On Linear Copositive Lyapunov Functions and the Stability of Switched Positive Linear Systems

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Abstract

We consider the problem of common linear copositive function existence for positive switched linear systems. In particular, we present a necessary and sufficient condition for the existence of such a function for switched systems with two constituent linear time-invariant (LTI) systems. A number of applications of this result are also given.

Key Words: Switched linear systems; Positive linear systems; Copositive Lyapunov functions

I. Introduction

An outstanding problem in systems theory concerns the basic stability properties of dynamic systems whose states are confined to the positive orthant. Such systems are generally referred to as positive systems and arise frequently in a number of important applications in Biology, Communications, Probability, Economics and in other fields. In particular, many applications in Communication networks involve algorithms that lead to extremely complex positive systems, typically involving significant nonlinearity, abrupt parameter switching, and state resets. These applications, which include networks employing TCP and other

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congestion control applications [10], synchronisation problems [5], wireless power control applications [8], and applications of learning automata to distributed coloring problems [6], typically require advanced analysis tools to prove their stability and convergence properties. Given the widespread application of positive systems, it is surprising that only recently has the stability of switched and nonlinear positive system become a topic of major interest to the systems theory community [3]. We continue this line of work in the current paper. Specifically, we consider the question of the existence of *copositive* linear Lyapunov functions, defined below, for a class of switched positive systems. We give an elegant necessary and sufficient condition for determining when such a function exists and provide a number of applications of this condition to special cases.

II. Notation and Mathematical Background

Throughout, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers and $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices with real entries. For x in $\mathbb{R}^n$, x_i denotes the i^{th} component of x, and the notation $x \succ 0$ ($x \succeq 0$) means that $x_i > 0$ ($x_i \geq 0$) for $1 \leq i \leq n$. The notations $x \prec 0$ and $x \preceq 0$ are defined in the obvious manner. $\mathbb{R}^n_+$ denotes the closed positive orthant of $\mathbb{R}^n$, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \succeq 0\}$, and $\mathrm{Int}(\mathbb{R}^n_+)$ denotes its interior, $\{x \in \mathbb{R}^n : x \succ 0\}$. Similarly, for a matrix A in $\mathbb{R}^{n\times n}$, a_{ij} denotes the element in the (i, j) position of A, and $A \succ 0$ ($A \succeq 0$) means that $a_{ij} > 0 (a_{ij} \geq 0)$ for $1 \leq i, j \leq n$.

We write A^T for the transpose of A and we shall occasionally abuse notation by writing A^{-T} for the inverse of A^T. For P in $\mathbb{R}^{n\times n}$ the notation $P > 0$ means that the matrix P is positive definite.

The spectral radius of a matrix A is the maximum modulus of the eigenvalues of A and is denoted by $\rho(A)$. Also we shall denote the maximal real part of any eigenvalue of A by $\mu(A)$. If $\mu(A) < 0$ (all the eigenvalues of A are in the open left half plane) A is said to be *Hurwitz*.

For a real number x we define the function $\mathrm{sign}(x)$ by

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Note that if a matrix $A \in \mathbb{R}^{n \times n}$ is Hurwitz, then $\mathrm{sign}(\det(A)) = (-1)^n$.

Throughout this paper, we shall be concerned with the stability of switched positive linear systems $\dot{x} = A(t)x$, $A(t) \in \{A_1, \ldots, A_m\}$ constructed by switching between positive LTI systems. Before proceeding, we shall now recall some basic facts about positive LTI systems.

Positive LTI systems and Metzler matrices

The LTI system

$$\Sigma_A : \dot{x}(t) = Ax(t), \qquad x(0) = x_0$$

is said to be positive if $x_0 \succeq 0$ implies that $x(t) \succeq 0$ for all $t \geq 0$. Basically, if the system starts in the non-negative orthant of $\mathbb{R}^n$, it remains there for all time. See [2] for a description of the basic theory and several applications of positive linear systems.

It is well-known [2] that the system Σ_A is positive if and only if the off-diagonal entries of the matrix A are non-negative. Matrices of this form are known as *Metzler* matrices. If A is Metzler we can write $A = N - \alpha I$ for some non-negative N and a scalar $\alpha \geq 0$. Note that if the eigenvalues of N are $\lambda_1, \ldots \lambda_n$, then the eigenvalues of $N - \alpha I$ are $\lambda_1 - \alpha, \ldots \lambda_n - \alpha$. Thus the Metzler matrix $N - \alpha I$ is Hurwitz if and only if $\alpha > \rho(N)$.

