Exponential stability of families of linear delay systems

F. Wirth
Zentrum für Technomathematik
Universität Bremen
28334 Bremen, Germany
fabian@math.uni-bremen.de

Keywords: Stability, delay systems, discrete inclusions.

Abstract

We analyze exponential stability concepts for families of linear delay systems. The main result states that for such families with coefficients varying in compact convex sets various characteristic exponents characterizing different concepts of stability coincide. This result is obtained using methods from the theory of discrete inclusions in Banach spaces. As a further application of this approach a new result on slowly-varying discrete inclusions is applied to the delay equation setting. This shows that just as in the standard case of time-varying systems the exponential behavior of a slowly time-varying system is determined by the limit family.

1 Introduction

In this work we are concerned with stability properties of families of time-varying linear delay equations. The motivation for the paper is to explain how abstract results in the theory of discrete inclusions in Banach spaces can be employed to obtain in a rather straightforward way results in more "concrete" situations. This approach has also been advocated by Przyłuski, see e.g. [9], [10], [11]
The motivation to study families of time-varying linear delay equations lies in the fact that they may be interpreted as a model for time-varying uncertainty in a time-invariant system. As such discrete inclusions are a very general way of describing time-varying uncertainty.
Our basic tool are the results from [13] where exponential growth of discrete inclusions on reflexive Banach spaces are studied. Infinite dimensional discrete inclusions have been studied to a great extent in [4], [5]. Robustness of stability was studied for discrete time systems in infinite dimensions in [12].
In Section 2 we introduce the class of delay systems we are considering. Their corresponding Lyapunov and Bohl exponents are introduced and the basic discretization method is presented. After this discussion of characteristic exponents we turn to the study of discrete inclusions in the following Section 3. For discrete inclusions on reflexive Banach spaces it is known that the supremal Lyapunov exponent, the supremal Bohl exponent and a further "uniform" exponential growth rate coincide. In Section 4 it is shown that similar results hold in the delay equation case. In order to prove this results we simply have to show that the setup for the delay equations can be transferred to the discrete inclusion setting.
In Section 5 we present a new result for discrete inclusions, that is concerned with slowly varying inclusions, that is time-varying inclusions that are given by (strongly) convergent sets, in analogy to the commonly employed term in the theory of time-varying systems. It is shown that the stability behavior is completely determined by the limit set. A result which extends the well known results for time-varying systems. It is explained how this result has immediate consequence for slowly-varying families of time-varying delay systems.

2 Delay Systems

In this section we introduce the class of systems we intend to study and present some basic prerequisites.
Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators from X to X. The norm on X and the induced operator norm on $\mathcal{L}(X)$ are both denoted by $\|\cdot\|$.
A Banach space X is called reflexive if the range of the natural embedding of X into X^{**} is X^{**}. A standard result is that X is reflexive iff the unit ball in X is compact in the weak topology which is in turn equivalent to the weak compactness of the unit ball in $\mathcal{L}(X)$ (see [3] Theorem V.4.7 and Exercise VI.9.6). Recall that a net $\{A_n\} \subset \mathcal{L}(X)$ converges weakly to A iff for all $x \in X$ and $f \in X^*$ it holds that $< A_n x, f >$ converges to $< Ax, f >$. Weak convergence is denoted by $\mathrm{w} - \lim_n A_n = A$. So the weak topology in $\mathcal{L}(X)$ is not to be confused with the topology that comes from the interpretation of $\mathcal{L}(X)$ as a Banach space and is induced by $\mathcal{L}(X)^*$ the space of bounded linear functionals on $\mathcal{L}(X)$. A set $\mathcal{M} \subset \mathcal{L}(X)$ is called weakly compact if it is compact with respect to the weak topology.
We are interested in defining a class of delay systems with distributed delay. Fix $1 < p < \infty$, $n, m \in \mathbb{N}$ and delay times $0 < h_1 < \ldots < h_m \leq h$. Let $\emptyset \neq U_i \subset \mathbb{R}^{n\times n}$, $i = 0, \ldots, m$ be compact and convex, and $\emptyset \neq V \subset L^q([-h, 0], \mathbb{R}^{n\times n})$ be weakly compact and convex, where

