ECE 567: Solutions of homework 2

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Problem 1

a) At the equilibrium:

$$\dot{x_r} = 0 \Rightarrow x_r = w_r/q_r$$

and

$$\dot{p_l} = 0$$

which yields

(i) $p_l = 0$ if $y_l \leq c_l$

(ii) $p_l > 0$ if $y_l = c_l$.

If we choose $U_r(x_r) = w_r \log x_r$, then, from the first condition, we get

$$U'(x_r) = q_r,$$

and the second condition is equivalent to

$$p_l(y_l - c_l) = 0, \text{ for all } l.$$

Therefore, the pair (x, p) satisfies the KKT conditions for the utility maximization problem with $U_r(x_r) = w_r \log x_r$, which is a proportional fairness resource allocation.

b)

i)

$$\begin{aligned}
\dot{V} &= \sum_r \frac{\partial V}{\partial x_r}\dot{x_r} + \sum_l \frac{\partial V}{\partial p_l}\dot{p_l} \\
&= \sum_r 2(x_r - \hat{x}_r)(\frac{w_r}{x_r} - q_r) + \sum_l 2(p_l - \hat{p}_l)(y_l - c_l)^+_{p_l} \\
&\leq \sum_r 2(x_r - \hat{x}_r)(\frac{w_r}{x_r} - q_r) + \sum_l 2(p_l - \hat{p}_l)(y_l - c_l)
\end{aligned}$$

But, if $\hat{p}_l > 0$, $\hat{y}_l = c_l$, and therefore $(p_l - \hat{p}_l)(y_l - c_l) = (p_l - \hat{p}_l)(y_l - \hat{y}_l)$, and if $\hat{p}_l = 0$, $\hat{y}_l \leq c_l$ and hence $(p_l - \hat{p}_l)(y_l - c_l) \leq (p_l - \hat{p}_l)(y_l - \hat{y}_l)$. So

$$\begin{aligned}
\dot{V} &\leq \sum_r 2(x_r - \hat{x}_r)(\frac{w_r}{x_r} - q_r) + \sum_l 2(p_l - \hat{p}_l)(y_l - \hat{y}_l) \\
&= \sum_r 2(x_r - \hat{x}_r)(\frac{w_r}{x_r} - q_r) + \sum_r 2(q_r - \hat{q}_r)(x_r - \hat{x}_r) \\
&= \sum_r 2(x_r - \hat{x}_r)(\frac{w_r}{x_r} - \hat{q}_r) \\
&= \sum_r 2(x_r - \hat{x}_r)(\frac{w_r}{x_r} - \frac{w_r}{\hat{x}_r}) \\
&= 2\sum_r w_r \frac{-(x_r - \hat{x}_r)^2}{x_r \hat{x}_r} < 0.
\end{aligned}$$

ii) Note that $\dot{V} < 0$ (strictly negative) for $x_r \neq \hat{x}_r$, and $\dot{V} = 0$ iff $x_r(t) = \hat{x}_r$ which results in $y_l(t) = \hat{y}_l$, and consequently $p_l(t) = \hat{p}_l$.

Problem 2

a)

$$\begin{aligned} V(k+1)-V(k) &= \frac{1}{2}\sum_l p_l^2(k+1)-p_l^2(k) \\ &= \frac{1}{2}\sum_l ((p_l(k)+\epsilon(y_l-c_l))^+)^2-p_l^2(k) \\ &\leq \frac{1}{2}\sum_l (p_l(k)+\epsilon(y_l-c_l))^2-p_l^2(k) \\ &= \frac{1}{2}\sum_l 2p_l(k)\epsilon(y_l-c_l)+\epsilon^2(y_l-c_l)^2 \\ &= \frac{\epsilon^2}{2}\sum_l (y_l-c_l)^2+\epsilon\sum_l p_l(k)(y_l-c_l) \end{aligned}$$

Noting that $x_r \leq X_{max}$, and $y^* \leq c_l$, for any feasible solution x_r^*, yields

$$\begin{aligned} V(k+1)-V(k) &\leq K\epsilon^2+\epsilon\sum_l p_l(k)(y_l-y_l^*) \\ &= K\epsilon^2+\epsilon\sum_r q_r(k)(x_r-x_r^*) \end{aligned}$$

where K is chosen to be a constant greater than $\sum_l (y_{lmax}-c_l)^2$, where $y_{lmax}=\sum_{r:l\in r} X_{max}$.