There are a number of equivalent conditions for a Metzler matrix to be Hurwitz [4], [1]. The following result records two of these conditions which are relevant for the work of this paper.

Theorem 2.1: Let $A \in \mathbb{R}^{n \times n}$ be Metzler. Then the following are equivalent:

(i) A is Hurwitz;

(ii) There is some vector $v \succ 0$ in $\mathbb{R}^n$ with $Av \prec 0$;

(iii) $A^{-1} \preceq 0$.

Convex Cones and Separation Theorems

Much of the work presented later in the paper is concerned with determining conditions for the intersection of two convex cones in $\mathbb{R}^n$. Recall that a set Ω in $\mathbb{R}^n$ is a *convex cone* if for all $x, y \in \Omega$, and all $\lambda \geq 0, \mu \geq 0$ in $\mathbb{R}$, $\lambda x + \mu y$ is in Ω. The convex cone Ω is said to be *open* (*closed*) if it is open (closed) with respect to the usual Euclidean topology on $\mathbb{R}^n$. For an open convex cone Ω, we denote the closure of Ω by $\overline{\Omega}$.

Given a set of points, $\{x_1, \dots, x_m\}$ in $\mathbb{R}^n$, we shall use the notation $CO(x_1, \dots, x_m)$ to denote the convex hull of $x_1, \dots, x_m$. Formally:

$$CO(x_1, \dots, x_m) = \{\sum_{i=1}^{m} \alpha_i x_i : \alpha_i \geq 0, 1 \leq i \leq m, \text{ and } \sum_{i=1}^{m} \alpha_i = 1\}.$$

The theory of finite-dimensional convex sets is a well established branch of mathematical analysis [9]. In the next section, we shall make use of the following special case of more general results [9] on the existence of separating hyperplanes for disjoint convex cones.

Theorem 2.2: Let Ω_1, Ω_2 be open convex cones in $\mathbb{R}^n$. Suppose that $\overline{\Omega_1} \cap \overline{\Omega_2} = \{0\}$. Then there is some vector $v \in \mathbb{R}^n$ such that

$$v^T x < 0 \text{ for all } x \in \Omega_1$$

and

$$v^T x > 0 \text{ for all } x \in \Omega_2.$$

III. Preliminaries on Linear Copositive Lyapunov Functions

The linear function $V(x) = v^T x$ defines a linear copositive Lyapunov function for the positive LTI system Σ_A is and only if the vector $v \in \mathbb{R}^n$ satisfies:

(i) $v \succ 0$;
(ii) $A^T v \prec 0$.

It follows from Theorem 2.1 that a positive LTI system is asymptotically stable if and only if it has a linear copositive Lyapunov function. The primary contribution of this paper is to derive a simple algebraic necessary and sufficient condition for a pair of asymptotically stable positive LTI systems, Σ_{A_1}, Σ_{A_2} to have a common linear copositive Lyapunov function $V(x) = v^T x$, where $v \succ 0$ and $A_i^T v \prec 0$ for $i = 1, 2$. This condition is given in Theorem 4.1 below and our derivation will be based on the following preliminary result.

Theorem 3.1: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler, Hurwitz matrices such that there exists no non-zero vector $v \succeq 0$ with $A_i^T v \preceq 0$ for $i = 1, 2$. Then there exist $w_1 \succ 0$, $w_2 \succ 0$ in $\mathbb{R}^n$ such that

$$A_1 w_1 + A_2 w_2 = 0.$$

Proof: For $i = 1, 2$, let $\mathcal{V}_{A_i}$ be given by

$$\mathcal{V}_{A_i} = \{v \succ 0 : A_i^T v \prec 0\}. \tag{1}$$

Then $\mathcal{V}_{A_1}$, $\mathcal{V}_{A_2}$ are open convex cones and it follows from Theorem 2.1 that

$$\mathcal{V}_{A_i} = -A_i^{-T}(\mathrm{Int}(\mathbb{R}^n_+)) \tag{2}$$

for $i = 1, 2$.

