Zubov's method for perturbed differential equations*

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Abstract

We present a generalization of Zubov's method to perturbed differential equations. The goal is to characterize the domain of attraction of a set which is uniformly locally asymptotically stable under all admissible time varying perturbations. We show that in this general setting the straightforward generalization of the classical Zubov's equations has a unique viscosity solution which characterizes the robust domain of attraction as a suitable sublevel set.

1 Introduction

The domain of attraction of an asymptotically stable fixed point has been one of the central objects in the study of continuous dynamical systems. The knowledge of this object is important in many applications modeled by those systems like e.g. the analysis of power systems [1] and turbulence phenomena in fluid dynamics [3, 9, 18]. Several papers and books discuss theoretical [20, 21, 7, 11] as well as computational aspects [19, 12, 1, 10] of this problem.

A generalization of the concept of a stable fixed point is a locally asymptotically stable compact set. This may be a periodic limit cycle (as considered e.g. in [2]), a compact attractor or some other forward invariant set with a suitable uniform attractivity property. Of course, also for these objects the question of the domain of attraction is interesting.

Taking into account that usually mathematical models of complex systems contain model errors and that exogenous perturbations are ubiquitous it is natural to consider systems with deterministic time varying perturbations and look for domains of attraction that are robust under all these perturbations. Here we consider systems of the form

$$\dot{x}(t) = f(x(t), a(t)), \quad x \in \mathbb{R}^n$$

where a is an arbitrary measurable function with values in some compact set $A \subset \mathbb{R}^m$. Under the assumption that $D \subset \mathbb{R}^n$ is a locally asymptotically stable compact set for all admissible perturbation functions a we try to find the set of points which are attracted to D under all these perturbations a.

For the special case of D being just one fixed point this set has been considered e.g. in [13, 14, 5, 8], for the case where D is a periodic orbit see e.g. [2]. The present paper follows the approach of [5], where a generalization of Zubov's classical method [21] has been developed in the framework of viscosity solutions for the characterization of the domain of attraction of an exponentially stable fixed point of a perturbed system. Based on this approach, in this paper we show the necessary modifications for non-exponential attraction and arbitrary compact uniformly attracting sets.

The main result we obtain that way is the formulation of a first order partial differential equation which possesses a unique viscosity solution which characterizes the domain of attraction as a suitable sublevel set. In addition, this function is a robust Lyapunov function for D on its domain of attraction.

It might be worth noting that in particular our approach is applicable to the classical Zubov equation (i.e. for unperturbed systems) and hence provides a way to characterize domains of attraction of compact sets also for unperturbed systems. Furthermore, the regularization technique from [6] also applies here and gives rise to a numerical approximation of the solution.

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This paper is organized as follows: In Section 2 we give the setup and collect some facts about robust domains of attraction. In Section 3 we formulate and prove the main result, and finally, Section 4 gives some further properties of the solution to our equation.

2 Robust domains of attraction

We consider systems of the following form

$$\begin{cases} \dot{x}(t) = f(x(t), a(t)), & t \in [0, \infty), \\ x(0) = x_0, \end{cases} \tag{1}$$

with solutions denoted by $x(t, x_0, a)$. Here $a(\cdot) \in \mathcal{A} = L^\infty([0, +\infty), A)$ and A is a compact subset of $\mathbb{R}^m$, f is continuous and bounded in $\mathbb{R}^n \times A$ and Lipschitz in x uniformly in $a \in A$.

We assume that there exists a compact and connected set $D \subset \mathbb{R}^n$ which is uniformly locally asymptotically stable for system (1), i.e.

(H1) there exists a constant $r > 0$ and a function β of class $\mathcal{KL}$ such that $\mathrm{dist}(x(t, x_0, a), D) \leq \beta(\mathrm{dist}(x_0, D), t)$ for any $x_0 \in B(D, r)$, any $a \in \mathcal{A}$, and all $t \geq 0$.

