A Spectral Characterization of Exponential Stability for Linear Time-Invariant Systems on Time Scales

Christian Pötzsche
Institut für Mathematik
Universität Augsburg, 86135 Augsburg, Germany
christian.poetzsche@math.uni-augsburg.de

Stefan Siegmund
Center for Dynamical Systems and Nonlinear Studies
Georgia Institute of Technology
Atlanta, Georgia 30332, U.S.A.
stefan.siegmund@math.gatech.edu

Fabian Wirth
Zentrum für Technomathematik
Universität Bremen, 28334 Bremen, Germany
fabian@math.uni-bremen.de

Abstract

We prove a necessary and sufficient condition for the exponential stability of time-invariant linear systems on time scales in terms of the eigenvalues of the system matrix. In particular, this unifies the corresponding characterizations for finite-dimensional differential and difference equations. To this end we use a representation formula for the transition matrix of Jordan reducible systems in the regressive case. Also we give conditions under which the obtained characterizations can be exactly calculated and explicitly calculate the region of stability for several examples.
2000 Mathematics Subject Classification. 34C11, 39A10, 37B55, 34D99
Keywords. Time Scale, Linear Dynamic Equation, Exponential Stability

1 Introduction

It is well-known that exponential decay of the solution of a linear autonomous ordinary differential equation (ODE) $\dot{x}(t) = Ax(t)$, $t \in \mathbb{R}$, or of an autonomous difference equation (OΔE) $x_{t+1} = Ax_t$, $t \in \mathbb{Z}$, can be characterized by spectral properties of A. Namely, the solutions tend to 0 exponentially as $t \to \infty$, if and only if all the eigenvalues of $A \in \mathbb{C}^{d \times d}$ have negative real parts or a modulus smaller than 1, respectively (cf. Hahn [6, p. 14], Agarwal [1, p. 227]). In the present paper we generalize this classical result to linear time-invariant dynamic equations $x^{\Delta} = Ax$ on arbitrary time scales. Here the problem is more subtle due to the possible inhomogeneity of the time scale and so far only sufficient conditions for the exponential decay of solutions are available.

The first result concerning the case of general time scales was obtained by Aulbach & Hilger [2, Theorem 13] and it contains a condition for the boundedness of solutions on time scales with bounded graininess. Although it unifies the time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, $h > 0$, its assumptions are often too pessimistic, e.g. on asymptotically homogeneous time scales (cf. Example 20), since the maximal graininess is involved. More detailed results are presented in Keller [10, p. 29, Satz 2.5.8], including criteria for asymptotic stability or instability. A totally different approach to the asymptotic stability of linear dynamic equations using Lyapunov functions can be found in Hilger & Kloeden [9, Theorem 3] and Pötzsche [11, Abschnitt 2.1] provides sufficient conditions for the uniform exponential stability in (infinite-dimensional) Banach spaces, as well as spectral stability conditions for time-varying systems on time scales.

As a thorough introduction into dynamic equations on time scales we refer to the paper Hilger [8] or the monograph Bohner & Peterson [3]. The paper Aulbach & Hilger [2] presents the theory with a focus on linear systems.

This paper is organized as follows. In Section 2 we introduce the class of systems we wish to study and define the concepts of exponential, uniform exponential and robust exponential stability. In Section 3 we completely analyze the case of scalar systems and use this characterization to define the set of exponential stability. We discuss some basic properties of this set and present several cases in which this set is easily calculated. To make the step to higher dimensions we consider in Section 4 the case of Jordan reducible time-varying systems. We introduce the notion of monomials on a time scale and show growth conditions for such monomials under the condition of uniform graininess. These results are used in Section 5 to study exponential stability for regressive matrices. The general case is investigated in Section 6.

First, however, we fix some notation. In the following $\mathbb{K}$ denotes the real ($\mathbb{K} = \mathbb{R}$) or the complex ($\mathbb{K} = \mathbb{C}$) field. For a complex number $z \in \mathbb{C}$ we denote by $\Re z$ and $\Im z$ the real and the imaginary part, respectively, and $B_\varepsilon(z)$ is the open ball with center z and radius $\varepsilon > 0$ in the complex plane. As usual, $\mathbb{K}^{d\times d}$ is the space of square matrices with d rows, I_d is the identity mapping on the d-dimensional space $\mathbb{K}^d$ over $\mathbb{K}$ and $\sigma(A) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix $A \in \mathbb{K}^{d\times d}$.

We also introduce some notions which are specific for the calculus on time scales. A *time scale* $\mathbb{T}$ is a non-empty, closed subset of the reals $\mathbb{R}$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa := \mathbb{T} \setminus \{m\}$ and otherwise $\mathbb{T}^\kappa := \mathbb{T}$. If $\mathbb{T}$ is unbounded above then $\mathbb{T}^\kappa = \mathbb{T}$. On the subset $\mathbb{T}^\kappa$ the *graininess* is defined as $\mu^*(t) := \inf\{s \in \mathbb{T} : t < s\} - t$. A time scale $\mathbb{T}$ which is unbounded above is called *homogeneous* if the graininess is constant. If $\lim_{t\to\infty} \mu^*(t)$ exists, then $\mathbb{T}$ is said to be *asymptotically homogeneous*. The space of rd-continuous, regressive mappings from $\mathbb{T}^\kappa$ to $\mathbb{K}^{d\times d}$ is denoted by $\mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{K}^{d\times d})$. Furthermore, given a function $\lambda \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{C})$, then

$$(\hat{\Re}\lambda)(t) := \lim_{s \searrow \mu^*(t)} \frac{|1 + s\lambda(t)| - 1}{s} \quad \text{for } t \in \mathbb{T}^\kappa$$

is the *Hilger real part* of λ, and we have the inclusion $\hat{\Re}\lambda \in \mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R})$, where

$$\mathcal{C}^+_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R}) := \{\alpha \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{R}) : 1 + \mu^*(t)\alpha(t) > 0 \text{ for } t \in \mathbb{T}^\kappa\}.$$

2 Preliminaries

In this section we define the class of systems we consider and several notions of stability associated to these systems. We show by example that these notions do not coincide. To begin with we work with time-varying systems as our first statements are also applicable in this case.

Let $A : \mathbb{T}^\kappa \to \mathbb{K}^{d\times d}$ be rd-continuous and consider the d-dimensional linear system of dynamic equations

$$\boxed{x^\Delta = A(t)x.} \tag{1}$$

Let $\Phi_A : \{(t,\tau) \in \mathbb{T}^\kappa \times \mathbb{T}^\kappa : t \geq \tau\} \to \mathbb{K}^{d\times d}$ denote the *transition matrix* corresponding to (1), that is, $\Phi_A(t,\tau)\xi$ solves the initial value problem (1) with initial condition $x(\tau) = \xi$ for $\xi \in \mathbb{K}^d$ and $t, \tau \in \mathbb{T}$ with $t \geq \tau$. The classical examples for this setup are the following.

Example 1. If $\mathbb{T} = \mathbb{R}$ we consider linear time-varying systems of the form $\dot{x}(t) = A(t)x(t)$. If $\mathbb{T} = h\mathbb{Z}$ then (1) reduces to $(x(t+h)-x(t))/h = A(t)x(t)$ or equivalently $x(t+h) = [I_d + hA(t)]x(t)$.

We are interested in the stability of the equilibrium position $x^* = 0$ of system (1) and introduce the following definitions.

Definition 2 (Exponential stability). Let $\mathbb{T}$ be a time scale which is unbounded above. We call system (1)

(i) *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $t_0 \in \mathbb{T}$ there exists a $K = K(t_0) \geq 1$ with

$$\|\Phi_A(t, t_0)\| \leq Ke^{-\alpha(t-t_0)} \quad \text{for } t \geq t_0 \;, \tag{2}$$

(ii) *uniformly exponentially stable* if K can be chosen independently of t_0 in the definition of exponential stability,

(iii) *robustly exponentially stable* if there is an $\varepsilon > 0$ such that the exponential stability of (1) implies the exponential stability of $x^\Delta = B(t)x$ for any rd-continuous $B : \mathbb{T} \to \mathbb{K}^{d\times d}$ with $\sup_{t\in\mathbb{T}} \|B(t) - A(t)\| \leq \varepsilon$. In particular, if A is constant we call (1) robustly exponentially stable if for all matrices B in a suitable neighborhood of A the corresponding system is exponentially stable.

Remark 3. (i) To the purist it may seem inadequate to define exponential stability for system (1) via the standard real exponential function instead of the exponential function $e_\alpha(t, t_0)$ on time scales, since the real exponential function has no intrinsic meaning on a general time scale. Although this may be the case we argue that our characterization gives a strong description of the asymptotic behavior of a solution which we believe to be of interest. The methods we employ are closely tied to our definition, which also makes us believe that the definition is right, as it is fruitful. Also the use of the real exponential function makes our result accessible to readers, who are not familiar with the "time scale calculus." Finally, let us point out that we deduce a criterion for exponential stability in Theorem 18 involving $e_\alpha(t, t_0)$.

