ECE 567: Solutions of Problem Set 1

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Problem 1

Proof: $\{\hat{x}_r\}$ be a max-min fair allocation $\Leftrightarrow$ every source has at least one bottleneck.

$\Rightarrow$: proof by contradiction

Assume we have max-min fairness allocation $\{\hat{x}_r\}$. Assume that $\exists$ a user r that does not have a bottleneck link. Thus either $y_l < c_l$ $\forall l \in r$, or for all link $l \in r$ such that $y_l = c_l$, $\exists s$ such that $l \in s$ and $\hat{x}_s > \hat{x}_r$. In either cases, we can increase $\hat{x}_r$ by a small amount $\epsilon > 0$ either without changing the rates for any other sources (first case) or by decreasing the rates of only those users s such that they share a link with r and have $\hat{x}_s > \hat{x}_r$ (second case). Thus, $\{\hat{x}_r\}$ cannot be a max-min fair allocation.

$\Leftarrow$: Let $\{\hat{x}_r\}$ be an allocation such that each user has at least one bottleneck link. Thus, every user r has a link l such that $y_\ell = c_\ell$ and $\hat{x}_s \leq \hat{x}_r \forall$ s s.t. $s \in \ell$. We increase the rate $\hat{x}_r$ and we look at the effect on its bottleneck link. There will be a user s s.t. $\hat{x_s} \leq \hat{x_r}$ and $x_s < \hat{x_s}$, where x_s is the new rate for user s. This is by definition max-min allocation.

Problem 2

Part a

Each iteration in the algorithm serves the network users with the lowest fair share. The first iteration will divide the full capacity of any link with the minimum fair share equally among all the users using the link. Thus we have a situation where $x_s = x_r$ and $y_\ell = c_\ell$ for all users using the link. This is the same bottle neck scenario discussed in question 1.

Every subsequent iteration fixes the users with the next lowest rates. Thus it holds for every new user that is fixed has a bottle neck, since every user fixed in the same iteration or any previous iteration will have $x_s \leq x_r$ and $y_\ell = c_\ell$. So, every user will have a bottleneck.

The problem is reduced to that of Problem 1.

Part b

Consider the following example: We have seven flows $(f_1, f_2, ..., f_7)$, We have three links in sequence (l_1, l_2, l_3) all with capacity 1. The links are used as follows:

1. $f_1 : l_1$
2. $f_2 : l_1$

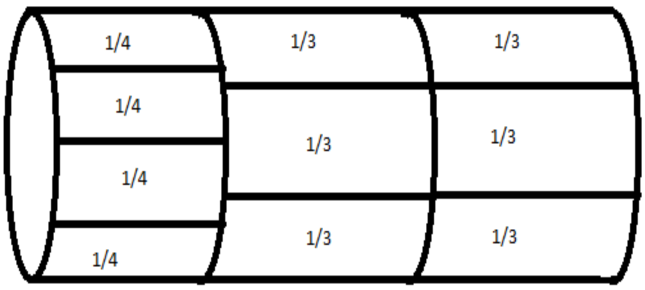

Figure 1: *Iteration 1.*

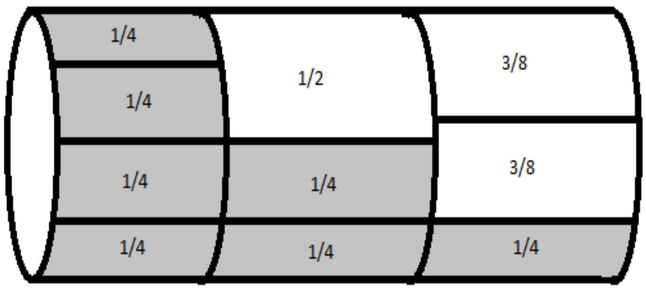

Figure 2: *Iteration 2.*

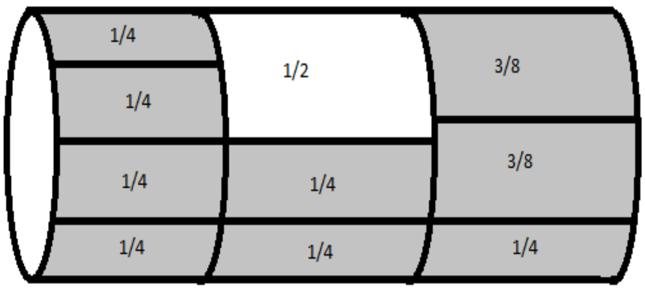

Figure 3: *Iteration 3.*

3. $f_3 : l_1 \rightarrow l_2$

4. $f_4 : l_1 \rightarrow l_2 \rightarrow l_3$

5. $f_5 : l_2$

6. $f_6 : l_3$

7. $f_7 : l_3$

Iteration 1:
See figure 1: Iteration = 1
$f_1(1) = 1/4,\ f_2(1) = 1/3,\ f_3(1) = 1/3$
$z_r(1) = min_{l:l \in r} f_l(i) = 1/4$ for users 1 through 4, 1/3 for users 5-7.
Fix all users with $z_r(1) = 1/4$.
Remove users and fixed capacities from system.

Iteration 2:
See figure 2: Iteration 2
$f_1(2) = 0/0,\ f_2(2) = 1/2,\ f_3(2) = 3/8$
$z_r(2) = min_{l:l \in r} f_l(2) = 1/2$ for user 5, 3/8 for users 6-7.
Fix all users with $z_r(2) = 3/8$.
Remove users and fixed capacities from system.

Iteration 3:
See figure 3: Iteration 3
$f_1(3) = 0/0,\ f_2(3) = 1/2,\ f_3(3) = 0/0$
$z_r(3) = min_{l:l \in r} f_l(3) = 1/2$ for user 5.
Fix all users with $z_r(3) = 1/2$.

Problem 3

Part a

We have $W(x) = \sum_r (x_r - \hat{x_r})^2$ which is $> 0\ \forall x \neq \hat{x}$ and $= 0\ \forall x = \hat{x}$. Take $V(x) = \sum_r U_r(x_r) - \sum_l B_l(y_l)$ and $V(\hat{x}) = \max_x V(x)$ as described in class. Since V(x) is concave, we know that:

$$V(\hat{x}) \leqslant V(x) + \nabla V(x)(\hat{x} - x) \Rightarrow 0 \leqslant V(\hat{x}) - V(x) \leqslant \nabla V(x)(\hat{x} - x),$$

where equality holds only at $x = \hat{x}$. Since $\dot{x}_r = \frac{\partial V}{\partial x_r}$, we get that

$$\dot{W} = \sum_r 2\dot{x}_r(x_r - \hat{x_r}) = -2[\nabla V^T(x)(\hat{x} - x)] < 0\ \forall x \neq \hat{x} \text{ and } = 0 \text{ for } x = \hat{x}.$$

This shows that the controller is globally asymptotically stable.

Part b

If $\kappa_r(x) \neq 1$ take the Lyapunov function to be $W(x) = \sum_r \int_{\hat{x}_r}^{x_r} \frac{u_r - \hat{x}_r}{\kappa_r(u_r)} du_r$. Now since $\kappa_r(x) > 0$ we have $W(x) > 0 \; \forall x \neq \hat{x}$ and $W(x) = 0$ for $x = \hat{x}$. We differentiate W(x) and obtain the following:

$$\dot{W} = \sum_r \frac{\partial W}{\partial x_r} \dot{x}_r = \sum_r \frac{x_r - \hat{x}_r}{\kappa_r(x_r)} \kappa_r(x_r) \frac{\partial V}{\partial x_r} = \sum_r (x_r - \hat{x}_r) \frac{\partial V}{\partial x_r}.$$

For the same reasons as Part a, this implies that $\dot{W} < 0 \; \forall x \neq \hat{x}$ and $= 0$ for $x = \hat{x}$. Note that for this Lyapunov function to work, we have to have the additional condition that $\kappa_r(x)$ should evolve in a way that the Lyapunov function goes to ∞ as $\|x\| \to \infty$.