Here $B(D, r) := \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, D) < r\}$ denotes the set of points with distance less than r from D. As usual in stability analysis, we call a function α of class $\mathcal{K}^\infty$ if it is a homeomorphism of $[0, \infty)$ (i.e. $\alpha(0) = 0$ and α is strictly increasing to infinity) and we call a continuous function β with two real nonnegative arguments of class $\mathcal{KL}$ if it is of class $\mathcal{K}^\infty$ in the first and decreasing to zero in the second argument.

It is known (see [16]) that for any $\beta \in \mathcal{KL}$ there exist two functions $\alpha_1, \alpha_2 \in \mathcal{K}^\infty$ such that

$$\beta(r, t) \leq \alpha_2(\alpha_1(r) e^{-t}). \tag{2}$$

Note that (H1) implies forward invariance of D, but not necessarily backward invariance, i.e. there might be trajectories leaving D in backward time and entering D in forward time. Hence here the situation is more general compared to [5] where the attracting set was assumed to be a (forward and backward invariant) singular fixed point.

The following sets describe domains of attraction for the set D of the system (1).

Definition 2.1 *For the system* (1) *satisfying (H1) we define the* robust domain of attraction *as*

$$\mathcal{D} = \left\{ x_0 \in \mathbb{R}^n \ : \ \begin{array}{l} \mathrm{dist}(x(t, x_0, a), D) \to 0 \text{ as} \\ t \to \infty \text{ for any } a \in \mathcal{A} \end{array} \right\},$$

and the uniform robust domain of attraction *by*

$$\mathcal{D}_0 = \left\{ x_0 \in \mathbb{R}^n \ : \ \begin{array}{l} \text{there exists a function } \gamma(t) \to 0 \\ \text{as } t \to \infty \text{ such that} \\ \mathrm{dist}(x(t, x_0, a), D) \leq \gamma(t) \\ \text{for all } t > 0,\ a \in \mathcal{A} \end{array} \right\}.$$

In particular, the setup in the present paper allows to relax in a certain sense the assumption of [5] that the fixed point (taken to be 0) is invariant under all perturbations, i.e. $f(0, a) = 0, \forall a \in A$. If we assume that 0 is locally asymptotically stable for the system $\dot{x} = f(x, a_0)$ for a particular $a_0 \in A$, then we may consider a local Lyapunov function W for this system. We now regard the sublevel sets $D_r := \{x \in \mathbb{R}^n \mid W(x) \leq r\}$. If the perturbations in $\mathcal{A}$ are sufficiently small, then for some $r > 0$, $D = D_r$ will satisfy assumption (H1). The interpretation of the domains $\mathcal{D}, \mathcal{D}_0$ would then be the set of points that are still (uniformly) attracted "close" to the fixed point of the unperturbed system, even though locally the fixed point moves under perturbation, or undergoes a bifurcation, which is a common scenario in many applications.

The next proposition summarizes some properties of (uniform) robust domains of attraction. As the proofs are straightforward generalizations of the proofs of [5, Proposition 2.4] we omit them here. Observe that several of these properties are very similar to those of the domain of attraction of an asymptotically stable fixed point of a time-invariant system, compare [11, Chap. IV].

Proposition 2.2 *Consider system (1) and assume (H1), then*

(i) $\mathrm{cl}B(D, r) \subset \mathcal{D}_0$.

(ii) $\mathcal{D}_0$ *is an open, connected, invariant set.* $\mathcal{D}$ *is a pathwise connected, invariant set.*

(iii) $\sup_{a \in \mathcal{A}}\{t(x, a)\} \to +\infty$ *for* $x \to x_0 \in \partial\mathcal{D}_0$ *or* $\|x\| \to \infty$, *where* $t(x, a) := \inf\{t > 0 : x(t, x, a) \in B(D, r)\}$.

(iv) $\mathrm{cl}\mathcal{D}_0$, $\mathrm{cl}\mathcal{D}$ *are forward invariant sets.*

(v) If for every $x \in \partial\mathcal{D}_0$ *there exists* $a \in \mathcal{A}$ *such that* $x(t, x, a) \in \partial\mathcal{D}_0$ *for all* $t \geq 0$ *then* $\mathcal{D} = \mathcal{D}_0$.

