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ON THE RATE OF CONVERGENCE OF INFINITE HORIZON DISCOUNTED OPTIMAL VALUE FUNCTIONS

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Abstract: In this paper we investigate the rate of convergence of the optimal value function of an infinite horizon discounted optimal control problem as the discount rate tends to zero. Using the Integration Theorem for Laplace transformations we provide conditions on averaged functionals along suitable trajectories yielding at most quadratic pointwise convergence. Under appropriate controllability assumptions from this we derive criteria for at most linear uniform convergence on control sets. Applications of these results are given and an example is discussed in which both linear and slower rates of convergence occur.

Keywords: Nonlinear optimal control, optimal value functions, rate of convergence

AMS Classification: 49L05, 41A25

1 Introduction

The question of convergence of optimal value functions of infinite horizon discounted optimal control problems has been considered by various authors during the last years, see e.g. [13], [6], [14], [16], [4], [1], [2], [12], [3] and the references therein. See also [5] for a related problem. Roughly summarized, these papers state that under appropriate controllability conditions the value function uniformly converges to the optimal value of an average time optimal control problem at least on certain subsets of the state space. The main motivation for obtaining such results is the fact that the optimal value functions of discounted optimal control problems have certain nice properties (e.g. it is characterized as the solution of a Hamilton-Jacobi-Bellman equation, it is numerically computable), which are not shared by the averaged time optimal value functions.

However, up to now little has been reported in the literature about the corresponding *rate of convergence*. In the discrete-time Markovian case the results in [16] can be used to obtain immediate estimates for the rate of convergence. The assumptions in this reference, however, exclude the deterministic case. Convergence results for the maxima of discounted value functions have been shown in [15]. This paper presents results for continuous time deterministic systems deriving rates for pointwise and uniform convergence, and is organized as follows: In Section 2 we start by describing the general setup. In Section 3 we develop appropriate estimates for corresponding discounted and averaged functionals based on the Integration Theorem for Laplace Transformations and we translate these results to the optimal value functions, thus obtaining a criterion for at most quadratic pointwise convergence. In Section 4 we characterize situations in which — for suitable compact subsets of the state space — at most linear uniform convergence holds by properties of optimal trajectories. Afterwards, in Section 5, we discuss two optimal control problems in which these properties are satisfied and finally, in Section 6, we provide an example illustrating that for one and the same control system both linear and slower rates of convergence may hold depending on the cost function defining the functional to be minimized.

2 Setup

We consider nonlinear optimal control problems for which the dynamics are given by control systems of the type

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1}$$

on some Riemannian manifold M where

$$u(\cdot) \in \mathcal{U} := \{u : \mathbb{R} \to U \,|\, u(\cdot) \text{ measurable}\}$$

and $U \subset \mathbb{R}^m$ is compact. We assume that f is continuous and $f(\cdot, u)$ is locally Lipschitz for every $u \in U$. By compactness of U it follows that the Lipschitz constants may be chosen uniformly in u. For a given initial value $x_0 \in M$ at time $t = 0$ and a given control function $u(\cdot) \in \mathcal{U}$ we denote the trajectories of (2.1) by $\varphi(t, x_0, u(\cdot))$ which we assume to exist for all $t \geq 0$. Let

$$g : M \times \mathbb{R}^m \to \mathbb{R} \tag{2.2}$$

be a cost function which is continuous and bounded, i.e. $|g(x, u)| \leq M_g$ for some constant M_g.

For a positive discount rate $\delta > 0$ we define the discounted functional

$$J_\delta(x_0, u(\cdot)) := \delta \int_0^\infty e^{-\delta s} g(\varphi(s, x_0, u(\cdot)), u(s)) ds \tag{2.3}$$

and the optimal value function for the corresponding minimization problem is defined by

$$v_\delta(x_0) := \inf_{u(\cdot) \in \mathcal{U}} J_\delta(x_0, u(\cdot)) \tag{2.4}$$

(Note that the corresponding maximization problem is obtained by simply replacing g by $-g$.) In order to characterize the convergence properties for $\delta \to 0$ we also need to define the averaged functionals

$$J_0^t(x_0, u(\cdot)) := \frac{1}{t} \int_0^t g(\varphi(s, x_0, u(\cdot)), u(s)) ds \quad \text{and} \quad J_0(x_0, u(\cdot)) := \limsup_{t \to \infty} J_0^t(x_0, u(\cdot))$$

and the averaged minimal value function

$$v_0(x) := \inf_{u(\cdot) \in \mathcal{U}} J_0(x, u(\cdot))$$

3 Rates of pointwise convergence

In this section we derive estimates for the rates of pointwise convergence for the discounted functionals and optimal value functions. For this purpose we first discuss the relation between discounted and averaged functionals. A direct approach to this problem has been given e.g. in [11]. Instead, here we will use a theorem from the theory of Laplace transformations as the starting point of our analysis. With this approach we avoid a lot of technical work and furthermore obtain sharper estimates. After that we state an immediate consequence from this relation to the discounted optimal value function and provide a useful estimate which will be used in what follows.

Theorem 3.1 Let $q : \mathbb{R} \to \mathbb{R}$ be a measurable function bounded by M_q. Then

$$\delta \int_0^\infty e^{-\delta t} q(t) dt = \delta^2 \int_0^\infty e^{-\delta t} \int_0^t q(s) ds\, dt$$

Proof: See e.g. [9, Theorem 8.1] □

We use Theorem 3.1 in order to obtain the following relation between the rate of convergence of discounted and average time functionals.