$p^{-1}+q^{-1}=1$. We consider the following type of delay systems:

$$\begin{aligned}\dot{x}(t) &= A_0(t)x(t)+\sum_{i=1}^{m}A_i(t)x(t-h_i)+\\ &\quad \int_{-h}^{0}\eta(t,\tau)x(t+\tau)d\tau\,,\ a.e. \qquad (1)\\ x(0) &= r\in\mathbb{R}^n\\ x([-h,0]) &= \phi\in L^p([-h,0],\mathbb{R}^n)\end{aligned}$$

where $A_i:\mathbb{R}\to U_i$, $i=0,\dots,m$ is measurable and $\eta:\mathbb{R}\to V$ is strongly measurable. Denote the set

$$\Gamma:=\{(A_0(\cdot),\dots,A_m(\cdot),\eta(\cdot))\ ;\ A_i:\mathbb{R}\to U_i,\ \text{measurable}\,,$$

$$i=0,\dots,m,\quad \eta:\mathbb{R}\to V\ \text{strongly measurable}\}\,.$$

Then we may interpret (1) as a family of time-varying delay systems indexed by the set Γ. In the following we will always implicitly use the natural isomorphism between $L^p([-h,0],\mathbb{R}^n)$ and $L^p([0,h],\mathbb{R}^n)$ given by simply shifting the interval of definition.
The solution of (1) corresponding to an initial value $x(0)=x_0\in\mathbb{R}^{n\times n}$, an initial function $x=\phi\in L^p([-h,0],\mathbb{R}^n)$ and a particular choice $\gamma\in\Gamma$ is denoted by $x(\cdot;x_0,\phi,\gamma)$. It is given by

$$x(t,x_0,\phi,\gamma)=\Phi_0(t,0)x_0+$$

$$\sum_{i=1}^{m}\int_0^t\Phi_0(t,s)A_i(t)x(s-h_i,x_0,\phi,\gamma)ds+$$

$$\int_0^t\Phi_0(t,s)\int_{-h}^{0}\eta(s,\tau)x(s+\tau,x_0,\phi,\gamma)d\tau ds\,,$$

where $\Phi_0(t,s)$ denotes the evolution operator generated by $\dot{x}=A_0(t)x$.
For the general theory of time-varying delay equations we refer to [6], Chapter 6 and [2], Chapter XII. Here however we refrain from considering delay equations with the continuous functions as initial conditions as our results depend heavily on the fact that the system has a representation on a reflexive Banach space. This is the next point to be discussed.
We introduce the Banach space

$$X:=\mathbb{R}^n\times L^p([0,h])\,.$$

Under the conditions we have mentioned it may be seen that for every choice $\gamma\in\Gamma$ the system (1) induces an evolution system on X. We omit the details. For the general theory of evolution systems see [7], [1], [8]. That is, we have a family of operators $U(t,s)\in\mathcal{L}(X)$, $t\geq s\geq 0$ satisfying

$$\begin{gathered}U_\gamma(s,s)=I_X\,,\\ U_\gamma(t,r)U_\gamma(r,s)=U_\gamma(t,s)\,,\ t\geq r\geq s\geq 0\,,\\ (t,s)\mapsto U(t,s)\ \text{is strongly continuous in each argument}\,,\end{gathered}$$

such that for any $x\in X$ the function $t\mapsto U_\gamma(t,s)x$, $t\geq s$ is a mild solution to (1) for the particular delay equation determined by γ. We are interested in the exponential behavior of all solutions of the family of delay systems indexed by the set Γ. In particular, we want to answer the question if the existence of exponential bounds of the form

$$\|U_\gamma(t,s)x\|<M_{\gamma,x}e^{\beta(t-s)}$$

implies the existence of a universal constant M such that

$$\|U_\gamma(t,s)x\|<Me^{\beta(t-s)}\,,\quad \forall\gamma\in\Gamma\,.$$

Note that for a particular choice of γ this conclusion would be false, see [1]. This is due to the fact that Bohl exponent and maximal Lyapunov exponent of an evolution system may differ. Let us recall the relevant definitions. Given an evolution system $\mathbf{U}:=(U(t,s))_{t\geq s\geq 0}$ in $\mathcal{L}(X)$ the *(upper) Bohl exponent* is

$$\begin{aligned}\beta(\mathbf{U})=\inf\{\beta\in\mathbb{R}\,;\exists c_\beta\geq 1:t\geq s\geq 0\Rightarrow\\ \|U(t,s)\|\leq c_\beta e^{\beta(t-s)}\}\,,\end{aligned}\qquad (2)$$