By hypothesis, $\overline{\mathcal{V}_{A_1}} \cap \overline{\mathcal{V}_{A_2}} = \{0\}$. Thus, from Theorem 2.2 there is some vector $v \in \mathbb{R}^n$ with $v^T A_1^{-T} w < 0$ and $v^T A_2^{-T} w > 0$ for all $w \succ 0$. But this implies that $w_1 = -A_1^{-1} v$, $w_2 = A_2^{-1} v$ are both positive, $w_1 \succ 0$, $w_2 \succ 0$, and that

$$A_1 w_1 + A_2 w_2 = -v + v = 0.$$

IV. Main Results

Given $A \in \mathbb{R}^{n\times n}$ and an integer i with $1 \leq i \leq n$, $A^{(i)}$ denotes the i^{th} column of A. Thus, $A^{(i)}$ denotes the vector in $\mathbb{R}^n$ whose j^{th} entry is a_{ji} for $1 \leq j \leq n$.

For a positive integer n, we denote the set of all mappings $\sigma : \{1, \ldots, n\} \to \{1, 2\}$ by $\mathcal{C}_{n,2}$. Now, given two matrices A_1, A_2 in $\mathbb{R}^{n\times n}$ and a mapping $\sigma \in \mathcal{C}_{n,2}$, we define the matrix $A_\sigma(A_1, A_2)$ by:

$$A_\sigma(A_1, A_2) = (A^{(1)}_{\sigma(1)} A^{(2)}_{\sigma(2)} \ldots A^{(n)}_{\sigma(n)}). \tag{3}$$

Thus, $A_\sigma(A_1, A_2)$, is the matrix in $\mathbb{R}^{n\times n}$ whose i^{th} column is the i^{th} column of $A_{\sigma(i)}$ for $1 \leq i \leq n$. We shall denote the set of all matrices that can be formed in this way by $\mathcal{S}(A_1, A_2)$.

$$\mathcal{S}(A_1, A_2) = \{A_\sigma(A_1, A_2) : \sigma \in \mathcal{C}_{n,2}\}. \tag{4}$$

Theorem 4.1: Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then the following statements are equivalent:

(i) The positive LTI systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function;

(ii) The finite set $\mathcal{S}(A_1, A_2)$ consists entirely of Hurwitz matrices.

Proof:

(i) ⇒ (ii): As Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function, there is some vector $v \succ 0$ in $\mathbb{R}^n$ with $v^T A_i \prec 0$ for $i = 1, 2$. This immediately implies that $v^T A_i^{(j)} < 0$ for $i = 1, 2$, $1 \leq j \leq n$ and hence we have that

$$v^T A \prec 0 \text{ for all } A \in \mathcal{S}(A_1, A_2). \tag{5}$$

Now note that as A_1, A_2 are Metzler, all matrices belonging to the set $\mathcal{S}(A_1, A_2)$ are also Metzler. It follows immediately from (5) and Theorem 2.1 that each matrix in $\mathcal{S}(A_1, A_2)$ must be Hurwitz.

(ii) ⇒ (i): We shall show that if $\Sigma_{A_1}, \Sigma_{A_2}$ do not have a common linear copositive Lyapunov function, then at least one matrix belonging to the set $\mathcal{S}(A_1, A_2)$ must be non-Hurwitz.

First of all, we make the stronger assumption (than non-existence of a common linear copositive Lyapunov function) that there is no non-zero vector $v \succeq 0$ with $v^T A_i \preceq 0$ for $i = 1, 2$. It follows from Theorem 3.1 that there are vectors w_1, w_2 such that $w_1 \succ 0$, $w_2 \succ 0$ and

$$A_1 w_1 + A_2 w_2 = 0. \tag{6}$$

As $w_1 \succ 0$, $w_2 \succ 0$, there is some positive definite diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ in $\mathbb{R}^{n\times n}$ with $w_2 = D w_1$. It follows from (6) that, for this D,

$$\det(A_1 + A_2 D) = 0. \tag{7}$$

Now, for the n-tuple, $(d_1, \ldots, d_n)^T \in \mathbb{R}^n$ and a mapping $\sigma \in \mathcal{C}_{n,2}$, we shall use $(d_1, \ldots, d_n)^{\sigma-1}$ to denote the product of $d_1, \ldots, d_n$ given by

$$(d_1, \ldots, d_n)^{\sigma-1} = \prod_{i=1}^{n} d_i^{\sigma(i)-1}. \tag{8}$$

In terms of this notation, we can now write

$$\det(A_1 + A_2 D) = \sum_{\sigma \in \mathcal{C}_{n,2}} \det(A_\sigma(A_1, A_2))(d_1, \ldots, d_n)^{\sigma-1}. \tag{9}$$

