On linear convergence of discounted optimal control problems with vanishing discount rate

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Abstract

In this paper we investigate the rate of convergence of the optimal value function of an infinite horizon discounted optimal control problem as the discount rate tends to zero. We provide several estimates along trajectories yielding results on the rate of convergence of the corresponding functional. Using appropriate controllability assumptions we derive a linear convergence theorem on control sets. Applications of these results are given and an example is discussed in which both linear and slower rates of convergence occur.

1 Introduction

The question of convergence of optimal value functions of infinite horizon discounted optimal control problems has been considered by various authors during the last years, see e.g. [10], [2], [12], [1], [13], [8] and the references therein. Roughly summarized, these papers state that under appropriate controllability conditions the value functions converge uniformly to the optimal value of an average yield optimal control problem at least on certain subsets of the state space. The main motivation for obtaining such results is the fact that the optimal value functions of discounted optimal control problems have certain nice properties (e.g. contrary to the average yield case they are characterized as the solution of a Hamilton-Jacobi-Bellman equation).

However, up to now little has been reported in the literature about the corresponding *rate of convergence*. In the discrete-time Markovian case the results in [13] can be used to obtain immediate estimates for the rate of convergence. The assumptions in this reference, however, exclude the deterministic case. This paper presents first results for continuous time deterministic systems (see also [11] for some related discrete time results). In Section 2 the precise problem formulation is presented. In Section 3 we develop appropriate estimates for corresponding discounted and averaged functionals based on the Integration Theorem for Laplace Transforms and we translate these results to the optimal value functions. In Section 4 a number of situations in which linear convergence holds are characterized. Afterwards, in Section 5, we discuss some cases where these properties are satisfied and finally, in Section 6, we provide an example illustrating that for one and the same control system linear convergence may or may not hold depending on the cost function defining the functional to be minimized. For the proofs to the statements given below we refer to [9].

2 Problem formulation

We consider nonlinear optimal control problems for which the dynamics are given by control systems of the type

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.1}$$

on some Riemannian manifold M where f is such that uniqueness and existence of solutions on $\mathbb{R}_+$ is satisfied for $u(\cdot) \in \mathcal{U} := \{u : \mathbb{R} \to U \,|\, u(\cdot) \text{ measurable}\}$ and $U \subset \mathbb{R}^m$ is compact. For a given initial value $x_0 \in M$ at time $t = 0$ and a given control function $u(\cdot) \in \mathcal{U}$ we denote the trajectories of (2.1) by $\varphi(t, x_0, u(\cdot))$. Let

$$g : M \times \mathbb{R}^m \to \mathbb{R} \tag{2.2}$$

be a cost function which is Lipschitz continuous and bounded, i.e. $|g(x,u)| \leq M_g$ for some constant M_g. For $\delta > 0$ we define the discounted functional

$$J_\delta(x_0, u(\cdot)) := \delta \int_0^\infty e^{-\delta s} g(\varphi(s, x_0, u(\cdot)), u(s)) ds \tag{2.3}$$

and the optimal value function for the corresponding minimization problem by

$$v_\delta(x_0) := \inf_{u(\cdot)\in\mathcal{U}} J_\delta(x_0, u(\cdot)) \tag{2.4}$$

(Note that the corresponding maximization problem is obtained by simply replacing g by $-g$.)

In order to characterize the convergence properties for $\delta \to 0$ we also need to define the averaged functionals

$$J_0^t(x_0, u(\cdot)) := \frac{1}{t}\int_0^t g(\varphi(s, x_0, u(\cdot)), u(s)) ds\,,$$

$$J_0(x_0, u(\cdot)) := \limsup_{t\to\infty} J_0^t(x_0, u(\cdot))\,.$$

and the averaged minimal value function

$$v_0(x) := \inf_{u(\cdot)\in\mathcal{U}} J_0(x, u(\cdot))\,.$$

3 Discounted and averaged functionals and value functions

In this section we discuss the relation between discounted and averaged functionals and value functions. Here we will use a theorem from the theory of Laplace transformations as the starting point of our analysis, (see e.g. [5, Theorem 8.1]).