(vi) If for all $x \in \mathcal{D}$ *the set* $\{f(x, a) : a \in \mathcal{A}\}$ *is convex then* $\mathcal{D}_0 = \mathcal{D}$.

3 Zubov's method for robust domains of attraction

In this section we discuss the following partial differential equation

$$\sup_{a \in A} \{Dv(x) f(x, a) + (1 - v(x)) g(x, a)\} = 0 \tag{3}$$

for $x \in \mathbb{R}^n$ whose solution—for suitable functions g—will turn out to characterize the uniform robust domain of attraction $\mathcal{D}_0$. This equation is a straightforward generalization of Zubov's equation [21]. In this generality, however, in order to obtain a meaningful result about solutions we have to work within the framework of viscosity solutions, which we recall for the convenience of the reader (for details about this theory we refer to [4]).

Definition 3.1 *Given an open subset Ω of $\mathbb{R}^n$ and a continuous function $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, we say that a lower semicontinuous (l.s.c.) function $u : \Omega \to \mathbb{R}$ (resp. an upper semicontinuous (u.s.c.) function $v : \Omega \to \mathbb{R}$) is a viscosity supersolution (resp. subsolution) of the equation*

$$H(x, u, Du) = 0 \qquad x \in \Omega \tag{4}$$

if for all $\phi \in C^1(\Omega)$ and $x \in argmin_\Omega(u - \phi)$ (resp., $x \in argmax_\Omega(v - \phi)$) we have

$$H(x, u(x), D\phi(x)) \geq 0 \quad \big(\text{resp., } H(x, v(x), D\phi(x)) \leq 0\big).$$

A continuous function $u : \Omega \to \mathbb{R}$ is said to be a viscosity solution of (4) *if u is a viscosity supersolution and a viscosity subsolution of* (4).

We now introduce the value function of a suitable optimal control problem related to (3).

Consider the following nonnegative, extended value functional $J : \mathbb{R}^n \times \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$

$$J(x, a) := \int_0^{+\infty} g(x(t), a(t))dt$$

and the optimal value function

$$v(x) := \sup_{a \in \mathcal{A}} 1 - e^{-J(x,a)}. \tag{5}$$

The function $g : \mathbb{R}^n \times A \to \mathbb{R}$ is supposed to be continuous and satisfies

(H2)

(i) For all $a \in A$, $g(x, a) \leq C\alpha_2^{-1}(\mathrm{dist}(x, D))$ for all $x \in \mathbb{R}^n$, α_2 from (2) and some $C > 0$, and $g(x, a) > 0$ for $x \notin D$.

(ii) There exists a constant $g_0 > 0$ such that $\inf_{x \notin B(D,r),\, a \in A} g(x, a) \geq g_0$.

(iii) For every $R > 0$ there exists a constant L_R such that $\|g(x, a) - g(y, a)\| \leq L_R\|x - y\|$ for all $\|x\|, \|y\| \leq R$, and all $a \in A$.

Since g is nonnegative it is immediate that $v(x) \in [0, 1]$ for all $x \in \mathbb{R}^n$. Furthermore, standard techniques from optimal control (see e.g. [4, Chapter III]) imply that v satisfies a dynamic programming principle, i.e. for each $t > 0$ we have

$$v(x) = \sup_{a \in \mathcal{A}} \{(1 - G(x, t, a)) + G(x, t, a)v(x(t, x, a))\} \tag{6}$$

with

$$G(t, x, a) := \exp\left(-\int_0^t g(x(\tau, x, a), a(\tau))d\tau\right). \tag{7}$$

Furthermore, a simple application of the chain rule shows

$$(1 - G(x, t, a)) = \int_0^t G(x, \tau, a)g(x(\tau, x, a), a(\tau))d\tau$$

implying

$$\begin{aligned} v(x) \;&=\; \sup_{a \in \mathcal{A}} \Big\{ \int_0^t G(x, \tau, a)g(x(\tau, x, a), a(\tau))d\tau \\ &\qquad + \; G(x, t, a)v(x(t, x, a))\Big\} \end{aligned} \tag{8}$$

The next proposition shows the relation between $\mathcal{D}_0$ and v, and the continuity of v.