Now if all matrices in the set $\mathcal{S}(A_1, A_2)$ were Hurwitz, then $\det(A_\sigma(A_1, A_2)) > 0$ for all $\sigma \in \mathcal{C}_{n,2}$ if n is even and $\det(A_\sigma(A_1, A_2)) < 0$ for all $\sigma \in \mathcal{C}_{n,2}$ if n is odd. In either case, this would contradict (7) which implies that, for the positive real numbers $d_1, \ldots, d_n$,

$$\sum_{\sigma \in \mathcal{C}_{n,2}} \det(A_\sigma(A_1, A_2))(d_1, \ldots, d_n)^{\sigma-1} = 0. \tag{10}$$

Hence, there must exist at least one $\sigma \in \mathcal{C}_{n,2}$ for which $A_\sigma(A_1, A_2)$ is non-Hurwitz.

For the remainder of the proof, we shall assume that the dimension n is even. In this case, for a Hurwitz $A \in \mathbb{R}^{n\times n}$, $\det(A) > 0$. The case of odd n follows in an identical manner.

We have shown that if $\overline{\mathcal{V}_{A_1}} \cap \overline{\mathcal{V}_{A_2}} = \{0\}$, then at least one matrix belonging to $\mathcal{S}(A_1, A_2)$ must be non-Hurwitz. In fact, as both A_1 and A_2 are Hurwitz, in this case, it follows from (9) and (7) that $\det(A) < 0$ for at least one A belonging to $\mathcal{S}(A_1, A_2)$. Next suppose that there is some non-zero $v \succeq 0$ in $\overline{\mathcal{V}_{A_1}} \cap \overline{\mathcal{V}_{A_2}}$ but that the intersection of the open cones

$$\mathcal{V}_{A_1} \cap \mathcal{V}_{A_2} \tag{11}$$

is empty.

Now, denote by $\mathbf{1}_n$ the matrix in $\mathbb{R}^{n\times n}$ consisting entirely of ones ($\mathbf{1}_n(i,j) = 1$ for $1 \leq i, j \leq n$) and for all $\epsilon > 0$, write $A_i(\epsilon) = A_i + \epsilon \mathbf{1}_n$ for $i = 1, 2$. Then it is straightforward to see that

$$\overline{\mathcal{V}_{A_1(\epsilon)}} \cap \overline{\mathcal{V}_{A_2(\epsilon)}} = \{0\}$$

for all $\epsilon > 0$. Thus, if we choose any $\epsilon > 0$ sufficiently small to ensure that $A_1(\epsilon)$ and $A_2(\epsilon)$ are Hurwitz and Metzler, it follows from the above argument that there must be at least one non-Hurwitz matrix in the set $\mathcal{S}(A_1(\epsilon), A_2(\epsilon))$. A limiting argument now shows that at least one matrix in the set $\mathcal{S}(A_1, A_2)$ is non-Hurwitz. This completes the proof of the theorem.

We now present a simple example to illustrate the use of the above theorem.

Example 4.1: Consider the Metzler, Hurwitz matrices in $\mathbb{R}^{2\times 2}$ given by

$$A_1 = \begin{pmatrix} -0.7125 & 0.7764 \\ 0.5113 & -0.9397 \end{pmatrix}, A_2 = \begin{pmatrix} -1.3768 & 0.8066 \\ 0.9827 & -1.3738 \end{pmatrix}.$$

Then it is easy to see that both of the matrices

$$\begin{pmatrix} -0.7125 & 0.8066 \\ 0.5113 & -1.3738 \end{pmatrix} \quad \begin{pmatrix} -1.3768 & 0.7764 \\ 0.9827 & -0.9397 \end{pmatrix}$$

are Hurwitz. It now follows from Theorem 4.1 that the systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function. In fact, for $v = (1.1499, \ \ 1.1636)^T$, it can be checked that $A_i^T v \prec 0$ for $i = 1, 2$.

