Input-Output Linearisation of Nonlinear Systems with Ill-Defined Relative Degree: The Ball & Beam Revisited

D.J.Leith, W.E.Leithead

Dept. Electronics & Electrical Engineering,
University of Strathclyde, 50 George St.,
Glasgow G1 1QE, U.K.
Email. doug@icu.strath.ac.uk, Tel. +44 141 548 2407,
Fax. +44 141 548 4203

Abstract

In this note it is established that for SISO systems lack of well-defined relative degree is *not* an obstacle to exact inversion. Sufficient conditions for the existence of exact linearising inputs are established. Exact tracking solutions of a number of example systems, including the ball and beam system studied by Hauser and others, are analysed in detail. It is shown that linearising inputs may be (i) highly non-unique, (ii) discontinuous and include impulses etc., (iii) fragile. A framework is established for approximate linearisation whereby exact linearising inputs are formulated as the limits of sequences of realisable, arbitrarily accurate linearising inputs. This framework is constructive in nature and of key importance when the exact linearising inputs are, for example, unrealisable. This is quite different from approximate linearisation approaches previously considered in the literature where the degree of accuracy is generally difficult to prescribe.

1. Introduction

The design of linearising controllers has received some attention in the literature. In particular, input-output linearisation (see, for example, Isidori 1995) is a widely advocated approach for accommodating plant nonlinearities. Unfortunately, current input-output linearisation methods are strictly confined to systems which are both minimum-phase and have well-defined relative degree. Many systems do not satisfy these conditions and this has motivated many attempts to extend linearisation methods to a wider class of systems.

Whilst the literature mainly concentrates on relaxing the minimum-phase requirement, the present paper addresses the linearisation of SISO systems with ill-defined relative degree. In SISO systems, lack of well-defined relative degree is associated with variations across the operating space in the number of differentiations require for the output to be directly coupled to the input. (In the MIMO case, relative degree may also be ill-defined due to the coupling matrix having less than full rank; when the rank is constant across the operating space this may be addressed using dynamic extension methods, Isidori 1995). With regard to SISO systems, Hauser *et al.* (1992) (and, more recently, Tomlin & Sastry 1997) consider applying input-output linearisation methods to a ball and beam system which does not have well-defined relative degree. The proposed approaches accommodate systems with ill-defined relative degree at the cost of achieving only *approximate* linearisation. Furthermore, the requirement for well-defined relative degree is so fundamental to the existing input-output linearisation theory that it is far from clear that exact linearisation is generally even possible for systems with ill-defined relative degree. The purpose of the present paper is, therefore, to investigate whether the lack of a well-defined relative degree is indeed a fundamental obstacle to achieving exact linearisation of the ball and beam system and other systems with ill-defined relative degree. The discussion is not, however, confined solely to establishing when an exact linearising input exists in specific cases. General existence conditions are sought and one main objective of the paper is to establish some of the new issues which must be resolved in order to achieve a solution to the ill-defined relative degree linearisation problem. It is noted that the cost incurred by adopting an approximate, rather than exact, linearisation strategy is frequently tolerable provided the accuracy of the approximation is effectively a design parameter and can be chosen to be arbitrarily good. However, this is not generally the case with the foregoing approaches for which the degree of accuracy achieved is difficult to prescribe. An additional purpose of this note is therefore to make a first step towards determining a systematic approach for achieving arbitrarily accurate linearisation of systems with ill-defined relative degree.

2. A first example

The requirement for well-defined relative degree is so fundamental to the existing input-output linearisation theory that it is far from clear that exact linearisation is even possible for systems with ill-defined relative degree. Consider, therefore, the nonlinear system

$$\dot{x} = -x + u, \quad y = x + g(u) \tag{1}$$

with

$$g(u) = \begin{cases} 0 & u \le -1 \\ 0.5u^2 + u + 0.5 & -1 < u < 0 \\ -0.5u^2 + u + 0.5 & 0 \le u < 1 \\ 1 & u \ge 1 \end{cases} \tag{2}$$

(This system is not in the control affine form normally considered in the input-output linearisation literature but becomes so when augmented with a suitable input filter). The relative degree of the system is 0 when $|u|<1$ and 1 otherwise, hence it is not globally well-defined. Nevertheless, owing to the relative simplicity of this system, it is possible to analytically derive an exactly linearising input. Namely, provided the initial condition of the system is compatible with initialising u to be $g^{-1}(v-x)$, an exact linearising input (such that y=v) is

$$u = \begin{cases} \dot{v} + x & v - x = 1 \;\&\; \dot{v} + x \ge 1 \\ g^{-1}(v - x) & 0 \le v - x \le 1 \;\&\; -1 \le \dot{v} \le 1 \\ \dot{v} + x & v - x = 0 \;\&\; \dot{v} + x \le -1 \end{cases} \qquad (3)$$

which corresponds to $u \geq 1$, $-1 \leq u \leq 1$ and $u \leq -1$ respectively. The switch in u from $\dot{v}+x$ to $g^{-1}(v\text{-}x)$ triggered by u reaching 1 from above or –1 from below is continuous, but the switch in u from $g^{-1}(v\text{-}x)$ to $\dot{v}+x$ triggered by u reaching 1 from below or –1 from above is *discontinuous* in general. In addition, while $u>1$, $\dot{x}=\dot{v}$ and so $v\text{-}x=k_1=1$; while $u<-1$, $\dot{x}=\dot{v}$ and $v\text{-}x=k_2=-1$.