Proposition 3.2 *Assume (H1), (H2). Then*

(i) $v(x) < 1$ *if and only if* $x \in \mathcal{D}_0$.

(ii) $v(x) = 0$ *if and only if* $x \in D$.

(iii) v *is continuous on* $\mathbb{R}^n$.

(iv) $v(x) \to 1$ *for* $x \to x_0 \in \partial\mathcal{D}_0$ *and for* $|x| \to \infty$.

Proof: We show

$$\sup_{a \in \mathcal{A}} J(x, a) < \infty \;\text{ for all }\; x \in B(D, r)$$

implying $v(x) < 1$ on $B(D, r)$. For this, for each $x \in B(D, r)$ and each $a \in A$ we can estimate

$$\begin{aligned} J(x, a) \;&\leq\; \int_0^\infty C\alpha_2^{-1}(\mathrm{dist}(x(t, x, a), D))dt \\ &\leq\; \int_0^\infty C\alpha_1(\mathrm{dist}(x, D))e^{-t}dt \\ &=\; C\alpha_1(\mathrm{dist}(x, D)) \end{aligned}$$

which is independent of a and hence implies the desired estimate.

Now all assertions follow as in the proof of [5, Proposition 3.1]. □

We now turn to the relation between v and equation (3). Recalling that v is locally bounded on $\mathbb{R}^n$ an easy application of the dynamic programming principle (6) (cp. [4, Chapter III]) shows that and v is a viscosity solution of (3). The more difficult part is to obtain uniqueness of the solution, since equation (3) exhibits a singularity on the set D.

In order to get a uniqueness result we use the following super- and suboptimality principles, which essentially follow from Soravia [17, Theorem 3.2 (i)], see [5, Proposition 3.5] for details.

Proposition 3.3

(i) Let w be a l.s.c. supersolution of (3) *in $\mathbb{R}^n$, then for any $x \in \mathbb{R}^n$*

$$\begin{aligned} w(x) \;&=\; \sup_{a \in \mathcal{A}} \sup_{t \geq 0} \{(1 - G(x, t, a)) \\ &\qquad + \; G(x, t, a)w(x(t))\}. \end{aligned} \tag{9}$$

(ii) Let u be a u.s.c. subsolution of (3) *in $\mathbb{R}^n$, and $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $u \leq \tilde{u}$. Then for any $x \in \mathbb{R}^n$ and any $T \geq 0$*

$$\begin{aligned} u(x) \;&\leq\; \sup_{a \in \mathcal{A}} \inf_{t \in [0,T]} \{(1 - G(x, t, a)) \\ &\qquad + \; G(x, t, a)\tilde{u}(x(t))\}. \end{aligned} \tag{10}$$

Remark 3.4 *If u is continuous or the set of the control functions $\mathcal{A}$ is replaced by the set of relaxed control laws $\mathcal{A}^r$, assertion (ii) can be strengthened to*

$$u(x) = \sup_{\mu \in \mathcal{A}^r} \inf_{t \geq 0} \{(1 - G(x,t,\mu)) + G(x,t,\mu)u(x(t))\},$$

which follows from [17, Theorem 3.2(iii)].

We can now apply these principles to the generalized version of Zubov's equation (3) in order to obtain comparison principles for sub- and supersolutions.

Proposition 3.5 *Let w be a bounded l.s.c. supersolution of* (3) *on $\mathbb{R}^n$ with $w(x) = \gamma \geq 0$ for all $x \in D$. Then $w \geq v$ for v as defined in* (5).