b) Choose $x^*=\hat{x}$. Since x_r is the maximizer of $U_r(x_r)-q_r(k)x_r$,

$$U_r(x_r)-q_r(k)x_r \geq U_r(\hat{x}_r)-q_r(k)\hat{x}_r,$$

or equivalently

$$q_r(k)(x_r-\hat{x}_r) \leq U_r(x_r)-U_r(\hat{x}_r).$$

Replacing the above expression in the result of part (a) yields

$$V(k+1)-V(k) \leq K\epsilon^2+\epsilon\sum_r U_r(x_r)-U_r(\hat{x}_r).$$

c) Summing the inequality of part (b) for $k=0,1,...,N$ yields

$$V(N)-V(0) \leq NK\epsilon^2+\epsilon\sum_{k=1}^{N}\sum_r U_r(x_r)-U_r(\hat{x}_r)$$

Dividing the both sides by N, we get

$$\frac{V(N)-V(0)}{N} \leq K\epsilon^2 + \epsilon\frac{1}{N}\sum_{k=1}^{N}\sum_{r} U_r(x_r) - U_r(\hat{x}_r)$$

But $V(0)$ is finite and $V(N) \geq 0$, so $\lim_{N\to\infty} \frac{V(N)-V(0)}{N} \geq 0$, and thus

$$0 \leq K\epsilon + \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{r} U_r(x_r) - U_r(\hat{x}_r)$$

Therefore

$$\begin{aligned}
\sum_r U_r(\hat{x}_r) &\leq K\epsilon + \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\sum_{r} U_r(x_r(k)) \\
&\leq K\epsilon + \sum_r U_r(\lim_{N\to\infty}\sum_{k=1}^{N} x_r(k)) \\
&= K\epsilon + \sum_r U_r(\bar{x_r})
\end{aligned}$$

where the last inequality follows from concavity of $U_r(.)$.

Problem 3

i

For any $i \neq j$, if $i > j$, the probability that the system goes from state i to state j in $|i-j|$ steps is $\mu_i\mu_{i-1}\ldots\mu_{j+1} > 0$; if $i < j$, the probability that the system goes from state i to state j in $|i-j|$ steps is $\lambda_i\lambda_{i+1}\ldots\lambda_{j-1} > 0$. Thus, the system is irreducible.

Since $P_{ii} > 0$ for all i, the period of state i is 1, and the system is aperiodic.

ii

Let $\nu_i = \frac{\lambda_i}{\mu_i}$. Consider the local balance condition:

$$\begin{aligned}
&\pi_i P_{i,i+1} = \pi_{i+1}P_{i+1,i} \\
\Rightarrow &\pi_i\lambda_i = \pi_{i+1}\mu_i \\
\Rightarrow &\pi_{i+1} = \nu_i\pi_i
\end{aligned}$$

Let $\pi_i = \prod_{k=0}^{i-1} \nu_k$, then $[\pi_i]$ satisfies the local balance condition. If $\sum_{i=0}^{\infty} \pi_i = \sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \nu_k = \infty$, then the Markov Chain is not positive recurrent. If $\sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \nu_k < \infty$, then the Markov Chain is positive recurrent. Both by guessing theorem.

Problem 4

Consider the local balance condition: $\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$. In this setting, $P_{i,i+1}$ is $\mu(1-\mu)$ if $i \leq B$, and is $\lambda(1-\mu)$ if $i > B$. $P_{i+1,i}$ is $\mu(1-\mu)$ if $i+1 \leq B$, and is $\mu(1-\lambda)$ if $i+1 > B$.

Let $\nu_i = \frac{P_{i,i+1}}{P_{i+1,i}}$. The value of ν_i is 1 if $i < B$, $\frac{1-\mu}{1-\lambda}$ if $i = B$, and $\frac{\lambda(1-\mu)}{\mu(1-\lambda)}$ if $i > B$. Similar to Problem 3, the Markov Chain is positive recurrent if and only if $\sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \nu_k < \infty$. Since $\nu_i = \frac{\lambda(1-\mu)}{\mu(1-\lambda)}$ for $i > B$, we have:

$$
\begin{aligned}
&\sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \nu_k < \infty \\
\Leftrightarrow &\sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \frac{\lambda(1-\mu)}{\mu(1-\lambda)} < \infty \\
\Leftrightarrow &\sum_{i=0}^{\infty} [\frac{\lambda(1-\mu)}{\mu(1-\lambda)}]^i < \infty \\
\Leftrightarrow &\frac{\lambda(1-\mu)}{\mu(1-\lambda)} < 1 \\
\Leftrightarrow &\lambda < \mu
\end{aligned}
$$

So, the system is positive recurrent if and only if $\lambda < \mu$.