Of course, the initial conditions for the system may be incompatible with initialising u to $g^{-1}(v\text{-}x)$ and/or computational inexactitude in the solution to (1) is sure to cause k_1 to depart from 1 and k_2 to depart from –1. Hence, the linearising system, (3), lacks robustness. Disturbances and mismatches in the initial conditions may be accommodated by augmenting the ideal input (3) to

$$u = \begin{cases} \dot{v} + x & v - x = 1 \;\&\; \dot{v} + x \ge 1 \\ g^{-1}(v - x) & 0 \le v - x \le 1 \;\&\; -1 \le \dot{v} \le 1 \\ \dot{v} + x & v - x = 0 \;\&\; \dot{v} + x \le -1 \\ u_{err} & \text{otherwise} \end{cases} \qquad (4)$$

where the additional term, u_{err}, is selected to ensure that the required trajectory, v, is attractive; for example,

$$u_{err} = \begin{cases} \dot{v} + x - \lambda_0(y - v) & \dot{v} + x - \lambda_0(v - y) \le -1, \; \dot{v} + x - \lambda_0(v - y) \le 1 \\ x + \dot{v} - (\lambda_1 + |x| + |\dot{v}|)\,\mathrm{sgn}(y - v) & \text{otherwise} \end{cases} \qquad (5)$$

where λ_0 and λ_1 are positive constants with $\lambda_1>1$. The associated closed-loop error dynamics

$$\left.\begin{array}{ll} \dot{y} - \dot{v} + \lambda_0(y - v) = 0 & \dot{v} + x - \lambda_0(v - y) \le -1, \; \dot{v} + x - \lambda_0(v - y) \ge 1 \\ \dot{y} - \dot{v} + (\lambda_1 + |x| + |\dot{v}|)\,\mathrm{sgn}(y - v) = 0 & \text{otherwise} \end{array}\right\} \qquad (6)$$

confirm that the reference trajectory is attractive as required. Hence, it is clearly demonstrated by the above example that the lack of a well-defined relative degree does not, in itself, prevent the exact input-output linearisation of a nonlinear system.

Remark 2.1 <u>*Discontinuity of exact linearising input*</u>. It should be noted that the exact linearising input in this example is generally *discontinuous* for trajectories in an operating region where the relative degree is ill-defined. This observation provides considerable insight into the restriction of conventional input-output linearisation theory to systems with well-defined relative degree. It appears to be associated with requiring the linearising inputs to be continuous and can be relaxed by extending consideration to include more general types of input.

Remark 2.2 <u>*Discontinuity implies generalised functions*</u>. By augmenting the above system with integrators at the input, it follows immediately that inverting inputs for systems with ill-defined relative degree may involve impulses and higher derivatives of steps. Of course, such inputs are not physically realisable. However, similarly to the situation with linear systems, this issue can be addressed provided the inverting input can be formulated as the limit of an appropriate sequence of realisable inputs such that arbitrarily accurate linearisation may be achieved using a realisable input. Note that this situation is quite different from the approximate linearisation approaches previously considered in the literature where the degree of accuracy achieved is generally difficult to prescribe.

3. Exactly linearisable systems with ill-defined relative degree

Consider the SISO nonlinear system

$$\Sigma: \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \quad y = h(\mathbf{x}) \qquad (7)$$

with solution $\mathbf{x}$ defined on time interval [0,T] (for clarity, the analysis which follows assumes a finite time interval but it may be readily extended to the infinite interval subject to the additional condition that the internal "zero" dynamics are stable). The relative degree at an operating point at which $\mathbf{x}$ equals $\mathbf{x_1}$ is $r(\mathbf{x_1})$. Let $\Phi \subseteq \Re^n$ denote the operating region of interest and let $\underline{r} = \inf_{\mathbf{x}\in\Phi} r(\mathbf{x})$ and $\bar{r} = \sup_{\mathbf{x}\in\Phi} r(\mathbf{x})$. Clearly, $0 < \underline{r} \leq \bar{r}$ and it is assumed that $\bar{r}$ is finite (which simply corresponds to a requirement that the output of the nonlinear system is indeed coupled to the input at every operating point in Φ, Isidori 1995). The relative degree is well-defined in Φ when $r(\mathbf{x})$ is the same at every operating point; that is, $\underline{r} = \bar{r}$. Conventional input-output linearisation theory is strictly confined to operating regions within which the relative degree is well defined and involves differentiating the output, y, the minimum number of times necessary for the input to appear directly in the output equation (namely, differentiating the output $\underline{r} = \bar{r}$ times). In the present context, however, the requirement for the relative degree to be well-defined is relaxed; that is, consideration is extended to accommodate operating regions within which $\underline{r}$ need not be equal $\bar{r}$. Since the relative degree may vary but nevertheless has an upper bound, $\bar{r}$, consider the $\bar{r}^{\,th}$ derivative of the output, y.

It follows by differentiating (7) that $y^{(\bar{r})}$ is of the form (explicit expressions are given by Lamnabhi-Lagarrigue & Crouch 1988)

$$y^{(\bar{r})} = \mathbf{L}_{\mathbf{f}}^{\bar{r}} h + \sum_{j=0}^{\bar{r}-\underline{r}} c_j(\mathbf{x}) u^{(j)} + F(\mathbf{x}, u, \cdots, u^{(\bar{r}-\underline{r}-1)}) \qquad (8)$$

where $c_j(\bullet)$, $F(\bullet,\bullet, \ldots,\bullet)$ are nonlinear functions; for example,

$$y^{(\underline{r}+1)} = \mathbf{L}_{\mathbf{f}}^{\underline{r}+1} h + \left(\mathbf{L}_{\mathbf{g}} \mathbf{L}_{\mathbf{f}}^{\underline{r}} h + \mathbf{L}_{\mathbf{f}} \mathbf{L}_{\mathbf{g}} \mathbf{L}_{\mathbf{f}}^{\underline{r}} h\right) u + \mathbf{L}_{\mathbf{g}} \mathbf{L}_{\mathbf{f}}^{\underline{r}} h \,\dot{u} + \mathbf{L}_{\mathbf{g}}^2 \mathbf{L}_{\mathbf{f}}^{\underline{r}} h\, u^2 \qquad (9)$$