Note that the inequality $0 \leqslant V(\hat{x}) - V(x) \leqslant \nabla V(x)(\hat{x} - x)$ does not necessarily hold for individual x_r's, since we don't necessarily have $0 \leqslant V(\hat{x}_r) - V(x_r)$. In order to better understand this issue, we can look at the following example.

Let f be defined as the function of two variables in the following way: $f(x) = -x^T A x$ where $A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. We can verify that A is a positive definite matrix, and therefore f is a concave function. This function has its maximum at $\hat{x} = [0, 0]^T$. Now pick $x = [3, -1]^T$. The gradient is $[-2x_1 - x_2, -x_1 - 2x_2]^T$.
We have $\nabla f(x)^T (\hat{x} - x) = [-5, -1][-3, 1]^T = 14 > 0$, however, $\frac{\partial f(x)}{\partial x_2}(\hat{x}_2 - x_2) = (-x_1 - 2x_2)(\hat{x}_2 - x_2) = (-3 + 2)(0 + 1) = -1 < 0$. Therefore we can see that the inequality is not necessarily in the same direction for all the components.

Problem 4

$$\begin{aligned}
\dot{x} &= k_r(x_r)(U_r^{'}(x_r) - q_r) \\
&= k_r(x_r)(U_r^{'}(x_r) - 1 + 1 - q_r) \\
&= k_r(x_r)\left(1 - q_r - \left(1 - U_r^{'}(x_r)\right)\right) \\
&= k_r(x_r)\left[\prod_l (1 - p_l) - (1 - U_r^{'}(x_r))\right].
\end{aligned}$$

Now, let

$$V(x) = \sum_l \int_0^{y_l} ln(1 - p_l(y))dy - \sum_r \int_0^{x_r} ln(1 - U_r^{'}(x))dx$$

Since $p_l(y)$ is increasing in y, $ln(1 - p_l(y))$ is decreasing in y. Thus, the first term of the summation is concave in x as it is a composition of a concave function with a linear function of x. Similarly, $U_r^{'}(x)$ is decreasing in x and hence $ln(1 - U_r^{'}(x))$ is increasing in x. This implies that second term (with negative sign) is strictly concave in x. Let $\hat{x}$ be the global maximizer of $V(x)$. Define $W(x) = V(\hat{x}) - V(x)$. Thus, $W(x) > 0$ for $x \neq \hat{x}$ and zero when $x = \hat{x}$. We use $W(x)$ as Lyapunov function. With this, $\frac{\partial W(x)}{\partial x_r} = -\frac{\partial V(x)}{\partial x_r}$.

Now,

$$\frac{\partial V(x)}{\partial x_r} = \sum_{l\in r} ln(1-p_l) - ln\left(1-U_r^{'}(x_r)\right)$$
$$= ln\left(\prod_{l\in r}(1-p_l)\right) - ln\left(1-U_r^{'}(x_r)\right),$$

implying that at $\hat{x}$, $\prod_{l\in r}(1-p_l) = 1-U_r^{'}(\hat{x}_r)$, and hence $\hat{x}$ is the stable point of the state dynamics too.

This implies that,

$$\dot{W} = -\sum_r k_r(x_r)\left[ln\left(\prod_{l\in r}(1-p_l)\right) - ln\left(1-U_r^{'}(x_r)\right)\right]\left[\prod_{l\in r}(1-p_l) - \left(1-U_r^{'}(x_r)\right)\right] \leq 0,$$

since for any $a>0, b>0,\ (a-b)(ln(a)-ln(b)) \geq 0$, with equality holding only when $a=b$. Thus, $\dot{W}=0$ only if $\prod_{l\in r}(1-p_l) = \left(1-U_r^{'}(x_r)\right)$, which is the equilibrium condition. Thus, this system is asymptotically stable.

For the analysis to make sense, we have to assume that $1-U_r^{'}(x_r) > 0$. Also, $k_r(x_r)$, $U_r^{'}(x_r)$ and p_l are such that if $x>0$, then $x_r(t) \neq 0\ \forall r,t$, and that $W(x)\to\infty$ as $\|x\|\to\infty$.

Problem 5

Part a

We establish this by showing that composition of a strictly concave function with a linear function need not be strictly concave.