Remarks:

(i) Note that the result of Theorem 4.1 relates the existence of a common Lyapunov function for a pair of positive LTI systems, and the asymptotic stability of the associated switched linear system, to the stability of a finite set of positive LTI systems. Formally, the existence of a common linear copositive Lyapunov function for Σ_{A_1}, Σ_{A_2} is equivalent to the stability of each of the 2^n positive LTI systems, Σ_A for $A \in \mathcal{S}(A_1, A_2)$. Of course, it follows that the asymptotic stability of this finite family of systems is sufficient for the asymptotic stability of the switched system $\dot{x} = A(t)x$, $A(t) \in \{A_1, A_2\}$.

(ii) A common linear copositive Lyapunov function for Σ_{A_1}, Σ_{A_2} will also define a linear copositive Lyapunov function for each of the systems Σ_A with $A \in \mathcal{S}(A_1, A_2)$.

(iii) In the proof of Theorem 4.1, the non-existence of a common linear copositive Lyapunov function is related to the existence of a diagonal matrix $D > 0$ such that $A_1 + A_2 D$ is singular. It is interesting to compare this with the recent result in [7], which established that the non-existence of a common diagonal Lyapunov function for a pair of positive LTI systems implied the existence of a diagonal $D > 0$ such that $A_1 + DA_2D$ is singular. The precise relationship between copositive Lyapunov functions, diagonal Lyapunov functions and quadratic Laypunov functions for general switched positive linear systems is in itself an interesting question, and the above result may prove useful in clarifying this relationship.

The next result follows easily from the above remarks and Theorem 4.1.

Corollary 4.1: Let A_1, A_2 be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$. Then the following statements are equivalent:

(i) There exists a common linear copositive Lyapunov function for the systems Σ_{A_1}, Σ_{A_2};

(ii) There is a common linear copositive Lyapunov function for the set of systems

$$\{\Sigma_A : A \in CO(\mathcal{S}(A_1, A_2))\};$$

(iii) All matrices in the convex hull $CO(\mathcal{S}(A_1, A_2))$ are Hurwitz;

(iv) All matrices in $\mathcal{S}(A_1, A_2)$ are Hurwitz.

The previous corollary shows that the Hurwitz-stability of the finite collection of matrices $\mathcal{S}(A_1, A_2)$ is sufficient to ensure the asymptotic stability under arbitrary switching of the system

$$\dot{x} = A(t)x \quad\quad A(t) \in CO(\mathcal{S}(A_1, A_2)).$$

Also, the equivalence of points (iii) and (iv) above means that the Hurwitz-stability of the set $\mathcal{S}(A_1, A_2)$ is necessary and sufficient for the Hurwitz-stability of its convex hull.

A close examination of the proof of Theorem 4.1 shows that the following characterisation of linear copositive Lyapunov function existence also holds.

Corollary 4.2: Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be Metzler, Hurwitz matrices. Then the systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function if and only if

$$\mathrm{sign}(\det(A)) = (-1)^n$$

for all $A \in \mathcal{S}(A_1, A_2)$.

V. Applications to systems differing by rank one

We next present two corollaries to Theorem 4.1 for the special case of a pair of Hurwitz, Metzler matrices A_1, A_2 in $\mathbb{R}^{n\times n}$ with $\mathrm{rank}(A_2 - A_1) = 1$. Before we formally state the following simple corollaries to Theorem 4.1 recall that for a matrix B in $\mathbb{R}^{n\times n}$ and an integer $i \in \{1, \ldots, n\}$, we write $B^{(i)}$ for the column vector given by the i^{th} column of B.

Corollary 5.1: Let A_1, $A_2 = A_1 + B$ be Hurwitz, Metzler matrices in $\mathbb{R}^{n\times n}$ with $\mathrm{rank}(B) = 1$. Furthermore, suppose that there is some i with $1 \leq i \leq n$ such that $B^{(i)}$ is the only non-zero column of B. Then the positive LTI systems Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function.