Whilst the expression for $y^{(\bar{r})}$ is, in general, rather complex, it is nevertheless noted that the highest derivative of u involved is $u^{(\bar{r}-\underline{r})}$ and, furthermore, the nonlinear function F is independent of $u^{(\bar{r}-\underline{r})}$. It follows immediately from (8) that the solutions (if any) to

$$c_{\bar{r}-\underline{r}}(\mathbf{x}) u^{(\bar{r}-\underline{r})} = v^{(\bar{r})} - (\mathbf{L}_{\mathbf{f}}^{\bar{r}} h + \sum_{j=0}^{\bar{r}-\underline{r}-1} c_j(\mathbf{x}_i) u^{(j)} + F(\mathbf{x}, u, \cdots, u^{(\bar{r}-\underline{r}-1)})) \qquad (10)$$

linearise the system (7) in the sense that

$$y^{(\bar{r})} = v^{(\bar{r})} \qquad (11)$$

The c_j, $j=0\ldots\bar{r}-\underline{r}$ are neither uniformly zero nor uniformly non-zero. Hence, (10) is a type of singularly perturbed differential equation for which the determination of solutions is generally not straightforward. In particular, it should be noted from the previous discussion that it may generally be necessary to consider non-classical types of weak solution involving discontinuities, impulses and so on. (While such solutions may be well defined, they are weak in the sense that they are not classically differentiable. Note that theoretical results by, for example, Hirschorn & Davis (1987) are restricted to situations where the linearising input is classically differentiable).

Since these solutions are not physically realisable, the requirement is to formulate them as the limit of an appropriate sequence of realisable inputs such that arbitrarily accurate inversion can be achieved using a realisable input. This issue is addressed by the following result.

Theorem Suppose that for a target trajectory, v, a solution (in the piecewise sense), u_i, exists satisfying

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \mathbf{g}(\mathbf{x}_i) u_i$$

$$\hat{y}_i^{(\bar{r})} = \mathbf{L}_{\mathbf{f}}^{\bar{r}} h + \sum_{j=0}^{\bar{r}-\underline{r}-1} c_j(\mathbf{x}_i) u_i^{(j)} + c_{\bar{r}-\underline{r}}^i(\mathbf{x}_i) u_i^{(\bar{r}-\underline{r})} + F(\mathbf{x}_i, u_i, \cdots, u_i^{(\bar{r}-\underline{r}-1)}) \qquad (12)$$

$$c_{\bar{r}-\underline{r}}^i(\mathbf{x}_i) u_i^{(\bar{r}-\underline{r})} = v^{(\bar{r})} - (\mathbf{L}_{\mathbf{f}}^{\bar{r}} h + \sum_{j=0}^{\bar{r}-\underline{r}-1} c_j(\mathbf{x}_i) u_i^{(j)} + F(\mathbf{x}_i, u_i, \cdots, u_i^{(\bar{r}-\underline{r}-1)})) \qquad (13)$$

with

$$c_{\bar{r}-\underline{r}}^i(\mathbf{x}) = \begin{cases} c_{\bar{r}-\underline{r}}(\mathbf{x}) & |c_{\bar{r}-\underline{r}}(\mathbf{x})| \geq \varepsilon_i \\ \varepsilon_i \operatorname{sgn} c_{\bar{r}-\underline{r}}(\mathbf{x}) & |c_{\bar{r}-\underline{r}}(\mathbf{x})| < \varepsilon_i \end{cases} \qquad (14)$$

where $c_{\bar{r}-\underline{r}}(\mathbf{x})$ is defined by (8) and ε_i is a positive finite constant from some sequence $\{\varepsilon_i : \varepsilon_i \to 0 \text{ as } i \to \infty\}$. Suppose, in addition, that there are a finite number of intervals $[t_j, t_{j+1}]$, j=1,2,... on which $c_{\bar{r}-\underline{r}}(\mathbf{x}_i) - c_{\bar{r}-\underline{r}}^i(\mathbf{x}_i)$ is non-zero and that

$$\sum_j \int_{t_j}^{t_{j+1}} \cdots \int_{t_j}^{\tau_2} |u_i^{(\bar{r}-\underline{r})}(\tau_1)| \, d\tau_1 \cdots d\tau_{\bar{r}} \qquad (15)$$

is bounded uniformly in i. Let y_i denote the solution to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) u_i$, $y_i = h(\mathbf{x})$ with initial conditions $y_i^{(k)}(t_o) = v^{(k)}(t_o)$, $k=0,1,\ldots\bar{r}$. Under the foregoing conditions, the tracking error $|y_i - v| \to 0$ as $\varepsilon_i \to 0$. That is, an arbitrarily accurate linearising input can be found.

Proof The relative degree of the system (12) is well-defined and equal to $\bar{r}$. Consequently, an input-output linearising input for this system is defined by (13). A sequence of systems (12) and associated sequence of inputs $\{u_i\}$ are defined as $\varepsilon_i \to 0$. It follows from (8), (12) and (13) that

$$y_i^{(\bar{r})} - v^{(\bar{r})} = \left(c_{\bar{r}-\underline{r}}(\mathbf{x}) - c_{\bar{r}-\underline{r}}^i(\mathbf{x})\right) u_i^{(\bar{r}-\underline{r})} \qquad (16)$$

On integrating (16),

$$\begin{aligned} |y_i(t) - v(t)| &= |\sum_{k=0}^{\bar{r}} \left(y_i^{(k)}(t_o) - v^{(k)}(t_o)\right) \frac{(t-t_o)^k}{k!} \\ &\quad + \int_{t_o}^{t} \cdots \int_{t_o}^{\tau_2} \left(c_{\bar{r}-\underline{r}}(\mathbf{x}) - c_{\bar{r}-\underline{r}}^i(\mathbf{x})\right) u_i^{(\bar{r}-\underline{r})}(\tau_1) d\tau_1 \cdots d\tau_{\bar{r}}| \\ &\leq \sum_{k=0}^{\bar{r}} \left|y_i^{(k)}(t_o) - v^{(k)}(t_o)\right| \frac{|t-t_o|^k}{k!} \\ &\quad + \varepsilon_i \sum_j \int_{t_j}^{t_{j+1}} \cdots \int_{t_j}^{\tau_2} |u_i^{(\bar{r}-\underline{r})}(\tau_1)| d\tau_1 \cdots d\tau_{\bar{r}} \end{aligned} \qquad (17)$$

The result now follows immediately.