Theorem 3.1 Let $q : \mathbb{R} \to \mathbb{R}$ be a measurable function bounded by M_q. Then

$$\delta \int_0^\infty e^{-\delta t} q(t) dt = \delta^2 \int_0^\infty e^{-\delta t} \int_0^t q(s) ds\, dt$$

We use Theorem 3.1 in order to obtain the following relation between the rate of convergence of discounted and average time functionals.

Proposition 3.2 Consider a point $x \in M$. Let $A \in \mathbb{R}$ and $T > 0$ and assume there exist sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $T_k \to \infty$ as $k \to \infty$ such that

$$J_0^t(x, u_k(\cdot)) \leq \lambda + \frac{A}{t} \text{ for all } t \in [T, T_k].$$

Then there exist $\varepsilon_k(\delta) \to 0$ for each fixed δ as $k \to \infty$ such that

$$J_\delta(x, u_k(\cdot)) \leq \lambda + A\delta + \delta^2 M_g T^2 + \varepsilon_k(\delta)$$

Conversely if there exists a $\delta > 0$ and $u(\cdot) \in \mathcal{U}$ such that

$$J_\delta(x, u(\cdot)) \leq \lambda + A\delta$$

then for each $\varepsilon > 0$ there exists a time $t(\delta, \varepsilon) \geq \sqrt{\varepsilon / \delta M_g}$ such that

$$J_0^{t(\delta,\varepsilon)}(x, u(\cdot)) \leq \lambda + \frac{A + \varepsilon}{t(\delta, \varepsilon)}$$

Both assertions also hold for the converse inequality, where in the first assertion "$+\delta^2 M_g T^2$" is replaced by "$-\delta^2 M_g T^2$" and in the second "$+\varepsilon$" is replaced by "$-\varepsilon$".

In what follows we will also need the following estimate, which can be shown by a straightforward calculation.

Lemma 3.3 Let $J_0^t(x, u(\cdot)) \leq \sigma$ for all $t \in [0, T]$. Then $J_\delta(x, u(\cdot)) \leq \sigma + e^{-\delta T} 2M_g$.

4 A linear convergence Theorem

We will now use the estimates from the preceding section in order to deduce results on linear convergence by imposing assumptions on the optimal trajectories. Also, we are going to use certain reachability and controllability properties of the system, and will start this section by defining the necessary objects and properties.

Definition 4.1 The *positive orbit of $x \in M$ up to the time T* is defined by

$$O_T^+(x) := \{y \in M \mid \exists 0 \leq t \leq T, u(\cdot) \in \mathcal{U}, \text{ such that } \varphi(t, x, u(\cdot)) = y\}.$$

The *positive orbit* of $x \in M$ is defined by $O^+(x) := \bigcup_{T \geq 0} O_T^+(x)$. The negative orbits $O_T^-(x)$ and $O^-(x)$ are defined similarly by using the time reversed system.

Definition 4.2 $D \subseteq M$ is called a *control set*, if:

(i) $D \subseteq \overline{O^+(x)}$ for all $x \in D$,

(ii) for every $x \in D$ there is $u(\cdot) \in \mathcal{U}$ such that the corresponding trajectory $\varphi(t, x, u(\cdot)) \in D\ \forall t \geq 0$,

(iii) D is maximal with the properties (i) and (ii)

A control set C is called *invariant*, if $\overline{C} = \overline{O^+(x)}\ \ \forall x \in C$.
Note that this (usual) definition of control sets demands only approximate reachability; a convenient way to avoid assumptions about the speed of this asymptotic reachability (as they are imposed e.g. in [1]) is to assume local accessibility. If local accessibility holds we have exact controllability in the interior of control sets, i.e. $\mathrm{int} D \subset O^+(x)$ for all $x \in D$, cp. e.g. [2].
Using the notion of control sets we are now able to characterize situations in which linear convergence holds. Our first result is that v_δ is constant except for a term linear in δ on compact subsets of the interior of control sets.