Proposition 3.2 Let $r : \mathbb{R}_0^+ \to \mathbb{R}$ be a nonnegative, monotone decreasing function and define

$$\hat{r}(\delta) := \delta^2 \int_0^\infty e^{-\delta t} t r(t) dt$$

Consider a point $x \in M$, let $T \geq 0$ and assume there exist sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $T_k \to \infty$ as $k \to \infty$ such that

$$J_0^t(x, u_k(\cdot)) \leq \lambda + r(t) \text{ for all } t \in [T, T_k].$$

Then

$$J_\delta(x, u_k(\cdot)) \leq \lambda + \hat{r}(\delta) + \delta^2 T^2 (M_g + r(T)) + \varepsilon_k(\delta)$$

with $\varepsilon_k(\delta)$ depending on δ, T_k and M_g, and $\varepsilon_k(\delta) \to 0$ for each fixed δ as $k \to \infty$.

Conversely, if there exists a $\delta > 0$ and $u(\cdot) \in \mathcal{U}$ such that

$$J_\delta(x, u(\cdot)) < \lambda + \hat{r}(\delta)$$

then for each $\varepsilon > 0$ there exists a time $t \geq \varepsilon / 3\delta M_g$ such that

$$J_0^t(x, u(\cdot)) < \lambda + r(t) + \varepsilon$$

Both assertions also hold for the converse inequality if we assume that $r(t) < 2M_g$ for all $t \geq 0$, in the first assertion replace "$+\delta^2 T^2 (M_g + r(T)) + \varepsilon_k(\delta)$" by "$-\delta^2 T^2 3M_g - \varepsilon_k(\delta)$" and in the second "$+\varepsilon$" by "$-\varepsilon$".

Proof: We abbreviate $q_k(t) = g(\varphi(t,x,u_k(\cdot),u_k(t)) - \lambda$. Note that each $|q_k|$ is bounded by $M_q \le 2M_g$. We define $\tilde{q}_k(t) := q_k(t)$ for $t \in [0,T_k]$ and $\tilde{q}_k(t) := 0$ for $t > T_k$. Observe that for each $\delta > 0$ then

$$\varepsilon_k(\delta) := \left| \delta \int_0^\infty e^{-\delta s} q_k(s)ds - \delta \int_0^\infty e^{-\delta s}\tilde{q}_k(s)ds \right| \to 0 \tag{3.1}$$

as $k \to \infty$ because $T_k \to \infty$.

For the proof of the first assertion pick $T \ge 0$ such that the assumption is satisfied for all $t \in [T,T_k]$ and fix $k \in \mathbb{N}$. Then

$$\int_0^t \tilde{q}_k(s)ds \le tr(t) \tag{3.2}$$

for all $t \ge T$. Let $t^* \in [0,T]$ be minimal such that (3.2) is satisfied for all $t \ge t^*$. Then

$$\int_0^{t^*} \tilde{q}_k(s)ds = t^*r(t^*) \quad \text{and} \quad \int_{t^*}^t \tilde{q}_k(s)ds \le tr(t) - t^*r(t^*)$$

is implied for all $t \ge t^*$. From this we can conclude

$$\begin{aligned}
\delta\int_{t^*}^\infty e^{-\delta t}\tilde{q}_k(t)dt &= e^{-\delta t^*}\delta\int_0^\infty e^{-\delta s}\tilde{q}_k(t^*+s)ds \\
&= e^{-\delta t^*}\delta^2\int_0^\infty e^{-\delta t}\int_0^t \tilde{q}_k(t^*+s)ds \\
&\le e^{-\delta t^*}\delta^2\int_0^\infty e^{-\delta t}\left((t^*+t)r(t^*+t) - t^*r(t^*)\right)dt \\
&= \delta^2\int_{t^*}^\infty e^{-\delta t}tr(t)dt - e^{-\delta t^*}\delta^2\int_0^\infty e^{-\delta t}t^*r(t^*)dt \\
&\le \hat{r}(\delta) - e^{-\delta t^*}\delta t^* r(t^*) \\
&\le \hat{r}(\delta) - \delta t^* r(t^*) + \delta^2 t^{*2} r(t^*)
\end{aligned} \tag{3.3}$$

for all $\delta > 0$ where we used Theorem 3.1 in the second step and the inequality $e^{-\delta s} \ge 1 - \delta s$ in the last step. Using this inequality again also

$$\delta\int_0^{t^*}(1-e^{-\delta s})M_q ds \le \delta\int_0^{t^*}\delta s M_q ds \le \delta^2 t^{*2} M_g$$

is implied. Thus we obtain

$$\begin{aligned}
\delta\int_0^{t^*} e^{-\delta s}\tilde{q}_k(s)ds &= \delta\int_0^{t^*}\tilde{q}_k(s)ds - \delta\int_0^{t^*}(1-e^{-\delta s})\tilde{q}_k(s)ds \\
&\le \delta t^* r(t^*) + \delta\int_0^{t^*}(1-e^{-\delta s})M_q ds \\
&\le \delta t^* r(t^*) + \delta^2 t^{*2} M_g
\end{aligned}$$

Now (3.3) together with (3.1) implies the first assertion by the monotonicity of r.

The converse inequality is shown the same way reversing the inequalities and the appropriate signs, and observing that the second last inequality of estimate (3.3) may be reversed if we add the term $-\delta^2 \int_0^{t^*} e^{-\delta t} t r(t) dt$ which by the assumption on $r(t)$ is bounded from below by $-\delta^2 t^{*2} M_g$.

For the second assertion fix an arbitrary $\varepsilon > 0$. Assume contrary to the assertion that

$$J_0^t(x, u(\cdot)) > \lambda + \varepsilon + r(t)$$

for all $t \geq T = \varepsilon / 3\delta M_g$. Note that without loss of generality we may assume $r(t) \leq 2M_g$ for all these t, since otherwise the above inequality will be immediately false. Thus we can use the first assertion for the opposite inequality with $\lambda = \lambda + \varepsilon$ yielding

$$J_\delta(x, u(\cdot)) \geq \lambda + \varepsilon + \hat{r}(\delta) - \delta^2 T^2 3M_g > \lambda + \hat{r}(\delta)$$

which contradicts the assumption, and thus implies the assertion for some $t \geq T$.