Remark 3.1 Following the conventional input-output linearisation approach, the requirement that the initial conditions, $y_i^{(k)}(t_o) - v^{(k)}(t_o)$, $k=0,1,\ldots\bar{r}$, are zero can be relaxed provided v is selected to ensure asymptotic tracking of the reference trajectory.

Remark 3.2 The condition, (15), is a generalised boundedness requirement. It should is emphasised that there is *no* requirement for the u_i themselves to be uniformly bounded. For example, exact linearising inputs may involve impulses, in accordance with Remark 2.2.

Remark 3.3 Following common practice in the switched system literature, consideration is, for simplicity, confined to a piece-wise solution to (13). This restricts attention to initial conditions for the state and trajectories, v, such that there are a finite number of intervals $[t_j, t_{j+1}]$, j=1,2,... on which $c_{\bar{r}-\underline{r}}(\mathbf{x}) - c_{\bar{r}-\underline{r}}^i(\mathbf{x})$ is non-zero. However, this may be relaxed by employing a more general type of solution definition; for example, Filippov (1964).

Remark 3.4 When the relative degree is well-defined, then this theorem reduces to classical input-output linearisation.

Many of the conditions in the foregoing theorem are, unfortunately, difficult to test explicitly at present. Despite the considerable body of literature relating to discontinuous (especially switched) systems, non-conservative existence conditions are lacking. With regard to the present context, uniform satisfaction of the boundedness condition, (15), is at present also difficult to test in general. Nevertheless, the foregoing results are *constructive* in nature and may therefore be investigated numerically by, for example, simulation. With regard to the example in section 2, it is straightforward to confirm by simulation that the sequence of inputs defined by (12)-(13) converges numerically to the exact inverting input, (3) and this example is not pursued further. In the remainder of this paper, insight into the nature of these conditions is sought via some illuminating examples.

4. Ball and beam re-visited: Exact tracking

The ball and beam an interesting nonlinear system which is widely used as a benchmark example. The analysis of this system is, consequently, of considerable interest in its own right and has been the subject of numerous papers. In particular, the ball and beam is a SISO system with ill-defined relative degree and previously studied by Hauser *et al*. (1992) and, more recently, Tomlin & Sastry 1997. Consider the ball and beam system illustrated in figure 1. The beam is made to rotate by applying a torque to the beam at its centre of rotation and the ball is free to roll along the beam with dynamics described (Hauser *et al*. 1992) by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ B\left(x_1 x_4^2 - G\sin x_3\right) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ u \end{bmatrix}, \quad y = x_1 \qquad (18)$$

where $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} r & \dot{r} & \theta & \dot{\theta} \end{bmatrix}^T$, r is the position of the ball, θ the angle of the beam, u is related to the torque by a non-singular transformation and the values of the parameters B and G are, respectively, 0.7143 and 9.81. It is assumed that the ball remains in contact with the beam at all angles; that is, in practical terms there exists guides ensuring contact (Hauser *et al* 1992 require a restriction on angle and acceleration to ensure contact). It should be noted that it is not the physical behaviour of an actual ball and beam, which anyway the equation (18) only approximates, that is of interest here but the properties of the mathematical model, (18), itself.

On differentiating (18)

$$\begin{aligned} \dot{y} &= x_2 \\ \ddot{y} &= B\left(x_1 x_4^2 - G\sin x_3\right) \qquad (19) \\ y^{(3)} &= Bx_2 x_4^2 + 2Bx_1 x_4 u - BGx_4\cos x_3 \end{aligned}$$

Evidently, in a neighbourhood of each operating point at which $x_1 \neq 0$ and $x_4 \neq 0$, the coefficient of u in the $y^{(3)}$ equation is non-zero. Hence, at these operating points the ball and beam system has a well-defined relative degree of three and it follows immediately from conventional input-output linearisation theory (see, for example, Isidori 1995) that choosing

$$u = \frac{v^{(3)} - \left(Bx_2 x_4^2 - BGx_4\cos x_3\right)}{2Bx_1 x_4} \qquad (20)$$

linearises the system in the sense that

$$y^{(3)} = v^{(3)}$$

However, at operating points at which the ball position, x_1, and/or beam angular velocity, x_4, are zero, the relative degree is not well-defined since it is not constant within any open neighbourhood of such an operating point. Consequently, conventional input-output linearisation methods are not applicable at these operating points. Moreover, since the ball and beam system is not involutive, exact feedback linearisation methods cannot be applied (Hauser *et al*. 1992).

In order to address the aforementioned difficulty, Hauser *et al*. (1992) propose dropping the term involving u in the expression for $y^{(3)}$ to obtain the approximate output equation

$$\hat{y}^{(3)} = Bx_2 x_4^2 - BGx_4\cos x_3 \qquad (21)$$

On differentiating (21),

$$\hat{y}^{(4)} = B^2 x_1 x_4^4 + BG(1-B)x_4^2\sin x_3 + \left(2Bx_2 x_4 - BG\cos x_3\right)u \qquad (22)$$

Assuming that the coefficient of u in (22) is non-zero in the relevant operating envelope, it can be seen that this approximate system has well-defined relative degree of four. Hence, it follows from conventional input-output linearisation theory that

$$u = \frac{v^{(4)} - \left(B^2 x_1 x_4^4 + BG(1-B)x_4^2\sin x_3\right)}{2Bx_2 x_4 - BG\cos x_3} \qquad (23)$$

linearises the approximate system in the sense that

$$\hat{y}^{(4)} = v^{(4)} \qquad (24)$$

Hauser *et al*. (1992) apply the input defined by (23) to the original ball and beam system and present simulation results demonstrating that approximate tracking of a required trajectory is achieved. In a similar vein, Tomlin & Sastry (1997), propose a switched controller which utilises an approximately linearising choice of input in a neighbourhood about the operating points at which x_1 and/or x_4 are zero, and elsewhere switches to the exact linearising input defined by (20).

