w}^i$ dynamics are stable. Assume, in addition, that the unforced sub-system involving the state, $\mathbf{z}$, is exponentially stable. Under these conditions, the tracking error velocity, $\dot{\mathbf{y}} - \dot{\mathbf{v}}$, decays exponentially to zero provided $\mathbf{C}(\rho)$ is uniformly bounded. When the initial conditions are selected such that $\mathbf{z}(0)$ is zero, $\dot{\mathbf{y}} - \dot{\mathbf{v}}$ is identically zero. When, in addition, $\mathbf{y}(0)$ equals $\mathbf{v}(0)$, it follows that $\mathbf{y}(t)$-$\mathbf{v}(t)$ is identically zero. Hence, the nonlinear system, (45)-(47), (with appropriate initial conditions) is, indeed, a realisation of the inverse of the MIMO nonlinear system, (40).

Remark 1 Full state information is not required in order to implement the inverse system obtained. Instead, it is only necessary to measure, or estimate, the scheduling variable, ρ, which frequently depends on only a small number of elements of the input and state vectors. Indeed, in the case of purely linear systems, there is no scheduling variable and, consequently, plant measurements are not required to implement the pole-zero inverse (in contrast to an Input-Output Linearisation controller which always requires full state information; for example, Isidori 1995). Since the input, $\mathbf{r}$, to the plant is the output of the inverse system, plant measurements are also not required with nonlinear systems for which the scheduling variable, ρ, depends only on $\mathbf{r}$; that is, when $\nabla_{\mathbf{x}}\rho$ is identically zero. This is discussed further in section 5 below.

Remark 2 The tracking error, $\mathbf{y}$-$\mathbf{v}$, is

$$\mathbf{y}(t) - \mathbf{v}(t) = \int_0^t \mathbf{C}(\rho)\mathbf{z}(s)ds + \mathbf{y}(0) - \mathbf{v}(0) \tag{60}$$

Owing to the pure integration in (60), with initial conditions other than the above, the tracking error need not decay to zero. However, under mild conditions, $\mathbf{y}$-$\mathbf{v}$ does tend to a limit asymptotically (see appendix A); that is, as $t\rightarrow\infty$, the output, $\mathbf{y}$, follows the input, $\mathbf{v}$, with a steady offset. This steady offset may be readily removed by integral feedback. Let

$$\mathbf{v}(t) = \int_0^t \Lambda(\mathbf{v}_r(s) - \mathbf{y}(s))ds \tag{61}$$

where $\Lambda=\lambda\mathbf{I}$ with λ positive and $\mathbf{v}_r$ is the new input. With this choice of $\mathbf{v}$,

$$\dot{\mathbf{y}}(t) + \lambda\mathbf{y}(t) = \lambda\mathbf{v}_r + \mathbf{C}(\rho)\mathbf{z}(s) \tag{62}$$

for which the solution is

$$\mathbf{y}(t) = e^{-\lambda t}\mathbf{y}(0) + \int_o^t e^{-\lambda(t-s)}(\lambda\mathbf{v}_r(s) + \mathbf{C}(\rho(s))\mathbf{z}(s))ds \tag{63}$$

Since $|\mathbf{C}(\rho)\,\mathbf{z}|$ tends to zero exponentially,

$$\mathbf{y}(t) \rightarrow \int_o^t \lambda e^{-\lambda(t-s)}\mathbf{v}_r(s)ds \text{ as } t\rightarrow\infty \tag{64}$$

and $\mathbf{y}$ is related to $\mathbf{v}_r$ by linear first-order dynamics with real poles at $s=-\lambda$. From linear theory, provided λ is sufficiently large compared to the frequencies at which the spectrum of $\mathbf{v}_r$ has significant energy, $\int_o^t \lambda e^{-\lambda(t-s)}\mathbf{v}_r(s)ds$ and so the output, $\mathbf{y}$, asymptotically tracks the input, $\mathbf{v}_r$ (without steady offset).

Example 1 - Nonlinear SISO system with well-defined point-wise relative degree of zero

Consider the SISO nonlinear system, depicted in figure 3,

$$\dot{x} = G(\rho), \quad y = x + G(\rho) \tag{65}$$

where $\rho = r-x$, $G(s)=\tanh(s)+0.01s$. The velocity-based linearisation associated with the operating points at which ρ equals ρ_1 is

$$\dot{x} = w \tag{66}$$

$$\dot{w} = -\nabla G(\rho_1)w + \nabla G(\rho_1)\dot{r} \tag{67}$$

$$\dot{y} = (1-\nabla G(\rho_1))w + \nabla G(\rho_1)\dot{r} \tag{68}$$

and this defines a velocity-based linearisation family as ρ_1 varies throughout its range. The velocity-based linearisation has transfer function representation

$$\dot{Y}(s) = \nabla G(\rho_1)\frac{s+1}{s+\nabla G(\rho_1)}\dot{R}(s) \tag{69}$$

where $\dot{R}(s)$ and $\dot{Y}(s)$ denote, respectively, the Laplace transfer functions of $\dot{r}(t)$ and $\dot{y}(t)$. Since $\nabla G(\rho_1)\neq 0\ \forall\rho_1$, the members of the velocity-based linearisation family have uniformly zero relative degree. It follows from the foregoing analysis that the velocity-based linearisation family of a corresponding inverse system is defined by

$$\dot{x}^i = w^i \tag{70}$$

$$\dot{w}^i = -w^i - \dot{v} \tag{71}$$

$$\dot{r} = \frac{1-\nabla G(\rho_1)}{\nabla G(\rho_1)}w^i + \frac{1}{\nabla G(\rho_1)}\dot{v} \tag{72}$$

as ρ_1 varies throughout its range. It should be noted that the velocity-form of the inverse system is obtained when ρ is permitted to vary in (70)-(72). The velocity-based linearisation (70)-(72) has transfer function representation

$$\dot{R}(s) = \frac{1}{\nabla G(\rho_1)}\frac{s+\nabla G(\rho_1)}{s+1}\dot{V}(s) \tag{73}$$

where $\dot{V}(s)$ denotes the Laplace transfer function of $\dot{v}(t)$. Clearly, the transfer functions (73), is the reciprocal of the transfer function, defined by (69), of the corresponding velocity-based linearisation associated with the original system. The velocity-based linearisation family of the cascade connection of (65) with the inverse system is defined by

$$\begin{bmatrix}\dot{x}\\ \dot{x}^i\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}z\\ w^i\end{bmatrix} \tag{74}$$

$$\begin{bmatrix}\dot{z}\\ \dot{w}^i\end{bmatrix} = \begin{bmatrix}-\nabla G(\rho_1) & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}z\\ w^i\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}\dot{v} \tag{75}$$

$$\dot{y} = \begin{bmatrix} 1 - \nabla G(\rho_1) & 0 \end{bmatrix} \begin{bmatrix} z \\ w^i \end{bmatrix} + \dot{v} \tag{76}$$

Owing to the direct relationship between the velocity-based linearisation family and the velocity-form of a nonlinear system, see figure 1, the nonlinear system obtained by allowing ρ to vary as a function of x and r in (74)-(76) is dynamically equivalent to the cascade connection of (65) with the inverse system corresponding to (70)-(72). Hence, it can be deduced immediately that the internal w^i dynamics are linear and stable. Moreover, since $\nabla G(\rho) > 0$, it follows from Lyapunov theory that the z dynamics are exponentially stable as required (for example, a Lyapunov function is $V=z^2$).

Example 2 - Nonlinear MIMO system with well-defined point-wise relative degree of zero

Consider the MIMO nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -(1+x_2^2)(x_1+r_1) \\ -x_2 + r_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + (1+x_2^2)r_1 \\ x_2 + r_2 \end{bmatrix} \tag{77}$$

for which the associated velocity formulation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{78}$$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -(1+x_2^2) & -2x_2(x_1+r_1) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} -(1+x_2^2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} \tag{79}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2x_2 r_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} (1+x_2^2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} \tag{80}$$

The members of the velocity-based linearisation family are simply the "frozen" forms of (78)-(80). It can be seen that the members of the velocity-based linearisation family have relative degree zero. It follows from the foregoing analysis that the velocity form of a corresponding inverse system is

$$\begin{bmatrix} \dot{x}_1{}^i \\ \dot{x}_2{}^i \end{bmatrix} = \begin{bmatrix} w_1{}^i \\ w_2{}^i \end{bmatrix} \tag{81}$$

$$\begin{bmatrix} \dot{w}_1{}^i \\ \dot{w}_2{}^i \end{bmatrix} = \begin{bmatrix} -x_2^2 & -2x_2x_1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} w_1{}^i \\ w_2{}^i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} \tag{82}$$

$$\begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+x_2^2} & \dfrac{2x_2 r_1}{1+x_2^2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1{}^i \\ w_2{}^i \end{bmatrix} + \begin{bmatrix} \dfrac{1}{1+x_2^2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} \tag{83}$$

■

3.4 Zero dynamics

The zero dynamics of the nonlinear system, (40), are defined as the internal dynamics arising when the system is driven by a non-trivial input, **r**, such that its output, **y**, is identically zero (see, for example, Isidori 1995 although the discussion therein concerns a subset of the class of nonlinear systems of the form (40)). Consider the cascade connection of the nonlinear system, (40), with the inverse system, (45)-(47),