The converse inequality is proved analogously with reversed inequalities and signs, where the bound on $r(t)$ here is already given by the assumption. □

Observe that both assertions remain true when the *whole* right hand side of each assertion is multiplied by -1 and the inequalities are reversed. This is easily seen by replacing g by $-g$.

Remark 3.3 In order to see what kinds of rates of convergence of the discounted functional are possible we give explicit estimates for $\hat{r}(\delta)$ for some special cases.

(i) If $r(t) \to 0$ as $t \to \infty$ then $\hat{r}(\delta) \to 0$ as $\delta \to 0$, i.e. convergence to 0 of $\hat{r}$ is implied.

(ii) If $r(t) \leq A/t^c$ for some $A \geq 0$, some $c \in (0, 2)$ and all $t \geq 0$ then $\hat{r}(\delta) \leq \delta^c \Gamma(2-c)$ for all $\delta > 0$. Since the Gamma function $\Gamma(2-c)$ with $c \in (0, 2)$ is bounded by $\max\{1, 1/(2-c)\}$ the rate of convergence of $r(t) \to 0$ in $1/t$ carries over to the rate of convergence of $\hat{r}(\delta) \to 0$ in δ.

(iii) If $r(t) \leq A/t(t+1)^{c-1}$ for some $c \in (0, 2)$ and all $t \geq 0$ then from (ii) we can conclude that $\hat{r}(\delta) \leq e^\delta \delta^c \max\{1, 1/(2-c)\}$, i.e. the same rates as in (ii).

(iv) If $r(t) \leq A/t(t+1)^{c-1}$ for some $c > 2$ and all $t \geq 0$ then $\hat{r}(\delta) \leq \delta^2 \int_1^\infty A t^{1-c} dt = \delta^2 A/(c-2)$, thus quadratic convergence is implied. In any case, if $r(t)$ is positive on some set with measure greater that 0, we can estimate $\hat{r}(\delta) \geq \delta^2 C$ for some constant $C > 0$ which is independent of δ for all $\delta > 0$ sufficiently small. Hence for nontrivial $r(t)$ a convergence rate faster than quadratic is impossible.

(v) If $r(t) \leq A/t(t+1)$ then for any $\varepsilon > 0$ we have $r(t) \leq A/t(t+1)^{1-\varepsilon}$ and thus by (iii) we can conclude $\hat{r}(\delta) \leq e^\delta \delta^{2-\epsilon} A/\epsilon$ which by choosing $\varepsilon = -1/\ln(\delta)$ implies $\hat{r}(\delta) \leq e^{1+\delta} \delta^2 \ln(1/\delta) A$, i.e. quadratic convergence up to a logarithmic factor. Conversely, if $r(t) \geq A/t(t+1)$ then $\hat{r}(\delta) \geq \delta^2 A \int_1^{1/\delta} e^{-\delta t} t^{-1} dt \geq \delta^2 A e^{-1} \int_1^{1/\delta} t^{-1} dt = \delta^2 A e^{-1} \ln(1/\delta)$, i.e. quadratic convergence of $r(t)$ does not imply quadratic convergence of $\hat{r}(\delta)$.

(vi) If $r(t) \leq A\ln(1+t)/t$ then for each $\varepsilon > 0$ the inequality $r(t) \leq A/(\varepsilon e^1 t(t+1)^{-\varepsilon})$ is implied, hence also by (iii) we obtain $\hat{r}(\delta) < e^\delta \delta^{1-\varepsilon} A/(\varepsilon e^1)$ which again by setting $\varepsilon = -1/\ln(\delta)$ implies $\hat{r}(\delta) < e^\delta \delta \ln(1/\delta) A$.

The following corollary on the pointwise rate of convergence for discounted optimal value functions is now an easy consequence of Proposition 3.2.

Corollary 3.4 Consider the optimal control problem (2.1)–(2.4). Assume there exists a point $x_0 \in M$, a time $T \geq 0$, a function $r : \mathbb{R}_0^+ \to \mathbb{R}$ and sequences of times $T_k \to \infty$ and control functions $u_k(\cdot) \in \mathcal{U}$ such that

$$J_0^t(x_0, u_k(\cdot)) \leq \lambda + r(t) \text{ for all } t \in [T, T_k]$$

Then

$$v_\delta(x_0) \leq \lambda + \hat{r}(\delta) + \delta^2 T^2 (M_g + r(T))$$

holds for the function $\hat{r}(\delta)$ from Proposition 3.2.

Conversely, if for any control function $u(\cdot) \in \mathcal{U}$ the inequality

$$J_0^t(x_0, u(\cdot)) \geq \lambda - r(t) \text{ for all } t \in [T, \infty)$$

holds then the inequality

$$v_\delta(x_0) \leq \lambda + \hat{r}(\delta) + \delta^2 T^2 (M_g + r(T))$$

is implied for this function $\hat{r}(\delta)$.

Proof: Immediately from Proposition 3.2. □

We end this section with an estimate for finite time trajectories that will be useful in the next section.

Lemma 3.5 Let $J_0^t(x, u(\cdot)) \leq \sigma$ for all $t \in [0, T]$. Then $J_\delta(x, u(\cdot)) \leq \sigma + e^{-\delta T} 2M_g$.