(ii) The notion of exponential stability for linear time-varying systems is defined in different ways according to different authors. For example our notion of uniform exponential stability is called *exponential asymptotic stability* in Agarwal [1, p. 240, Definition 5.4.1(xi)], whereas our exponential stability is not defined in that book. Cesari [4] avoids the concept of exponential stability but introduces the difference between uniform and nonuniform asymptotic stability of linear time-varying systems, which is the distinction that we want to emphasize. In the terminology of DALECKII & KREIN [5] exponential stability means negativity of the maximal Lyapunov exponent, whereas uniform exponential stability means negativity of the maximal Bohl exponent.

(iii) It is well-known that the three notions of stability from Definition 2 coincide in the autonomous case for ODEs and OΔEs. As Example 4 below demonstrates, this fails to be true on inhomogeneous time scales. Consequently, it is advantageous to distinguish between uniformly exponentially stable and only exponentially stable time-invariant dynamic equations, which are our main topic in Section 5. Indeed the main result of this paper (Theorem 21) is based on an estimate of the type (2), where K is allowed to depend on t_0.

(iv) It can be shown that uniform exponential stability of a linear system implies robust exponential stability [11, Abschnitt 1.3]. Thus there was no call for the definition of "robust uniform exponential stability".

Before we proceed with our analysis of properties characterizing exponential stability we will first present some examples showing that even for time-invariant systems the different notions need not coincide. Furthermore, we present a negative example pertaining to the question of linearization theory. In particular, we show by example that in the time-invariant case

(i) exponential stability does not imply uniform exponential stability,

(ii) exponential stability does not imply robust exponential stability,

(iii) exponential stability of a linearization is not sufficient for local asymptotic stability of a nonlinear system linearized at a fixed point.

The examples are given in the order of the list above.

Example 4. Let $\mathbb{K} = \mathbb{R}$ and $d = 1$. We define a sequence s_k recursively by

$$s_0 := 0\,, \quad s_{k+1} := s_k + 3k + 1\,, \quad k \in \mathbb{N}_0\,,$$

and the time scale $\mathbb{T}$ by the discrete set

$$\mathbb{T} := \{0, 1, 4, 5, 8, 11, 12, \dots, s_k, s_k + 3, \dots, s_k + 3k, s_{k+1}, \dots\}\,.$$

Consider on $\mathbb{T}$ the scalar system

$$x^{\Delta} = -x\,. \tag{3}$$

For $k \geq 1$ elementary calculations yield for $t \in \mathbb{T}, x_0 \in \mathbb{R}$ that

$$\varphi(t, s_k - 1, x_0) = 0,\, t > s_k - 1\,, \quad \text{and } \varphi(s_k + 3k, s_k, x_0) = (-2)^k x_0\,.$$

This shows that the system (3) is exponentially stable, as all trajectories reach 0 in finite time. On the other hand the system is not uniformly exponentially stable, as a solution starting in $x_0 = 1$ may become arbitrarily large depending on the initial time t_0. This completes the example showing claim (i) from above.
We now show that the system is also not robustly exponentially stable. To this end let $|\eta| < 1/4$ and consider the system

$$x^\Delta = (-1+\eta)x\,.$$

Then we have

$$\varphi(s_{k+1}, s_k, x_0) = (-2+3\eta)^k \eta x_0\,.$$

Now for every $|\eta| < 1/4, \eta \neq 0$ there exists a k_0 such that for all $k > k_0$ we have $|(-2+3\eta)^k\eta| > 2$. Hence all nonzero trajectories start to grow after time s_k. As no nonzero trajectory reaches 0 in finite time this shows exponential instability.

To show that a linearization principle does not hold we consider a slight modification of the previous example.

Example 5. Let $\{b_k\}_{k\in\mathbb{N}}$ be a sequence of positive integers such that

$$\sum_{k=0}^{\infty} \frac{b_k}{3^k} = \infty\,.$$

Now define the sequence s_k recursively by

$$s_0 := 0\,, \quad s_{k+1} := s_k + 3b_k + 1\,, \quad k \in \mathbb{N}_0\,,$$

and the time scale $\mathbb{T}$ by

$$\mathbb{T} := \{\dots, s_k, s_k+3, \dots, s_k+3b_k, s_{k+1}, \dots\}\,.$$

Finally, consider the system

$$x^\Delta = -x - x^3\,.$$

Here, we have $x(s_k) = -x(s_{k-1})^3$ and $x(t+3) = -2x(t) - 3x(t)^3$, for $t \in \mathbb{T}, t \neq s_k, k \in \mathbb{N}$. This shows that all transitions are diffeomorphisms, hence no trajectory reaches 0 in finite time. Also we have

$$|\varphi(s_{k+1}-1, s_k, x_0)| > 2^{b_k}|x_0|\,, \quad \text{hence } |\varphi(s_{k+1}, s_k, x_0)| > 2^{3b_k}|x_0|^3\,,$$

and inductively

$$|\varphi(s_{k+l}, s_k, x_0)| > |x_0|^{3^l} \prod_{j=0}^{l-1} 2^{3^{(l-j)} b_{k+j}} .$$

It suffices that $|\varphi(s_{k+l}, s_k, x_0)| > 1$ so that the trajectories remain bounded away from 0 for all $t > s_{k+l}$. Thus we have to consider the condition

$$|x_0|^{3^l} \prod_{j=0}^{l-1} 2^{3^{(l-j)} b_{k+j}} > 1 ,$$

or equivalently

$$\log_2(|x_0|) > -\sum_{j=k}^{l} \frac{b_j}{3^j} .$$

As the sum on the right diverges this shows that for all initial conditions (s_k, x_0), $x_0 \neq 0$, the trajectory remains bounded away from 0 for all t large enough, so that the system is not asymptotically stable. It is immediate from Example 4 that the linearized system $x^\Delta = -x$ is exponentially stable.

3 The Set of Exponential Stability

From now on let $\mathbb{T}$ be a time scale which is unbounded above. In this section we define the subset of the complex plane which is relevant for a spectral characterization of exponential stability for linear time-invariant systems

$$\boxed{x^\Delta = Ax,} \tag{4}$$

where $A \in \mathbb{K}^{d \times d}$. To motivate this definition we begin with the analysis of scalar systems.

Proposition 6. Let $\mathbb{T}$ be a time scale which is unbounded above and let $\lambda \in \mathbb{C}$. The scalar system

$$x^\Delta = \lambda x , \quad x \in \mathbb{C} \tag{5}$$

is exponentially stable if and only if one of the following conditions is satisfied for arbitrary $t_0 \in \mathbb{T}$

(i) $\gamma(\lambda) := \limsup_{T\to\infty} \frac{1}{T-t_0} \int_{t_0}^{T} \lim_{s \searrow \mu^*(t)} \frac{\log|1+s\lambda|}{s} \Delta t < 0$,

(ii) $\forall T \in \mathbb{T}: \; \exists t \in \mathbb{T}$ with $t > T$ such that $1 + \mu^*(t)\lambda = 0$,

where we use the convention $\log 0 = -\infty$ in (i).

Proof. ($\Rightarrow$) Assume that (5) is exponentially stable and that $1 + \mu^*(t)\lambda \neq 0$ for all $t > t_0$ and some $t_0 \in \mathbb{T}$. Then HILGER [8, Theorem 7.4(iii)] implies the following explicit presentation of the modulus of the (possibly complex) evolution operator of (5)

$$|e_\lambda(T, t_0)| = \exp\left(\int_{t_0}^{T} \lim_{s \searrow \mu^*(t)} \frac{\log|1 + s\lambda|}{s} \, \Delta t\right) \quad \text{for } T \geq t_0$$

and the estimate $|e_\lambda(T, t_0)| \leq K e^{-\alpha(T - t_0)}$ for $T \geq t_0$ with $K = K(t_0) \geq 1$ yields

$$\int_{t_0}^{T} \lim_{s \searrow \mu^*(t)} \frac{\log|1 + s\lambda|}{s} \, \Delta t \leq -\alpha(T - t_0) + \log K \quad \text{for } T \geq t_0 \, .$$

We therefore have

$$\limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu^*(t)} \frac{\log|1 + s\lambda|}{s} \, \Delta t \leq -\alpha < 0$$

and the claim follows.