Lemma 1. *Let $G(t): \mathcal{D}\subseteq\mathbb{R}\mapsto\mathbb{R}$ be a strictly concave function and for $\mathbf{t}=(t_1,t_2,\ldots,t_n)$ such that $\sum_{i=1}^n t_i\in\mathcal{D}$, let $H(\mathbf{t})=\sum_{i=1}^n t_i$. Then $G\circ H(\mathbf{t}) = G(H(\mathbf{t}))$ is concave with respect to $\mathbf{t}$ but not strictly concave.*

Proof. Concavity of $G\circ H(\mathbf{t})$ with respect to $\mathbf{t}$ follows from the standard result that composition of a concave function with an affine function is still concave. We just need to show that this is not strictly concave. Consider vectors $\mathbf{t}$ and $\mathbf{s}$ such that $\mathbf{t}\neq\mathbf{s}$, but $\sum_{i=1}^n t_i = \sum_{i=1}^n s_i\in\mathcal{D}$. Clearly, we can find such $\mathbf{t},\mathbf{s}$. Then $G\circ H(\mathbf{t}) = G\circ H(\mathbf{s})$. Let $\lambda\in(0,1)$. Then,

$$G\circ H(\lambda\mathbf{t}+(1-\lambda)\mathbf{s}) = G\left(\sum_{i=1}^n(\lambda t_i+(1-\lambda)s_i)\right),$$
$$= G\left(\lambda\left(\sum_{i=1}^n t_i\right)+(1-\lambda)\left(\sum_{i=1}^n s_i\right)\right),$$
$$= G\left(\sum_{i=1}^n t_i\right) = G\left(\sum_{i=1}^n s_i\right) = G\circ H(\mathbf{t}) = G\circ H(\mathbf{s}).$$

Thus, $G\circ H(\lambda\mathbf{t}+(1-\lambda)\mathbf{s}) = \lambda G\circ H(\mathbf{t}) + (1-\lambda)G\circ H(\mathbf{s})$. But, for strict concavity, we need $G\circ H(\lambda\mathbf{t}+(1-\lambda)\mathbf{s}) > \lambda G\circ H(\mathbf{t}) + (1-\lambda)G\circ H(\mathbf{s})$, $\forall\lambda\in(0,1)$. Thus, $G\circ H(\mathbf{t})$ is not a strict concave function of $\mathbf{t}$. □

Now, let $V(\mathbf{x}) = \sum_s U_s(z_s) - \sum_l \int_0^{y_l} dy$, where $\mathbf{x}$ is the vector of all routes for all users. Thus, we need to show that $V(\mathbf{x})$ need not be a strict concave function. Now, for each s, $U_s(z_s) = U_s(\sum_{r\in R(s)} x_r)$ is a composition of $U_s(.)$ with a linear function of $\mathbf{x}$, and hence not a strict concave function of $\mathbf{x}$. Similarly, for each l, if $F_l(t) = \int_0^t f_l(s)ds$ then $\int_0^{y_l} f_l(s)ds = F_l(\sum_{r:l\in r} x_r)$ is a composition of a convex function $F(.)$ with a linear function of $\mathbf{x}$, and hence not a strict convex function of $\mathbf{x}$. Thus, $V(\mathbf{x})$ being a sum of concave functions of $\mathbf{x}$ that are not strictly concave need not be a strict concave function.

Part b

Let

$$V(\mathbf{x}) = \sum_s U_s(z_s) - \sum_l \int_0^{y_l} dy + \epsilon \sum_r log x_r, \tag{1}$$

where $z_s = \sum_{r\in R(s)} x_r$ and $y_l = \sum_{r:l\in r} x_r$. $V(\mathbf{x})$ is strictly concave. Thus, there exists a unique maximizer $\hat{\mathbf{x}}$. Since $\hat{\mathbf{x}}$ is the maximizer of $V(\mathbf{x})$, it must satisfy

$$\frac{\partial V(\mathbf{x})}{\partial x_r} = 0 \text{ at } \mathbf{x} = \hat{\mathbf{x}},\ \forall r. \tag{2}$$