Proof: Let $q(t) := g(\varphi(t, x, u(\cdot)), u(t)) - \sigma$ for $t \in [0, T]$ and $q(t) = 0$ for $t > T$. Then

$$\frac{1}{t} \int_0^t q(s) ds \leq 0 \text{ for all } t > 0$$

and thus by Theorem 3.1 we obtain

$$\delta \int_0^\infty e^{-\delta s} q(s) ds \leq 0 \text{ for all } \delta > 0.$$

Since w.l.o.g. $|\sigma| \leq M_g$ we obtain

$$\delta \int_T^\infty e^{-\delta s} (g(\varphi(s, x, u(\cdot)), u(s)) - q(s)) ds \leq \sigma + e^{-\delta T} 2M_g$$

and the assertion follows. □

4 Rates of uniform convergence

We will now use the estimates from the preceding section in order to deduce results on the rates of uniform convergence by imposing assumptions on the optimal trajectories. Here we investigate those regions where v_δ uniformly converges to some constant function. As already noted e.g. in [6], [14] and [12], this can be guaranteed by suitable controllability assumptions on our system, furthermore the limiting function can be identified to be v_0. Also here we are going to use certain reachability and controllability properties of the system, and will start this section by defining the necessary objects and properties.

Definition 4.1 The *positive orbit of $x \in M$ up to the time T* is defined by

$$O_T^+(x) := \{y \in M \,|\, \text{there is } 0 \le t \le T \text{ and } u(\cdot) \in \mathcal{U}, \text{ such that } \varphi(t,x,u(\cdot)) = y\}.$$

The *positive orbit* of $x \in M$ is defined by

$$O^+(x) := \bigcup_{T \ge 0} O_T^+(x).$$

The negative orbits $O_T^-(x)$ and $O^-(x)$ are defined similarly by using the time reversed system.

For a subset $D \subset M$ we define $O_T^+(D) := \bigcup_{x \in D} O_T^+(x)$ and $O^+(D)$, $O_T^-(D)$, $O^-(D)$ analogously.

Definition 4.2 A subset $D \subseteq M$ is called a *control set*, if:

(i) $D \subseteq \overline{O^+(x)}$ for all $x \in D$

(ii) for every $x \in D$ there is $u(\cdot) \in \mathcal{U}$ such that the corresponding trajectory $\varphi(t,x,u(\cdot))$ stays in D for all $t \ge 0$

(iii) D is maximal with the properties (i) and (ii)

A control set C is called *invariant*, if

$$\overline{C} = \overline{O^+(x)} \;\; \forall x \in C.$$

Note that this (usual) definition of control sets demands only approximate reachability (i.e. existence of controls steering into any neighborhood of a given point); a convenient way to avoid assumptions about the speed of this asymptotic reachability (as they are imposed e.g. in [2]) is to assume local accessibility, i.e. that the positive and negative orbit for any point and arbitrary small times has nonvoid interior. This assumption is guaranteed e.g. by the following Lie-algebraic property: Let $L = \mathcal{LA}\{X(\cdot,u),\, u \in U\}$ denote the Lie-algebra generated by the vector fields $X(\cdot,u)$. Let Δ_L denote the distribution generated by L in TM, the tangent space of M and assume that

$$\dim \Delta_L(x) = \dim M \;\text{ for all }\; x \in M. \tag{4.1}$$

As a consequence of assumption (4.1) we have exact controllability in the interior of control sets, more precisely $\mathrm{int}D \subset O^+(x)$ for all $x \in D$, cp. e.g. [11].

Using this notion of control sets and assuming (4.1) we are now able to characterize situations in which uniform convergence holds. Although Remark 3.3 shows that the fastest possible rate of pointwise convergence is quadratic, the following result on the behavior of v_δ on control sets suggests that for uniform convergence an at most linear rate seems to be the more realistic situation, cp. the example in Section 6 with cost function g_2.

Proposition 4.3 Consider the optimal control problem (2.1)–(2.4) and assume (4.1). Let $D \subset M$ be a control set with nonvoid interior. Let $K \subset \mathrm{int}D$ be a compact set. Then there exists a constant C_K such that

$$|v_\delta(x) - v_\delta(y)| \leq \delta C_K M_g$$

for all x, $y \in K$.

Proof: By [11, Proposition 2.5] with $K_1 = K_2 = K$ there exists a time $T_K > 0$ such that for each two points x, $y \in K$ there exists a control function $u_{x,y}(\cdot) \in \mathcal{U}$ satisfying $\varphi(t_{x,y}, x, u_{x,y}(\cdot)) = y$ for some time $t_{x,y} \leq T_K$. Thus

$$\begin{aligned}
v_\delta(x) - v_\delta(y) &\leq \int_0^{t_{x,y}} e^{-\delta s} g(\varphi(s,x,u_{x,y}(\cdot)), u_{x,y}(s))ds + e^{-\delta t_{x,y}} v_\delta(y) - v_\delta(y) \\
&\leq \left|\int_0^{t_{x,y}} e^{-\delta s} g(\varphi(s,x,u_{x,y}(\cdot)), u_{x,y}(s))ds\right| + |e^{-\delta t_{x,y}} v_\delta(y) - v_\delta(y)| \\
&\leq \left|\int_0^{T_K} e^{-\delta s} M_g ds\right| + |(e^{-\delta T_K} - 1)M_g| = 2(1 - e^{\delta T_K})M_g \leq 2\delta T_K M_g
\end{aligned}$$

and by symmetry of this inequality in x and y the assertion holds with $C_K = 2T_K$. □

Remark 4.4 Note that by the same argument v_0 is constant in the interior of control sets.

By Proposition 4.3 we can now give a characterization of the uniform rate of convergence on compact subsets of the interior of control sets.