($\Leftarrow$) To prove the converse direction let $\tau \in \mathbb{T}$ be fixed. If $1 + \mu^*(t)\lambda = 0$ for some $t \geq \tau$, $t \in \mathbb{T}$, then trivially $\mu^*(t) > 0$ and

$$[x(t + \mu^*(t)) - x(t)]/\mu^*(t) = \lambda x(t)$$

or equivalently $x(t + \mu^*(t)) = 0$ and thus (5) is exponentially stable if for every $\tau \in \mathbb{T}$ there is a $\tau < t \in \mathbb{T}$ with this property. Now assume this is not the case so that $1 + \mu^*(T)\lambda \neq 0$ for all $T \geq \tau$ and some $\tau \in \mathbb{T}$ large enough, then

$$|e_\lambda(T, \tau)| = \exp\left(\int_{\tau}^{T} \lim_{s \searrow \mu^*(t)} \frac{\log|1 + s\lambda|}{s} \, \Delta t\right) \quad \text{for } T \geq \tau$$

and with

$$\alpha := -\limsup_{T \to \infty} \frac{1}{T - \tau} \int_{\tau}^{T} \lim_{s \searrow \mu^*(t)} \frac{\log|1 + s\lambda|}{s} \, \Delta t > 0$$

we obtain for any $\varepsilon > 0$ that there exists a constant $K = K(\tau) \geq 1$ such that

$$|e_\lambda(t, \tau)| \leq K e^{-(\alpha - \varepsilon)(t - \tau)} \quad \text{for } t \geq \tau \, .$$

In particular, if we choose $\varepsilon < \alpha$ we obtain exponential stability of (5). $\square$

In view of the previous definition the following notion appears to be appropriate.

Definition 7 (Set of exponential stability). Given a time scale $\mathbb{T}$ which is unbounded above we define for arbitrary $t_0 \in \mathbb{T}$

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) := \{\lambda \in \mathbb{C} \,:\, \limsup_{T\to\infty} \frac{1}{T-t_0} \int_{t_0}^{T} \lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda|}{s} \Delta t < 0\}$$

and

$$\mathcal{S}_{\mathbb{R}}(\mathbb{T}) := \{\lambda \in \mathbb{R} | \, \forall T \in \mathbb{T} : \exists t \in \mathbb{T} \text{ with } t > T \text{ such that } 1+\mu^*(t)\lambda = 0\} \,.$$

The *set of exponential stability* for the time scale $\mathbb{T}$ is then defined by

$$\mathcal{S}(\mathbb{T}) := \mathcal{S}_{\mathbb{C}}(\mathbb{T}) \cup \mathcal{S}_{\mathbb{R}}(\mathbb{T}) \,.$$

Remark 8. (i) Note that the definition of $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ is indipendent of t_0.
(ii) For any time scale $\mathbb{T}$ we have $\mathcal{S}_{\mathbb{C}}(\mathbb{T}) \subset \{\lambda \in \mathbb{C} \mid \Re\lambda < 0\}$ because $\Re\lambda \geq 0$ implies that $|1+s\lambda| \geq 1$ for all nonnegative $s \in \mathbb{R}$. Thus, if $\Re\lambda \geq 0$ the function appearing under the integral is nonnegative. Likewise, it is easy to see that $\mathcal{S}_{\mathbb{R}}(\mathbb{T}) \subset (-\infty, 0)$. Furthermore $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ is symmetric with respect to the real axis, as $|1+s\lambda| = |1+s\bar{\lambda}|$ for real s. As s is not only real but also positive, this implies that $|1+s\lambda_1| < |1+s\lambda_2|$ if $\Re\lambda_1 = \Re\lambda_2$ and $0 \leq \Im\lambda_1 < \Im\lambda_2$. This shows that if $\lambda \in \mathcal{S}_{\mathbb{C}}(\mathbb{T})$ then the segment $\{\Re\lambda + i\alpha\Im\lambda \mid \alpha \in [-1,1]\} \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$. In particular, the connected components of $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ are simply connected.
(iii) It is evident from the definition that $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ is an open subset of $\mathbb{C}$. On the other hand, given a time scale $\mathbb{T}$ the set $\mathcal{S}_{\mathbb{R}}(\mathbb{T})$ is at most countable, because the condition $\lambda \in \mathcal{S}_{\mathbb{R}}(\mathbb{T})$ implies that the time scale $\mathbb{T}$ has infinitely many "gaps" of length $|\lambda|^{-1}$. In every such gap there exists a rational number q_λ. If there were uncountable many $\lambda \in \mathcal{S}_{\mathbb{R}}(\mathbb{T})$, then there would exist an uncountable number of distinct rational numbers q_λ. This is impossible.
(iv) For regressive $\lambda \in \mathbb{C}$, we have

$$\lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda|}{s} = \begin{cases} \frac{\log|1+\mu^*(t)\lambda|}{\mu^*(t)} & \text{for } \mu^*(t) > 0 \\ \Re\lambda & \text{for } \mu^*(t) = 0 \end{cases} .$$

In general, the set $\mathcal{S}_{\mathbb{C}}$ is awkward to calculate because of the limit superior involved in the definition. We therefore present some criteria which allow for an easier calculation of $\gamma(\lambda)$.

Lemma 9. Let $\mathbb{T}$ be a time scale which is unbounded above and let $\lambda \in \mathbb{C}$.

(i) If $a := \lim_{t\to\infty} \lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda|}{s}$ exists then $\gamma(\lambda) = a$.

(ii) If there are $t_0 \in \mathbb{T}$, $p > 0$ such that for all $k \in \mathbb{N}_0$ we have $t_0+kp \in \mathbb{T}$ and

$$a_p := \frac{1}{p} \lim_{k\to\infty} \int_{t_0+kp}^{t_0+(k+1)p} \lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda|}{s} \Delta t$$

exists, then $\gamma(\lambda) = a_p$.

(iii) Let X be a compact metric space and $T : X \to X$ be a mapping that is uniformly ergodic with ergodic measure η. Let $\rho_1 : X \to (0,\infty)$ be continuous with image $[a,b]$ and $\rho_2 : X \to [0,\infty)$ be continuous. For every $x_0 \in X$ define a time scale $\mathbb{T}(x_0)$ by

$$\bigcup_{m\geq 0} \left[\sum_{k=0}^{m} \rho_1(T^k x_0) + \sum_{k=0}^{m-1} \rho_2(T^k x_0), \sum_{k=0}^{m} (\rho_1(T^k x_0) + \rho_2(T^k x_0)) \right] .$$

Then for every $\lambda \in \mathbb{C} \setminus [-a^{-1}, -b^{-1}]$ we have

$$\gamma(\lambda) = a_e := \frac{\int_X \rho_2(x)\Re\lambda + \log|1+\rho_1(x)\lambda| d\eta(x)}{\int_X \rho_1(x) + \rho_2(x) d\eta(x)} . \tag{6}$$

Remark 10. An illustrative interpretation of the time scale defined in (iii) is that there are continuous intervals of length $\rho_2(T^k x_0)$ alternating with "gaps" of length $\rho_1(T^k x_0)$. In particular, we can construct purely discrete time scales in this manner by choosing $\rho_2 \equiv 0$.

Proof. (i) and (ii) follow from easy calculations. To show (iii) we appeal to [13, Theorem 6.19] which shows in particular that unique ergodicity implies that for every initial condition $x_0 \in X$ and every continuous function $f : X \to \mathbb{R}$ we have that

$$\frac{1}{m} \sum_{k=0}^{m-1} f(T^k x_0) \to \int_X f(x) d\eta(x) .$$

Now the function $x \mapsto \rho_2(x)\Re\lambda$ is clearly continuous for all $\lambda \in \mathbb{C}$ and $x \mapsto \log|1+\rho_1(x)\lambda|$ is continuous for those $\lambda \in \mathbb{C}$ such that $1+\rho_1(x)\lambda \neq 0$ for all $x \in X$, that is in particular for $\lambda \notin [-a^{-1}, -b^{-1}]$. Then for $T_m = \sum_{k=0}^{m-1} (\rho_1(T^k x_0) + \rho_2(T^k x_0))$ we have

$$\frac{1}{T_m} \int_0^{T_m} \lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda|}{s} \Delta t =$$

$$= \frac{m}{T_m} \left(\frac{1}{m} \sum_{k=0}^{m-1} \rho_2(T^k x_0) \Re\lambda + \frac{1}{m} \sum_{k=0}^{m-1} \log |1 + \rho_1(T^k x_0)\lambda| \right) .$$

By continuity of $\rho_1 + \rho_2$ the expression T_m/m converges to $\int_X \rho_1 + \rho_2 \, d\eta$ and the limit is nonzero as ρ_1 is strictly positive. Thus also m/T_m converges and we obtain that the expression to the right converges to a_e for $m \to \infty$. For those $T \in \mathbb{T}(x_0)$ that are not of the form T_m we have at least that for some $m \in \mathbb{N}_0$ it holds that $0 \leq T - T_m \leq \max_{x \in X} \rho_2(x)$. Using this fact an easy calculation shows that indeed $\gamma(\lambda) = a_e$. This concludes the proof. $\square$

We note the following examples in order to show the applicability of the previous lemma.

Example 11. (i) Consider the time scale $\mathbb{T} = h\mathbb{Z}$, $h > 0$, with $\mathcal{S}_{\mathbb{R}}(h\mathbb{Z}) = \{-\frac{1}{h}\}$. An application of Lemma 9 (i) shows $\mathcal{S}(h\mathbb{Z}) = B_{\frac{1}{h}}(-\frac{1}{h})$, as expected.