Now,

$$\frac{\partial V(\mathbf{x})}{\partial x_r} = U_s^{'}(z_s) - \sum_{l:l\in r} f_l(y_l) + \frac{\epsilon}{x_r}. \tag{3}$$

For each source s and each $r \in R(s)$, let the state dynamics be

$$\dot{x_r} = k_r(x_r)\left(U_s^{'}(z_s) - \sum_{l:l\in r} f_l(y_l) + \frac{\epsilon}{x_r}\right), \tag{4}$$

where $k_r(x_r) > 0$. Let $W(\mathbf{x}) \triangleq V(\hat{\mathbf{x}}) - V(\mathbf{x})$. Thus, $W(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \hat{\mathbf{x}}$, and is equal to zero at $\mathbf{x} = \hat{\mathbf{x}}$. We use $W(\mathbf{x})$ as a Lyapunov function for showing that state dynamics (4) converges to $\hat{\mathbf{x}}$. We assume that $U_s(.), k_r(.)$ and $f_l(.)$ are such that $W(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$.

$$\begin{aligned}
\dot{W} &= \sum_r \left(\frac{\partial W(\mathbf{x})}{\partial x_r} k_r(x_r)\left(U_s^{'}(z_s) - \sum_{l:l\in r} f_l(y_l) + \frac{\epsilon}{x_r}\right)\right), \\
&= \sum_r \left(-k_r(x_r)\frac{\partial V(\mathbf{x})}{\partial x_r}\frac{\partial V(\mathbf{x})}{\partial x_r}\right), \\
&= -\sum_r k_r(x_r)\left(\frac{\partial V(\mathbf{x})}{\partial x_r}\right)^2 \le 0
\end{aligned}$$

where, $\dot{W} = 0$ implies $\frac{\partial V(\mathbf{x})}{\partial x_r} = 0$, which is true only when $\mathbf{x} = \hat{\mathbf{x}}$. Thus, system dynamics given by (4) converges to $\hat{\mathbf{x}}$.

Problem 6

Part a

The delay differential equation for TCP-Reno is

$$\dot{x}(t) = x(t-T)\left[\frac{(1-p(t-T_b))}{T^2 x(t)} - \frac{p(t-T_b)x(t-T)}{2}\right], \tag{5}$$

where symbols have their usual meanings as in the lecture notes. Here, $p(t-T_b) = f(x(t-T_b-T_f)) = f(x(t-T))$. Let $\hat{x}$ be the equilibrium point and define $y(t) \triangleq x(t)-\hat{x}$, where $y(t) << \hat{x}$. Equilibrium point $\hat{x}$ satisfies

$$\frac{1-f(\hat{x})}{T^2\hat{x}} - \frac{f(\hat{x})\hat{x}}{2} = 0. \tag{6}$$

Using $x(t) = \hat{x} + y(t)$ in (5), we get

$$\dot{y}(t) = [\hat{x} + y(t-T)]\left[\frac{1 - f\left(\hat{x} + y(t-T)\right)}{T^2(\hat{x}+y(t))} - \frac{f\left(\hat{x}+y(t-T)\right)\left(\hat{x}+y(t-T)\right)}{2}\right]. \tag{7}$$

Using first-order approximation, we have $(\hat{x}+y(t))^{-1} \approx \frac{1}{\hat{x}}\left(1 - \frac{y(t)}{\hat{x}}\right)$ and $f\left(\hat{x}+y(t-T)\right) \approx f(\hat{x}) + y(t-T)f^{'}(\hat{x})$. With this, (7) simplifies to