Theorem 4.5 Consider the optimal control problem (2.1)–(2.4) satisfying (4.1). Let $D \subset M$ be a control set with nonvoid interior. Assume there exists a point $x_0 \in \mathrm{int}D$, a time $T \geq 0$, a function $r : \mathbb{R}_0^+ \to \mathbb{R}$ and sequences of times $T_k \to \infty$ and control functions $u_k(\cdot) \in \mathcal{U}$ such that

$$J_0^t(x_0, u_k(\cdot)) \leq \lambda + r(t) \text{ for all } t \in [T, T_k]$$

Then for each compact subset $K \subset \mathrm{int}D$ there exist constants $B_K > 0$ and $\delta_0 > 0$ such that

$$v_\delta(x) \leq \lambda + \hat{r}(\delta) + \delta B_K$$

holds for all $x \in K$ and the function $\hat{r}(\delta)$ from Proposition 3.2 and all $\delta \leq \delta_0$.

Conversely, if for any control function $u(\cdot) \in \mathcal{U}$ the inequality

$$J_0^t(x_0, u(\cdot)) \geq \lambda - r(t) \text{ for all } t \in [T, \infty)$$

holds then the inequality

$$v_\delta(x_0) \leq \lambda + \hat{r}(\delta) + \delta B_K$$

is implied for this function $\hat{r}(\delta)$.

If both assumptions are true for the same value λ and $r(t) \to 0$ as $t \to \infty$ then $\lambda = v_0(x)$, and thus convergence with rate $\hat{r}(\delta) + \delta B_K$ to the averaged value function is implied.

Proof: The first two assertions follow immediately from Corollary 3.4 and Proposition 4.3 with $B_K = A_K M_g + \delta_0 T^2(M_g + r(T))$.

The third assertion follows from the definition of v_0. □

Although this theorem gives quite precise estimates on the rates of convergence the assumption on the function $r(t)$ here might be difficult to check. Thus we are now going to develop geometrical conditions on the optimal trajectories guaranteeing linear convergence on control sets. For this purpose we start by deriving estimates for finite time averaged functionals along trajectories staying in some compact subset of a control set. We introduce the following notation: Given a set $K \subset M$ and $x \in K$ denote by $\mathcal{U}_{x,K} \subset \mathcal{U}$ the set of all control functions $u(\cdot)$ satisfying $\varphi(t, x, u(\cdot)) \in K$ for all $t \geq 0$.

Proposition 4.6 Consider the optimal control problem (2.1)–(2.4) and assume (4.1). Let $D \subset M$ be a control set with nonvoid interior. Let $K \subseteq D$ be a compact set. Then

(i) For each $x \in \text{int}K$ there exists a constant $A = A(x) > 0$ and a time $T = T(x)$ such that

$$J_0^t(x, u(\cdot)) \geq v_0(x) - \frac{A}{t}$$

for all $u(\cdot) \in \mathcal{U}_{x,K}$ and all $t > T$.

(ii) There exist a point $x^* \in K$ and sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $t_k \to \infty$ such that

$$J_0^t(x^*, u_k(\cdot)) \leq \inf_{x \in K} \inf_{u(\cdot) \in \mathcal{U}_{x,K}} J_0(x, u(\cdot)) + \varepsilon_k(T)$$

for all $T > 0$ and all $t \in [0, \min\{T, t_k\}]$ where $\varepsilon_k(T) \to 0$ for $k \to \infty$ and each fixed $T > 0$.

Proof: (i) First note that by [11, Proposition 2.5] for each $x \in \text{int}K \subseteq \text{int}D$ there exists a time $T_x > 0$ such that for any point $y \in K$ there exists a control function $u_y(\cdot) \in \mathcal{U}$ with $\varphi(t_y, y, u_y(\cdot)) = x$ for some $t_y \leq T_x$.

Now let $x \in \mathrm{int} K$ and assume contrary to the assertion that for each constant $A > 0$ and each $T > 0$ there exists a control function $u(\cdot) \in \mathcal{U}_{x,K}$ such that

$$J_0^t(x, u(\cdot)) < v_0(x) - \frac{A}{t}$$

for some $t \geq T$. Since the point $y = \varphi(t, x_0, u(\cdot))$ lies in K we find a control $u_y(\cdot)$ steering y to x_0 in a time $t_y \leq T_x$. Letting $\tilde{u}(\cdot)$ be the concatenation of $u(\cdot)|_{[0,t)}$ and $u_y(\cdot)|_{[0,t_y)}$ we obtain for $t_1 = t + t_y$ and sufficiently large $t > 0$

$$J_0^{t_1}(x, \tilde{u}(\cdot)) \leq \frac{t}{t_1} v_0(x) - \frac{A}{t_1} + \frac{M_g t_y}{t_1} \leq v_0(x_0) - \frac{A}{t_1} + \frac{2M_g t_y}{t_1} < v_0(x_0)$$

and $\varphi(t_1, x, \tilde{u}(\cdot)) = x$. Thus we can continue periodically with this control which yields

$$J_0^{nt_1}(x, \tilde{u}(\cdot)) < v_0(x)$$

for each $n \in \mathbb{N}$ and consequently also

$$J_0(x, \tilde{u}(\cdot)) < v_0(x)$$

which contradicts the definition of v_0.