where $\inf\emptyset=\infty$. Given furthermore an initial condition $(t_0,x_0)\in\mathbb{R}_+\times(X\setminus\{0\})$ the *Lyapunov exponent* corresponding to (t_0,x_0,ϕ_0) is defined by

$$\begin{aligned}\lambda(t_0,x_0,\phi_0) &= \inf\{\lambda\in\mathbb{R};\exists c_\lambda\geq 1:t\geq t_0\Rightarrow\\ &\quad \|U(t,t_0)x_0\|\leq c_\lambda e^{\lambda(t-t_0)}\|x_0\|\|\phi_0\|\}.\end{aligned}\qquad (3)$$

Furthermore we define the supremal Lyapunov exponent by

$$\kappa(\mathbf{U}):=\sup\{\lambda(t_0,x_0);(t_0,x_0)\in\mathbb{N}\times(X\setminus\{0\})\}.$$

It should be noted that Lyapunov and Bohl exponents do not characterize asymptotic stability. Also, in general $\kappa(\mathbf{U})<\beta(\mathbf{U})$. For the family of time-varying delay systems we introduce the exponential growth constants

$$\begin{aligned}\bar{\kappa}(\Gamma) &:= \sup\{\kappa(\mathbf{U}_\gamma)\ ;\ \gamma\in\Gamma\}\,,\\ \bar{\beta}(\Gamma) &:= \sup\{\beta(\mathbf{U}_\gamma)\ ;\ \gamma\in\Gamma\}\,,\\ \bar{\delta}(\Gamma) &:= \limsup_{t\to\infty}\frac{1}{t}\log\sup_{\gamma\in\Gamma}\|U_\gamma(t,0)\|\,.\end{aligned}\qquad (4)$$

Let us now convert our data to a discrete-time setting. For fixed $\gamma\in\Gamma$ an equation of the form (1) induces an operator $A_\gamma\in\mathcal{L}(X)$ given by

$$A_\gamma\begin{bmatrix}r\\ \phi\end{bmatrix}:=U_\gamma(h,0)\begin{bmatrix}r\\ \phi\end{bmatrix}=\begin{bmatrix}x(h;r,\phi,\gamma)\\ x(\cdot;r,\phi,\gamma)\end{bmatrix}.$$

Thus we may consider the discrete inclusion

$$x(t+1)\in\{Ax(t)\ ;\ A\in\mathcal{M}\}\quad t\in\mathbb{N}\,.\qquad (5)$$

where the set $\mathcal{M}$ is now given by

$$\mathcal{M}:=\{U_\gamma(h,0)\ :\ \gamma\in\Gamma\}\qquad (6)$$

A sequence $\{x(t)\}_{t\in\mathbb{N}}$ is called solution of (5) with initial condition $x_0\in X$ if $x(0)=x_0$ and for all $t\in\mathbb{N}$ there exists an $A(t)\in\mathcal{M}$ such that $x(t+1)=A(t)x(t)$.
In the following section we briefly collect the facts we need from the theory of discrete inclusions in Banach spaces.

3 Discrete inclusions in Banach spaces

Let us briefly recall the basic notions of stability of importance for discrete-time time-varying systems in Banach spaces. We consider time-varying linear discrete-time systems of the form

$$x(t+1) = A(t)x(t), \quad t \in \mathbb{N}\,, \tag{7}$$

where $A(\cdot) = (A(t))_{t\in\mathbb{N}} \in \mathcal{L}(X)^{\mathbb{N}}$ is a sequence of bounded linear operators on X. The evolution operator associated to this system is defined by

$$\Phi_A(t,t) = I_X\,,\ \Phi_A(t,s) = A(t-1)\cdots A(s),\ t > s.$$

System (7) is called *exponentially stable*, if there are $c, \beta > 0$ such that $\|\Phi(t,s)\|_{\mathcal{L}(\mathcal{X})} \leq c\, e^{-\beta(t-s)}, \qquad s,t \in \mathbb{N}, \quad t \geq s$ holds. Let X be a Banach space and $\mathcal{M} \subset \mathcal{L}(X)$ be a bounded set. We consider the discrete inclusion