Whilst Hauser *et al.* (1992) and Tomlin & Sastry (1997) both consider specific choices of input which *approximately* input-output linearise the ball and beam system, neither address the fundamental issue of whether exact input-output linearisation is, in fact, possible nor do they consider systematic methods for prescribing the accuracy of an approximate linearisation.

4.1 Existence of exact tracking inputs

Initially, following Hauser *et al.* (1992), consider the task of achieving exact output tracking of the trajectory, y_{ref}= 3cos(πt/5). Since the required trajectory passes through zero, exact tracking of this trajectory necessarily requires consideration of an operating region within which the relative degree is ill-defined. Owing to the relative simplicity of the ball and beam example, it is possible to derive explicit information concerning the form of input which ensures exact tracking of the trajectory. It is evident from (18) that the beam angle, θ (*i.e.* state x_3), is related to the input, u, simply by a double integrator. Consider, therefore employing θ as a virtual input to the system. This change of input focuses attention on the primary dynamics of the ball and beam. Assume, for the moment, that the output, y, equals y_{ref}. It follows from the ball and beam equations, (18), that along this output trajectory the virtual input, θ, must be a solution, if one exists, of

$$By\dot{\theta}^2 = -\left(\frac{\pi}{5}\right)^2 y + BG\sin\theta \qquad (25)$$

with

$$\dot{y}^2 = \left(\frac{\pi}{5}\right)^2 \left(9 - y^2\right) \qquad (26)$$

The issue of the existence of a solution to the exact tracking problem is thus reformulated as the existence of a solution, θ, to (25) and (26) for all values of output, y, corresponding to the reference trajectory, y_{ref}. The equations (25) and (26) specify a second-order nonlinear differential equation and so can be investigated using standard phase-plane techniques. A detailed phase plane analysis is provided in the Appendix. Evidently, the analysis establishes that exact tracking inputs do exist for the ball and beam system. Moreover, the analysis highlights a number of more general issues relating to the conditions in the theorem proved in section 3.

Remark 4.1 Non-uniqueness of linearising inputs. Not only do exact tracking inputs exist for the ball and beam system, there exists *infinitely many* such inputs some of which are *continuous* and others which are *discontinuous*. The input derived by Hauser *et al.* (1992) approximates that based on the continuous solution marked 'A' in figure 2b.

Remark 4.2 Non-uniqueness implies possible non-smooth convergence. Although the analysis in section 3 establishes conditions under which the output sequence $\{y_i\}$ converges smoothly (to v), the corresponding input sequence $\{u_i\}$ need *not* converge in the same manner. Rather, since there may exist many exact tracking inputs, the u_i may switch from approximating one particular input to another (and indeed this is observed in numerical simulations).

Remark 4.3 Characterisation of exact linearising inputs. Non-smooth convergence makes it difficult at present to establish a general characterisation of the limiting set of (exact tracking/linearising) inputs and this remains an open problem. Characterisation of the limiting set is, nevertheless, possible for certain classes of system. For example, let Σ denote any set of systems for which the linearising inputs may contain step discontinuities (e.g. Example 1 above and/or the Hauser ball and beam) and consider the class of systems formed by augmenting the memebre of Σ with integrators at the input. It follows immediatedly that the linearising inputs for this class are just the distributions (including steps, impulses etc.). Note that the generalised boundedness requirement in theorem of section 3 does not preclude such unbounded inputs. Such inputs are not physically realisable but this issue can be addressed provided the inverting input is formulated, as in the present analysis, as the limit of an appropriate sequence of realisable inputs such that arbitrarily accurate linearisation may be achieved using a realisable input.

5. Ball & beam continued: fragility of tracking

Phase plane analysis establishes that the lack of well-defined relative degree is not an obstacle to exact tracking of the ball and beam. However, these results apply to an idealised situation where the initial conditions of the system ensure that the response to an appropriate input exactly tracks the trajectory. Even when considering a computer simulation where modelling errors and measurement noise can be eliminated, the system may be numerically perturbed away from the required trajectory and, therefore, robust tracking which can accommodate inexact initial conditions and perturbations is required. It follows from (18) that when there exists a solution, θ, to

$$By\dot{\theta}^2 = \ddot{v} + BG\sin\theta \qquad (27)$$

then the ball and beam system is linearised in the sense that

$$\ddot{y} = \ddot{v} \qquad (28)$$

Owing to the square term on the left-hand side of (27), it is clear that there can only exist solutions when the sign of $\ddot{v} + BG\sin\theta$ is the same as the sign of y. It follows from inspection of the phase portraits in figure 2 that $\ddot{v} + BG\sin\theta$ and $\dot{\theta}$ are both zero at the extreme points of the required trajectory. At these times, the system is at a boundary point and the exact tracking solution is extremely sensitive to any perturbation in $\ddot{v}$. Indeed, at such boundary points an infinitesimally small change in $\ddot{v}$ can result in the sign of the right hand side of (27) being opposite to that of y in which case a solution ceases to exist. (Solution here refers to an input which achieves exact tracking of the reference trajectory. In practical terms, non-existence of such a solution implies that only inaccurate tracking is achieved). It should be noted that this situation is exacerbated by the existence of inputs which, whilst ensuring tracking of the required trajectory, cease to exist before the output magnitude reaches 3; for example, the solution marked 'B' in figure 2b. Hence, although there exists an input which ensures exact tracking of the reference trajectory under ideal conditions, the tracking cannot be expected to be non-fragile.