(ii) If $\cup_{x \in K} \mathcal{U}_{x,K} = \emptyset$ there is nothing to show. Otherwise let $\gamma := \inf_{x \in K} \inf_{u(\cdot) \in \mathcal{U}_{x,K}} J_0(x, u(\cdot))$. Then there exist sequences of points $x_l \in K$ and control functions $u_l(\cdot) \in \mathcal{U}_{x_l,K}$ such that

$$J_0(x_l, u_l(\cdot)) \to \gamma \text{ as } l \to \infty$$

By the definition of J_0 these sequences may be chosen such that there also exists a sequence of times $t_l \to \infty$ satisfying

$$J_0^t(x_l, u_l(\cdot)) \leq \gamma + \frac{1}{l+1} \quad \text{for all} \quad t \geq t_l\,.$$

For each $l \in \mathbb{N}$ let $s_l > 2M_g l^2$. Then [12, Lemma 3.8] implies the existence of times $s_l^* > 0$ with $s_l^* - s_l > l$ such that

$$J_0^s(\varphi(s_l^*, x_l, u_l(\cdot)), u_l(s_l^* + \cdot)) \leq \gamma + \frac{2}{l+1}$$

for all $s \in [0, s_l^* - s_l]$. We set $x_l^* := \varphi(s_l^*, x_l, u_l(\cdot))$ and $u_l^*(\cdot) := u_l(s_l^* + \cdot)$. Since $\{x_l^*\} \subset K$ we may assume that $x_l^* \to x^* \in K \subset \mathrm{int} D$. For any fixed $T > 0$ the functional $J_0^t(\cdot, u(\cdot))$ is continuous in $x \in K$ uniformly for all $u(\cdot) \in \mathcal{U}$ and for all $t \in [0, T]$ (as a consequence of the uniform Lipschitz continuity of f on $K \times U$) and hence we obtain for all $t \in [0, T]$

$$J_0^t(x^*, u_l^*(\cdot)) \leq J_0^t(x_l^*, u_l^*(\cdot)) + \bar{\varepsilon}_l(T) \leq \gamma + \bar{\varepsilon}_l(T) + \frac{2}{l+1}$$

for all $l > 0$ for which $t \in [0, l]$. Here $\bar{\varepsilon}_l(T) \to 0$ as $l \to \infty$.

Thus the assertion follows with $u_k = u_l^*(\cdot)$ and $\varepsilon_k(T) = \bar{\varepsilon}_l(T) + \frac{2}{l+1}$ for $k = l$. □

Now we combine the Propositions 3.2 and 4.6 in order to obtain our main theorem on geometric conditions for linear convergence.

Theorem 4.7 Consider the optimal control problem (2.1)–(2.4) and assume (4.1). Let $D \subset M$ be a control set with nonvoid interior. Assume that there exist a compact subset $K_0 \subset \text{int}D$ and sequences of points $x_k \in K_0$ and control functions $u_k(\cdot) \in \mathcal{U}$ such that $\varphi(t, x_k, u_k(\cdot)) \in K_0$ for all $k \in \mathbb{N}$ and all $t \geq 0$ and

$$J_0(x_k, u_k(\cdot)) \to v_0|_{\text{int}D}$$

Then for each compact subset $K \subset \text{int}D$ there exist constants $A_K > 0$ and $\delta_0 > 0$ such that

$$v_\delta(x) \leq v_0|_{\text{int}D} + \delta A_K \text{ for all } x \in K \text{ and all } \delta \leq \delta_0.$$

Conversely, if there exists $x_0 \in$ intD and a compact subset $K_1 \subseteq D$ such that for all sufficiently small $\delta > 0$ there exist optimal trajectories for v_δ starting in x_0 and staying in K_1 then for each compact subset $K \subset \text{int}D$ there exist constant $B_K > 0$ and $\delta_0 > 0$ such that

$$v_\delta(x) \geq v_0|_{\text{int}D} - \delta B_K \text{ for all } x \in K \text{ and all } \delta \leq \delta_0.$$

Proof: Under the first assumption we can apply Proposition 4.6(ii) and obtain a point $x^* \in \text{int}D$, a new sequence of control functions $u_l(\cdot) \in \mathcal{U}$ and a sequence of times $t_l \to \infty$ such that

$$J_0^t(x^*, u_l(\cdot)) \leq v_0|_{\text{int}D} + \varepsilon_l(T)$$

for all $t \in [0, \min\{T, t_l\}]$ where $\varepsilon_l(T) \to 0$ for $l \to \infty$. Now fix an arbitrary sequence of times $T_k \to \infty$ and an arbitrary constant $A > 0$. For each $k \in \mathbb{N}$ we pick a value $l_k \in \mathbb{N}$ such that $t_{l_k} > T_k$ and $\varepsilon_{l_k}(T_k) < A/T_k$. Applying the first part of Proposition 3.2 to the sequences $u_{l_k}(\cdot)$ and T_k (with $T = 0$) yields $v_\delta(x^*) < v_0|_{\text{int}D} + A\delta$. Since A was arbitrary we can conclude $v_\delta(x^*) < v_0|_{\text{int}D}$ since $x^* \in \text{int}D$ the first assertion follows by Proposition 4.3 with $A_K = C_K M_g$.

For the second assertion assume that for each $B > 0$ and each $\delta_0 > 0$ there exists $\delta \in [0, \delta_0]$ such that

$$v_\delta(x_0) \leq v_0|_{\text{int}D} - B\delta.$$

Then for some arbitrary but fixed $\varepsilon > 0$ the assumption and the second part of Proposition 3.2 yield the existence of a time $t(\delta, \varepsilon)$ such that

$$J_0^{t(\delta,\varepsilon)}(x_0, u(\cdot)) \leq v_0 - \frac{B - \varepsilon}{t(\delta, \varepsilon)}$$

and the corresponding trajectory stays inside K_1. Since B and δ_0 were arbitrary and $t(\delta, \varepsilon) \to \infty$ as $\delta \to 0$ this contradicts Proposition 4.6(i) for $B - \varepsilon > A$ and thus Proposition 4.3 yields the assertion. □

Remark 4.8 Note that under the first assumption we have indeed proved the existence of a point $x^* \in \text{int}D$ with $v_\delta(x^*) \leq v_0(x^*)$ for all $\delta > 0$.

Using the invariance property of invariant control sets we can conclude the following corollary from the theorems in this section.