Proof: First observe that the lower semicontinuity of w and the assumption $w(x) = \gamma \geq 0$ for all $x \in D$ imply that for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$w(x) \geq -\epsilon \text{ for all } x \in \mathbb{R}^n \text{ with } \operatorname{dist}(x, D) \leq \delta. \tag{11}$$

Furthermore, the upper optimality principle (9) implies

$$w(x_0) \geq \sup_{a \in \mathcal{A}} \inf_{t \geq 0} \{1 + G(x_0,t,a)(w(x(t,x_0,a)) - 1)\}. \tag{12}$$

Now we distinguish two cases:
(i) $x_0 \in \mathcal{D}_0$: In this case we know that for each $a \in \mathcal{A}$ we have $\operatorname{dist}(x(t,x_0,a), D) \to 0$ as $t \to \infty$. Thus from (11) and (12), and using the definition of v we can conclude

$$w(x_0) \geq \sup_{a \in \mathcal{A}} \left\{ \lim_{t \to \infty} (1 - G(x_0,t,a)) \right\} = v(x_0).$$

which shows the claim.
(ii) $x_0 \notin \mathcal{D}_0$: Since $v(x) \in [0,1]$ for all $x \in \mathbb{R}^n$ it is sufficient to show that $w(x_0) \geq 1$. Now consider the time $t(x,a)$ as defined in Proposition 2.2(iii). By the definition of $\mathcal{D}_0$ we know that for each $T > 0$ there exists $a_T \in \mathcal{A}$ such that $t(x_0, a_T) > T$, which implies $G(x_0, T, a_T) \leq \exp(-Tg_0)$ which tends to 0 as $T \to \infty$. Thus denoting the bound on $|w|$ by $M > 0$ the inequality (12) implies

$$w(x_0) \geq (1 - \exp(-Tg_0)) - \exp(-Tg_0)M$$

for every $T > 0$ and hence $w(x_0) \geq 1$. □

Proposition 3.6 *Let u be a bounded u.s.c. subsolution of* (3) *on $\mathbb{R}^n$ with $u(x) = \gamma \leq 0$ for all $x \in D$. Then $u \leq v$ for v defined in* (5).

Proof: By the upper semicontinuity of u and $u(0) \leq 0$ we obtain that for every $\epsilon > 0$ there exists a $\delta > 0$ with $u(x) \leq \epsilon$ for all $x \in \mathbb{R}^n$ with $\operatorname{dist}(x, D) \leq \delta$. Thus for each $\epsilon > 0$ we find a bounded and continuous function $\tilde{u}_\epsilon : \mathbb{R}^n \to \mathbb{R}$ with

$$\tilde{u}_\epsilon(x) < \epsilon \text{ for all } x \in D \text{ and } u \leq \tilde{u}_\epsilon. \tag{13}$$

Now the lower optimality principle (10) implies for every $t \geq 0$ that

$$u(x_0) \leq \sup_{a \in \mathcal{A}} \{1 + G(x_0,t,a)(\tilde{u}_\epsilon(x(t,x_0,a)) - 1)\}. \tag{14}$$

Again, we distinguish two cases:
(i) $x_0 \in \mathcal{D}_0$: In this case $\operatorname{dist}(x(t,x_0,a), D) \to 0$ as $t \to \infty$ uniformly in $a \in \mathcal{A}$. Hence for each $\epsilon > 0$ there exists $t_\epsilon > 0$ such that

$$\tilde{u}_\epsilon(x(t_\epsilon,x_0,a)) \leq \epsilon \text{ and } |G(x_0,t_\epsilon,a) - G(x_0,\infty,a)| \leq \epsilon$$

for all $a \in \mathcal{A}$. Thus from (13) and (14), and using the definition of v we can conclude

$$\begin{aligned} u(x_0) &\leq \sup_{a \in \mathcal{A}} \{1 - (1-\epsilon)G(x_0,t_\epsilon,a)\} \\ &\leq v(x_0) + \epsilon(1 - v(x_0)) + \epsilon, \end{aligned}$$