$$x(t+1) \in \{Ax(t)\ ;\ A \in \mathcal{M}\} \quad t \in \mathbb{N}\,. \tag{8}$$

The notion of solution for this kind of system is the one given at the end of Section 2. Note furthermore that in a completely analogous fashion we may introduce Lyapunov and Bohl exponents for time-varying systems by just replacing $U(t,s)$ with $\Phi(t,s)$ in (2), (3), which yields

$$\begin{aligned}\beta(\Phi) = \inf\{\beta \in \mathbb{R}; \exists c_\beta \geq 1 : t \geq s \geq 0 \Rightarrow \\ \|\Phi(t,s)\| \leq c_\beta e^{\beta(t-s)}\}\,,\end{aligned} \tag{9}$$

$$\begin{aligned}\lambda(t_0\,,x_0) \quad = \quad \inf\{\lambda \in \mathbb{R}; \exists c_\lambda \geq 1 : t \geq t_0 \Rightarrow \\ \|\Phi(t,t_0)x_0\| \leq c_\lambda e^{\lambda(t-t_0)}\|x_0\|\}.\end{aligned} \tag{10}$$

Furthermore we define the supremal Lyapunov exponent by

$$\kappa(\Phi) := \sup\{\lambda(t_0\,,x_0); (t_0\,,x_0) \in \mathbb{N} \times (X \setminus \{0\})\}\,.$$

and thus also for (8) the quantities defined in (4) are well defined. We denote them by $\bar{\kappa}(\mathcal{M}), \bar{\beta}(\mathcal{M}), \bar{\delta}(\mathcal{M})$ (see also [13]). Note that the following inequality is an immediate consequence of the definition.

$$\bar{\kappa}(\mathcal{M}) \leq \bar{\beta}(\mathcal{M}) \leq \bar{\delta}(\mathcal{M})\,. \tag{11}$$

In [13] it is in fact shown that the three quantities in (11) are in fact equal for discrete inclusions on reflexive Banach spaces using the following basic result.

Theorem 3.1 *[13] Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. If for all $x_0 \in X$, $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ it holds that*

$$\lim_{t\to\infty} \Phi_{A(\cdot)}(t,0)x_0 = 0\,,$$

then there exists a constant $c_{\mathcal{M}} \in \mathbb{R}$ such that

$$\sup\{\|\Phi_{A(\cdot)}(t,0)\|\ ;\ t \in \mathbb{N}, A(\cdot) \in \mathcal{M}^{\mathbb{N}}\} < c_{\mathcal{M}}\,. \tag{12}$$

Using Theorem 3.1 we obtain the following result on growth bounds of discrete inclusions:

Theorem 3.2 *[13] Let X be a reflexive Banach space and assume that $\mathcal{M} \subset \mathcal{L}(X)$ is weakly compact, then*

$$\bar{\kappa}(\mathcal{M}) = \bar{\beta}(\mathcal{M}) = \bar{\delta}(\mathcal{M})\,. \tag{13}$$

That is a discrete inclusion given by a weakly compact set $\mathcal{M}$ has exponentially decaying trajectories iff it is exponentially stable iff it is uniformly exponentially stable.

Corollary 3.3 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. For every $\varepsilon > 0$ there exists a constant $c_{\mathcal{M},\varepsilon}$ such that*

$$\|\Phi_{A(\cdot)}(t,s)\| \leq c_{\mathcal{M},\varepsilon} e^{(\bar{\beta}(\mathcal{M})+\varepsilon)(t-s)}\,, \tag{14}$$

for all $A(\cdot) \in \mathcal{M}^{\mathbb{N}}$ and all $t \geq s \in \mathbb{N}$.

Corollary 3.4 *Let X be a reflexive Banach space and let $\mathcal{M} \subset \mathcal{L}(X)$ be weakly compact. If the discrete inclusion (5) given by $\mathcal{M}$ has a positive uniform growth rate, then there exists a trajectory with positive exponential growth rate. In particular, there exists an unbounded trajectory.*

In the following section we will explore how these results translate to the delay equation problem.

4 Stability indices of delay equations

In order to use the results of the previous section note first that our base space $X = \mathbb{R}^n \times L^p([0,h])$ is clearly reflexive. Thus we need an equivalence of exponential growth rates of the discrete inclusion (5) with those of the evolution system (2). Furthermore we have to check that the set $\mathcal{M}$ defined by (6) is weakly compact.
We have the following Lemma.