Corollary 4.9 Consider the optimal control problem (2.1)–(2.4) and assume (4.1). Let $C \subset M$ be a compact invariant control set with nonvoid interior. Assume that one of the following conditions is satisfied

(i) There exist a compact subset $K_0 \subset \mathrm{int}C$ and sequences of points $x_k \in K_0$ and control functions $u_k(\cdot) \in \mathcal{U}$ such that $\varphi(t, x_k, u_k(\cdot)) \in K$ for all $k \in \mathbb{N}$ and all $t \geq 0$ and

$$J_0(x_k, u_k(\cdot)) \to v_0|_{\mathrm{int}C}$$

(ii) There exist $x_0 \in \mathrm{int}C$, $T \geq 0$ and sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $T_k \to \infty$ as $k \to \infty$ such that the inequality

$$J_0^t(x_0, u_k(\cdot)) \leq v_0(x_0) + \frac{A}{t}$$

holds for some constant $A \geq 0$ and all $t \in [T, T_k]$.

Then for each compact subset $K \subset \mathrm{int}D$ there exist constant $B_K > 0$ and $\delta_0 > 0$ such that

$$|v_\delta(x) - v_0(x)| \leq \delta B_K \ \text{ for all } \ x \in K \text{ and all } \delta \leq \delta_0.$$

Proof: The invariance of C immediately implies that the second assumption from Theorem 4.7 is always satisfied with $K_1 = C$. Thus Theorem 4.7 and Theorem 4.5 with $r(t) = A/t$, respectively, yield the assertion. □

5 Applications

In this section we will highlight two situations in which linear convergence can be concluded from the results in this paper.

The first situation is given by completely controllable systems on compact manifolds. More precisely the following corollary holds.

Corollary 5.1 Consider an optimal control system (2.1)–(2.4) on a compact manifold M satisfying (4.1). Assume the system is completely controllable, i.e. there exists an invariant control set $C = M$. Then there exists a constant $K > 0$ such that

$$\|v_\delta - v_0\|_\infty < K\delta.$$

Proof: Follows immediately from Corollary 4.9 by the fact that $M = \mathrm{int}M$ is compact. □

Note that this setup coincides with the one in [10]; in fact there is a strong relation between this result and the periodicity result there since in both cases the values of trajectory pieces have to be estimated. The techniques, however, used in order to obtain these results are rather different.

The second application of our results is somewhat more specific. Here we consider the problem of the approximation of the top Lyapunov exponent of a semilinear control system

$$\dot{x}(t) = A(u(t))x(t), \quad x \in \mathbb{R}^d \tag{5.1}$$

This problem is the continuous time analogon to the one considered in [15]. Note that here we consider the maximization problem so all results are applied with inverted inequalities. Also, since here we are going to derive an estimate for the supremum of v_δ, i.e. for one specific point, we will use Corollary 3.4 and Proposition 4.6 instead of the "uniform" Theorems 4.5 and 4.7.

We will briefly collect some facts about this problem, for detailed information we refer to [7] and [8].

The Lyapunov exponent of a solution $x(t, x_0, u(\cdot))$ of (5.1) is defined by

$$\lambda(x_0, u(\cdot)) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t, x_0, u(\cdot))\|$$

which for $\|x_0\| = 1$ can also be expressed as an averaged integral by

$$\lambda(x_0, u(\cdot)) = J_0(x_0, u(\cdot)) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi(s, x_0, u(\cdot)), u(s))ds$$

where $\varphi(t, x_0, u(\cdot))$ denotes the solution of the system projected to $M = \mathbb{S}^{d-1}$ — which satisfies $\dot{s}(t) = (A(u(t)) - s(t)^T Au(t)s(t) \cdot \mathrm{Id})s(t)$ — and g is a suitable function meeting our general assumptions. For simplicity here we will embed $\mathbb{S}^{d-1}$ into $\mathbb{R}^d$, thus any $x \in \mathbb{R}^d$ with $\|x\| = 1$ is an element of the sphere and vice versa any element $s \in \mathbb{S}^{d-1}$ can be considered as an element of $\mathbb{R}^d$ with $\|s\| = 1$.

Since the Lyapunov exponent does not depend on the length $\|x_0\|$ this averaged integral indeed gives all possible Lyapunov exponents of (5.1) depending on x_0 and $u(\cdot)$. Thus the top Lyapunov exponent can be defined on $\mathbb{S}^{d-1}$ via

$$\kappa := \sup_{x_0\in\mathbb{S}^{d-1}} \sup_{u(\cdot)\in\mathcal{U}} \lambda(x_0, u(\cdot)).$$

It characterizes the stability of the solutions of (5.1) under all possible functions $u(\cdot)$, and can also be used to define a stability radius of (5.1) analogous to the discrete time setting in [15].

It already follows from the arguments in [12] that $\sup_{x\in\mathbb{S}^{d-1}} v_\delta(x)$ converges to κ as $\delta \to 0$. Now it remains to determine the rate of convergence.

We assume (4.1) for the projected system. Under this condition the projected system possesses a unique invariant control set C with nonvoid interior. Furthermore, the top Lyapunov exponent can be realized from any initial value $x_0 \in \mathbb{S}^{d-1}$, hence in particular from any point $x_0 \in \mathrm{int}C$. Thus Proposition 4.6(ii) with $K = C$ yields the existence of a point $x^* \in C$ and sequences of control functions $u_l(\cdot) = u(t_{k_l} + \cdot)$ and times t_l satisfying

$$J_0^t(x^*, u_l(\cdot)) \geq \kappa - \varepsilon_l(T) \;\text{ for all }\; t \in [0, \min\{T, t_l\}].$$

As in the proof of Theorem 4.7 (cp. also Remark 4.8) we can thus conclude that $v_\delta(x^*) \geq \kappa$ for all $\delta > 0$. It thus remains to find an upper bound for $\sup_{x\in\mathbb{S}^{d-1}} v_\delta(x)$.