(ii) If $\mathbb{T} = \mathbb{R}$ we obtain $\mathcal{S}_{\mathbb{R}}(\mathbb{R}) = \emptyset$ and from Lemma 9 (i) that $\mathcal{S}(\mathbb{R}) = \{\lambda \in \mathbb{C} : \Re\lambda < 0\}$.

(iii) Consider the time scale $\mathbb{T} = \{t_n\}_{n \in \mathbb{N}}$ of so-called *harmonic numbers* $t_n := \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}$, which is unbounded above. The graininess is given by $\mu^*(t_n) = \frac{1}{n+1}$. Using methods from elementary calculus it can be shown that $\lim_{x \to \infty} x \log \left|1 + \frac{\lambda}{x}\right| = \Re\lambda$ for $\lambda \in \mathbb{C}$ and consequently

$$\lim_{t \to \infty} \lim_{s \searrow \mu^*(t)} \frac{\log |1 + s\lambda|}{s} = \lim_{n \to \infty} (n+1) \log \left| 1 + \frac{\lambda}{n+1} \right| = \Re\lambda \quad \text{for } \lambda \in \mathbb{C}.$$

Now from Lemma 9 (i) we obtain $\{\lambda \in \mathbb{C} : \Re\lambda < 0\} = \mathcal{S}_{\mathbb{C}}(\mathbb{T}) = \mathcal{S}(\mathbb{T})$. Note that no gap occurs an infinite number of times, so that $\mathcal{S}_{\mathbb{R}}(\mathbb{T}) = \emptyset$.

(iv) Let $\mathbb{T}_\sigma = \bigcup_{k \in \mathbb{N}_0} [k, k + \sigma]$, $\sigma \in (0, 1)$, be a union of closed intervals. To calculate the set of exponential stability for this time scale we observe that

$$\begin{aligned} \int_k^{k+1} \lim_{s \searrow \mu^*(t)} \frac{\log |1 + s\lambda|}{s} \Delta t &= \int_k^{k+\sigma} \Re\lambda \, dt + \log |1 + (1 - \sigma)\lambda| = \\ &= \sigma\Re\lambda + \log |1 + (1 - \sigma)\lambda| \quad \text{for } k \in \mathbb{N}_0 \end{aligned}$$

and consequently by Lemma 9 (ii) with $t_0 = 0$, $p = 1$ we have

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma) = \{\lambda \in \mathbb{C} : \sigma\Re\lambda + \log |1 + (1 - \sigma)\lambda| < 0\} .$$

Also it is clear that $\mathcal{S}_{\mathbb{R}}(\mathbb{T}_\sigma) = \{(\sigma - 1)^{-1}\} \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma)$. This representation includes the limit cases $\mathcal{S}(\mathbb{T}_0) = \mathcal{S}(\mathbb{Z})$ and $\mathcal{S}(\mathbb{T}_1) = \mathcal{S}(\mathbb{R})$. In

Figure 1 we show the stability region for the examples. In each picture the set $\mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma)$ is given by the hatched area. Note in particular that for the value $\sigma = 0.21$ the stability region is disconnected. Let us briefly discuss for which values of σ there are disconnected stability regions in this example. By Remark 8 (ii) we have that $\lambda \in \mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma)$ implies $\Re\lambda \in \mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma)$ so that we only have to investigate the question for which $\sigma \in (0,1)$ the set $\mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma) \cap (-\infty, 0)$ is disconnected.

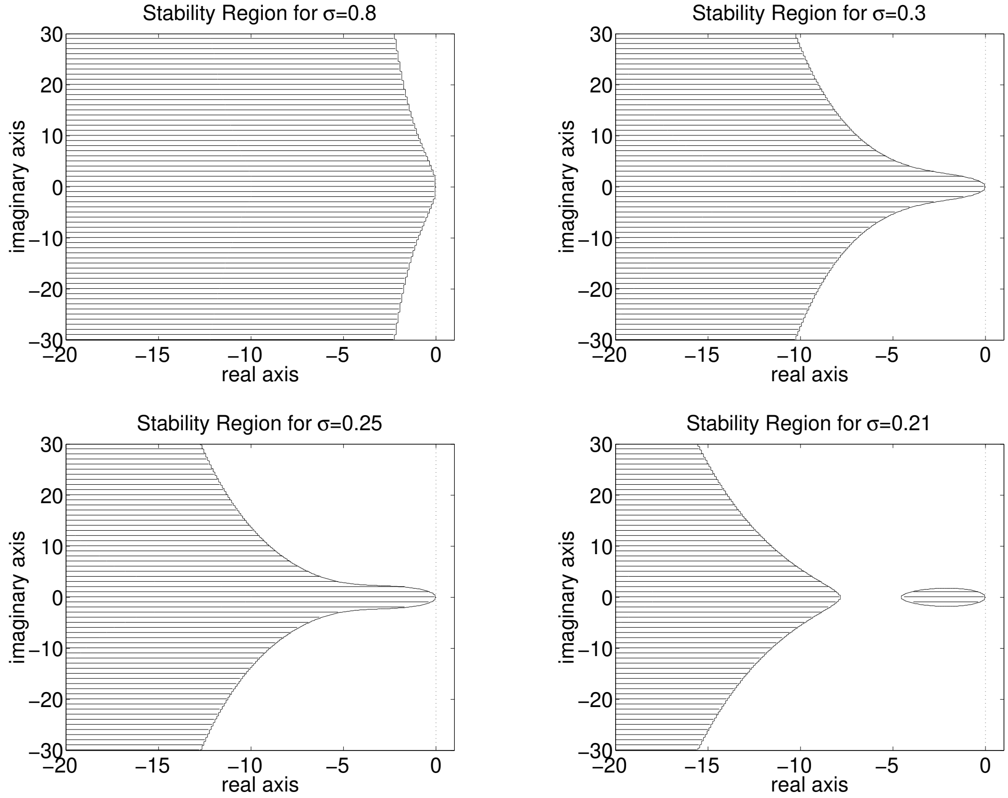

Figure 1: Stability regions as described in Example 11 (iv) for different values of σ

For $\lambda \in (-\infty, 0)$ to be in the set $\mathcal{S}_{\mathbb{C}}(\mathbb{T}_\sigma)$ it is necessary and sufficient that $|1+(1-\sigma)\lambda| < e^{-\sigma\lambda}$. If $\lambda \in [(\sigma-1)^{-1}, 0)]$ it is easy to see that this is always the case. So that we now consider the case $\lambda \in J_\sigma := (-\infty, (\sigma - 1)^{-1}]$. Here we have to satisfy the inequality $(\sigma - 1)\lambda - 1 < e^{-\sigma\lambda}$. It is clear that this is satisfied for negative λ with $|\lambda|$ large enough. However, by using standard calculus it is easy to see that for $\sigma \in (0, 1/2)$ there is a unique local maximum of the function $f_\sigma(\lambda) := (\sigma - 1)\lambda - 1 - e^{-\sigma\lambda}$ at $\lambda_\sigma = \sigma^{-1}\log(\sigma(1-\sigma)^{-1})$. The requirement that $\lambda_\sigma \in J_\sigma$ or equivalently, $\sigma/(\sigma - 1) > \log(\sigma/(1-\sigma))$ implies that $\sigma \in (0, a)$ with a constant $a \approx 0.361896$. Now we are interested in the question for which σ we have $f_\sigma(\lambda_\sigma) \geq 0$. This leads to the condition $(\sigma-1)^{-1} \geq \log(\sigma/(1-\sigma))$ which is true for $\sigma \in (0, b]$ with $b \approx 0.2178117$. In all we have shown that the stability region is disconnected if and only if $\sigma \in (0, b]$.

(v) It is known that for $\alpha \in [0,1] \setminus \mathbb{Q}$ the map

$$x \mapsto x + \alpha \mod 1$$

is uniquely ergodic. As the continuous functions ρ_1, ρ_2 we choose $\rho_1(x) = 1 + (x - 1/2)^2$, $\rho_2(x) = \sin(\pi x)$ and by Lemma 9 (iii) we have that

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \frac{2}{\pi}\Re\lambda + \int_0^1 \log \left| 1 + \lambda + \left(x - \frac{1}{2}\right)^2 \lambda \right| dx < 0 \right\}.$$

If we choose $\rho_2 \equiv 0$, then we obtain

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \int_0^1 \log \left| 1 + \lambda + \left(x - \frac{1}{2}\right)^2 \lambda \right| dx < 0 \right\}.$$

In Figure 2 a sketch of the stability regions corresponding to the different choices of ρ_2 is shown.