Proposition 4.3 Consider a locally accessible optimal control problem of the form (2.1)–(2.4). Let $D \subset M$ be a control set with nonvoid interior. Let $K \subset \mathrm{int} D$ be a compact set. Then there exists a constant C_K such that

$$|v_\delta(x) - v_\delta(y)| \leq \delta C_K M_g \text{ for all } x, y \in K.$$

The next step in the analysis of the rate of convergence of optimal value functions on control sets is to derive estimates for finite time averaged functionals along trajectories staying in some compact subset of a control set. To this end for $x \in K \subset M$ we denote by $\mathcal{U}_{x,K} \subset \mathcal{U}$ the set of all control functions $u(\cdot)$ satisfying $\varphi(t, x, u(\cdot)) \in K$ for all $t \geq 0$.

Proposition 4.4 Consider the optimal control problem (2.1)–(2.4) and assume that (2.1) is locally accessible. Let $D \subset M$ be a control set with nonvoid interior and $K \subseteq D$ be compact. Then
(i) For each $x \in \mathrm{int} K$ there exists a constant $A = A(x) > 0$ and a time $T = T(x)$ such that

$$J_0^t(x, u(\cdot)) \geq v_0(x) - \frac{A}{t}, \text{for all } u(\cdot) \in \mathcal{U}_{x,K},\ t > T.$$

(ii) There exist a point $x^* \in K$ and sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $t_k \to \infty$ such that

$$J_0^t(x^*, u_k(\cdot)) \leq \inf_{x \in K} \inf_{u(\cdot) \in \mathcal{U}_{x,K}} J_0(x, u(\cdot)) + \varepsilon_k(T)$$

for all $t \in [0, \min\{T, t_k\}]$ where $\varepsilon_k(T) \to 0$ for $k \to \infty$.

Now we can combine Propositions 3.2 and 4.4 in order to obtain our main theorem.

Theorem 4.5 Consider the optimal control problem (2.1)–(2.4) and assume that (2.1) is locally accessible. Let $D \subset M$ be a control set with nonvoid interior. Assume that one of the following conditions is satisfied
(i) There exist a compact subset $K_0 \subset \mathrm{int}D$ and sequences of points $x_k \in K_0$ and control functions $u_k(\cdot) \in \mathcal{U}_{x_k,K_0}$ such that

$$J_0(x_k, u_k(\cdot)) \to v_0|_{\mathrm{int}D}$$

(ii) There exist $x_0 \in \mathrm{int}D$, $T \geq 0$ and sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $T_k \to \infty$ as $k \to \infty$ such that the inequality

$$J_0^t(x_0, u_k(\cdot)) \leq v_0|_{\mathrm{int}D} + \frac{A}{t}$$

holds for some constant $A \geq 0$ and all $t \in [T, T_k]$, $k \in \mathbb{N}$. Then for each compact subset $K \subset \mathrm{int}D$ there exist constants $A_K > 0$ and $\delta_0 > 0$ such that

$$v_\delta(x) \leq v_0|_{\mathrm{int}D} + \delta A_K \text{ for all } x \in K \text{ and all } \delta \leq \delta_0.$$

Conversely, if the following assumption is valid
(iii) There exists $x_0 \in$ intD and a compact subset $K_1 \subseteq D$ such that for all sufficiently small $\delta > 0$ there exist optimal trajectories for v_δ in $\mathcal{U}_{x_0,K_1}$
then for each compact subset $K \subset \mathrm{int}D$ there exist constant $B_K > 0$ and $\delta_0 > 0$ such that

$$v_\delta(x) \geq v_0|_{\mathrm{int}D} - \delta B_K \text{ for all } x \in K \text{ and all } \delta \leq \delta_0.$$

Using the invariance property of invariant control sets we can conclude the following corollary.