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} \tag{84}$$

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\rho) & 0 \\ 0 & \mathbf{A}^i(\rho) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{B}^i(\rho) \end{bmatrix} \dot{\mathbf{v}} \tag{85}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{C}(\rho) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \dot{\mathbf{v}} \tag{86}$$

When $\mathbf{z}(0)=0$, it follows from (84)-(86) that $\mathbf{z}(t)$ is identically zero. Hence, when $\dot{\mathbf{v}}=0$, $\mathbf{z}(0)=0$, $\mathbf{y}(0)=0$ the output, $\mathbf{y}(t)$, is identically zero. It should be noted that whilst the input, $\mathbf{v}$, to the cascade system is constant ($\dot{\mathbf{v}}$ is zero), the input, $\mathbf{r}$, to the nonlinear system, (40), satisfies

$$\dot{\mathbf{r}} = \mathbf{C}^i(\rho^i)\mathbf{w}^i \tag{87}$$

Clearly, $\mathbf{r}$ is dependent, through the evolution of the state $\mathbf{w}^i$, on the dynamics of the inverse system. Since $\mathbf{z}(t)$ is identically zero,

$$\mathbf{w} = -\mathbf{w}^i \tag{88}$$

and the internal dynamics of (40) satisfy

$$\dot{\mathbf{x}} = \mathbf{w} \tag{89}$$

$$\dot{\mathbf{w}} = \mathbf{A}^i(\rho)\mathbf{w} \tag{90}$$

where it is noted that ρ may depend on both the state, $\mathbf{x}$, and the input, $\mathbf{r}$. The internal dynamics, (89)-(90), are, by definition, the zero dynamics of the nonlinear system, (40). As might be expected by analogy with linear system theory, it can be seen that the zero dynamics of the nonlinear system, (40), are precisely the dynamics of the corresponding inverse system, (45)-(47).

3.5 Integrability

The terms $\nabla\mathbf{f}(\rho)\,\nabla_\mathbf{x}\rho$, $\nabla\mathbf{f}(\rho)\,\nabla_\mathbf{r}\rho$, $\nabla\mathbf{g}(\rho)\,\nabla_\mathbf{x}\rho$ and $\nabla\mathbf{g}(\rho)\,\nabla_\mathbf{r}\rho$ in the velocity-based linearisations of the nonlinear system, (40), are simply the derivatives of the nonlinear mappings, $\mathbf{f}(\rho)$ and $\mathbf{g}(\rho)$, evaluated at every point in the operating space. Hence, it follows immediately that the solutions, $\hat{\mathbf{f}}(\rho)$ and $\hat{\mathbf{g}}(\rho)$, to

$$\nabla\hat{\mathbf{f}}(\rho)\begin{bmatrix}\nabla_\mathbf{x}\rho\\ \nabla_\mathbf{r}\rho\end{bmatrix}=\begin{bmatrix}\phi_\mathbf{x}\\ \phi_\mathbf{r}\end{bmatrix},\quad \nabla\hat{\mathbf{g}}(\rho)\begin{bmatrix}\nabla_\mathbf{x}\rho\\ \nabla_\mathbf{r}\rho\end{bmatrix}=\begin{bmatrix}\gamma_\mathbf{x}\\ \gamma_\mathbf{r}\end{bmatrix} \tag{91}$$

where

$$\phi_\mathbf{x} = \nabla\mathbf{f}(\rho)\,\nabla_\mathbf{x}\rho,\quad \phi_\mathbf{r} = \nabla\mathbf{f}(\rho)\,\nabla_\mathbf{r}\rho,\quad \gamma_\mathbf{x} = \nabla\mathbf{g}(\rho)\,\nabla_\mathbf{x}\rho,\quad \gamma_\mathbf{r} = \nabla\mathbf{g}(\rho)\,\nabla_\mathbf{r}\rho \tag{92}$$

exist and are, respectively, $\mathbf{f}(\rho)$ and $\mathbf{g}(\rho)$. However, this integrability property requires a strong constraint on the terms in the velocity-based linearisation family which, whilst quite natural, is certainly not satisfied by every choice of $\phi_\mathbf{x}$, $\phi_\mathbf{r}$, $\gamma_\mathbf{x}$ and $\gamma_\mathbf{r}$. For example, a sufficient condition for integrability is that $\phi_\mathbf{x}$, $\phi_\mathbf{r}$, $\gamma_\mathbf{x}$ and $\gamma_\mathbf{r}$ are continuous and differentiable functions with

$$\begin{aligned}&\nabla_{\rho_k}(\phi_\mathbf{x}(\rho))_j = \nabla_{\rho_j}(\phi_\mathbf{x}(\rho))_k,\ \nabla_{\rho_k}(\phi_\mathbf{r}(\rho))_j = \nabla_{\rho_j}(\phi_\mathbf{r}(\rho))_k \qquad \forall k\neq j\\ &\nabla_{\rho_k}(\gamma_\mathbf{x}(\rho))_j = \nabla_{\rho_j}(\gamma_\mathbf{x}(\rho))_k,\ \nabla_{\rho_k}(\gamma_\mathbf{r}(\rho))_j = \nabla_{\rho_j}(\gamma_\mathbf{r}(\rho))_k \qquad \forall k\neq j\end{aligned} \tag{93}$$

and ρ_j, $(\phi_\mathbf{x}(\rho))_j$, $(\phi_\mathbf{r}(\rho))_j$, $(\gamma_\mathbf{x}(\rho))_j$ and $(\gamma_\mathbf{r}(\rho))_j$ denote, respectively the j^{th} elements of ρ, $\phi_\mathbf{x}$, $\phi_\mathbf{r}$, $\gamma_\mathbf{x}$ and $\gamma_\mathbf{r}$ (see, for example, Vidyasagar 1993 p382). It should be noted that the analysis in the foregoing sections does *not* require that the velocity-based inverse system is integrable. The class of nonlinear systems with the velocity-based form

$$\dot{\mathbf{x}} = \mathbf{w} \tag{94}$$

$$\dot{\mathbf{w}} = \phi_\mathbf{x}(\rho)\,\mathbf{w} + \phi_\mathbf{r}(\rho)\,\dot{\mathbf{r}} \tag{95}$$

$$\dot{\mathbf{y}} = \gamma_\mathbf{x}(\rho)\,\mathbf{w} + \gamma_\mathbf{r}(\rho)\,\dot{\mathbf{r}} \tag{96}$$

is somewhat richer than the class of nonlinear systems, (40), and the velocity-based inverse system need not, in general, belong to the latter class. Indeed, whilst the analysis in the foregoing sections takes the nonlinear system, (40), as its starting point, thereby implicitly assuming that the plant velocity-based linearisation family is integrable, this assumption is not used at any point in the analysis and the results are, therefore, directly applicable to the general class of velocity-based systems, (94)-(96), including those which are not integrable.

3.6 Realisation of inverse system

The velocity-form of the inverse system is obtained when ρ is permitted to vary in the corresponding velocity-based linearisation family, see figure 1. The velocity-form has a differentiation operator at the input (to obtain $\dot{v}$ from v) and an integration at the output (to obtain y from $\dot{y}$). When the inverse system is augmented with integral action, the differentiation at the input can be formally combined with the integrator. It is emphasised that this reformulation is purely formal in nature: no unstable pole-zero cancellation is involved. The velocity-form of the inverse system can now be realised directly with the integral action appearing explicitly at the output. For instance, the direct realisation of the velocity-form of the inverse system in Example 1, augmented with integral action, is shown in figure 4. It should be noted that, in the context of control systems, integral action is almost always required in order to meet performance requirements. When integral action cannot be employed, realisation of the inverse system in direct form imposes certain integrability constraints on the class of allowable inverse system velocity-based linearisation families in order to ensure the existence of a bounded transformation from the velocity-form to the direct form (see section 3.5). This issue is discussed in detail in Leith & Leithead (1996, 1998a) and the implementation approach there may be readily extended to the class of systems considered here.

4. Extension to systems with non-zero point-wise relative degree

The analysis of section 3 generalises the linear frequency-domain pole-zero inversion approach to square nonlinear systems with well-defined point-wise relative degree of zero. In this section, the extension of these results to nonlinear systems with non-zero point-wise relative degree is considered. In the linear case, when the relative degree is non-zero, the pole-zero inverse system is improper and, therefore, not realisable. However, this difficulty can be resolved by formally augmenting the original system with additional zeroes, such that its relative degree is zero, and a corresponding pole-zero inverse can then determined. Alternatively, the (unrealisable) inverse of the original system can be augmented with additional poles such that the augmented inverse is proper. By placing the additional poles sufficiently far left in the complex plane, the augmented inverse system approximates the exact inverse system arbitrarily accurately.

A nonlinear analogue of the first approach is to formally augment the input or output of the system with a number of differentiators. Provided the point-wise relative degree of the augmented system is well-defined and zero, the analysis of section 3 can then be employed to determine an inverse of the augmented system. It should be emphasised that the differentiators are purely formal in nature and need not be implemented. When the foregoing inverse is connected in cascade with the augmented system, the output of the augmented system follows the input to the inverse system (after some initial transients). Hence, when this inverse is connected in cascade with the original (unaugmented) system, the output of this system is linearly related, via a chain of integrators, to the input to the inverse system. However, since this approach involves repeated differentiation of the output, the direct relationship between the direct formulation and the velocity formulation (figure 1) is obscured. Hence, this approach is not pursued further here.