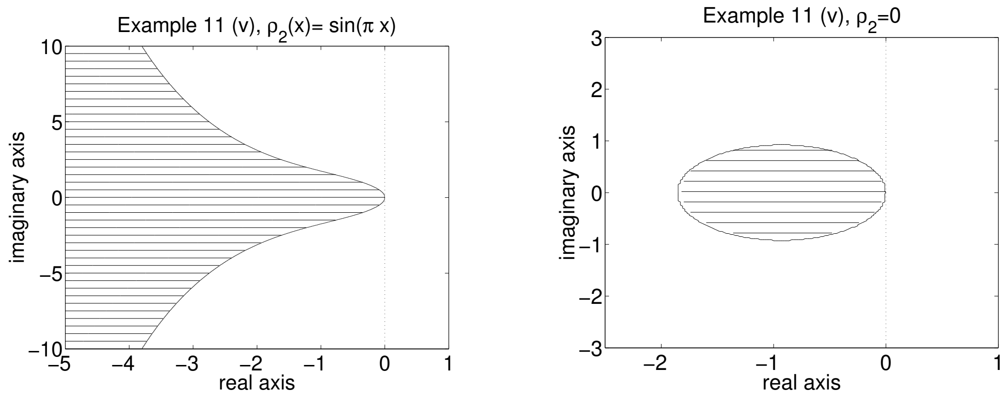

Figure 2: Two stability regions as described in Example 11 (v) with our without continuous intervals

If we choose $\alpha \in \mathbb{Q}$ the map $x \mapsto x + \alpha \mod 1$ is periodic. And so if we consider the time scales described in Lemma 9 (iii) given by the maps ρ_1 from above and $\rho_2 \equiv 0$ the stability region can be calculated by virtue of Lemma 9 (ii). In principle, this region now depends on the initial condition. For the choice $\alpha = 1/2$ this difference is easily noticeable and the stability regions for $x_0 = 0.0099$ and $x_0 = 0.7382$ are shown in Figure 3. For the choice $\alpha = 1/20$ however, we were not able to produce pictures that give any noticeable difference (although it exists of course).

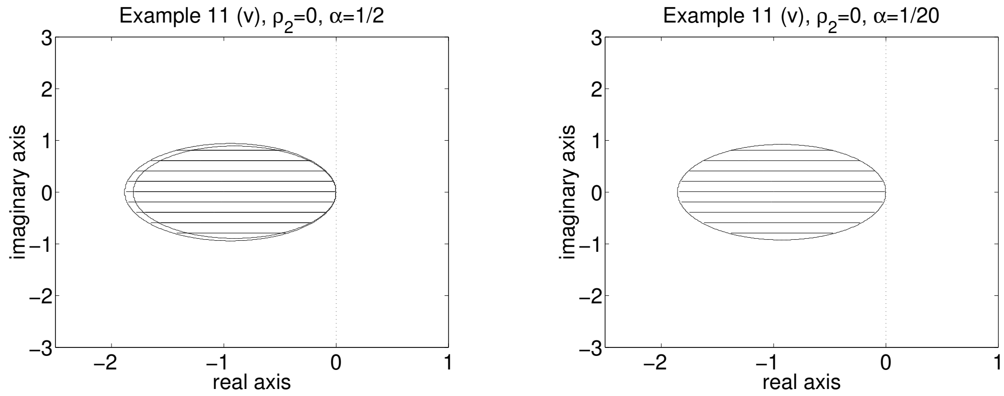

Figure 3: Two stability regions as described in Example 11 (v) with rational α.

(vi) Finally, let $\mathbb{T}$ be obtained by gluing together identical Cantor sets. That is, if M_C denotes the standard Cantor set obtained as the limit (in the Hausdorff topology) of the compact sets M_k recursively defined by

$$M_0 := [0,1]\,, \quad M_k := M_{k-1} \setminus \bigcup_{j=0}^{2^{k-1}-1} \left(\frac{6j+1}{3^k}, \frac{6j+2}{3^k} \right)\,,$$

then we define $\mathbb{T}$ by $t \in \mathbb{T} \Leftrightarrow t - n \in M_C$ for some $n \in \mathbb{N}_0$. This time scale is clearly periodic so that we may apply Lemma 9 (ii) to obtain that

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \sum_{k=1}^{\infty} 2^{k-1} \log \left| 1 + \frac{1}{3^k}\lambda \right| < 0 \right\}\,,$$

because there are always 2^{k-1} gaps of length 3^{-k} for $k = 1, 2, \ldots$. An approximation of this set is shown in Figure 4. As the Cantor set itself has measure 0 the points $t \in \mathbb{T}$ with $\mu^*(t) = 0$ do not contribute to the definition of the set of exponential stability. Moreover, since $(-\infty, 0) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$, Remark 8 (ii) yields $\mathcal{S}_{\mathbb{R}}(\mathbb{T}) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T}) = \mathcal{S}(\mathbb{T})$.

In the remainder of the article we discuss higher dimensional systems.

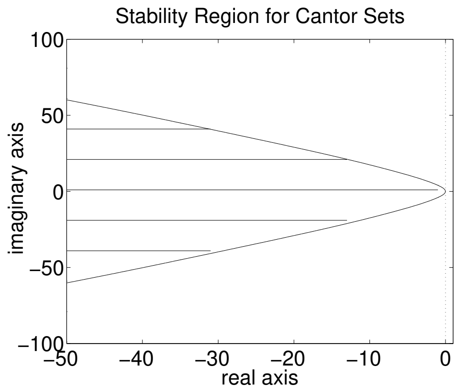

Figure 4: The stability region for repeated Cantor sets

4 Jordan Reducible Systems

Let $A : \mathbb{T}^\kappa \to \mathbb{K}^{d\times d}$ be an rd-continuous mapping. In this section we consider d-dimensional time-varying linear systems (1), which are *Jordan reducible*, i.e. there exist (constant) invertible matrices $S \in \mathbb{C}^{d\times d}$ such that

$$S^{-1}A(t)S = \begin{pmatrix} J_1(t) & & \\ & \ddots & \\ & & J_n(t) \end{pmatrix} =: J(t) \quad \text{for } t \in \mathbb{T}^\kappa, \tag{7}$$

where each $J_i(t) \in \mathbb{C}^{d_i\times d_i}$, $d_1 + \ldots + d_n = d$, $1 \le i \le n \le d$, is a Jordan block

$$J_i(t) := \begin{pmatrix} \lambda_i(t) & 1 & 0 & \ldots & 0 \\ & \lambda_i(t) & 1 & \ldots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda_i(t) \end{pmatrix} \quad \text{for } t \in \mathbb{T}^\kappa.$$

Evidently time-invariant systems (1) are Jordan reducible and in this case λ_i is a constant eigenvalue of A. We first note that Jordan reducibility allows for a block decomposition of the transition matrix.

Theorem 12. Suppose $A \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{K}^{d\times d})$ is such that (1) is Jordan reducible. Then the transition matrix of $x^\Delta = A(t)x$ is given by

$$\Phi_A(t,\tau) = S \begin{pmatrix} \Phi_{J_1}(t,\tau) & & \\ & \ddots & \\ & & \Phi_{J_n}(t,\tau) \end{pmatrix} S^{-1} \quad \text{for } t,\tau \in \mathbb{T}^\kappa, \tag{8}$$

where we have used the notation introduced in (7). If A is not regressive then the representation (8) holds for $t \geq \tau \in \mathbb{T}$.

Proof. For the matrix function $\Psi(t) := S\Phi_J(t,\tau)S^{-1}$ the identity

$$\Psi^\Delta(t) = SJ(t)\Phi_J(t,\tau)S^{-1} \overset{(7)}{=} SS^{-1}A(t)S\Phi_J(t,\tau)S^{-1} = A(t)\Psi(t)$$

for $t \in \mathbb{T}^\kappa$ holds and because of $\Psi(\tau) = I_d$ we obtain the assertion. This calculation can be performed without further assumptions on A if $t \geq \tau$, so that $\Phi_J(t,\tau)$ is well defined. This proves the second statement. $\square$

It is our goal to give an expression for the transition matrix of Jordan reducible equations. This, however, needs some preparation.

Definition 13 (Monomials). For each $n \in \mathbb{N}_0$ and $\lambda \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{C})$ the mappings $m_\lambda^n : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{C}$, recursively defined by

$$m_\lambda^0(t,\tau) :\equiv 1, \qquad m_\lambda^{n+1}(t,\tau) := \int_\tau^t \frac{m_\lambda^n(s,\tau)}{1+\mu^*(s)\lambda(s)}\,\Delta s \quad \text{for } n \in \mathbb{N}_0, \tag{9}$$

are called *monomials of degree n*.

Example 14. On homogeneous time scales with graininess $\mu^*(t) \equiv h \geq 0$ and for regressive constants $\lambda \in \mathbb{C}$ we obtain $m_\lambda^n(t,\tau) = \frac{(t-\tau)^n}{n!(1+h\lambda)^n}$ for $t,\tau \in \mathbb{T}$.

Lemma 15. Consider a mapping $\lambda \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{C})$ which is *uniformly regressive*, i.e. there exists a $\gamma > 0$ such that

$$\gamma^{-1} \leq |1+\mu^*(t)\lambda(t)| \quad \text{for } t \in \mathbb{T}^\kappa. \tag{10}$$

Then the estimate $|m_\lambda^n(t,\tau)| \leq \gamma^n(t-\tau)^n$ holds for $t \geq \tau$ and $n \in \mathbb{N}_0$.