Corollary 4.6 Consider the optimal control problem (2.1)–(2.4) and assume that (2.1) is locally accessible. Let $C \subset M$ be a compact invariant control set with nonvoid interior. Assume that one of the following conditions is satisfied
(i) There exist a compact subset $K_0 \subset \mathrm{int}C$ and sequences of points $x_k \in K_0$ and control functions $u_k(\cdot) \in \mathcal{U}$ such that $\varphi(t, x_k, u_k(\cdot)) \in K$ for all $k \in \mathbb{N}$ and all $t \geq 0$ and

$$J_0(x_k, u_k(\cdot)) \to v_0|_{\mathrm{int}C}$$

(ii) There exist $x_0 \in \mathrm{int}C$, $T \geq 0$ and sequences of control functions $u_k(\cdot) \in \mathcal{U}$ and times $T_k \to \infty$ as $k \to \infty$ such that the inequality

$$J_0^t(x_0, u_k(\cdot)) \leq v_0(x_0) + \frac{A}{t}$$

holds for some constant $A \geq 0$ and all $t \in [T, T_k]$.
Then for each compact subset $K \subset \mathrm{int}D$ there exist constant $B_K > 0$ and $\delta_0 > 0$ such that

$$|v_\delta(x) - v_0(x)| \leq \delta B_K \text{ for all } x \in K \text{ and all } \delta \leq \delta_0.$$

Proof: The invariance of C immediately implies that assumption (iii) of Theorem 4.5 is always satisfied (with $K_1 = C$). □

5 Applications

In this section we will highlight two situations in which linear convergence can be concluded from the theorems in the preceding section.
The first situation is given by completely controllable systems on compact manifolds. More precisely the following corollary is an immediate consequence of Corollary 4.6.

Corollary 5.1 Consider a locally accessible optimal control system (2.1)–(2.4) on a compact manifold M. Assume the system is completely controllable, i.e. there exists an invariant control set $C = M$. Then there exists a constant $K > 0$ such that

$$\|v_\delta - v_0\|_\infty < K\delta.$$

Note that this setup coincides with the one in [6]; in fact there is a strong relation between this result and the periodicity result there since in both cases the values of trajectory pieces have to be estimated. The techniques, however, used in order to obtain these results are rather different.
The second application of our results is somewhat more specific. Here we consider the problem of the approximation of the top Lyapunov exponent of a semi-linear control system

$$\dot{x}(t) = A(u(t))x(t), \quad x \in \mathbb{R}^d \tag{5.1}$$

This problem is the continuous time analogue to the one considered in [11]. Note that here we consider the maximization problem so all results are applied with inverted inequalities. Also, since here we are going to derive an estimate for the supremum of v_δ we will directly use Propositions 3.2 and 4.4 instead of Theorem 4.5.
We will briefly collect some facts about this problem, for detailed information we refer to [3] and [4].
The Lyapunov exponent of a solution $x(t, x_0, u(\cdot))$ of (5.1) is defined by

$$\lambda(x_0, u(\cdot)) = \limsup_{t\to\infty} \frac{1}{t} \ln \|x(t, x_0, u(\cdot))\|$$

which for $\|x_0\| = 1$ can also be expressed as an averaged integral by

$$\lambda(x_0, u(\cdot)) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t g(\varphi(s, x_0, u(\cdot)), u(s))ds$$

where $\varphi(t, x_0, u(\cdot))$ denotes the solution of the system projected to $M = \mathbb{S}^{d-1}$ — which satisfies $\dot{s}(t) = (A(u(t)) - s(t)^T Au(t)s(t) \cdot \mathrm{Id})s(t)$ — and g is a suitable function meeting our general assumptions. The top Lyapunov exponent can be defined on $\mathbb{S}^{d-1}$ via

$$\kappa := \sup_{x_0\in\mathbb{S}^{d-1}} \sup_{u(\cdot)\in\mathcal{U}} \lambda(x_0, u(\cdot)).$$

It characterizes the stability of the solutions of (5.1) under all possible functions $u(\cdot)$, and can also be used to define a stability radius of (5.1) analogously to [11].

It already follows from the arguments in [8] that $\sup_{x\in\mathbb{S}^{d-1}} v_\delta(x)$ converges to κ as $\delta \to 0$. Now it remains to determine the rate of convergence.