5.1 Hauser ball & beam

Adopting the approach in section 3, in simulations (with $v^{(4)}=d^4/dt^4(3\cos(\pi t/5))$) the tracking error, even between $y^{(4)}$ and $v^{(4)}$, is found to rapidly increase after a short initial period of accurate tracking. It is necessary to augment the system, with an outer feedback loop by selecting $v^{(4)}$ according to

$$v^{(4)} = y_{ref}^{(4)} + \lambda_3(y_{ref}^{(3)} - y^{(3)}) + \lambda_2(\ddot{y}_{ref} - \ddot{y}) + \lambda_1(\dot{y}_{ref} - \dot{y}) + \lambda_o(y_{ref} - y) \tag{29}$$

The outer feedback loop is required to be strong with the coefficients selected so that the poles of the nominal error dynamics are placed at s=-10. Time histories of the output tracking error, $y-y_{ref}$, are shown in figure 5 for decreasing values of perturbation, ε. On closer inspection of figure 5, it can be seen that the magnitude of the tracking error generally decreases as ε decreases, as might be anticipated, except for the rather peculiar "disturbances" which can be seen at times near 4, 10, 14 and 19 seconds and which determine the peak tracking error observed (in particular, between $y^{(4)}$ and $v^{(4)}$). The rapid increase in tracking error previously observed in the absence of the restoring action of the outer feedback loop is associated with these "disturbances". It is straightforward to verify that the "disturbances" observed correspond to periods when $\ddot{v} + BG\sin\theta$ is of opposite sign to y and thus during which there exists no exact tracking solution for the Hauser ball and beam system. These periods occur when a solution boundary is reached and, in accordance with the previous discussion, the fragile nature of the exact tracking solution combined with the approximate nature of the numerical solution subsequently leads to the boundary being violated.

Remark 5.1 Generalised boundedness condition. This is an example where the generalised boundedness condition in the theorem of section three is violated and illustrates the non-trivial nature of this condition.

5.2. Bilinear ball & beam

The non-convergence of the sequence of approximate linearising inputs in the case of the Hauser ball and beam is associated with the singular nature of the system whereby there does not exist in the phase plane an open set of trajectories which includes $3\cos(\pi t/5)$ and for which each member of the set can be exactly tracked by this system. This is evidently a singular situation. To clarify the analysis, and confirm that the difficulty lies with the square term in the dynamic equations of the Hauser ball and, beam, consider a modified ball and beam system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ B\left(x_1x_4 - G\sin x_3\right) \\ x_4 \\ 0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\u\end{bmatrix}, \quad y = x_1 \tag{30}$$

This system is identical to the Hauser ball and beam except for the term x_1x_4 replacing $x_1x_4^2$. The issues associated with the singular nature of the square term on the left hand side of (27) are thereby avoided. Nevertheless, on differentiating (30) it can be seen that

$$\begin{aligned} \dot{y} &= x_2 \\ \ddot{y} &= B\left(x_1x_4 - G\sin x_3\right) \\ y^{(3)} &= Bx_2x_4 + Bx_1u - BGx_4\cos x_3 \end{aligned} \tag{31}$$

Hence, similarly to the original ball and beam system, the relative degree of the modified system is still ill-defined in operating regions where x_1 may be zero.

For the cosine trajectory as considered previously, phase plane analysis again establishes that (infinitely many) exact tracking inputs exist. The tracking is now non-fragile since the singularity associated with the square term in the original ball and beam is no longer present. Adopting the approach in section 3, the output tracking error between $y^{(4)}$ and $v^{(4}$ is plotted versus ε in figure 4. It can be seen that the tracking error (as measured by the integral of the magnitude of the point-wise error) decreases monotonically as ε decreases. As usual, to remove the influence of initial conditions and ensure that y asymptotically tracks v (in addition to $y^{(4)}$ tracking $v^{(4)}$), it is necessary to augment the system with an outer feedback loop. However, this feedback can be very weak since it is no longer required remediate breaking of the solution boundary as in 5.1.

6. Conclusion

Previous work on SISO systems with ill-defined relative degree has been confined to approximate linearisation methods, with little provision to prescribe the approximation error nor consideration of whether exact linearising inputs may in fact exist. The contribution of this note is four-fold, namely

1. It is established that for SISO systems lack of well-defined relative degree is *not* an obstacle to exact inversion.
2. In the case of general SISO control-affine nonlinear systems, sufficient conditions for the existence of an arbitrarily accurate linearising system are established. While the conditions involved are presently difficult to test analytically, the derivation is constructive in nature and the conditions may therefore be evaluated numerically.

3. Exact tracking solutions of a number of example systems, including the ball and beam system studied by Hauser and others, are analysed in detail. In view of the widespread use of, in particular, the ball and beam as a benchmark example, this analysis is of considerable interest in its own right. Valuable insight is gained into the foregoing linearisability conditions and the nature of linearising inputs for ill-defined relative degree systems generally including that inputs may be (i) highly non-unique, (ii) discontinuous and include impulses etc., (iii) fragile.
4. A framework is established for approximate linearisation whereby exact linearising inputs are formulated as the limits of sequences of realisable, arbitrarily accurate linearising inputs. This framework is constructive in nature and of key importance when the exact linearising inputs are, for example, unrealisable. Note that it is quite different from approximate linearisation approaches previously considered in the literature where the degree of accuracy is generally difficult to prescribe.

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Appendix

Before constructing the phase portrait of the ball and beam solution in detail, the following features can be deduced immediately.