Proof. The proof is obtained using an easy induction argument. Trivially the desired estimate holds for $n = 0$. The induction step $n \to n+1$ follows from

$$\begin{aligned}
\left|m_\lambda^{n+1}(t,\tau)\right| &\overset{(9)}{\leq} \int_\tau^t \left|\frac{m_\lambda^n(s,\tau)}{1+\mu^*(s)\lambda(s)}\right| \Delta s \leq \int_\tau^t \frac{\gamma^n(s-\tau)^n}{|1+\mu^*(s)\lambda(s)|}\Delta s \leq \\
&\overset{(10)}{\leq} \gamma^{n+1}\int_\tau^t (s-\tau)^n\,\Delta s \leq \gamma^{n+1}\int_\tau^t (t-\tau)^n\,\Delta s = \\
&= \gamma^{n+1}(t-\tau)^{n+1} \quad \text{for } t \geq \tau\,,
\end{aligned}$$

as desired. $\square$

Lemma 16. If $\lambda \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}^\kappa, \mathbb{C})$ and if $J_\lambda : \mathbb{T}^\kappa \to \mathbb{C}^{d\times d}$,

$$J_\lambda(t) := \begin{pmatrix} \lambda(t) & 1 & 0 & \dots & 0 \\ & \lambda(t) & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda(t) \end{pmatrix}$$

denotes a mapping with values in (complex) Jordan canonical form, then the transition matrix of $x^\Delta = J_\lambda(t)x$ is given by

$$\Phi_{J_\lambda}(t,\tau) = e_\lambda(t,\tau) \begin{pmatrix} 1 & m_\lambda^1(t,\tau) & \dots & m_\lambda^{d-1}(t,\tau) \\ & 1 & \dots & m_\lambda^{d-2}(t,\tau) \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \quad \text{for } t,\tau \in \mathbb{T}^\kappa.$$

Proof. Obviously $\Phi_{J_\lambda}(\tau,\tau) = I_d$ by Definition 13, and for arbitrary $\tau \in \mathbb{T}^\kappa$ an elementary calculation using the product rule (cf. Hilger [8, Theorem 2.6(ii)]) yields the identity $\Phi_{J_\lambda}(\cdot,\tau)^\Delta(t) = J(t)\Phi_{J_\lambda}(t,\tau)$ for $t \in \mathbb{T}^\kappa$. □

Lemma 17. Consider mappings $\alpha \in \mathcal{C}_{rd}^{+}\mathcal{R}(\mathbb{T},\mathbb{R})$, $\lambda \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T},\mathbb{C})$ on a time scale $\mathbb{T}$ with bounded graininess and which is unbounded above. Under the assumption

$$\exists T \in \mathbb{T} : 0 < \inf_{t\in[T,\infty)} \big[\alpha(t) - (\hat{\Re}\lambda)(t)\big]$$

it holds that

$$\lim_{t\to\infty} m_\lambda^n(t,\tau) e_{\lambda\ominus\alpha}(t,\tau) = 0 \quad \text{for } \tau \in \mathbb{T},\ n \in \mathbb{N}_0.$$

Proof. Using the decomposition from Hilger [8, Theorem 7.4(ii)] it suffices to show $\lim_{t\to\infty} m_\lambda^n(t,\tau)e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau) = 0$. To do this we proceed by mathematical induction over $n \in \mathbb{N}_0$. For $n = 0$ we have $m_\lambda^0(t,\tau) = 1$ (cf. (9)) and the assertion follows by Hilger [7, p. 59, Satz 9.2], namely

$$\lim_{t\to\infty} e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau) = 0 \quad \text{for } \tau \in \mathbb{T}.$$

Now keeping $n \in \mathbb{N}_0$ fixed, by assumption the relation

$$\begin{aligned} 0 \ &\le \ \left| \frac{m_\lambda^n(t,\tau)e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau)}{(\alpha\ominus\hat{\Re}\lambda)(t)\,(1+\mu^*(t)\lambda(t))} \right| = \left| \frac{m_\lambda^n(t,\tau)e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau)}{\alpha(t)-(\hat{\Re}\lambda)(t)} \right| \le \\ &\le \ \left| \frac{m_\lambda^n(t,\tau)e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau)}{\inf_{t\in[T,\infty)}\big[\alpha(t)-(\hat{\Re}\lambda)(t)\big]} \right| \xrightarrow[t\to\infty]{} 0 \quad \text{for } \tau \in \mathbb{T} \end{aligned}$$

holds. Therefore the Theorem of de l'Hospital (cf. BOHNER & PETERSON [3, p. 48, Theorem 1.120]), applied separately to the real and imaginary part, leads to

$$\begin{aligned}
&\lim_{t\to\infty} m_\lambda^{n+1}(t,\tau)e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau) = \\
\overset{(9)}{=}\; &\lim_{t\to\infty}\left[\int_\tau^t \Re\frac{m_\lambda^n(s,\tau)}{1+\mu^*(s)\lambda(s)}\,\Delta s + \int_\tau^t \Im\frac{m_\lambda^n(s,\tau)}{1+\mu^*(s)\lambda(s)}\,\Delta s\right] e_{\alpha\ominus\hat{\Re}\lambda}(t,\tau)^{-1} = \\
= \;&\lim_{t\to\infty}\left[\Re\frac{m_\lambda^n(t,\tau)}{1+\mu^*(t)\lambda(t)} + \Im\frac{m_\lambda^n(t,\tau)}{1+\mu^*(t)\lambda(t)}\right]\frac{e_{\hat{\Re}\lambda\ominus\alpha}(t,\tau)}{(\alpha\ominus\hat{\Re}\lambda)(t)} = \\
= \;&\lim_{t\to\infty}\frac{m_\lambda^n(t,\tau)e_{\hat{\Re}\lambda\ominus\alpha(t,\tau)}}{(\alpha\ominus\hat{\Re}\lambda)(t)(1+\mu^*(t)\lambda(t))} = 0 \quad \text{for } \tau\in\mathbb{T},
\end{aligned}$$

which proves our lemma. □

Theorem 18. Suppose $A \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T},\mathbb{K}^{d\times d})$ and that (1) is Jordan reducible on a time scale $\mathbb{T}$ with bounded graininess and which is bounded above. If there exists an $\alpha \in \mathcal{C}_{rd}^+\mathcal{R}(\mathbb{T},\mathbb{R})$ such that

$$\exists T\in\mathbb{T}:\ \forall i\in\{1,\dots,n\}:\ 0<\inf_{t\in[T,\infty)}\big[\alpha(t)-(\hat{\Re}\lambda_i)(t)\big] \tag{11}$$

holds, then the transition matrix of (1) satisfies

$$\lim_{t\to\infty} e_\alpha(t,\tau)\Phi_A(t,\tau) = 0 \quad \text{for } \tau\in\mathbb{T}.$$

Proof. By Theorem 12 and in particular by the representation (8) we only have to show $\lim_{t\to\infty} e_\alpha(t,\tau)\Phi_{J_i}(t,\tau) = 0$ for $i\in\{1,\dots,n\}$. This is immediate from Lemma 16, since the assumption (11) allows for the application of Lemma 17. □

Remark 19. Theorem 18 can be used to show the stability of linear time-varying systems using time-dependent eigenvalues. Here the assumption of a Jordan reducible mapping A is essential as classical examples for ODEs show (cf. HAHN [6, p. 307]).

Example 20. Consider an asymptotically homogeneous time scale $\mathbb{T}$ with $h := \lim_{t\to\infty}\mu^*(t) > 0$ and assume that $A \in \mathbb{K}^{d\times d}$ is a regressive matrix such that $\sigma(A) \subset B_{\frac{1}{h}}\left(-\frac{1}{h}\right)$. Then for the time-invariant system $x^\Delta = Ax$ the inequality $\frac{|1+h\lambda|-1}{h} < 0$ holds for $\lambda\in\sigma(A)$ and there exists a $T\in\mathbb{T}$ such that $0<\inf_{t\in[T,\infty)}\big[-(\hat{\Re}\lambda)(t)\big]$. Therefore $\Phi_A(t,t_0)$, $t_0\in\mathbb{T}$, tends to 0 as $t\to\infty$.

5 The Regressive Case

In this section we assume that $A \in \mathbb{K}^{d\times d}$ is regressive, so that we may freely use all the results obtained in Section 4. Moreover the eigenvalues $\lambda \in \sigma(A)$ are regressive (see Bohner & Peterson [3, Exercise 5.6, p. 190]).

Theorem 21 (Characterization of exponential stability). Let $\mathbb{T}$ be a time scale which is unbounded above and let $A \in \mathbb{K}^{d\times d}$ be regressive. Then the following holds:

(a) If the system (4) is exponentially stable, then $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$.

(b) If (10) holds for all eigenvalues λ of A and if $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$, then (4) is exponentially stable.