The second approach, in the linear case, is to augment the (unrealisable) inverse of the original system with additional dynamics such that the augmented inverse is realisable. Consider the velocity-based linearisation family, (41)-(43), associated with the nonlinear system, (40). In order to facilitate the analysis, assume that $\mathbf{D}(\rho)$ is identically zero (and so the relative degree is at least one): this involves no loss of generality since it can always be achieved by augmenting the system with a suitable low-pass input filter. The relative degree, $\{r_1(\boldsymbol{\rho_1}),\ldots,r_p(\boldsymbol{\rho_1})\}$, of the member of the family corresponding to ρ equal to $\boldsymbol{\rho_1}$ is defined by

$$\mathbf{C}_j(\rho_1)\mathbf{A}^k(\rho_1)\mathbf{B}(\rho_1) = 0 \quad \forall k < r_j(\rho_1) - 1 \tag{97}$$

$$\mathbf{C}_j(\rho_1)\mathbf{A}^{r_j(\rho_1)-1}(\rho_1)\mathbf{B}(\rho_1) \neq 0 \tag{98}$$

where $\mathbf{C}_j(\rho_1)$ denotes the j^{th} row of $\mathbf{C}(\rho_1)$ and j=1..m. There exists an infinitesimally small perturbation, ε_d, such that $\mathbf{D}(\boldsymbol{\rho_1})+\varepsilon_d$ is non-singular and for which, therefore, the relative degree is zero. Similarly, perturbing $\mathbf{B}(\boldsymbol{\rho_1})$ to $\mathbf{B}(\boldsymbol{\rho_1})+\boldsymbol{\varepsilon_b}$ and $\mathbf{C}(\boldsymbol{\rho_1})$ to $\mathbf{C}(\boldsymbol{\rho_1})+\boldsymbol{\varepsilon_c}$, it is straightforward to show that there exists infinitesimally small $\boldsymbol{\varepsilon_b}$ and $\boldsymbol{\varepsilon_c}$ for which $\mathbf{B}(\boldsymbol{\rho_1})+\boldsymbol{\varepsilon_b}$ and $\mathbf{C}(\boldsymbol{\rho_1})+\boldsymbol{\varepsilon_c}$ violate the equality constraints, (97). Moreover, the perturbations are additive at the input and output of the system as depicted in block diagram form in figure 5. This latter observation is unsurprising since the perturbation is augmenting the zeroes of the linearisation whilst leaving the poles unchanged; for example, it can be seen that in the linear canonical form, (25), the zeroes are determined solely by the elements of **B** and **D**.

By analogy with the linear case, consider perturbing the nonlinear system in the above manner such that the relative degrees of the members of the perturbed velocity-based linearisation family are uniformly well-defined and zero. Since the perturbed system then has well-defined point-wise relative degree of zero, its inverse can be readily determined using the analysis of section 3. Similarly to the linear situation, provided the perturbations employed are sufficiently small and that a smooth inverse does indeed exist, the inverse of the perturbed system is an arbitrarily accurate, realisable, inverse of the original nonlinear system, (40) (Appendix B).

Example 3 - Nonlinear SISO system with non-zero point-wise relative degree

Consider the SISO nonlinear system

$$\dot{x} = G(\rho), \quad y = x \tag{99}$$

$$G(s)=\tanh(s)+s \tag{100}$$

where ρ = r-x. The velocity-based linearisation associated with the operating points at which ρ equals ρ_1 is

$$\dot{x} = w \tag{101}$$

$$\dot{w} = -\nabla G(\rho_1)w + \nabla G(\rho_1)\dot{r} \tag{102}$$

$$\dot{y} = w \tag{103}$$

and a velocity-based linearisation family is defined as ρ_1 varies throughout its range. The velocity-based linearisation has transfer function representation

$$\dot{Y}(s) = \frac{\nabla G(\rho_1)}{s + \nabla G(\rho_1)}\dot{R}(s) \tag{104}$$

where $\dot{R}(s)$ and $\dot{Y}(s)$ denote, respectively, the Laplace transfer functions of $\dot{r}(t)$ and $\dot{y}(t)$. The members of the velocity-based linearisation family have relative degree one. It follows from the foregoing analysis that the velocity-based linearisation family of a corresponding approximate inverse system is defined by

$$\dot{x}^i = w^i \tag{105}$$

$$\dot{w}^i = -\nabla G(\rho_1)\left(\frac{1+\varepsilon_d}{\varepsilon_d}\right)w^i - \frac{\nabla G(\rho_1)}{\varepsilon_d}\dot{v} \tag{106}$$

$$\dot{r} = \frac{1}{\varepsilon_d}w^i + \frac{1}{\varepsilon_d}\dot{v} \tag{107}$$

The velocity-based linearisation, (105)-(107), has transfer function representation

$$\dot{R}(s) = \frac{s + \nabla G(\rho_1)}{\varepsilon_d s + \varepsilon_d + \nabla G(\rho_1)}\dot{V}(s) \tag{108}$$

where $\dot{V}(s)$ denotes the Laplace transfer function of $\dot{v}(t)$. Evidently, as $\varepsilon_d \to 0$ the transfer function, (108), tend to the reciprocal of the transfer function, (104), of the corresponding velocity-based linearisation associated with the original system. Simulation results with v chosen to be sin2t and a range of values of ε_d are presented in figure 6: it can clearly be seen that the error, $\dot{y} - \dot{v}$, tends to zero as ε_d decreases.

Example 4 - Nonlinear MIMO system with non-zero point-wise relative degree

Consider the MIMO nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 - B(x_1) \\ -A(x_1) + r_1 + r_2 \\ -C(x_3) + r_2 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_3 \end{bmatrix} \tag{109}$$

for which the velocity-based linearisation associated with the operating points at which (x_1, x_3) equals (x_{1_2}, x_{3_1}) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{110}$$

$$\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{bmatrix}=\begin{bmatrix}-\nabla B(x_{1_1}) & 1 & 0\\ -\nabla A(x_{1_1}) & 0 & 0\\ 0 & 0 & -\nabla C(x_{3_1})\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\dot{r}_1\\ \dot{r}_2\end{bmatrix} \tag{111}$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix} \tag{112}$$

where A(s)=(10+sin s) s, B(s)=(5+s^2)s, C(s)=tanh(s). The input and output of the velocity-based linearisation is related, in transfer function form, by

$$\begin{bmatrix}\dot{Y}_1(s)\\ \dot{Y}_2(s)\end{bmatrix}=\begin{bmatrix}\dfrac{\dot{R}_1(s)+\dot{R}_2(s)}{s^2+\nabla B(x_{1_1})s+\nabla A(x_{1_1})}\\ \dot{Y}_1(s)+\dfrac{\dot{R}_2(s)}{\left(s+\nabla C(x_{3_1})\right)}\end{bmatrix}=\begin{bmatrix}\dfrac{\dot{R}_1(s)+\dot{R}_2(s)}{s^2+\nabla B(x_{1_1})s+\nabla A(x_{1_1})}\\ \dfrac{\dot{R}_1(s)}{s^2+\nabla B(x_{1_1})s+\nabla A(x_{11})}+\dfrac{s^2+(1+\nabla B(x_{1_1}))s+(\nabla A(x_{1_1})+\nabla C(x_{3_1}))}{\left(s+\nabla C(x_{3_1})\right)\left(s^2+\nabla B(x_{1_1})s+\nabla A(x_{1_1})\right)}\dot{R}_2(s)\end{bmatrix} \tag{113}$$

where $\dot{R}_i(s)$ and $\dot{Y}_i(s)$ (i=1, 2) denote, respectively, the Laplace transfer functions of $\dot{r}_i(t)$ and $\dot{y}_i(t)$. The relative degree of the velocity-based linearisations is 2 for output, y_1, and unity for output, y_2. Following the foregoing analysis that the velocity-based linearisation family of a corresponding approximate inverse system is defined by

$$\begin{bmatrix}\dot{x}_1^{\,i}\\ \dot{x}_2^{\,i}\\ \dot{x}_3^{\,i}\end{bmatrix}=\begin{bmatrix}w_1^{\,i}\\ w_2^{\,i}\\ w_3^{\,i}\end{bmatrix} \tag{114}$$

$$\begin{bmatrix}\dot{w}_1^{\,i}\\ \dot{w}_2^{\,i}\\ \dot{w}_3^{\,i}\end{bmatrix}=\begin{bmatrix}-\nabla B(x_{1_1}) & 1 & 0\\ -\nabla A(x_{1_1})-\dfrac{2}{\varepsilon_d} & 0 & -\dfrac{1}{\varepsilon_d}\\ -\dfrac{1}{\varepsilon_d} & 0 & -\nabla C(x_{3_1})-\dfrac{1}{\varepsilon_d}\end{bmatrix}\begin{bmatrix}w_1^{\,i}\\ w_2^{\,i}\\ w_3^{\,i}\end{bmatrix}+\begin{bmatrix}0 & 0\\ -\dfrac{1}{\varepsilon_d} & -\dfrac{1}{\varepsilon_d}\\ 0 & -\dfrac{1}{\varepsilon_d}\end{bmatrix}\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\end{bmatrix} \tag{115}$$

$$\begin{bmatrix}\dot{r}_1\\ \dot{r}_2\end{bmatrix}=\begin{bmatrix}\dfrac{1}{\varepsilon_d} & 0 & 0\\ \dfrac{1}{\varepsilon_d} & 0 & \dfrac{1}{\varepsilon_d}\end{bmatrix}\begin{bmatrix}w_1^{\,i}\\ w_2^{\,i}\\ w_3^{\,i}\end{bmatrix}+\begin{bmatrix}\dfrac{1}{\varepsilon_d} & 0\\ 0 & \dfrac{1}{\varepsilon_d}\end{bmatrix}\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\end{bmatrix} \tag{116}$$