$$\begin{aligned}
\dot{y}(t) &\approx [\hat{x}+y(t-T)]\left[\frac{1-f(\hat{x})-y(t-T)f^{'}(\hat{x})}{T^2\hat{x}}\left(1-\frac{y(t)}{\hat{x}}\right) - \frac{\left(f(\hat{x})+y(t-T)f^{'}(\hat{x})\right)\left(\hat{x}+y(t-T)\right)}{2}\right],\\
&\approx [\hat{x}+y(t-T)]\left[\frac{1-f(\hat{x})}{T^2\hat{x}} - \frac{f(\hat{x})\hat{x}}{2} - \frac{(1-f(\hat{x}))y(t)}{T^2\hat{x}^2} - y(t-T)\left(\frac{f^{'}(\hat{x})}{T^2\hat{x}} + \frac{f(\hat{x})+\hat{x}f^{'}(\hat{x})}{2}\right)\right],\\
&= [\hat{x}+y(t-T)]\left[-y(t)\left(\frac{1-f(\hat{x})}{T^2\hat{x}^2}\right) - y(t-T)\left(\frac{f^{'}(\hat{x})}{T^2\hat{x}} + \frac{f(\hat{x})+\hat{x}f^{'}(\hat{x})}{2}\right)\right],\\
&\approx -y(t)\left(\frac{1-f(\hat{x})}{T^2\hat{x}}\right) - y(t-T)\left(\frac{f^{'}(\hat{x})}{T^2} + \frac{\hat{x}\left(f(\hat{x})+\hat{x}f^{'}(\hat{x})\right)}{2}\right),
\end{aligned}$$

where, the second line is obtained by removing quadratic terms in $y(.)$ and rearranging remaining terms, third line is by using (6), and last line is again by removing quadratic terms in $y(.)$.

Thus, the linear delay-differential equation of TCP-Reno is given by

$$\dot{y}(t) = -y(t)\left(\frac{1-f(\hat{x})}{T^2\hat{x}}\right) - y(t-T)\left(\frac{f^{'}(\hat{x})}{T^2} + \frac{\hat{x}\left(f(\hat{x})+\hat{x}f^{'}(\hat{x})\right)}{2}\right). \tag{8}$$

Part b

Let $a = \frac{1-f(\hat{x})}{T^2\hat{x}}$ and $b = \frac{f^{'}(\hat{x})}{T^2} + \frac{\hat{x}\left(f(\hat{x})+\hat{x}f^{'}(\hat{x})\right)}{2}$. Since $f(.)$ is a loss probability, we have $0 \leq f(\hat{x}) < 1$. Also, since $f(.)$ is an increasing function, $f^{'}(\hat{x}) \geq 0$, in addition to $\hat{x} > 0$. We assume that not both $f(\hat{x})$ and $f^{'}(\hat{x})$ are zero at the same time. With this, we have $a > 0$ and $b > 0$.

Taking Laplace transformation of (8), we get

$$\begin{aligned}
& sY(s) - y(0) = -aY(s) - be^{-sT}Y(s), \\
& \Leftrightarrow (s + a + be^{-sT})Y(s) = y(0), \\
& \Leftrightarrow \frac{Y(s)}{y(0)} = \frac{1}{s + a + be^{-sT}}.
\end{aligned}$$

Thus, for the linear system to be stable, solutions of equation $s + a + be^{-sT} = 0$ must lie in the left half plane. Let $\sigma + j\omega$ be a solution. Assume that $\sigma \geq 0$. Then we must have

$$\sigma + j\omega + a + be^{-\sigma T}(cos(\omega T) - jsin(\omega T)) = 0. \tag{9}$$

This implies,

$$cos(\omega T) = -\frac{\sigma + a}{be^{-\sigma T}}, \tag{10}$$

and

$$\frac{sin(\omega T)}{\omega T} = \frac{1}{Tbe^{-\sigma T}}. \tag{11}$$

Since $a > 0, b > 0, \sigma \geq 0$, (10) implies that $\omega T \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. However, in this range, $\frac{sin(\omega T)}{\omega T} < \frac{2}{\pi}$. Thus, (11) gives $bTe^{-\sigma T} > \frac{\pi}{2}$, implying $bT > \frac{\pi}{2}$. Thus, if $bT \leq \frac{\pi}{2}$ then $\sigma < 0$. Thus, a sufficient condition for the linearized system to be stable is

$$bT \leq \frac{\pi}{2} \Longleftrightarrow \left(\frac{f^{'}(\hat{x})}{T^2} + \frac{\hat{x}\left(f(\hat{x}) + \hat{x}f^{'}(\hat{x})\right)}{2}\right)T \leq \frac{\pi}{2}. \tag{12}$$