Proof. To begin with, we choose an invertible matrix $S \in \mathbb{C}^{d\times d}$ such that $J := S^{-1}AS$ is in Jordan canonical form and let the matrix A have the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $n \le d$. Throughout this proof we use the induced matrix norm

$$\|A\| := \max_{i\in\{1,\dots,d\}} \sum_{j=1}^{d} |a_{ij}| \tag{12}$$

of $A = (a_{ij})_{i,j\in\{1,\dots,d\}}$; since all norms on $\mathbb{K}^{d\times d}$ are equivalent, this is sufficient for our purpose. Now suppose $t_0 \in \mathbb{T}$ is fixed.

(a) Assume that (4) is exponentially stable. Then if v is an eigenvector corresponding to $\lambda \in \sigma(A)$, by Bohner & Peterson [3, p. 198, Theorem 5.30] we have that

$$|e_\lambda(t,\tau)|\|v\| = \|\Phi_A(t,\tau)v\| \le K_\tau e^{-\alpha(t-\tau)}\|v\|\,, \quad t \ge \tau\,,$$

for suitable constants $K_\tau, \alpha > 0$. This shows that $|e_\lambda(t,\tau)| \le K_\tau e^{-\alpha(t-\tau)}$ and Proposition 6 implies that $\lambda \in \mathcal{S}_{\mathbb{C}}(\mathbb{T})$. As $\lambda \in \sigma(A)$ was arbitrary, this completes the proof.

(b) Since the eigenvalues of $A \in \mathbb{K}^{d\times d}$ are assumed to be uniformly regressive, there exists a $\gamma > 0$ such that

$$\gamma^{-1} \le |1 + \mu^*(t)\lambda_i| \quad \text{for } t \in \mathbb{T},\ i \in \{1,\dots,n\}\,.$$

Then for arbitrary $i \in \{1,\dots,n\}$ it follows as above

$$|e_{\lambda_i}(T,t_0)| = \exp\left(\int_{t_0}^{T} \lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda_i|}{s}\,\Delta t\right) \quad \text{for } T \ge t_0$$

and with $\alpha := \frac{1}{2}\min\{\alpha_1, \dots, \alpha_n\}$, where

$$\alpha_i := -\limsup_{T\to\infty} \frac{1}{T-t_0} \int_{t_0}^{T} \lim_{s\searrow\mu^*(t)} \frac{\log|1+s\lambda_i|}{s}\,\Delta t > 0,$$

we obtain the estimate

$$|e_{\lambda_i}(t,t_0)| \le K_1 e^{-\alpha(t-t_0)} \quad \text{for } t \ge t_0$$

with some real $K_1 = K_1(t_0) \ge 1$. On the other hand our Theorem 12 implies $\|\Phi_A(t,t_0)\| \le \|S\|\,\|S^{-1}\|\,\|\Phi_J(t,t_0)\|$ and since all the non-zero entries of the matrix $\Phi_J(t,t_0)$ are of the type $m_{\lambda_i}^{j_i}(t,t_0)e_{\lambda_i}(t,t_0)$ for some integer $j_i \in \{0,\dots,d_i-1\}$, Lemma 15 implies

$$\left|m_{\lambda_i}^{j_i}(t,t_0)e_{\lambda_i}(t,t_0)\right| \le K_1\gamma^{j_i}(t-t_0)^{j_i}e^{-\alpha(t-t_0)} \quad \text{for } t \ge t_0.$$

Elementary calculus leads to the existence of some $K_2 = K_2(K_1,\alpha,t_0,j_i) \ge 1$ such that

$$\left|m_{\lambda_i}^{j_i}(t,t_0)e_{\lambda_i}(t,t_0)\right| \le K_2 e^{-\frac{\alpha}{2}(t-t_0)} \quad \text{for } t \ge t_0\,.$$

As each non-zero entry of the matrix $\Phi_J(t,t_0)$ satisfies such an estimate, we have also that $\Phi_J(t,t_0)$ is norm-wise exponentially bounded, i.e., we have $\|\Phi_J(t,t_0)\| \le K_3 e^{-\frac{\alpha}{2}(t-t_0)}$ for all $t \ge t_0$ and some $K_3 \ge 1$ depending in particular on d. □

An immediate consequence of the previous result is the following characterization of robust exponential stability.

Corollary 22. Let $A \in \mathbb{K}^{d\times d}$ be regressive. Then the following holds:

(a) If the system (4) is robustly exponentially stable, then $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$,

(b) if (10) holds for all eigenvalues λ of A on a time scale with bounded graininess, and if $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$, then (4) is robustly exponentially stable.

Proof. (a) If (4) is robustly exponentially stable then it is in particular exponentially stable, hence $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$.

(b) The set $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ is clearly open and by the continuous dependence of the spectrum of a matrix on its entries (cf. e.g. Stewart and Sun [12, Theorem IV.1.1]), there is a neighborhood $V \subset \mathbb{K}^{d\times d}$ of $A \in \mathbb{K}^{d\times d}$ such that $\sigma(B) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$ for $B \in V$. It remains to show that each $\lambda \in \sigma(B)$ satisfies (10). By assumption there exists a $\gamma > 0$ such that

$$\gamma^{-1} \le |1+\mu^*(t)\mu| \quad \text{for } t \in \mathbb{T},\ \mu \in \sigma(A)$$

and we may choose V small enough so that for each $\lambda \in \sigma(B)$ there exists a $\mu \in \sigma(A)$ with $|\mu - \lambda| < \frac{1}{2H\gamma}$, where we abbreviate $H := \sup_{t\in\mathbb{T}} \mu^*(t)$. Now the estimate

$$\gamma^{-1} \leq |1 + \mu^*(t)\lambda| + \mu^*(t)\,|\mu - \lambda| < |1 + \mu^*(t)\lambda| + (2\gamma)^{-1} \quad \text{for } t \in \mathbb{T}$$

leads to $(2\gamma)^{-1} \leq |1 + \mu^*(t)\lambda|$ and therefore $\lambda \in \sigma(B)$ is uniformly regressive. As λ was arbitrary, this completes the proof. □

6 The General Case

In this section we treat the case of not necessarily regressive matrices $A \in \mathbb{K}^{d\times d}$. We are therefore not able to use the results of Section 4. The following lemma provides an alternative way to conclude for exponential stability of Jordan blocks. This will be used in the proof of the main result of this section (Theorem 24).

Lemma 23. Let $\mathbb{T}$ be a time scale which is unbounded above and with bounded graininess. For $\lambda \in \mathbb{C}$ consider the Jordan block $J_\lambda \in \mathbb{C}^{d\times d}$ given by

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \lambda & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}$$

If the scalar system

$$x^\Delta = \lambda x \tag{13}$$

is uniformly exponentially stable then the system

$$x^\Delta = J_\lambda x \tag{14}$$

is exponentially stable.

Proof. We show the assertion by constructing explicit bounds for individual solutions with initial condition $x(\tau) = \xi \in \mathbb{K}^d$, $\tau \in \mathbb{T}$. Again we will use without loss of generality the norm $\|x\| := \max\{|x_1|, \dots, |x_d|\}$ for $x = (x_1, \dots, x_d) \in \mathbb{K}^d$ in our considerations.

Assume that for the solutions of (13) we have bounds of the form $\|x(t)\| \leq Ke^{-\alpha(t-\tau)}\|\xi\|$ for suitable constants $\alpha > 0, K \geq 1$ and all $\tau \in \mathbb{T}$. Fix $\tau \in \mathbb{T}$

and $-\alpha < \beta < 0$. Now choose a sequence $-\alpha = \beta_d < \beta_{d-1} < \ldots < \beta_2 < \beta_1 = \beta$. We will prove by induction on $j = d, \ldots, 1$ that there exists constants K_j such that the j-th component of the solution of (14) is exponentially bounded by

$$|x_j(t)| \leq K_j e^{\beta_j (t-\tau)} \, \|\xi\| \, .$$

For $j = d$ the assertion follows from the assumption as the d-th entry of $x(t)$ is a solution of (13) and hence

$$|x_d(t)| \leq K e^{-\alpha(t-\tau)} |\xi_d| = K e^{\beta_d (t-\tau)} |\xi_d| \leq K e^{\beta_d (t-\tau)} \, \|\xi\| \, .$$

So assume the assertion is shown for some index $d \geq (j+1) \geq 2$. By construction the j-th component of the solution satisfies the equation

$$x_j^{\Delta}(t) = \lambda x_j(t) + x_{j+1}(t) \quad \text{for } t \in \mathbb{T} .$$

Thus by the variation of constants formula (which is shown in the general non-regressive case in [11, Abschnitt 1.3]) we have the representation

$$x_j(t,\tau) = e_\lambda(t,\tau)\xi_j + \int_\tau^t e_\lambda(t, s + \mu^*(s)) x_{j+1}(s) \Delta s \, .$$