The input and output of the velocity-based linearisation, (114)-(116), of the approximate inverse system are related, in transfer function form, by

$$\begin{bmatrix}\dot{R}_1(s)\\ \dot{R}_2(s)\end{bmatrix}=\begin{bmatrix}\dfrac{-\dot{R}_2(s)+(s^2+\nabla B(x_{1_1})s+\nabla A(x_{1_1}))\dot{V}_1(s)}{\varepsilon_d s^2+\varepsilon_d\nabla B(x_{1_1})s+\varepsilon_d\nabla A(x_{1_1})+1}\\ \dfrac{\left(s+\nabla C(x_{3_1})\right)\left(\varepsilon_d\dot{R}_1(s)-\dot{V}_1(s)+\dot{V}_2(s)\right)}{\varepsilon_d s+\varepsilon_d\nabla C(x_{3_1})+1}\end{bmatrix} \tag{117}$$

where $\dot{V}_i(s)$ (i=1,2) denotes the Laplace transfer function of $\dot{v}_i(t)$. Evidently, as $\varepsilon_d\rightarrow 0$ the transfer function, (117), tend to the inverse of the transfer function, (113), of the corresponding velocity-based linearisation associated with the original system. Owing to the direct relationship between the velocity-based linearisation family and the velocity-form of a nonlinear system, the nonlinear system obtained by allowing x_1 and x_3 to vary with time in (114)-(116) is a realisation of the inverse system. Simulation results with v_1 chosen to be sin(t) and v_2 chosen to be cos(2t)-1 are presented in figure 7 for a range of values of ε_d : it can clearly be seen that the error, $\dot{\mathbf{y}}-\dot{\mathbf{v}}$, tends to zero as ε_d decreases.

■

5. Dynamic inversion without plant measurements

The foregoing velocity-based dynamic inversion approach is a direct generalisation of the linear frequency-domain pole-zero inverse approach. Whilst full state information is not required to implement the velocity-based inverse system, a measurement or estimate of the scheduling variable, ρ, is needed. Of course, in the case of purely linear systems, there is no scheduling variable and, consequently, plant measurements are not required to implement the pole-zero inverse (in contrast to an Input-Output Linearisation controller which always requires full state information). In addition, since the input, $\mathbf{r}$, to the plant is the output of the inverse system, plant measurements are not required with nonlinear systems for which the scheduling variable, ρ, depends only on $\mathbf{r}$; that is, when $\nabla_{\mathbf{x}}\rho$ is identically zero. However, in general ρ depends on the state of the plant and, although ρ typically involves only a small subset of the elements in the full state vector, it remains attractive to consider dynamic inverses which require a minimal number of plant measurements.

The state, $\mathbf{x}$, of the plant, (40), is related to velocity state, $\mathbf{w}$, by

$$\mathbf{w} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\rho) = \mathbf{F}(\mathbf{x}, \mathbf{r}) \tag{118}$$

Assume that $\mathbf{F}(\bullet,\bullet)$ is invertible in the sense that

$$\mathbf{x} = \mathbf{F}^{-1}(\mathbf{w}, \mathbf{r}) \tag{119}$$

This ensures that the velocity transformation relating the direct and velocity forms of the nonlinear system (see figure 1) is algebraic. It follows from the analysis of section 3 that the state, $\mathbf{w}^i$, of the velocity-based inverse system tends asymptotically to the negative, $-\mathbf{w}$, of the state of the plant. Consider, therefore, using the estimated scheduling variable

$$\rho^i = \nabla_{\mathbf{x}}\rho\ \tilde{\mathbf{x}} + \nabla_{\mathbf{r}}\rho\ \mathbf{r} \tag{120}$$

where

$$\tilde{\mathbf{x}} = \mathbf{F}^{-1}(-\mathbf{w}^i, \mathbf{r}) \tag{121}$$

in the inverse system, (45)-(47), rather than the exact scheduling variable, $\rho=\nabla_{\mathbf{x}}\rho\mathbf{x} + \nabla_{\mathbf{r}}\rho\ \mathbf{r}$. The resulting system utilises *no* plant measurements. Assuming for simplicity that the point-wise relative degree is zero, the velocity-based form of the cascade combination of the plant with the resulting inverse is defined by

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} \tag{122}$$

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\rho\ ) & \mathbf{B}(\rho_1)\mathbf{C}^i(\rho^i\ ) + \mathbf{A}^i(\rho^i\ ) - \mathbf{A}(\rho\ ) \\ 0 & \mathbf{A}^i(\rho^i) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \begin{bmatrix} \mathbf{B}(\rho\ )\mathbf{D}^i(\rho^i) + \mathbf{B}^i(\rho^i) \\ \mathbf{B}^i(\rho^i) \end{bmatrix} \dot{\mathbf{v}} \tag{123}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{C}(\rho\ ) & \mathbf{D}(\rho)\mathbf{C}^i(\rho^i\ ) - \mathbf{C}(\rho\ ) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \mathbf{D}(\rho\ )\mathbf{D}^i(\rho^i)\dot{\mathbf{v}} \tag{124}$$

where

$$\mathbf{z} = \mathbf{w} + \mathbf{w}^i \tag{125}$$

$$\mathbf{A}^i(\rho^i)=\mathbf{A}(\rho^i)-\mathbf{B}(\rho^i)\mathbf{D}^{-1}(\rho^i)\mathbf{C}(\rho^i),\ \ \mathbf{B}^i(\rho^i) = -\mathbf{B}(\rho^i)\mathbf{D}^{-1}(\rho^i),\ \ \mathbf{C}^i(\rho^i) = \mathbf{D}^{-1}(\rho^i)\mathbf{C}(\rho^i),\ \mathbf{D}^i(\rho^i)=\mathbf{D}^{-1}(\rho^i) \tag{126}$$

This velocity-based system may be reformulated as

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}^i \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} \tag{127}$$

$$\begin{bmatrix} \dot{\mathbf{z}} \\ \dot{\mathbf{w}}^i \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\rho\ ) & \xi_1 \\ 0 & \mathbf{A}^i(\rho^i\ ) \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + \begin{bmatrix} \xi_2 \\ \mathbf{B}^i(\rho^i) \end{bmatrix} \dot{\mathbf{v}} \tag{128}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} \mathbf{C}(\rho\ ) & \xi_3 \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{w}^i \end{bmatrix} + (\mathbf{I} + \xi_4)\dot{\mathbf{v}} \tag{129}$$

where

$$\xi_1 = \mathbf{B}(\rho)\left(\mathbf{C}^i(\rho^i) - \mathbf{C}^i(\rho)\right) + \left(\mathbf{A}^i(\rho^i) - \mathbf{A}^i(\rho)\right),\ \xi_2 = \mathbf{B}(\rho)\left(\mathbf{D}^i(\rho^i) - \mathbf{D}^i(\rho)\right) + \left(\mathbf{B}^i(\rho^i) - \mathbf{B}^i(\rho)\right) \tag{130}$$

$$\xi_3 = \mathbf{D}(\rho\ )\left(\mathbf{C}^i(\rho^i\ ) - \mathbf{C}^i(\rho)\right),\quad \xi_4 = \mathbf{D}(\rho\ )\left(\mathbf{D}^i(\rho^i) - \mathbf{D}^i(\rho)\right) \tag{131}$$

It follows from (125), (121) and (120) that when $\mathbf{z}(0)$ is zero, ρ^i equals ρ and the ξ_i are zero. Hence, owing to the direct relationship between the velocity-based linearisation family and the velocity-form of a nonlinear system, see figure 1, it can be deduced immediately that with initial condition $\mathbf{z}(0)=0$, the $\mathbf{z}$

dynamics in (128) are unforced and $\mathbf{z}$ is identically zero. Consequently, the ξ_i vanish and $\dot{\mathbf{y}} - \dot{\mathbf{v}}$ is identically zero as required. More generally, $\dot{\mathbf{y}} - \dot{\mathbf{v}}$ tends to zero asymptotically provided the $\mathbf{z}$ dynamics are asymptotically stable so that transients associated with other initial conditions decay to zero. (Sufficient conditions for stability of the $\mathbf{z}$ dynamics are derived in Appendix C).