1. The singular points, at which many trajectories intersect on the phase portrait for (25) and (26) are indicated in figure 2a. They are
$$\begin{aligned}&(\theta_o+2\pi n,0),(\pi+2\pi n,0),\\&(\theta_o+2\pi n,3),(\pi-\theta_o+2\pi n,3),\\&(\pi+\theta_o+2\pi n,-3),(2\pi-\theta_o+2\pi n,-3): n\in \mathrm{Z}\end{aligned} \qquad (32)$$
with $\theta_o = \sin^{-1}\left(\frac{3}{BG}\left(\frac{\pi}{5}\right)^2\right)$.
2. Owing to the square term on the left hand sides of (25) and (26), a solution exists only when the signs of the right hand sides are appropriate. It is known *a priori* that $3\cos(\pi t/5)$ is a solution to (26) with $-3\le y\le 3$ and thus the right hand side of (26) non-negative as required. However, it is evident that a solution to (25) can only exist in the region on the phase plane within which the sign of $-\left(\frac{\pi}{5}\right)^2 y + BG\sin\theta$ is identical to the sign of y. This constraint arises from the square term on the left hand side of (25). The region in the phase plane within which a solution to (25) and (26) cannot exist is indicated in grey in figure 2a. The boundary is the curve
$$-\left(\frac{\pi}{5}\right)^2 y + BG\sin\theta = 0,\ y\in\{0,\ 3,-3\} \qquad (33)$$
3. Any solution with $y=3\cos(\pi t/5)$ must traverse continuously from singular points with y=-3 to singular points with y=3 and vice versa via singular points with y=0.
4. Owing to the square terms in both (25) and (26), it is necessary to consider solutions corresponding to both the appropriate positive and the negative square roots. This leads to four possible combinations; that is, $\dot{y}$ and $\dot{\theta}$ both positive, $\dot{y}$ and $\dot{\theta}$ both negative, $\dot{y}$ positive and $\dot{\theta}$ negative, $\dot{y}$ negative and $\dot{\theta}$ positive.

The phase portrait of the solutions to (25) and (26) for $0\le\theta\le\pi$ with $\dot{y}$ and $\dot{\theta}$ both positive has the following features

(i) There is a single trajectory connecting the singular point (0,0) to the singular point $(\theta_o,3)$.
(ii) All the trajectories to the left of the trajectory described in (i) connect the singular point (0,0) to a point on (33) with $0<y<3$.
(iii) To the right of the trajectory described in (i), there are infinitely many trajectories connecting the singular point (0,0) to the singular point $(\pi-\theta_o,3)$.
(iv) All the trajectories below those described in (iii) connect the singular point (0,0) to a point on (33) with $0<y<3$.

The phase portrait for $-\pi\le\theta\le 0$ is obtained from the portrait for $0\le\theta\le\pi$ by the transformation $(\theta,y)\to(-\theta,-y)$ and reversing the arrows. Some trajectories are depicted in figure 2b. To obtain the solutions with $\dot{y}$ and $\dot{\theta}$ both negative the arrows in figure 2b are simply reversed.

The phase portrait of the solutions to (25) and (26) for $0 \le \theta \le \pi$ with $\dot{y}$ negative and $\dot{\theta}$ positive has the following features

(i) There is a single trajectory connecting the stationary point $(\pi-\theta_o,3)$ to the singular point $(\pi,0)$.

(ii) All the trajectories to the right of the trajectory described in (i) connect the singular point $(\pi,0)$ to a point on (33) with $0<y<3$.

(iii) To the left of the trajectory described in (i), there are infinitely many trajectories connecting the singular point $(\theta_o,3)$ to the singular point $(\pi,0)$.

(iv) All the trajectories below those described in (iii) connect a point on (33) with $0<y<3$ to the singular point $(\pi,0)$.

The phase portrait for $-\pi \le \theta \le 0$ is obtained from the portrait for $0 \le \theta \le \pi$ by the transformation $(\theta,y) \rightarrow (-\theta,-y)$ and reversing the arrows. Some trajectories are depicted in figure 2c. To obtain the solutions with $\dot{y}$ positive and $\dot{\theta}$ negative the arrows in figure 2c are simply reversed. While the phase portrait in figures 2b and 2c is only shown for the region $-\pi \le \theta \le \pi$, owing to the sinusoidal term in (25), it is periodic with period 2π (see figure 2a).

It can be seen immediately, by inspection of the phase portrait, that there exist an infinite number of inputs, θ, for which the output, y, exactly tracks any quarter period of the trajectory $3\cos(\pi t/5)$. The phase portrait only depicts solution fragments and these must, of course, be pieced together in order to obtain a solution over a full period of the reference trajectory owing to the singular points in the phase portrait. To ensure that the actual input, u, is continuous, $\ddot{\theta}$ must be continuous at all points including those for which y=0 and y=±3. Since,

$$\ddot{\theta} = \frac{d^2\theta}{dy^2}\dot{y}^2 + \frac{d\theta}{dy}\ddot{y} \qquad (34)$$

continuity at y=±3 requires continuity of $d\theta/dy$. In addition, continuity at y=0 requires continuity of $d^2\theta/dy^2$. The input derived by Hauser *et al.* (1992) and plotted in figure 3 can be seen to approximate that based on the continuous solution marked ‘A’ in figure 2b.