which shows the claim since v is bounded and $\epsilon > 0$ was arbitrary.
(ii) $x_0 \notin \mathcal{D}_0$: Since in this case $v(x_0) = 1$ (by Proposition 3.2(i)) it is sufficient to show that $u(x_0) \leq 1$. By (i) we know that $u(y) \leq v(y) < 1$ for each $y \in \mathcal{D}_0$, hence analogous to (13) for each $\epsilon > 0$ we can conclude the existence of a continuous $\tilde{u}_\epsilon$ with $u \leq \tilde{u}_\epsilon$ and $\tilde{u}_\epsilon(y) \leq 1 + \epsilon$ for each $y \in \mathcal{D}_0$. Since u is bounded by assumption, we may choose $\tilde{u}_\epsilon$ such that $M_\epsilon := \sup_{x \in \mathbb{R}^n} \tilde{u}_\epsilon(x) < \infty$. If $M_\epsilon \leq 1$ for some $\epsilon > 0$ we are done. Otherwise fix $\epsilon > 0$ and consider a sequence $t_n \to \infty$. Then (14) implies that there exists a sequence $a_n \in \mathcal{A}$ with

$$u(x_0) - \epsilon \leq 1 + G(x_0,t_n,a_n)(\tilde{u}_\epsilon(x(t_n,x_0,a_n)) - 1).$$

If $x(t_n,x_0,a_n) \in \mathcal{D}_0$ we know that $\tilde{u}_\epsilon(x(t_n,x_0,a_n)) \leq 1 + \epsilon$, and since $G \leq 1$ we obtain $u(x_0) - \epsilon \leq 1 + \epsilon$. If $x(t_n,x_0,a_n) \notin \mathcal{D}_0$ then $G(x_0,t_n,a_n) \leq \exp(-g_0 t_n)$, thus

$$\begin{aligned} &1 + G(x_0,t_n,a_n)(\tilde{u}_\epsilon(x(t_n,x_0,a_n)) - 1) \\ &\leq 1 + \exp(-g_0 t_n)(M_\epsilon - 1). \end{aligned}$$

Thus for each $n \in \mathbb{N}$ we obtain

$$u(x_0) \leq 2\epsilon + 1 + \exp(-g_0 t_n)(M_\epsilon - 1),$$

which for $n \to \infty$ implies $u(x_0) \leq 1 + 2\epsilon$. This proves the assertion since $\epsilon > 0$ was arbitrary.

□

Using these propositions we can now formulate an existence and uniqueness theorem for the generalized version of Zubov's equation (3).

Theorem 3.7 *Consider the system* (1) *and a function $g : \mathbb{R}^n \times A \to \mathbb{R}$ such that (H1) and (H2) are satisfied. Then* (3) *has a unique bounded and continuous viscosity solution v on $\mathbb{R}^n$ satisfying $v(x) = 0$ for all $x \in D$.*

This function coincides with v from (5). *In particular the characterization $\mathcal{D}_0 = \{x \in \mathbb{R}^n \mid v(x) < 1\}$ holds.*

Proof: This is immediate from Propositions 3.5 and 3.6. □

The following theorem is an immediate consequence of Theorem 3.7. It shows that we can restrict ourselves to a proper open subset $\mathcal{O}$ of the state space and still obtain our solution v, provided $\mathcal{D}_0 \subseteq \mathcal{O}$. This is in particular important for our computational approach as we will not be able to approximate v on the whole $\mathbb{R}^n$.

Theorem 3.8 *Consider the system* (1) *and a function* $g : \mathbb{R}^n \times A \to \mathbb{R}$. *Assume (H1) and (H2). Let* $\mathcal{O} \subset \mathbb{R}^n$ *be an open set containing* D, *and let* $v : \mathrm{cl}\mathcal{O} \to \mathbb{R}$ *be a bounded and continuous function which is a viscosity solution of* (3) *on* $\mathcal{O}$ *and satisfies* $v(x) = 0$ *for all* $x \in D$ *and* $v(x) = 1$ *for all* $x \in \partial\mathcal{O}$.

Then v *coincides with the restriction* $v|_{\mathcal{O}}$ *of the function* v *from* (5). *In particular the characterization* $\mathcal{D}_0 = \{x \in \mathbb{R}^n \mid v(x) < 1\}$ *holds.*

Proof: Any solution $\tilde{v}$ meeting the assumption can be continuously extended to a viscosity solution of (3) on $\mathbb{R}^n$ by setting $\tilde{v}(x) = 1$ for $x \in \mathbb{R}^n \setminus \mathcal{O}$. Hence the assertion follows. □

4 Further properties of the solution

In this section we show two properties of the solution v from Theorem 3.7. First, we show that v is a robust Lyapunov function on $\mathcal{D}_0$ and second we give conditions on g which guarantee (global) Lipschitz continuity of v.