6. Comparison with Input-Output Linearisation

The velocity-based gain-scheduling approach is quite general and directly supports the design of feedback configurations for which the closed-loop dynamics are nonlinear. However, the discussion in this paper concentrates on the design of velocity-based controllers which, when combined with a nonlinear plant, attain linear closed-loop dynamics. Of course, the design of linearising controllers has received considerable attention in the literature. In particular, Input-Output Linearisation (see, for example, Isidori 1995) is a widely advocated approach for accommodating plant nonlinearities. It is, therefore, necessary to compare the velocity-based approach with that of Input-Output Linearisation. At a number of points in the preceding sections, specific features of the velocity-based linearising approach are briefly compared with the corresponding aspects of the Input-Output Linearisation approach. Nevertheless, in order to facilitate the comparison, it is appropriate to collect these observations together in the present section.

The velocity-based approach investigated in this paper is a direct generalisation to nonlinear systems of the classical frequency-domain pole-zero inversion approach for linear systems. In contrast, the Input-Output Linearisation approach is a direct extension of the state-feedback inversion approach for linear systems studied by Silverman (1969). These approaches are quite distinct, even in the linear case. Similarly to the purely linear case, the velocity-based nonlinear pole-zero inversion approach leads to a dynamic inverse which requires both the nonlinear system concerned and its inverse to be, in an appropriate sense, stable. Input-Output Linearisation, on the other hand, is essentially based on static state feedback and only requires stability of the internal dynamics rendered unobservable by the feedback. Of course, state feedback requires the measurement or estimation of the full state of the plant whereas the nonlinear pole-zero inverse requires only a measurement or estimate of the scheduling variable, ρ. Frequently, ρ depends on only a small number of elements of the state and/or input vectors. Indeed, in the case of purely linear systems, there is no scheduling variable and, consequently, plant measurements are not required to implement the pole-zero inverse (in contrast to an Input-Output Linearisation controller which always requires full state information).

With regard to the design task associated with each approach, the velocity-based approach decomposes the nonlinear design task into a number of straightforward linear sub-problems; that is, the methodology supports the divide and conquer philosophy and maintains continuity with well established linear methods. In this sense, it is closely related to the gain-scheduling methodology. In contrast, it is difficult to discern any relationship between the Input-Output Linearisation design procedure and linear control design. It is emphasised that, in practice, the importance of maintaining a degree of continuity with well established linear methods should not be underestimated; for example, safety certification procedures are typically based on experience with conventional linear methods and the cost of developing and assessing entirely new procedures is often prohibitive. Moreover, the Input-Output Linearisation approach involves the repeated analytic differentiation of the output of the nonlinear system concerned. Since this rapidly becomes intractable as the order of the system increases, the utility of the Input-Output Linearisation approach appears to be largely restricted to systems with relatively low order. When the nonlinear system concerned is described by a differential equation of the form, (40), the velocity-based linearisation approach also requires a formal differentiation step to obtain the corresponding velocity-based representation, (15)-(17). However, this is confined to a single formal differentiation for which the complexity increases relatively slowly with increasing system order. In addition, the velocity-based representation is often a natural one and may be identified directly from experimental data (see, for example, Leith & Leithead 1998d), in which case the formal differentiation step may be avoided completely.

It should be noted that the Input-Output Linearisation and nonlinear pole-zero inversion approaches are extreme cases in the sense that the former is essentially static and utilises full state feedback whilst the latter is dynamic and reduces to open-loop inversion in the linear case. Other dynamic inversion approaches, which lie between these extremes, may be derived by utilising a combination of Input-Output Linearisation and nonlinear pole-zero inversion. However, these are not considered further here.

7. Conclusions

The velocity-based analysis framework provides direct support for divide and conquer design approaches, such as the gain-scheduling control design methodology, whereby the design of a nonlinear system is decomposed into the design of an associated family of linear systems. Specifically, since a velocity-based linearisation family is associated with the nonlinear plant, a corresponding linear controller family can be obtained by designing a linear controller for each member of the plant family. The velocity-based gain-scheduling approach is quite general and directly supports the design of feedback configurations for which the closed-loop dynamics are nonlinear. However, the present paper concentrates on the velocity-based design of controllers which, when combined with a nonlinear plant, attain linear closed-loop dynamics.

The velocity-based approach to dynamic linearisation investigated in this paper

- is a direct generalisation to nonlinear systems of the classical frequency-domain pole-zero inversion approach.
- requires, in general, only a measurement or estimate of the scheduling variable, ρ. Frequently, ρ depends on only a small number of elements of the state and/or input vectors. Indeed, in the case of purely linear systems, there is no scheduling variable and, consequently, plant measurements are not required to implement the pole-zero inverse.
- decomposes the nonlinear design task into a number of straightforward linear sub-problems; that is, the methodology supports the divide and conquer philosophy and maintains continuity with well established linear methods. In this sense, it is closely related to the gain-scheduling methodology. However, it is emphasised that the velocity-based approach does *not* necessitate a slow variation requirement.

The velocity-based dynamic linearisation and Input-Output Linearisation approaches are, in many ways, complementary. In particular the former is dynamic and reduces to open-loop inversion in the linear case whilst the latter is essentially static and utilises full state feedback.

Acknowledgement
D.J.Leith gratefully acknowledges the generous support provided by the Royal Society for the work presented.

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Appendix A

The tracking error satisfies

$$\mathbf{y}(t)-\mathbf{v}(t)=\int_0^t \mathbf{C}(\rho)\mathbf{z}(s)ds+\mathbf{y}(0)-\mathbf{v}(0) \tag{132}$$

Assume that $\mathbf{C}(\rho)$ is uniformly bounded; that is, $|\mathbf{C}(\rho)| \leq \alpha \ \forall \rho$. Since the $\mathbf{z}$ dynamics are exponentially stable, it follows that

$$|\mathbf{C}(\rho(t))\mathbf{z}(t)| \leq \alpha\gamma e^{-at}|\mathbf{z}(0)| \tag{133}$$

where γ, a are positive finite constants. Let

$$\mathbf{x}_k=\mathbf{y}(kT_o)-\mathbf{v}(kT_o) \tag{134}$$

where T_o is a positive constant. It follows that

$$|\mathbf{x}_m-\mathbf{x}_n| \leq \left|\int_{nT_{oi}}^{mT_o} \mathbf{C}(\rho(s))\mathbf{z}(s)ds\right| \leq \frac{\alpha\gamma}{a}\left|e^{-anT_o}-e^{-amT_o}\right||\mathbf{z}(0)| \tag{135}$$

and so, provided $\mathbf{z}(0)$ is bounded, $|\mathbf{x}_m-\mathbf{x}_n| \to 0$ as $m,n \to \infty$. Hence, from Cauchy's criterion for uniform convergence (see, for example Sutherland 1995 Theorem 1.2.9), $\mathbf{x}_k$ tends to a limit, say $\bar{\mathbf{x}}$, as $k \to \infty$. Moreover, it follows from (132) and (133) that

$$|\bar{\mathbf{x}}| \leq \int_0^\infty |\mathbf{C}(\rho)\mathbf{z}(s)|ds+|\mathbf{y}(0)-\mathbf{v}(0)| \leq \frac{\alpha\gamma|\mathbf{z}(0)|}{a}+|\mathbf{y}(0)-\mathbf{v}(0)| \tag{136}$$

Hence, the limit, $\bar{\mathbf{x}}$, is finite provided $\mathbf{z}(0)$ and $\mathbf{y}(0)$-$\mathbf{v}(0)$ are both finite. Since

$$|\mathbf{y}(t)-\mathbf{v}(t)-\bar{\mathbf{x}}| \leq |\mathbf{y}(t)-\mathbf{v}(t)-\mathbf{x}_k|+|\mathbf{x}_k-\bar{\mathbf{x}}| \leq \frac{\alpha\gamma}{a}\left|e^{-at}-e^{-akT_o}\right||\mathbf{z}(0)|+|\mathbf{x}_k-\bar{\mathbf{x}}| \tag{137}$$

then provided $\mathbf{z}(0)$ is bounded, for any positive ε there exists a choice of t_o and k_o sufficiently large that

$$|\mathbf{y}(t)-\mathbf{v}(t)-\bar{\mathbf{x}}| \leq \varepsilon \quad \forall t \geq t_o, k \geq k_o \tag{138}$$

Consequently, $\mathbf{y}(t)$-$\mathbf{v}(t)$ tends to a finite limit, $\bar{\mathbf{x}}$, as $t \to \infty$

Appendix B

Consider the nonlinear system, (40), for which the velocity-based linearisation family is (41)-(43). In order to facilitate the analysis, assume that $\mathbf{D}(\rho)$ is identically zero: this involves no loss of generality since it can always be achieved by augmenting the system with a suitable low-pass input filter. Perturbing $\mathbf{D}(\rho)$, $\mathbf{B}(\rho)$, $\mathbf{C}(\rho)$ to ε_d, $\mathbf{B}(\rho)+\varepsilon_\mathbf{b}$, $\mathbf{C}(\rho)+\varepsilon_\mathbf{c}$, the velocity-based form of the resulting perturbed nonlinear system is