Proof: From Theorem 4.1, Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function if and only if all matrices belonging to the set $\mathcal{S}(A_1, A_2)$ are Hurwitz. However, under the hypotheses of the corollary, it is easy to see that

$$\mathcal{S}(A_1, A_2) = \{A_1, A_2\}. \tag{12}$$

The result now follows immediately.

The previous result establishes that for Metzler, Hurwitz matrices A_1, A_2 which differ in only one column, the associated LTI systems Σ_{A_1}, Σ_{A_2} must have a common linear copositive Lyapunov function. Moreover, it follows that the associate switched linear system

$$\dot{x} = A(t)x \quad\quad A(t) \in \{A_1, A_2\},$$

must be uniformly asymptotically stable under arbitrary switching. It might seem reasonable to expect that a similar result to Corollary 5.1 would also hold for the case of matrices differing by a general rank one matrix. However, the following example shows that this is unfortunately not the case.

Example 5.1: Consider the 3×3 Metzler, Hurwitz matrices, A_1, $A_2 = A_1 + bc^T$ where

$$A_1 = \begin{pmatrix} -1.4528 & 0.6435 & 0.7266 \\ 0.4983 & -1.5714 & 0.4120 \\ 0.2140 & 0.9601 & -1.1469 \end{pmatrix}, b = \begin{pmatrix} 0.0589 \\ -0.4251 \\ -0.1798 \end{pmatrix}, c = \begin{pmatrix} 1 \\ -1.1802 \\ -0.448 \end{pmatrix}.$$

It is simple to check that the matrix $(A_1^{(1)} A_2^{(2)} A_2^{(3)})$ is not Hurwitz and hence it follows from Theorem 4.1 that the systems Σ_{A_1}, Σ_{A_2} do not have a common linear copositive Lyapunov function.

The above example shows that two stable positive LTI systems whose system matrices differ by a rank one matrix need not in general have a common linear copositive Lyapunov function. However, the next corollary provides a simple sufficient condition for the existence of a common linear copositive Lyapunov function for this case.

Corollary 5.2: Let A_1, $A_2 = A_1+B$ be Metzler, Hurwitz matrices in $\mathbb{R}^{n\times n}$, with $\mathrm{rank}(B) = 1$. For each $i \in \{1,\dots,n\}$, let $T_i \in \mathbb{R}^{n\times n}$ be the matrix given by

$$T_i^{(j)} = \begin{cases} B^{(j)} & \text{if } j = i \\ A_1^{(j)} & \text{if } j \neq i \end{cases}$$

Then the positive LTI systems, Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function if for $1 \leq i \leq n$, either $\mathrm{sign}(\det(T_i)) = (-1)^n$ or $\mathrm{sign}(\det(T_i)) = 0$.

Proof: Suppose that for $1 \leq i \leq n$, either $\mathrm{sign}(\det(T_i)) = (-1)^n$ or $\mathrm{sign}(\det(T_i)) = 0$. As $\mathrm{rank}(B) = 1$, we can write $B = bc^T$ for column vectors $b, c \in \mathbb{R}^n$. (Thus, all columns of B are scalar multiples of each other.) It follows from this, and the linear dependence of the determinant function on each column, that for any $A \in \mathcal{S}(A_1, A_2)$, there is some set of indices $\{i_1,\dots,i_k\} \subset \{1,\dots,n\}$ such that

$$\det(A) = \det(A_1) + \sum_{j=1}^{k} \det(T_{i_j}). \tag{13}$$

Hence, as $\mathrm{sign}(\det(T_i))$ is either $(-1)^n$ or 0 for $1 \leq i \leq n$, it follows that $\mathrm{sign}(\det(A)) = (-1)^n$ for all $A \in \mathcal{S}(A_1, A_2)$. It now follows immediately from Corollary 4.2 that Σ_{A_1}, Σ_{A_2} have a common linear copositive Lyapunov function as claimed.

VI. Conclusions

In this paper we have presented a method for determining whether or not a given switched positive continuous time linear system is exponentially stable. Our approach is based upon determining verifiable conditions for the existence of a common copositive linear Lyapunov function for a pair of positive LTI systems. Future work will involve extending this result to arbitrary finite sets of such LTI systems, and developing synthesis procedures to exploit our result for the design of stable switched positive systems.

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