Lemma 4.1 *Let $U_i \subset \mathbb{R}^{n\times n}, i = 1,\ldots,m$ be compact and $V \subset L^p([-h,0],\mathbb{R}^{n\times n})$ be weakly compact, then $\mathcal{M}$ defined by (6) is weakly compact in $\mathcal{L}(X)$.*

The idea of the proof relies on standard arguments from functional analysis and is omitted here.

Lemma 4.2 *Let $U \subset \mathbb{R}^{n\times n}$ be compact and $V \subset L^p([-h,0],\mathbb{R}^{n\times n})$ be weakly compact, and consider the corresponding family of time-varying delay equations (1) and the discrete inclusion (5). Then*

$$\bar{\kappa}(\Gamma) = \bar{\kappa}(\mathcal{M})\,, \quad \bar{\beta}(\Gamma) = \bar{\beta}(\mathcal{M})\,, \quad \bar{\delta}(\Gamma) = \bar{\delta}(\mathcal{M})\,.$$

To prove this we have to show that in the interior of intervals $[kh,(k+1)h]$ no growth can occur, that is not noticed at the sampling points $kh,(k+1)h$. This can be shown using the compactness of the sets U_i and V, which yield immediate estimates for uniform growth constants. We sketch an proof

for the case $\bar{\kappa}(\Gamma) = \bar{\kappa}(\mathcal{M})$. It is clear that, $\bar{\kappa}(\Gamma) \geq \bar{\kappa}(\mathcal{M})$, as

$$\left\| \begin{bmatrix} x(kh; r, \phi, \gamma) \\ x_{|[(k-1)h,kh]}(\cdot; r, \phi, \gamma) \end{bmatrix} \right\| =$$

$$\sqrt{\|x(kh; r, \phi, \gamma)\|^2 + \|x_{|[(k-1)h,kh]}(\cdot; r, \phi, \gamma)\|_p^2}$$

$$\leq \sqrt{2} \max_{t \in (k-1)h,kh]} \|x(t; r, \phi, \gamma)\| .$$

To prove the converse direction fix $x_0 \in \mathbb{R}^n, \phi_0 \in L^p([-h, 0], \mathbb{R}^n), \gamma \in \Gamma$ and let $t_k, k \in \mathbb{N}$, $t_k \to \infty$ be a sequence such that

$$\lim_{k \to \infty} \frac{1}{t_k} \log \|x(t_k, r, \phi, \gamma)\| = \lambda(t_0, x_0, \phi_0) ,$$

where we have suppressed the dependence on γ. Let $t^* \in (0, h]$ be a limit point of $t_k \bmod h$ and define $x_0^* := x(t^*, r, \phi, \gamma), \phi_0^* = x_{|[t^*-h,t^*]}(\cdot, r, \phi, \gamma)$ and the time-varying system γ^* obtained by shifting all elements of γ by t^* to the left, that is e.g. $A_0^*(t) = A_0(t + t^*)$. Then it is easy to show that the Lyapunov exponent corresponding to these initial conditions with respect to the discrete time system (7) determined by $A_{\gamma^*}(\cdot)$ coincides with $\lambda(t_0, x_0, \phi_0)$.
Combining the two previous lemmas we obtain the main result of this paper.

Theorem 4.3 *Let $U \subset \mathbb{R}^{n \times n}$ be compact and $V \subset L^p([-h, 0], \mathbb{R}^{n \times n})$ be weakly compact, and consider the corresponding family of time-varying delay equations (1). Then*

$$\bar{\kappa}(\Gamma) = \bar{\beta}(\Gamma) = \bar{\delta}(\Gamma) .$$

Proof. Immediate from Lemmas 4.1 and 4.2 and Theorem 3.2. □

Now Corollaries 3.3 and 3.4 imply the following analogous statements for (2).

Corollary 4.4 *Consider the family of delay systems (1) given by Γ. For every $\varepsilon > 0$ there exists a constant $c_{\Gamma,\varepsilon}$ such that*

$$\|U_\gamma(t, s)\| \leq c_{\Gamma,\varepsilon} e^{(\bar{\beta}(\Gamma)+\varepsilon)(t-s)} , \tag{15}$$

for all $\gamma \in \Gamma$ and all $t \geq s \in \mathbb{R}$.