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{w}} \tag{139}$$

$$\dot{\tilde{\mathbf{w}}} = \mathbf{A}(\tilde{\rho})\,\tilde{\mathbf{w}} + \tilde{\mathbf{B}}(\tilde{\rho})\,\dot{\mathbf{r}} \tag{140}$$

$$\dot{\tilde{\mathbf{y}}} = \tilde{\mathbf{C}}(\tilde{\rho})\,\tilde{\mathbf{w}} + \tilde{\mathbf{D}}(\tilde{\rho})\,\dot{\mathbf{r}} \tag{141}$$

where

$$\tilde{\rho} = \nabla_\mathbf{x}\rho\,\tilde{\mathbf{x}} + \nabla_\mathbf{r}\rho\,\mathbf{r},\ \ \mathbf{A}(\tilde{\rho}) = \mathbf{A}+\nabla\mathbf{f}(\tilde{\rho})\,\nabla_\mathbf{x}\rho,\ \ \tilde{\mathbf{B}}(\tilde{\rho}) = \mathbf{B}+\nabla\mathbf{f}(\tilde{\rho})\,\nabla_\mathbf{r}\rho+\varepsilon_\mathbf{b} \tag{142}$$

$$\tilde{\mathbf{C}}(\tilde{\rho}) = \mathbf{C}+\nabla\mathbf{g}(\tilde{\rho})\,\nabla_\mathbf{x}\rho ++\varepsilon_\mathbf{c},\quad \tilde{\mathbf{D}}(\tilde{\rho}) = \varepsilon_d \tag{143}$$

It follows from the analysis of section 3 that a corresponding inverse system is

$$\dot{\mathbf{x}}^i = \mathbf{w}^i \tag{144}$$

$$\dot{\mathbf{w}}^i = \mathbf{A}^i(\rho^i)\,\mathbf{w}^i + \mathbf{B}^i(\rho^i)\,\dot{\mathbf{v}} \tag{145}$$

$$\dot{\mathbf{r}} = \mathbf{C}^i(\rho^i)\,\mathbf{w}^i + \mathbf{D}^i(\rho^i)\,\dot{\mathbf{v}} \tag{146}$$

where

$\rho^i = \rho$, $\mathbf{A}^i(\rho^i) = \mathbf{A}(\rho)\text{-}\tilde{\mathbf{B}}(\rho)\tilde{\mathbf{D}}^{-1}(\rho)\tilde{\mathbf{C}}(\rho)$, $\mathbf{B}^i(\rho^i) = -\tilde{\mathbf{B}}(\rho)\tilde{\mathbf{D}}^{-1}(\rho)$ (147)

$\mathbf{C}^i(\rho^i) = \tilde{\mathbf{D}}^{-1}(\rho)\tilde{\mathbf{C}}(\rho)$, $\mathbf{D}^i(\rho^i) = \tilde{\mathbf{D}}^{-1}(\rho)$ (148)

When employing the perturbed inverse system with the original nonlinear system, (40), let

$\rho^i = \rho$ (149)

The velocity-based form of the cascade combination of this inverse system with the original nonlinear system, (40), is

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{x}}^i\end{bmatrix} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \mathbf{w}^i\end{bmatrix} \quad (150)$$

$$\begin{bmatrix}\dot{\mathbf{z}}\\ \dot{\mathbf{w}}^i\end{bmatrix} = \begin{bmatrix}\mathbf{A}(\rho) & -\varepsilon_\mathbf{b}\tilde{\mathbf{D}}^{-1}(\rho)\tilde{\mathbf{C}}(\rho)\\ 0 & \mathbf{A}^i(\rho)\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \mathbf{w}^i\end{bmatrix} + \begin{bmatrix}-\varepsilon_\mathbf{b}\tilde{\mathbf{D}}^{-1}(\rho)\\ \mathbf{B}^i(\rho)\end{bmatrix}\dot{\mathbf{v}} \quad (151)$$

$$\dot{\mathbf{y}} = \begin{bmatrix}\mathbf{C}(\rho) & -\mathbf{C}(\rho)\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \mathbf{w}^i\end{bmatrix} \quad (152)$$

where

$\mathbf{z} = \mathbf{w} + \mathbf{w}^i$ (153)

By (151), the state $\mathbf{w}$, of the nonlinear system is related to the state $\mathbf{w}^i$, of the inverse system by the dynamics

$$\dot{\mathbf{z}} = \mathbf{A}(\rho)\mathbf{z} - \varepsilon_\mathbf{b}\tilde{\mathbf{D}}^{-1}(\rho)\left(\mathbf{C}(\rho)\mathbf{w}^i + \dot{\mathbf{v}}\right) \quad (154)$$

Assume that the dynamics, $\dot{\chi} = \mathbf{A}(\rho(t))\chi$, are exponentially stable. It follows (see, for example, Khalil 1992 Lemma 4.8) that the dynamics, (154), are bounded-input bounded-output stable with

$$|\mathbf{z}| \le \gamma e^{-at}|\mathbf{z}(0)| + \kappa\left\|\varepsilon_\mathbf{b}\tilde{\mathbf{D}}^{-1}(\rho)\left(\mathbf{C}(\rho)\mathbf{w}^i + \dot{\mathrm{v}}\right)\right\| \quad (155)$$

where γ, a and κ are positive constants and $\|\bullet\| = \sup|\bullet|$. In addition, assume that the inverse system is bounded-input bounded-output stable: since $\dot{\mathbf{v}}$ is bounded, it follows that $\mathbf{w}^i$ is bounded. It should be noted that these stability conditions are just those employed in section 3 for the case of zero relative degree. Selecting $\varepsilon_\mathbf{b}$ such that $\left|\varepsilon_\mathbf{b}\tilde{\mathbf{D}}^{-1}(\rho)\right| \to 0$ as $|\varepsilon_d| \to 0$, then $\mathbf{z}$ tends to zero asymptotically as $|\varepsilon_d| \to 0$ provided $\mathbf{C}(\rho)$ is bounded; that is, $\mathbf{w} \to -\mathbf{w}^i$ asymptotically as $|\varepsilon_d| \to 0$.

From (146),

$$\dot{\mathbf{r}}(t) = \tilde{\mathbf{D}}^{-1}(\rho(t))\left(\tilde{\mathbf{C}}(\rho(t))\mathbf{z}(t) - \tilde{\mathbf{C}}(\rho(t))\mathbf{w}(t) + \dot{\mathbf{v}}(t)\right) \quad (156)$$

Hence,

$$\begin{aligned}\dot{\mathbf{w}}(t) &= \mathbf{A}(\rho(t))\mathbf{w}(t) + \mathbf{B}(\rho(t))\dot{\mathbf{r}}\\ &= \left(\mathbf{A}(\rho(t)) - \mathbf{B}(\rho(t))\tilde{\mathbf{D}}^{-1}(\rho(t))\tilde{\mathbf{C}}(\rho(t))\right)\mathbf{w}(t) + \mathbf{B}(\rho(t))\tilde{\mathbf{D}}^{-1}(\rho(t))\left(\dot{\mathbf{v}}(t) + \tilde{\mathbf{C}}(\rho(t))\mathbf{z}(t)\right)\end{aligned} \quad (157)$$

Assume that there exists a bounded input, $\dot{\bar{\mathbf{r}}}(t)$, and associated bounded solution, $\bar{\mathbf{w}}(t)$, such that

$\dot{\bar{\mathbf{w}}}(t) = \mathbf{A}(\rho(t))\bar{\mathbf{w}}(t) + \mathbf{B}(\rho(t))\dot{\bar{\mathbf{r}}}(t)$ (158)

$\dot{\mathbf{y}}(t) = \mathbf{C}(\rho(t))\bar{\mathbf{w}}(t) = \dot{\mathbf{v}}(t) + \xi(t)$ (159)

where $\xi(t)$ decays asymptotically to zero as $t \to \infty$. The inversion task is then well-posed in the sense that a smooth solution does indeed exist. (The nonlinear relative degree, when its exists, is defined as the minimum number of times each output must be differentiated such that the inputs are directly coupled to the outputs. It follows that a sufficient condition for the existence of $\dot{\bar{\mathbf{r}}}$ is that the relative degree is uniform in the relevant operating region and the associated coupling is invertible). Adopting a fast/slow time-scale separation approach, let $\tau = t/\varepsilon$ be the fast time-scale and let

$\mathbf{w}(t) = \hat{\mathbf{w}}(\tau) + \bar{\mathbf{w}}(t)$ (160)

It follows that

$$\frac{d\hat{\mathbf{w}}(\tau)}{d\tau} = \varepsilon\frac{d\hat{\mathbf{w}}(\tau)}{dt} = \varepsilon\left(\mathbf{A}(\rho(\varepsilon\tau)) - \mathbf{B}(\rho(\varepsilon\tau))\tilde{\mathbf{D}}^{-1}(\rho(\varepsilon\tau))\tilde{\mathbf{C}}(\rho(\varepsilon\tau))\right)\hat{\mathbf{w}}(\varepsilon\tau) + \eta(\varepsilon\tau) \quad (161)$$

where

$$\eta = -\varepsilon\mathbf{B}(\rho(\varepsilon\tau))\dot{\bar{\mathbf{r}}}(\varepsilon\tau) + \varepsilon\mathbf{B}(\rho(\varepsilon\tau))\tilde{\mathbf{D}}^{-1}(\rho(\varepsilon\tau))\left(-\varepsilon_c\bar{\mathbf{w}}(\varepsilon\tau) - \xi(\varepsilon\tau) + \tilde{\mathbf{C}}(\rho(\varepsilon\tau))\mathbf{z}(\varepsilon\tau)\right) \quad (162)$$