We start by giving the Lyapunov function property, which in fact immediately follows from the optimality principle.

Proposition 4.1 *Assume (H1) and (H2) and consider the unique viscosity solution* v *of* (3) *with* $v(x) = 0$ *for all* $x \in D$. *Then the function* v *is a robust Lyapunov function for the system* (1). *More precisely we have*

$$\begin{aligned} & v(x(t, x_0, a)) - v(x_0) \\ & \le \left[1 - e^{-\int_0^t g(x(\tau), a(\tau))d\tau}\right] (v(x(t, x_0, a)) - 1) \\ & < \ 0 \end{aligned}$$

for all $x_0 \in \mathcal{D}_0 \setminus D$ *and all functions* $a \in \mathcal{A}$.

Proof: Follows immediately from (6). □

Now we turn to the Lipschitz property.

Proposition 4.2 *Assume (H1) and (H2) and consider the unique viscosity solution* v *of* (3) *with* $v(x) = 0$ *for all* $x \in D$.

If $f(\cdot, a)$ *and* $g(\cdot, a)$ *are uniformly Lipschitz continuous in* $\mathcal{D}_0$, *with constants* L_f, $L_g > 0$ *uniformly in* $a \in A$, *and if there exists an open neighborhood* N *of* D *such that for all* x, $y \in N$ *the inequality*

$$\begin{aligned} & |g(x, a) - g(y, a)| \\ & \le K\alpha_2^{-1}(\max\{\mathrm{dist}(x, D),\, \mathrm{dist}(y, D)\})^s \|x - y\| \end{aligned}$$

holds for some $K > 0$, $s > L_f$ *and* α_2 *from* (2), *then the function* v *is Lipschitz continuous in* $\mathbb{R}^n$ *for all* g *with* $g_0 > 0$ *from (H2) sufficiently large.*

Proof: It is sufficient to show that

$$V(x) := \sup_{a \in \mathcal{A}} J(x, a)$$

is (locally) Lipschitz on $\mathcal{D}_0$, since then the assertion follows as in the proof of [5, Proposition 4.4].

In order to see Lipschitz continuity of V observe that

$$\begin{aligned} & |V(x) - V(y)| \\ & \le \sup_{a \in \mathcal{A}} \int_0^\infty |g(x(t, x, a), a(t)) - g(x(t, y, a), a(t))| dt. \end{aligned}$$

By continuous dependence on the initial value for all $x \in \mathcal{D}_0$ and by the asymptotic stability of D there exists a time $T > 0$ and a neighborhood B such that $x(T + t, y, a) \in N$ for all $a \in \mathcal{A}$, all $y \in B$ and all $t \geq 0$. Abbreviating $x(t) = x(t, x, a)$ and $y(t) = x(t, y, a)$ we can conclude

$$\begin{aligned} & |V(x) - V(y)| \\ & \le \sup_{a \in \mathcal{A}} \int_0^T |g(x(t), a(t)) - g(y(t), a(t))| dt \\ & \quad + \sup_{a \in \mathcal{A}} \int_T^\infty |g(x(t), a(t)) - g(y(t), a(t))| dt \\ & \le \int_0^T L_g e^{L_f t} \|x - y\| dt \\ & \quad + \int_T^\infty K\alpha_1(\max\{\mathrm{dist}(x(T), D),\, \mathrm{dist}(y(T), D)\})^s \\ & \qquad e^{-s(t-T)} e^{L_f t} \|x - y\| dt \\ & \le \left(\frac{L_g e^{L_f T} - 1}{L_f} + \frac{K\alpha_1(C)^s e^{L_f T}}{s - L_f} \right) \|x - y\| \end{aligned}$$

where we assumed without loss of generality boundedness of N, i.e. $\sup_{x \in N} \mathrm{dist}(x, D) \le C < \infty$. This shows the Lipschitz property of V. □