Corollary 4.5 *If the family of delay systems (1) given by Γ has a positive uniform growth rate, then there exists a trajectory of (1) with positive exponential growth rate. In particular, there exists an unbounded trajectory.*

In the following section we consider the case of additional time-varying perturbations that vanish as time goes to infinity.

5 Slowly-varying Discrete Inclusions and Delay Systems

Let X be a Banach space and let for every $t \in \mathbb{N}$ the set $\emptyset \neq \mathcal{M}(t) \subset \mathcal{L}(X)$ be bounded. We consider the time-varying discrete inclusion

$$x(t + 1) \in \{Ax(t) \ ; \ A \in \mathcal{M}(t)\} \quad t \in \mathbb{N} . \tag{16}$$

A sequence $\{x(t)\}_{t \in \mathbb{N}}$ is called solution of (16) with initial condition $x_0 \in X$ if $x(0) = x_0$ and for all $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}(t)$ such that $x(t + 1) = A(t)x(t)$. Also we say that a sequence $A(\cdot) \in \mathcal{M}(\cdot)$ if $A(t) \in \mathcal{M}(t)$ for all $t \in \mathbb{N}$. Here we concentrate on the case when the sets $\mathcal{M}(t)$ converge to a set $\mathcal{M}_\infty$, where we speak of convergence if the following holds:
For every $\varepsilon > 0$ there exists a t_ε such that for all $t \geq t_\varepsilon$

$$\max\{ \sup_{A \in \mathcal{M}(t)} d(A, \mathcal{M}_\infty), \sup_{A \in \mathcal{M}_\infty} d(A, \mathcal{M}(t))\} < \varepsilon .$$

thus we consider the usual Hausdorff distance of sets. Again Lyapunov and Bohl exponents are defined as usual and the meaning of the growth rates $\bar{\kappa}(\mathcal{M}(\cdot)), \bar{\beta}(\mathcal{M}(\cdot)), \bar{\delta}(\mathcal{M}(\cdot))$ is given by an appropriate change of notation in (4). Then we have

Theorem 5.1 *Let X be a reflexive Banach space and let $\mathcal{M}(\cdot)$ be uniformly bounded. Assume furthermore that for each $t \in \mathbb{N}$ the set $\mathcal{M}(t)$ is weakly compact. Assume that $\lim_{t \to \infty} \mathcal{M}(t) = \mathcal{M}$, where $\mathcal{M}$ is weakly compact, then*

$$\bar{\kappa}(\mathcal{M}_\infty) = \bar{\beta}(\mathcal{M}(\cdot)) = \bar{\beta}(\mathcal{M}_\infty) =$$

$$\bar{\delta}(\mathcal{M}(\cdot)) = \bar{\delta}(\mathcal{M}_\infty) .$$

Proof. The equality $\bar{\kappa}(\mathcal{M}_\infty) = \bar{\beta}(\mathcal{M}_\infty) = \bar{\delta}(\mathcal{M}_\infty)$ is a direct consequence of Theorem 3.2. Let $A(\cdot) \in \mathcal{M}(\cdot)$, then there exists by assumption a sequence $B(\cdot) \in \mathcal{M}_\infty^{\mathbb{N}}$ such that

$$\lim_{t \to \infty} \|A(t) - B(t)\| = 0 .$$

Then it follows from [10, 11] that $\beta(\Phi_A) = \beta(\Phi_B)$. This implies $\bar{\beta}(\mathcal{M}(\cdot)) \leq \bar{\beta}(\mathcal{M}_\infty)$ and by symmetry the quantities are equal. By Theorem 5.1 in [13] we have furthermore, that $\bar{\beta}(\mathcal{M}(\cdot)) = \bar{\delta}(\mathcal{M}(\cdot))$. □

This result implies the following result for delay systems. For maps $t \mapsto U(t) \subset \mathbb{R}^{n \times n}$ and $t \mapsto V(t) \subset L^q([-h, 0])$, where $U(t)$ and $V(t)$ are (weakly) compact for all t, we use the (norm-wise) notion of set convergence as defined above.
Assume we are given upper-semicontinuous set-valued maps $U_i(\cdot)$, $i = 0, \ldots, m$, with $U_i(t) \subset \mathbb{R}^{n \times n}$ compact, convex and an strongly upper semicontinuous map $V(\cdot)$ with weakly compact, convex values in $L^q([-h, 0])$. Assume furthermore that $U_i(t)$, $V(t)$ are uniformly bounded in t. Then we may consider the delay system