Suppose $\lambda < \mu$, that is, the system is positive recurrent. The value of $\prod_{k=0}^{i-1} \nu_k$ is 1 if $i \leq B$ and $\frac{1-\mu}{1-\lambda}[\frac{\lambda(1-\mu)}{\mu(1-\lambda)}]^{i-B-1}$ if $i > B$. Then we have $\sum_{i=0}^{\infty} \prod_{k=0}^{i-1} \nu_k = B + \frac{\mu(1-\mu)}{\mu-\lambda}$, call this value M. Let $\pi_i = \frac{1}{M}$ if $i \leq B$ and $\frac{1-\mu}{1-\lambda}[\frac{\lambda(1-\mu)}{\mu(1-\lambda)}]^{i-B-1}/M$, if $i > B$. Then $[\pi_i]$ is stationary distribution.

The expected value of queue length is $\sum i\pi_i = \frac{1}{M}(\frac{B(B+1)}{2} + B\frac{\mu(1-\mu)}{\mu-\lambda} + \frac{\mu^2(1-\mu)}{(\mu-\lambda)^2})$.

Problem 5

(i) In steady state $\mathbf{E}(q(k+1) - q(k)) = 0$
$\mathbf{E}(q(k) + a(k) - s(k) + u(k) - q(k)) = 0$
$\mathbf{E}(u(k)) = \mathbf{E}(s(k)) - \mathbf{E}(a(k)) = \mu - \lambda$

(ii) In steady state $\mathbf{E}(V(q(k+1)) - V(q(k)) = 0$ where $V(q(k)) = \frac{q(k)^2}{2}$
$0 = \mathbf{E}(\frac{1}{2}(q+a-s+u)^2 - \frac{1}{2}q^2) = \frac{1}{2}\mathbf{E}((a-s)^2 + u^2 + 2u(q+a-s) + 2q(a-s))$
Now note that $\mathbf{E}(u(q+a-s)) = \mathbf{E}(-u^2) = -\mathbf{E}(u^2)$ because when $q+a-s \geq 0,\ u = 0$ and $u(q+a-s) = -u^2 = 0$ and when $q+a-s < 0,\ u = -(q+a-s)$ and $u(q+a-s) = -u^2$.
So back to the problem, $0 = \frac{1}{2}(m_2 + 2\mathbf{E}(q)(\lambda - \mu) - \mathbf{E}(u^2))$ where $m_2 = \mathbf{E}((a-s)^2)$

$$\boxed{\mathbf{E}(q)(\mu - \lambda) \geq \frac{1}{2}(m_2 - \mathbf{E}(u^2))}$$

Now in heavy traffic, $\lambda \to \mu$. Given that $s(k) \leq S_{max}$ and $u(k) \leq s(k) \leq S_{max}$ then $\mathbf{E}(u^2) \leq \mathbf{E}(u)S_{max}$. As $\lambda \to \mu$, $\mathbf{E}(u) = \mu - \lambda \to 0$ and $\mathbf{E}(u^2) \to 0$. So in heavy traffic $\mathbf{E}(q)(\mu - \lambda) \to \frac{1}{2}m_2$.

Problem 6

(a) $V(q(k)) = \frac{q(k)^2}{2},\ m_2 = \mathbf{E}(a(k) - s(k))^2 < \infty$.
Express $q(k+1)$ in the following way $q(k+1) = q(k) - s(k) + u(k) + a(k)$ and $u(k) \leq s(k) \leq S_{max}$. Note that, $\mathbf{E}(u(k)(q(k) - s(k))) = \mathbf{E}(-u(k)^2) = -\mathbf{E}(u(k)^2)$ because when $q(k) - s(k) \geq 0,\ u(k) = 0$ and $u(k)(q(k) - s(k)) = -u(k)^2 = 0$ and when $q(k) - s(k) < 0,\ u(k) = -(q(k) - s(k))$ and $u(k)(q(k) - s(k)) = -u(k)^2$.