Assume that $|\mathbf{B}(\rho)|$ and $|\mathbf{C}(\rho)|$ are uniformly bounded and select ε_c and ε such that $|\varepsilon_c|\to 0$ as $|\varepsilon_d|\to 0$ and $|\varepsilon\tilde{\mathbf{D}}^{-1}(\rho)|$ is uniformly bounded (it should be noted that ε_c and ε may depend on ε_d). Since $\bar{\mathbf{w}}(t)$ and $\dot{\bar{\mathbf{r}}}(t)$ are bounded, $\mathbf{z}(t)$ tends to zero asymptotically as $|\varepsilon_d|\to 0$ and $\xi(t)$ decays asymptotically to zero, it follows that η decays asymptotically to zero as $|\varepsilon_d|\to 0$. Hence, provided (161) is bounded-input bounded-output stable uniformly in ε, then $\hat{\mathbf{w}}(\tau)\to 0$ and $\mathbf{w}(t)\to\bar{\mathbf{w}}(t)$ as $|\varepsilon_d|\to 0$. It follows immediately that $\dot{\mathbf{v}} - \dot{\mathbf{y}} \to 0$ and the approximate inverse system is an arbitrarily accurate approximation to the exact inverse as $|\varepsilon_d|\to 0$.

Appendix C

Sufficient conditions for stability of the $\mathbf{z}$ dynamics may, for example, be derived as follows. Provided $\mathbf{A}^i(\rho)$, $\mathbf{B}^i(\rho)$, $\mathbf{C}^i(\rho)$ and $\mathbf{D}^i(\rho)$ are differentiable with Lipschitz continuous first derivatives, it follows from the Mean Value Theorem (see, for example, Khalil 1992 p68) that the magnitudes of the ξ_i are proportional to $|\rho^i-\rho|_2$ (where $|\bullet|_2$ denotes the Euclidean norm). In addition, provided $\mathbf{F}^{-1}(\bullet,\bullet)$ is differentiable with respect to its first argument and the derivative is Lipschitz continuous

$$|\rho^i-\rho|_2 = |\nabla_{\mathbf{x}}\rho\,\tilde{\mathbf{x}} - \nabla_{\mathbf{x}}\rho\mathbf{x}|_2 \le |\nabla_{\mathbf{x}}\rho|_2|\mathbf{F}^{-1}(-\mathbf{w}^i, \mathbf{r}) - \mathbf{F}^{-1}(\mathbf{w}, \mathbf{r})|_2 \le \beta_1|\mathbf{w}^i+\mathbf{w}|_2 = \beta_1|\mathbf{z}|_2 \quad (163)$$

where β_1 is a positive finite constant. The $\mathbf{z}$ dynamics in (128) are

$$\dot{\mathbf{z}} = \mathbf{A}(\rho)\mathbf{z} + \eta \quad (164)$$

where

$$\eta = \xi_1\mathbf{w}^i + \xi_2\dot{\mathbf{v}} \quad (165)$$

Under the foregoing conditions, it follows from (163) that

$$|\eta|_2 \le \beta_2|\mathbf{z}|_2\{|\mathbf{w}^i|_2+|\dot{\mathbf{v}}|_2\} \quad (166)$$

where β_2 is a positive finite constant. Assume that the dynamics, (164), are exponentially stable when η is zero. It follows (see, for example, Khalil 1992 Theorem 4.5) that there exists a Lyapunov function, V, satisfying

$$\alpha_1|\mathbf{z}|_2^2 \le V \le \alpha_2|\mathbf{z}|_2^2, \quad \frac{\partial V}{\partial \mathbf{z}}\mathbf{A}(\rho)\mathbf{z} + \frac{\partial V}{\partial t} \le -\alpha_3|\mathbf{z}|_2^2, \quad \left|\frac{\partial V}{\partial \mathbf{z}}\right|_2 \le \alpha_4|\mathbf{z}|_2 \quad (167)$$

where the α_i are positive constants. Hence, when η is non-zero, the derivative of V along the solution trajectories of (164) satisfies

$$\frac{dV}{dt} = \frac{\partial V}{\partial \mathbf{z}}\mathbf{A}(\rho)\mathbf{z} + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{z}}\eta \le -\alpha_3|\mathbf{z}|_2^2 + \alpha_4|\mathbf{z}|_2|\eta|_2 \le -\left(\alpha_3 - \alpha_4\beta_2\left\{|\mathbf{w}^i|_2+|\dot{\mathbf{v}}|_2\right\}\right)|\mathbf{z}|_2^2 \quad (168)$$

Assume that the $\mathbf{w}^i$ dynamics are bounded-input bounded-output stable with exponentially decaying transients; that is,

$$|\mathbf{w}^i(t)|_2 \le \alpha\gamma e^{-at}|\mathbf{w}^i(0)|_2 + \kappa\|\dot{\mathbf{v}}\| \quad (169)$$

where $\|\dot{\mathbf{v}}\| = \sup_t|\dot{\mathbf{v}}(t)|_2$. In addition, assume that the input, $\dot{\mathbf{v}}$, and initial condition, $\mathbf{w}^i(0)$, satisfy

$$\left\{\alpha\gamma|\mathbf{w}^i(0)|_2 + (\kappa+1)\|\dot{\mathbf{v}}\|\right\} < \frac{\alpha_3}{\alpha_4\beta_2} \quad (170)$$

Under these conditions, it follows from (168) that dV/dt is negative definite (with quadratic upper bound) and the function, V is also a Lyapunov function for the $\mathbf{z}$ dynamics, (164), when η is non-zero. Hence, from standard Lyapunov theory (see, for example, Khalil 1992, Corollary 4.2), $\mathbf{z}$ decays exponentially to zero as required .

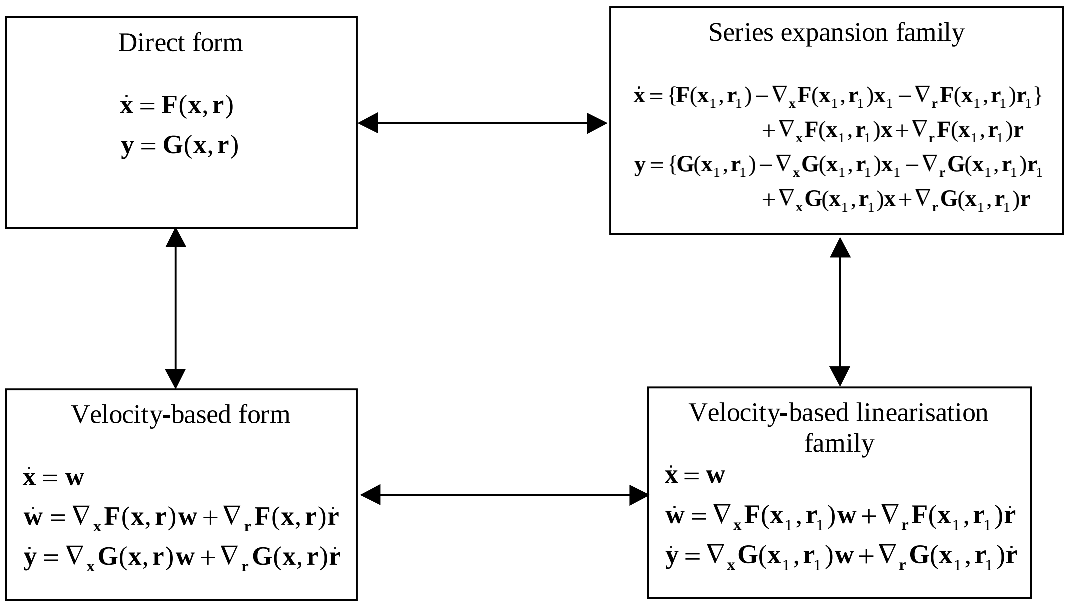


Figure 1 Alternative representations of a nonlinear system

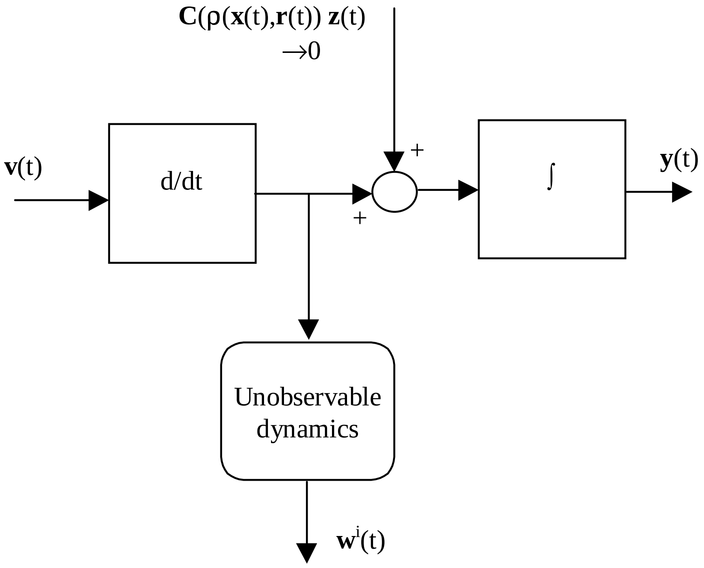


Figure 2 Structure of cascade system

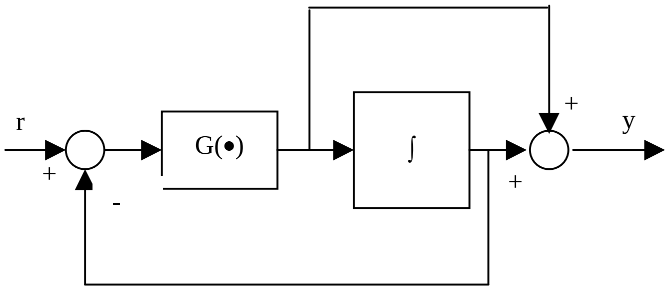


Figure 3 Nonlinear system with relative degree zero considered in Example 1.