Assuming local accessibility for the projected system there exists a invariant control set $C \subset \mathbb{S}^{d-1}$ with nonvoid interior (in $\mathbb{S}^{d-1}$). Furthermore the top Lyapunov exponent can be realized from any initial value $x_0 \in \mathbb{S}^{d-1}$, hence from any point $x_0 \in \mathrm{int}C$. Thus Proposition 4.4(ii) with $K = C$ yields the existence of a point $x^* \in C$ and sequences of control functions $u_l(\cdot) = u(t_{k_l} + \cdot)$ and times t_l satisfying

$$J_0^t(x^*, u_l(\cdot)) \geq \kappa - \varepsilon_l(T) \text{ for all } t \in [0, \min\{T, t_l\}].$$

We can conclude that $v_\delta(x^*) \geq \kappa$ for all $\delta > 0$ and it remains to find an upper bound for $\sup_{x\in\mathbb{S}^{d-1}} v_\delta(x)$. To this end consider a basis $x_1, \ldots, x_d$ of $\mathbb{R}^d$ such that $\|x_i\| = 1$ and $x_i \in \mathrm{int}C$ for all $i = 1, \ldots, d$. Then Proposition 4.4(i) with $K = C$ yields the existence of a constant $B > 0$ such that

$$J_0(x_i, u(\cdot)) \leq \kappa + \frac{B}{t}$$

for all $i = 1, \ldots, d$ and all $u(\cdot) \in \mathcal{U}$ and hence $\|x(t, x_i, u(\cdot))\| \leq e^B e^{\kappa t}$. By the compactness of $\mathbb{S}^{d-1}$ there exists a constant $\nu > 0$ such that any point $x_0 \in \mathbb{S}^{d-1}$ can be written as a linear combination $x_0 = \sum_{i=1}^d \mu_i(x_0) x_i$ with coefficients $|\mu_i(x_0)| \leq \nu$. Thus we obtain

$$\|x(t, x_0, u(\cdot))\| = \|\sum_{i=1}^d \mu_i(x_0) x(t, x_i, u(\cdot))\| \leq d\nu e^B e^{\kappa t}.$$

Thus with $A = B + \ln d\nu$ it follows that

$$J_0(x_0, u(\cdot)) \leq \kappa + \frac{A}{t}$$

for all $x_0 \in \mathbb{S}^{d-1}$ and all $u(\cdot) \in \mathcal{U}$. For any $\tilde{A} > A$ Proposition 3.2 (with $u_k(\cdot) = u(\cdot)$ for all k) yields $v_\delta(x_0) \leq \kappa + \delta\tilde{A}$ for all sufficiently small δ which finally yields the desired estimate

$$\sup_{x\in\mathbb{S}^{d-1}} v_\delta(x) \in [\kappa, \kappa + \delta\tilde{A}]\,.$$

6 An Example

Here we provide an example of a simple 1d control system with one (invariant) control set where linear convergence does or does not hold depending on the cost function. Consider the control system

$$\dot{x} = -ux|x| + (u-1)(x-1)|x-1| \qquad (6.1)$$

with $x \in \mathbb{R}$ and $u \in [0,1]$. It is easily seen that (6.1) possesses an (invariant) control set $C = [0,1]$. For the cost function $g_1(x,u) = |x|$ and initial values $x_0 \in C$ it is obviously optimal to steer to the left as fast as possible, i.e. the optimal control is $u \equiv 1$.

The solution for this constant control is given by $x(t) = \frac{x_0}{tx_0+1}$, thus $J_0^t(x_0, 1) = \frac{ln(tx_0+1)}{tx_0}$ does not converge linearly, and by the first assertion of Proposition 3.2 (for the converse inequality) the same holds for δv_δ.

Now we consider $g_2(x,u) = |x - 0.5|$. For the initial value $x_0 = 1/2$ we obtain with $u \equiv 1/2$ that $x(t, x_0, u) = x_0$ for all $t > 0$, hence $J_0^t(1/2, 1/2) = 0$ for all $t > 0$. Obviously here Condition (i) of Corollary 4.6 is satisfied, thus linear convergence follows. A similar argumentation is valid for all $\alpha \in (0,1)$.

7 Conclusions

Convergence rates of optimal value functions of discounted optimal control problems have been investigated. It has been shown that under appropriate assumptions linear convergence holds. These conditions are applied to problems from application implying linear convergence. However, an example shows that linear convergence is not always satisfied.

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