$$
\begin{aligned}
\mathbf{E}(V(q_{k+1}) - V(q_k)|q_k = q) &= \frac{1}{2}\mathbf{E}(((q - s(k))^+ + a(k))^2 - q^2) \\
&= \frac{1}{2}\mathbf{E}((q - s(k) + u(k) + a(k))^2 - q^2) \\
&= \frac{1}{2}\mathbf{E}((q - s(k) + a(k))^2 + u(k)^2 + 2u(k)(q - s(k)) \\
&\quad +2a(k)u(k) - q^2) \\
&= \frac{1}{2}\mathbf{E}((q - s(k) + a(k))^2 - u(k)^2 + 2a(k)u(k) - q^2) \\
&\leq \frac{1}{2}\mathbf{E}((a(k) - s(k))^2 + 2q(a(k) - s(k)) + 2a(k)u(k)) \\
&\leq \frac{1}{2}m_2 + \lambda\mu - q(\mu - \lambda)
\end{aligned}
$$

So $\mathbf{E}(V(q(k+1)) - V(q(k))|q(k) = q) \leq \frac{1}{2}m_2 + \lambda\mu - q(\mu - \lambda) < -\varepsilon$ for a sufficiently large q and for $\lambda < \mu$. Therefore, by the Foster-Lyapunov theorem the Markov chain q is positive recurrent and consequently has a stationary distribution (irreducibility and aperiodicity is assumed).

(b) In steady state $\mathbf{E}(V(q(k+1)) - V(q(k)) = 0$
$0 \leq \frac{1}{2}\mathbf{E}((a - s)^2 + 2q(a - s) + 2au)$ (part (a))
Now in steady state $\mathbf{E}(q(k+1) - q(k)) = 0$
$\mathbf{E}(q(k) + a(k) - s(k) + u(k) - q(k)) = 0$
$\mathbf{E}(u(k)) = \mathbf{E}(s(k)) - \mathbf{E}(a(k)) = \mu - \lambda$
So $\mathbf{E}(au) = \lambda(\mu - \lambda)$ (a and u are independent)
From part(a):
$\mathbf{E}(q)(\mu - \lambda) \leq \frac{1}{2}(m_2) + \mathbf{E}(au) = \frac{1}{2}(m_2) + \lambda(\mu - \lambda)$

$$\boxed{\mathbf{E}(q)(\mu - \lambda) \leq \frac{1}{2}(m_2) + \lambda(\mu - \lambda)}$$

(c) Very similar to concepts used in problem 5 part (*ii*).

In steady state,

$$\begin{aligned} 0 &= \mathbf{E}(V(q(k+1)) - V(q(k)) = \mathbf{E}(\frac{1}{2}(q-s+u+a)^2 - \frac{1}{2}q^2) \\ &= \frac{1}{2}\mathbf{E}((q-s)^2 + (u+a)^2 + 2(u+a)(q-s) - q^2) \\ &= \frac{1}{2}\mathbf{E}(s^2 + u^2 + a^2 - 2qs + 2ua + 2u(q-s) + 2a(q-s)) \end{aligned}$$

Now note that $\mathbf{E}(u(q-s)) = \mathbf{E}(-u^2) = -\mathbf{E}(u^2)$ because when $q - s \geq 0$, $u = 0$ and $u(q-s) = -u^2 = 0$ and when $q - s < 0$, $u = -(q-s)$ and $u(q-s) = -u^2$.
So back to the problem, $0 = \frac{1}{2}(m_2 + 2\mathbf{E}(q)(\lambda - \mu) - \mathbf{E}(u^2))$

$$\boxed{\mathbf{E}(q)(\mu - \lambda) \geq \tfrac{1}{2}(m_2 - \mathbf{E}(u^2))}$$

Now in heavy traffic, $\lambda \to \mu$. Given that $s(k) \leq S_{max}$ and $u(k) \leq s(k) \leq S_{max}$ then $\mathbf{E}(u^2) \leq \mathbf{E}(u)S_{max}$. As $\lambda \to \mu$, $\mathbf{E}(u) = \mu - \lambda \to 0$ and $\mathbf{E}(u^2) \to 0$. So in heavy traffic $\mathbf{E}(q)(\mu - \lambda) \to \frac{1}{2}(m_2)$.