Problem Set 2

ECE 534 | Fall 2009
R. Srikant | Due: Sept. 22

1. Consider the following congestion control algorithm:

$$\dot{x}_r = \kappa_r \left(\frac{w_r}{x_r} - q_r \right),$$
$$\dot{p}_l = h_l \left(y_l - c_l \right)^+_{p_l},$$

where $q_r = \sum_{l:l\in r} p_l$, $y_l = \sum_{r:l\in r} x_r$, and κ_r and h_l are positive constants. This algorithm is called the primal-dual algorithm for congestion control.

(a) Show that the equilibrium points of the above congestion control equation solve a utility maximization problem. What type of fairness property (such as max-min fairness, proportional fairness, etc.) does the equilibrium possess?

(b) Assume that the equilibrium point is unique and show that the congestion controller is globally asymptotically stable by using the Lyapunov function

$$V(x,p) = \sum_r \frac{(x_r - \hat{x}_r)^2}{\kappa_r} + \frac{\sum_l (p_l - \hat{p}_l)^2}{h_l},$$

where $(\hat{x}, \hat{p})$ denotes the equilibrium point. To do this, show that (i) $\dot{V} \leq 0$ and (ii) that $\dot{V} = 0$ implies $(x(t), p(t)) = (\hat{x}, \hat{p})$. The result then follows from LaSalle's invariance principle (see "Intro and Background" notes).

2. Consider the following discrete-time version of the dual congestion control algorithm: at each time slot k, each source chooses a transmission rate $x_r(k)$ which is the solution to

$$\max_{0\leq x_r \leq X_{max}} U_r(x_r) - q_r(k) x_r,$$

where X_{max} is the maximum rate at which any user can transmit. Each link l computes its price $p_l(k)$ according to the following update rule which is a discretization of the continuous-time algorithm used in the notes:

$$p_l(k+1) = \left(p_l(k) + \epsilon (y_l - c_l) \right)^+,$$

where $\epsilon > 0$ is a small step-size parameter. The variables y_l and q_r are defined as usual:

$$q_r(k) = \sum_{l:l\in r} p_l(k), \qquad y_l(k) = \sum_{r:l\in r} x_r(k).$$

We will show that, on average, the above discrete-time algorithm is nearly optimal in the sense that it approximately solves the utility maximization problem.

(a) Consider the Lyapunov function

$$V(k) = \frac{1}{2} \sum_l p_l^2(k).$$

Show that

$$V(k+1) - V(k) \leq K\epsilon^2 + \epsilon \sum_r q_r (x_r - x_r^*),$$

for some constant $K > 0$.

(b) Next, show that

$$V(k+1) - V(k) \leq K\epsilon^2 + \epsilon \sum_r (U_r(x_r) - U_r(\hat{x}_r)),$$

where $\hat{x}$ is the solution to the utility maximization problem

$$\max_{x \geq 0} \sum_r U_r(x_r), \qquad \text{subject to} \qquad \sum_{r:l \in r} x_r \leq c_l.$$

Assume that $X_{max} > \max_r \hat{x}_r$.

(c) Finally, show that

$$\sum_r U_r(\hat{x}_r) \leq \sum_r U_r(\bar{x}_r) + K\epsilon,$$

where

$$\bar{x}_r := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} x_r(k).$$

Assume that U_r is concave (but it doesn't have to be strictly concave for the results of this problem to hold).

3. Consider a Markov chain whose state space is the set of non-negative integers and whose transition matrix satisfies $P_{ij} = 0$ if $|i - j| > 1$. Such Markov chains are called *birth-death* chains. Consider a birth-death chain with $P_{i,i+1} = \lambda_i$ and $P_{i+1,i} = \mu_i$, and assume that $\lambda_i, \mu_i > 0$ for all i. Further assume that $P_{ii} > 0$ for all i.

(i) Show that the Markov chain is irreducible and aperiodic.

(ii) Obtain conditions under which the Markov chain is (a) positive recurrent and (b) not positive recurrent.

4. Consider the simple infinite-buffer queueing model of a wireless channel discussed in class with $\mu \in (0, 1)$. Instead the arrival rate being constant, assume that the number of arrivals per time slot is Bernoulli with mean μ when the queue length is less than or equal to B and is equal to λ otherwise. Compute the stationary distribution and the expected queue length in steady-state of this Markov chain when it is positive recurrent. Clearly identify the conditions under which the Markov chain is positive recurrent and the conditions under which it is not.

5. Consider the discrete-time queueing model discussed in class to obtain the Kingman bound:

$$q(k+1) = (q(k) + a(k) - s(k))^+ .$$

Another way to represent the above queueing dynamics is to define a non-negative random variable $u(k)$, which denotes unused service in a time slot, and rewrite the above equation as

$$q(k+1) = q(k) + a(k) - s(k) + u(k).$$

Note that $u(k) \leq s(k)$. For the rest of this problem, we will assume that the initial probability distribution for this system is the steady-state distribution, i.e., we consider the system in steady-state.

(i) Using the fact that, in steady-state, $E(q(k+1) - q(k)) = 0$, show that $E(u(k)) = \mu - \lambda$.

(ii) We now show that the Kingman upper bound on $E(q(k))$ obtained in class is tight in heavy-traffic under the assumption that $s(k) \leq S_{max}$ for all k, where S_{max} is some positive constant. Obtain a lower bound on $E(q(k))$ and show that the upper and lower bounds on $E(q(k))(\mu - \lambda)$ coincide when $\lambda \to \mu$.

6. In discrete-time systems, one can make different assumptions on the order in which arrivals and departures can occur. In this problem, we will assume

$$q(k+1) = (q(k) - s(k))^+ + a(k).$$

Thus, unlike in problem 5, we assume that departures occur first, followed by arrivals. As in class, assume that the arrivals and departures are i.i.d. over time, the mean arrival rate (λ) is less than the mean service rate (μ), and $E\left[(a(k) - s(k))^2\right]$ is finite.

(a) Show that the Markov chain q is positive recurrent and hence, has a stationary distribution. (As in class, we implicitly assume that the arrival and service processes are such that q is irreducible and aperiodic.)

(b) Now assume that the Markov chain is in steady-state and obtain an upper bound on the expected queue length.

(c) Assume $s(k) \leq S_{max}$ and show that the upper bound is tight in the heavy-traffic sense described in problem 5.