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t-h_i) + \\ &\quad \int_{-h}^{0} \eta(t,\tau)x(t+\tau)\,d\tau\,, \qquad (17) \\ x(0) &= r \in \mathbb{R}^n \\ x([-h,0]) &= \phi \in L^p([-h,0],\mathbb{R}^n) \end{aligned}$$

under the condition $A_i(t) \in U_i(t)$ a.e. and A_i measurable, $i = 0, \ldots, m$ and similarly $\eta(t) \in V(t)$ a.e. and η strongly measurable. For this system we denote the set of admissible functions by Γ'. The definition of $\bar{\kappa}(\Gamma')$, $\bar{\beta}(\Gamma')$ and $\bar{\delta}(\Gamma')$ is completely analogous to (4). Then we obtain the following result for time-varying delay systems

Proposition 5.2 *Assume the maps $U_i(\cdot)$, $i = 0, \ldots, m$ and $V(\cdot)$ are as above and that furthermore*

$$\lim_{t\to\infty} U_i(t) = U_i\,, \quad \lim_{t\to\infty} V(t) = V$$

where U_i, V satisfy the hypotheses of Theorem 4.3, then

$$\bar{\kappa}(\Gamma') = \bar{\beta}(\Gamma') = \bar{\delta}(\Gamma') = \bar{\beta}(\Gamma)\,.$$

So this result recaptures the well known result from time-varying systems that if the generators $A(t)$ converge, then the limit of the $A(t)$ determines the growth behavior of the system.

6 Conclusion

Using results from the theory of discrete inclusions in Banach spaces it has been shown that for families of time-varying delay equations the exponential growth can be defined in various equivalent ways via exponential growth of trajectories, Bohl exponents or uniform growth bounds. With a new result for discrete inclusions it has been shown that the slowly varying inclusion case yields an analogous result to the case of slowly-varying systems. This result is also directly applicable to the case of slowly-varying families of time-varying delay equations.

References

[1] J. L. Daleckii and M. G. Krein. *Stability of Solutions of Differential Equations in Banach Spaces*, volume 43 of *Translations of Mathematical Monographs*. A.M.S, Providence R.I., (1974).

[2] O. Diekman, S. A. van Gils, S. M. Verduyn Lunel, and H. Walther. *Delay Equations*, volume 110 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin-Heidelberg-New York, 1995.

[3] N. Dunford and J. Schwartz. *Linear Operators, Part I: General Theory*. Interscience, New York, (1958).

[4] L. Gurvits. Stability of discrete linear inclusions. *Lin. Alg. Appl.*, **231**:47-85 (1995).

[5] L. Gurvits. Stability of discrete linear inclusions - Part II. submitted.

[6] J.K. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*, volume 99 of *Applied Mathematical Science*. Springer-Verlag, Berlin-Heidelberg-New York, 1993.

[7] B. Jacob, V. Dragan, and A. Pritchard. Infinite dimensional time-varying systems with nonlinear output feedback. *Integral Equations and Operator Theory*, 22:440–462, 1995.

[8] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, 1983.

[9] K. Przyłuski. Infinite-dimensional discrete-time equations as models for linear systems with time delay. In *Preprints of the 2nd IFAC Symp. on Distributed Parameter Systems*, Warwick, June 1977.

[10] K. M. Przyłuski. Remarks on the stability of linear infinite-dimensional discrete-time systems. *J. Differ. Equations*, **72**(2):189–200, (1988).

[11] K. M. Przyłuski. Stability of linear infinite-dimensional systems revisited. *Int. J. Control*, **48**(2):513–523, (1988).

[12] F. Wirth and D. Hinrichsen. On stability radii of infinite dimensional time-varying discrete-time systems. *IMA Journal of Mathematical Control and Information*, **11**:253–276, (1994).

[13] F. Wirth. On stability of infinite-dimensional discrete inclusions (summary). *Journal Mathematical, Systems, Estimation and Control*, 8(4):507–510, (1998). Full electronic manuscript, 21 pp. 429 KB, Retrieval code 72701. ftp://trick.ntp.springer.de/pub/jmsec