By [15, Theorems 1 & 2, Proposition 3] it follows that if we add the assumption that $f(x, A)$ be convex for all $x \in \mathbb{R}^n$ then there exists a C^∞ Lyapunov function V on $\mathcal{D}_0$ (which is in this case equal to $\mathcal{D}$ by Proposition 2.2 (iv)). Assuming that $\omega : \mathcal{D} \to \mathbb{R}_{\geq 0}$ is an indicator function for D, that is ω is continuous, $\omega(x) = 0$ if and only if $x \in D$, and $\omega(x_n) \to \infty$ for any sequence $\{x_n\}$ with $\lim x_n \in \partial\mathcal{D}$ or $\lim \|x_n\| = \infty$, then V can be chosen in such a manner, that it has the following additional properties. There exist two class $\mathcal{K}^\infty$ functions η_1, η_2 such that

$$\eta_1(\omega(x)) \le V(x) \le \eta_2(\omega(x)) \tag{15}$$

and it holds that

$$\max_{a \in A} DV(x) f(x, a) \le -V(x)\,. \tag{16}$$

Using this result we can also obtain smooth solutions of Zubov's equation by a proper choice of g.

Proposition 4.3 *Assume (H1) and that $f(x, A)$ is convex for all $x \in \mathbb{R}^n$. Let $B \subset \mathcal{D}_0$ satisfy* $\operatorname{dist}(B, \partial\mathcal{D}_0) > 0$*, then there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ such that the corresponding solution v of* (3) *is C^∞ on a neighborhood of B.*

Proof: Given a smooth Lyapunov function V defined on $\mathcal{D}$ and defining $v(x) = 1 - e^{-V(x)}$ as before it suffices to define g on $\mathcal{D}$ by

$$g(x,a) := g(x) := -\sup_{a\in\mathcal{A}} \frac{Dv(x)f(x,a)}{1-v(x)} = \quad (17)$$

$$-\sup_{a\in\mathcal{A}} \frac{e^{-V(x)}DV(x)f(x,a)}{e^{-V(x)}} = -\sup_{a\in A} DV(x)f(x,a)\,.$$

Then a short calculation shows that the functions v and g thus defined solve the partial differential equation (3). The problem with this is that it is a priori unclear if g can be extended continuously to $\mathbb{R}^n$. Given a closed set $B \subset \mathcal{D}_0$, however, we can use the definition (17) on a neighborhood of B whose closure is contained in $\mathcal{D}_0$ and extend the function g continuously to $\mathbb{R}^n$ in some manner so that (H2) (ii) and (iii) are satisfied. This results in a solution v of (3) that is smooth on the chosen neighborhood of B. In order to guarantee that g satisfies condition (H2) (i) we will slightly modify V in a neighborhood of D. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be any C^∞ function that satisfies $\gamma(s) = 0, s \le 0$ and

$$\gamma'(s) \le \frac{\min\{\alpha_2^{-1}(\operatorname{dist}(x,D)) \mid V(x) = c\}}{s}$$

for $0 \le s \le r/2$ and furthermore $\gamma(s) = s$ for all s large enough. Then it is easy to see that $\gamma \circ V$ is a smooth Lyapunov function on $\mathcal{D}$, and using (16) it is easy to see that the function g defined by (17) using $\gamma\circ V$ satisfies (H2) (i). □

5 Conclusion

In this paper we have presented a generalization of Zubov's equation for the characterization of domains of attractions and the generation of Lyapunov functions. The generalization is made in three points: First, we allow for time varying deterministic perturbations, second, we do not assume exponential but only asymptotic stability, and third, we consider arbitrary compact and connected attracting sets instead of fixed points or periodic orbits.

As in this generality smooth solutions to the PDE under consideration cannot be expected we work within the framework of viscosity solutions using in particular methods from nonlinear optimal control. Under mild conditions on the coefficients appearing in the equation we can, however, ensure global Lipschitz continuity of the solution.

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