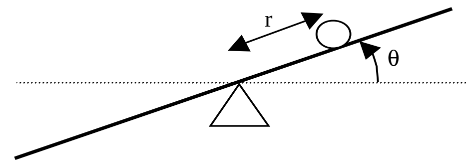
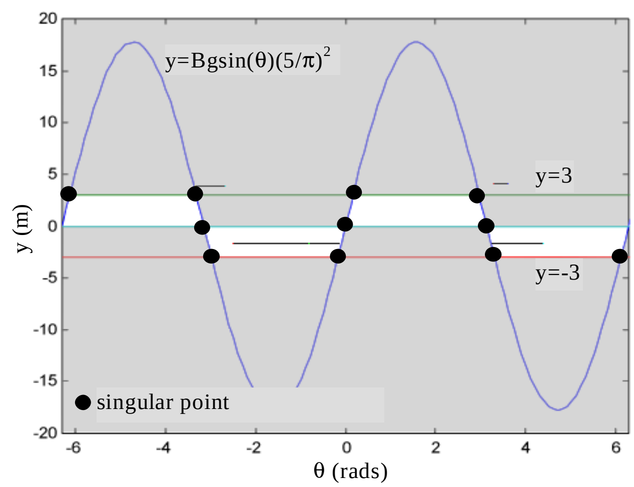
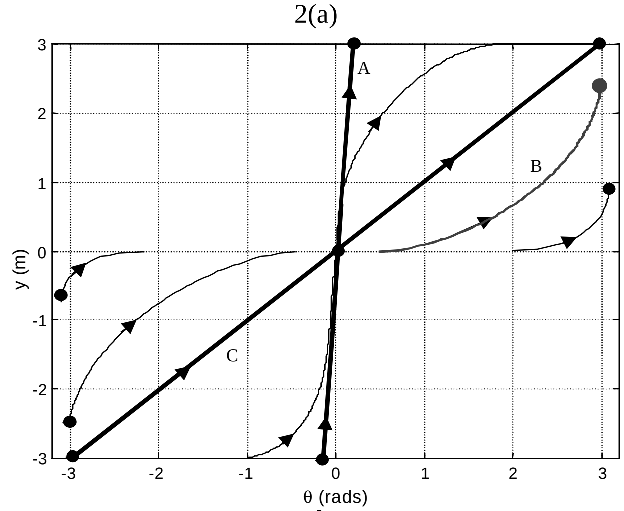


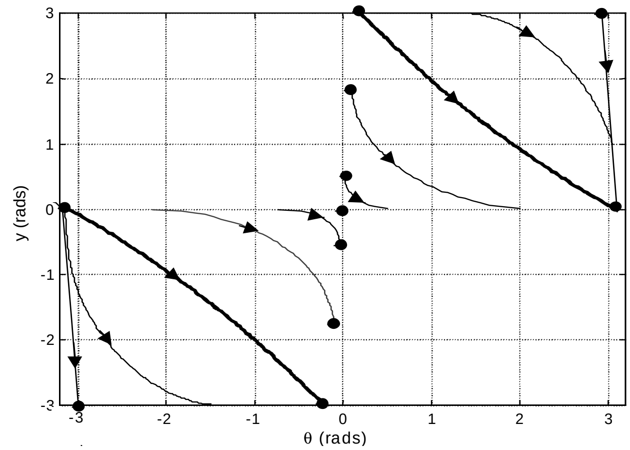
Figure 1 Schematic of ball and beam system



2(a)



2(b)



2(c)

Figure 2 Phase portraits of ball and beam along output trajectory 3cos(πt/5): (a) overview with regions within which no solution exists indicated in grey, (b) detailed trajectories with $\dot{y}, \dot{\theta}$ positive, (c) $\dot{y}$ negative, $\dot{\theta}$ positive. For increased clarity, th ebeginning/end of the trajectories are marked by • in (b) and (c).

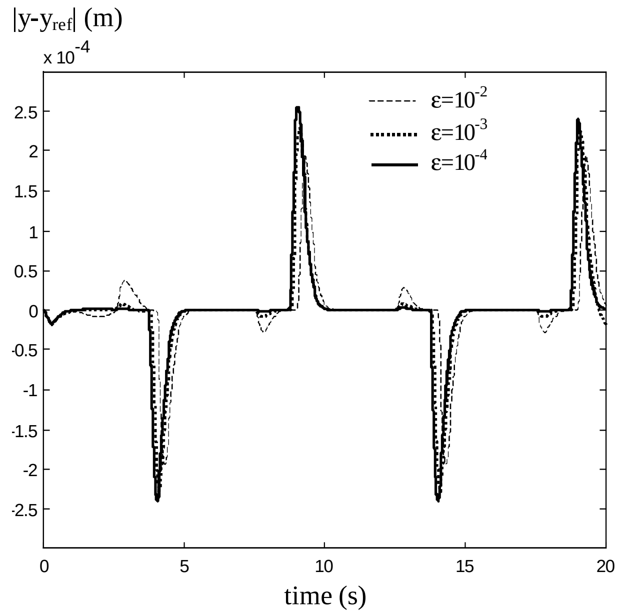


Figure 5 Tracking error in Hauser ball and beam example for decreasing perturbation, ε.

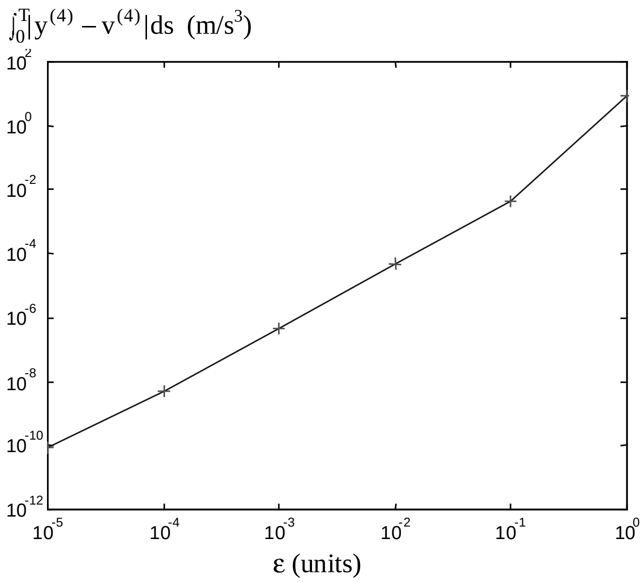


Figure 4 Tracking error vs. ε for modified (bilinear) ball & beam.