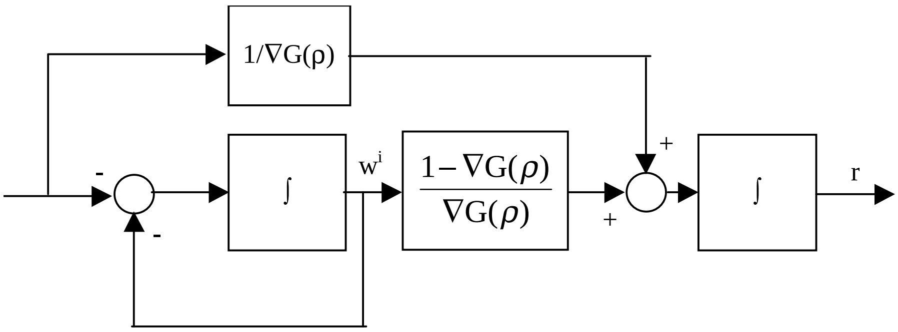


Figure 4 Direct realisation of velocity-form of inverse system in Example 1

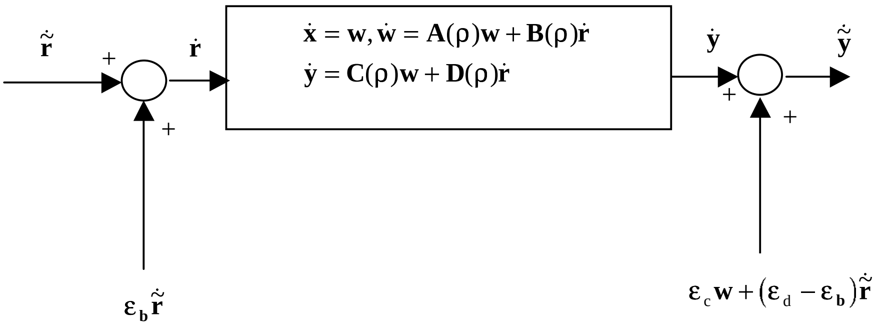


Figure 5 Structure of perturbed system

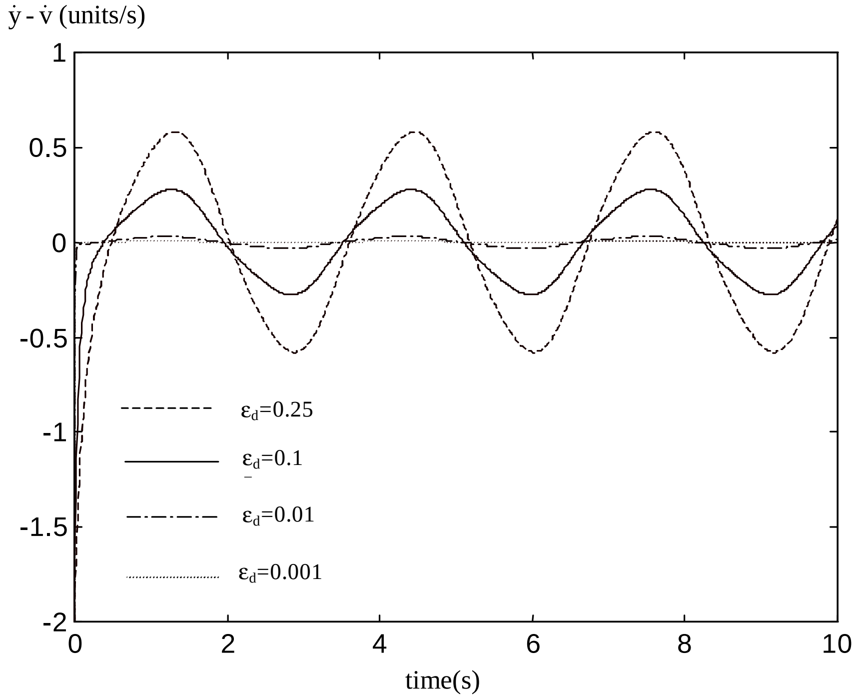
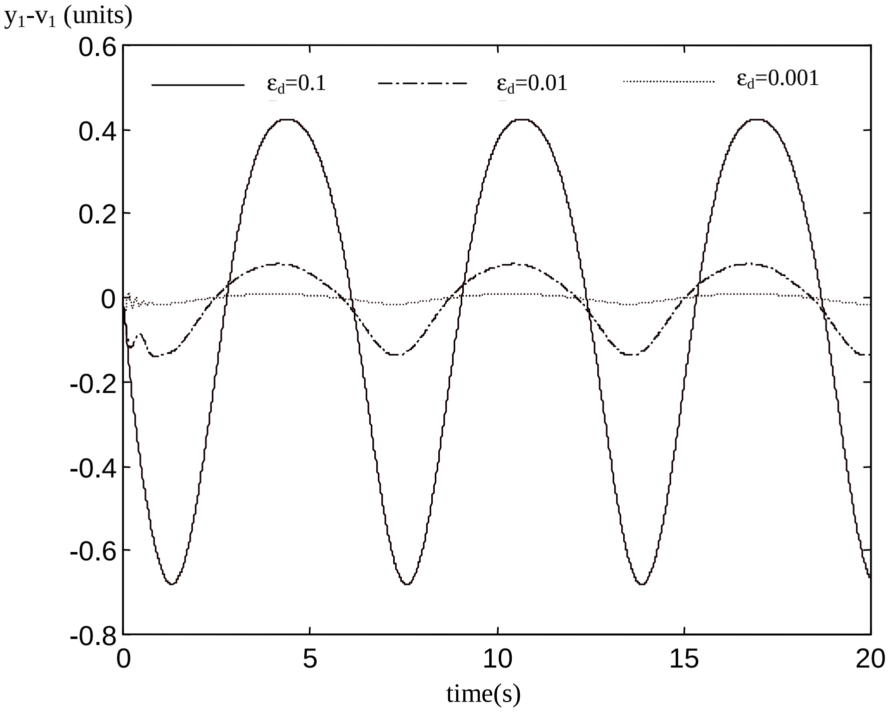
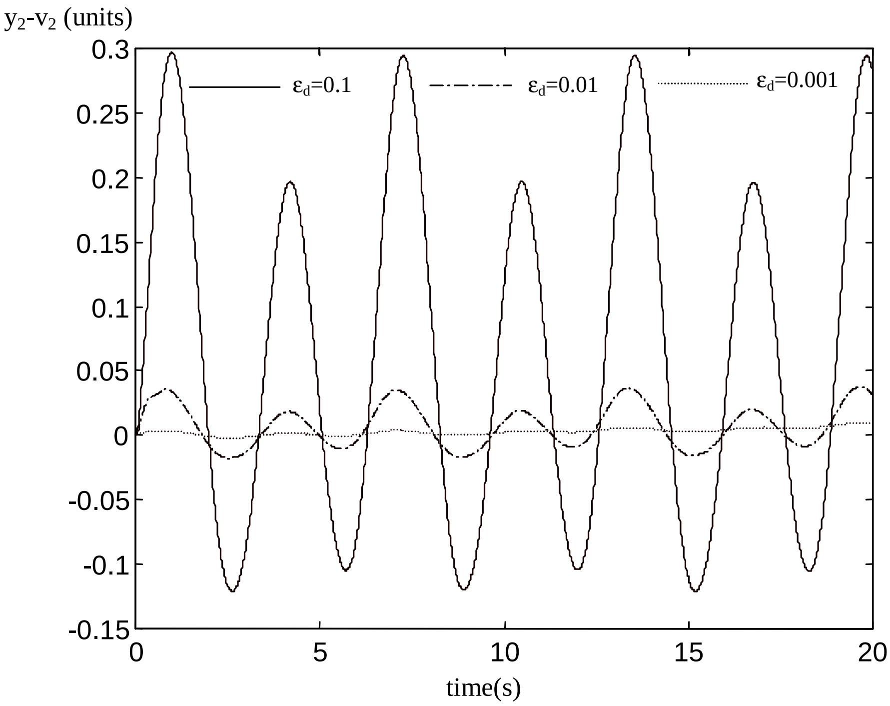


Figure 6 Variation of inversion error with ε_d (Example 3)



(a)



(b)

Figure 7 Variation of inversion error with ε_d (Example 4)