For this purpose consider a basis $x_1, \ldots, x_d$ of $\mathbb{R}^d$ such that $\|x_i\| = 1$ and $x_i \in \mathrm{int} C$ for all $i = 1, \ldots, d$. Then Proposition 4.6(i) with $K = C$ yields the existence of a constant $B > 0$ such that

$$J_0(x_i, u(\cdot)) \leq \kappa + \frac{B}{t}$$

for all $i = 1, \ldots, d$ and all $u(\cdot) \in \mathcal{U}$ and hence

$$\|x(t, x_i, u(\cdot))\| \leq e^B e^{\kappa t}.$$

By the compactness of $\mathbb{S}^{d-1}$ there exists a constant $\nu > 0$ such that any point $x_0 \in \mathbb{S}^{d-1}$ can be written as a linear combination $x_0 = \sum_{i=1}^d \mu_i(x_0)x_i$ with coefficients $|\mu_i(x)| \leq \nu$. Thus we obtain

$$\|x(t, x_0, u(\cdot))\| = \|\sum_{i=1}^d \mu_i(x_0)x(t, x_i, u(\cdot))\| \leq d\nu e^B e^{\kappa t}.$$

Thus with $A = B + \ln d\nu$ it follows that

$$J_0(x_i, u(\cdot)) \leq \kappa + \frac{A}{t}$$

for all $x_0 \in \mathbb{S}^{d-1}$ and all $u(\cdot) \in \mathcal{U}$. Thus for any $\tilde{A} > A$ Corollary 3.4 yields

$$v_\delta(x_0) \leq \kappa + \delta\tilde{A}$$

for all sufficiently small δ which finally yields

$$\sup_{x\in\mathbb{S}^{d-1}} v_\delta(x) \in [\kappa, \kappa + \delta\tilde{A}]$$

and thus the desired estimate.

In fact, with a similar argument one can also verify the assumption (i) of Corollary 4.9 and thus linear convergence follows not only for the supremum but also on any compact subset of $\mathrm{int} C$.

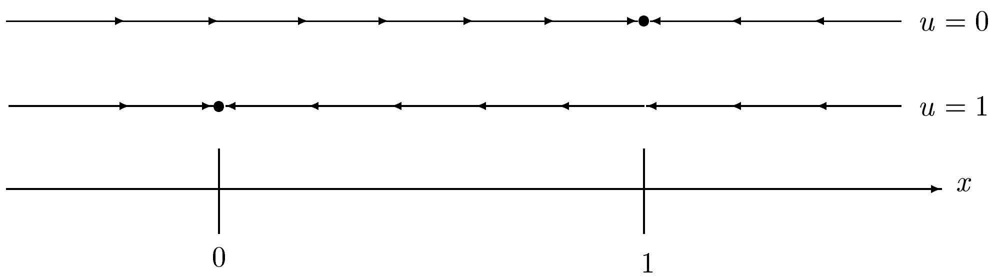
6 An Example

Here we provide an example of a simple 1d control system with one (invariant) control set where for one cost function g_1 the rate of convergence of v_δ is slower than linear but for a slightly modified g_2 it is indeed linear.

Consider the control system

$$\dot{x} = -ux|x| + (u - 1)(x - 1)|x - 1| \tag{6.1}$$

with $x \in \mathbb{R}$ and $u \in [0, 1]$. The vector fields are sketched in the following picture.



It is easily seen (cp. the figure) that (6.1) possesses an (invariant) control set $C = [0, 1]$. All solutions starting outside C can be steered to C but no trajectory can leave C.

For the cost function $g_1(x, u) = |x|$ and initial values $x_0 \in C$ it is obviously optimal to steer to the left as fast as possible, i.e. the optimal control is $u \equiv 1$.

The solution for this constant control can be computed explicitly, it is given by

$$x(t) = \frac{x_0}{tx_0 + 1}$$

Thus

$$J_0^t(x_0, 1) = \frac{1}{t} \int_0^t \frac{x_0}{sx_0 + 1} ds = \frac{ln(tx_0 + 1)}{tx_0}$$

does not converge linearly, and by the first assertion of Proposition 3.2 (for the converse inequality) the same holds for δv_δ, more precisely similar to Remark 3.3(vi) we obtain that $\hat{r}(\delta) \geq C\delta \ln(1/\delta)$.

Now we consider $g_2(x, u) = |x - 0.5|$. For the initial value $x_0 = 1/2$ we obtain with $u \equiv 1/2$ that $x(t, x_0, u) = x_0$ for all $t > 0$, hence $J_0^t(1/2, 1/2) = 0$ for all $t > 0$. Obviously here Condition (i) of Corollary 4.9 is satisfied, thus linear convergence follows. A closer look at the problem reveals that here we even obtain $v_\delta(0.5) = 0$ for all $\delta > 0$, nevertheless outside this point we have not more than linear convergence, again suggesting that — apart from exceptional situations — this is a kind of "natural" bound for the uniform rate of convergence.

In fact, with the same arguments linear convergence holds for all cost functions $g_\alpha(x, u) = |x - \alpha|$, $\alpha \in (0, 1)$ using the fact that $\alpha \in (0, 1)$ is a fixed point of (6.1) for $u = \frac{(\alpha^2-1)^2}{\alpha^2+(\alpha^2-1)^2} \in (0, 1)$.

7 Conclusions

Convergence rates of optimal value functions of discounted optimal control problems are investigated. Conditions on related averaged functionals are defined which imply at most pointwise quadratic and uniform linear convergence. Furthermore geometric conditions on optimal trajectories ensuring uniform linear convergence are given. These conditions are checked for two optimal control problems where linear convergence can be verified. However, an example shows that linear convergence is not always satisfied.

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