Using the exponential bound on $e_\lambda(t,\tau)$ and denoting by H the bound on the graininess of $\mathbb{T}$ we obtain

$$\begin{aligned}
|x_j(t)| &\leq |e_\lambda(t,\tau)\xi_j| + \int_\tau^t |e_\lambda(t, s+\mu^*(s)) x_{j+1}(s)| \Delta s \leq \\
&\leq K e^{-\alpha(t-\tau)} |\xi_j| + \int_\tau^t K e^{-\alpha(t-s)} e^{\alpha \mu^*(s)} K_{j+1} e^{\beta_{j+1}(s-\tau)} \Delta s \, \|\xi\| \leq \\
&\leq K e^{\beta_{j+1}(t-\tau)} |\xi_j| + \int_\tau^t K e^{\beta_{j+1}(t-s)} e^{\alpha H} K_{j+1} e^{\beta_{j+1}(s-\tau)} \Delta s \, \|\xi\| = \\
&= K e^{\beta_{j+1}(t-\tau)} |\xi_j| + K K_{j+1} e^{\alpha H} (t-\tau) e^{\beta_{j+1}(t-\tau)} \, \|\xi\|
\end{aligned}$$

and choosing K_j large enough and using $\beta_{j+1} < \beta_j$ we obtain

$$|x_j(t)| \leq K_j e^{\beta_j (t-\tau)} \, \|\xi\| \quad \text{for } t \geq \tau \, ,$$

as desired. As we have exponential decay of all components of the solution $x(t)$ this implies the assertion. $\square$

The main result of this section is now the following. Recall that an eigenvalue is called *defective* if it is not semi-simple, i.e. if geometric and algebraic multiplicity do not coincide.

Theorem 24. Let $\mathbb{T}$ be a time scale which is unbounded above. Let $A \in \mathbb{K}^{d\times d}$ and consider the linear system (4). Then the following assertions hold.

(i) If (4) is exponentially stable then $\sigma(A) \subset \mathcal{S}(\mathbb{T})$.

(ii) If $\sigma(A) \subset \mathcal{S}(\mathbb{T})$, the time scale $\mathbb{T}$ has bounded graininess and for all defective $\lambda \in \sigma(A)$ the scalar system (5) is uniformly exponentially stable, then system (4) is exponentially stable.

(iii) If A is diagonalizable then system (4) is exponentially stable if and only if $\sigma(A) \subset \mathcal{S}(\mathbb{T})$.

Proof. (i) Let $\lambda \in \sigma(A)$ and choose an associated eigenvector v. Then we have for $t \geq \tau \in \mathbb{T}$ that (cf. Bohner & Peterson [3, p. 198, Theorem 5.30])

$$|e_\lambda(t,\tau)|\|v\| = \|\Phi_A(t,\tau)v\| \leq K_\tau e^{-\alpha(t-\tau)}\|v\| \quad \text{for } t \geq \tau\,,$$

for suitable constants $K_\tau, \alpha > 0$. This shows that $|e_\lambda(t,\tau)| \leq K_\tau e^{-\alpha(t-\tau)}$ and Proposition 6 implies that $\lambda \in \mathcal{S}(\mathbb{T})$. As $\lambda \in \sigma(A)$ was arbitrary, this completes the proof of (i).

(ii) Let $S \in \mathbb{C}^{d\times d}$ be such that $J := S^{-1}AS$ is in Jordan canonical form with Jordan blocks $J_i, i = 1, \dots, n \leq d$. If for some i the Jordan block J_i is one dimensional the assumption on the spectrum of A and Proposition 6 immediately imply exponential stability of

$$x^\Delta = J_i x\,. \tag{15}$$

If $\dim J_i > 1$, that is, if the associated eigenvalue is defective then exponential stability of (15) is a consequence of the assumptions and Lemma 23. In total, we have exponential stability in each of the Jordan blocks and the assertion easily follows using Theorem 12.

(iii) This is immediate from (i) and (ii).

□

Corollary 25. Let $\mathbb{T}$ be a time scale which is unbounded above. Let $A \in \mathbb{K}^{d\times d}$ and consider the linear system (4). Then the following assertions hold.

(i) If (4) is robustly exponentially stable then $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$.

(ii) If $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$, the time scale $\mathbb{T}$ has bounded graininess and for all multiple eigenvalues $\lambda \in \sigma(A)$ the scalar system (5) is uniformly exponentially stable, then system (4) is robustly exponentially stable.

(iii) If A has d distinct eigenvalues then system (4) is robustly exponentially stable if and only if $\sigma(A) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$.

Proof. (i) By Theorem 24 (i) the assumption implies $\sigma(B) \subset \mathcal{S}(\mathbb{T})$ for all matrices B in a neighborhood of A. Again using [12, Theorem IV.1.1] this is equivalent to the statement that for a suitable neighborhood U of $\sigma(A)$ we have $U \subset \mathcal{S}(\mathbb{T})$. This implies that $\sigma(A) \subset \operatorname{int} \mathcal{S}(\mathbb{T}) = \mathcal{S}_{\mathbb{C}}(\mathbb{T})$.

(ii) As $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ is open and by continuous dependence of the eigenvalues on the entries of a matrix we have $\sigma(B) \subset \mathcal{S}_{\mathbb{C}}(\mathbb{T})$ for all matrices B in a neighborhood V of A. If $\lambda \in \sigma(A)$ has algebraic multiplicity greater than 1 we have by assumption that the scalar systems (5) is uniformly exponentially stable. By arguments similar to [11, Abschnitt 1.3] this implies that the scalar system

$$x^{\Delta} = \eta x \tag{16}$$

is uniformly exponentially stable for all $|\eta - \lambda| < \varepsilon$ for some $\varepsilon > 0$ small enough. Now [12, Theorem IV.1.4] guarantees that by choosing a sufficiently small neighborhood U of A we can ensure that any defective eigenvalue of a matrix $B \in U$ has to satisfy $|\eta - \lambda| < \varepsilon$ for some multiple eigenvalue $\lambda \in \sigma(A)$. Thus for all $B \in U \cap V$ the assumptions of Theorem 24 (ii) are satisfied which shows robust exponential stability of A.

(iii) This is immediate from (i) and (ii).

□

7 Conclusion

We have presented a domain of exponential stability which completely characterizes exponential stability of scalar systems. This immediately implies a characterization of the (generic) case of matrices with distinct eigenvalues. For the case of (defective) multiple eigenvalues we obtain some criteria in the regressive case under a uniform regressivity assumption on the eigenvalues of A. If the assumption of regressivity is dropped this can be replaced by a uniform exponential stability assumption on the scalar systems defined by defective eigenvalues.

The topic warrants further investigation. In particular, it should be examined if uniform exponential stability of certain $\lambda \in \sigma(A)$ is really necessary to prove Theorem 24 (ii). Also it would be interesting to know conditions for uniform exponential stability. Finally, the set of exponential stability is not completely understood. It seems clear, that it can have many connected components. It should be possible to construct examples of that type using some modification of Example 11 (iv) by introducing gaps of varying sizes. What is unclear is, if there are conditions that imply unboundedness of $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$. One could conjecture that for this to happen sufficiently many times $t \in \mathbb{T}$ with $\mu^*(t) < \varepsilon$ are needed for any $\varepsilon > 0$. But a precise statement remains obscure to us for the moment.

Concluding this paper we remark that all statements remain true with obvious modifications if one replaces the time scale $\mathbb{T}$ by an arbitrary measure chain (cf. Hilger [7, 8]).

References

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker Inc., New York, 1992.

[2] B. Aulbach and S. Hilger, *Linear dynamic processes with inhomogeneous time scale*, in Nonlinear Dynamics and Quantum Dynamical Systems, G. A. Leonov, V. Reitmann, W. Timmermann, ed., Mathematical Research Bd. 59, Akademie-Verlag, Berlin, 1990, pp. 9–20.

[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.

[4] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations.* Springer-Verlag, Berlin, 1971. 3rd edition.

[5] J. L. Daleckii and M. G. Krein, *Stability of Solutions of Differential Equations in Banach Spaces.* Number 43 in Translations of Mathematical Monographs. American Mathematical Society, Providence, Rhode Island, 1974.

[6] W. Hahn, *Stability of Motion*, Grundlehren der mathematischen Wissenschaften, 138, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[7] S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Dissertation, Universität Würzburg, 1988.

[8] ——, *Analysis on measure chains — A unified approach to continuous and discrete calculus*, Results in Mathematics, 18 (1990), pp. 18–56.

[9] S. Hilger and P. E. Kloeden, *Comparative time grainyness and asymptotic stability of dynamical systems*, Preprint, School of Computing and Mathematics, Deakin University, Australia, 1993.

[10] S. Keller, *Asymptotisches Verhalten invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen*, Dissertation, Universität Augsburg, 1999.

[11] C. Pötzsche, *Langsame Faserbündel dynamischer Gleichungen auf Maßketten*, Dissertation, Universität Augsburg, 2002.

[12] G. W. Stewart and J. Sun, *Matrix perturbation theory*, Computer Science and Scientific Computing. Academic Press, Boston, 1990.

[13] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1982.