Further results on the Lyapunov spectrum of time-varying linear discrete-time systems

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Keywords: Time-varying, discrete-time, Floquet exponents, Lyapunov exponents, generalized spectral radius.

Abstract

We continue our analysis of the Lyapunov and Floquet spectrum of discrete-time systems begun in [12]. In particular we settle an open question by presenting an example of an invariant control set in which Lyapunov exponents may be realized that are not contained in the closure of the Floquet spectrum corresponding to the control set. Furthermore we show that under weak conditions the top Lyapunov exponents behaves strictly monotone under monotone growth of the families of time-varying systems. This implies a strong linearization result for stability radii of nonlinear system with respect to time-varying perturbations.

1 Introduction

The question as to which exponential growth rates or Lyapunov exponents a family of time-varying linear system can exhibit has attracted much attention in recent years. For continuous time systems Floquet, Lyapunov and Morse spectra have been studied by Colonius and Kliemann, see [4, 5] and references therein. In discrete-time the largest Lyapunov exponent has received particular attention under the name of generalized spectral radius in a recent flourishing branch of literature on linear discrete inclusions [2, 3, 7, 6].

In the following section we introduce the concepts of exponential growth rates we will consider, then we recall some controllability concepts of nonlinear systems in Section 3. In Section 4 we recall how these relate to Lyapunov exponents and give a counterexample to an open problem in this area, by presenting a control set for which the closure of the corresponding Floquet spectrum does not contain the Lyapunov exponents realizable in the control set. In the final Section 5 we show that the top Lyapunov exponent is either constant or strictly increasing if strictly increasing maps of admissible values are considered. This result is applied to prove a stronger linearization result for nonlinear stability radii than the one obtained in [8].

2 Preliminaries

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\tilde{U} \subset \mathbb{K}^m$ be open and connected. For an analytic map $A : \tilde{U} \to \mathbb{K}^{n\times n}$ we consider a family of time-varying linear system of the form

$$x(t+1) = A(u(t))x(t)\,, \quad t \in \mathbb{N}\,,\ x(0) = x_0 \in \mathbb{K}^n\,, \quad (1)$$

where $u : \mathbb{N} \to U \subset \tilde{U}$. For $t \in \mathbb{N}$, U^t denotes the set of admissible finite control sequences $u = (u(0), \dots, u(t-1))$, while $U^{\mathbb{N}}$ is the set of infinite control sequences $u = (u(0), u(1), \dots)$. The evolution operator generated by a control sequence $u \in U^{\mathbb{N}}$ is defined by $\Phi_u(s,s) = I$ and

$$\Phi_u(t+1, s) = A(u(t))\Phi_u(t,s)\,, \quad t \geq s \in \mathbb{N}\,. \quad (2)$$

With this notation $\Phi_u(t,0)x_0$ is the solution of (1) corresponding to the initial value $x_0 \in \mathbb{K}^n$ and the control $u \in U^{\mathbb{N}}$ at time t.

To the linear system (1) we associate a system on projective space whose controllability properties determine the sets of characteristic exponents in a qualitative way. In the sequel $\mathbb{P}^{n-1}_{\mathbb{K}}$ denotes the $n-1$ dimensional projective space, and for $W \subset \mathbb{K}^n$, $\mathbb{P}W$ denotes the natural projection of $W \backslash \{0\}$ onto the projective space $\mathbb{P}^{n-1}_{\mathbb{K}}$. With this notation the projected system is given by

$$\begin{aligned} \xi(t+1) &= \mathbb{P}A(u(t))\xi(t)\,, \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in \mathbb{P}^{n-1}_{\mathbb{K}}\,, \quad u \in U^{\mathbb{N}}(\xi_0). \end{aligned} \quad (3)$$

Naturally for each point in projective space this only makes sense if $\xi(t) \not\subset \operatorname{Ker} A(u(t))$. We therefore define the admissible control values for ξ by $U(\xi) := \{u \in U; A(u)x \neq 0, x \neq 0, \mathbb{P}x = \xi\}$. The sets of admissible control sequences are denoted by $U^t(\xi), U^{\mathbb{N}}(\xi)$. The solution of (3) corresponding to an initial value $\xi_0 \in \mathbb{P}^{n-1}_{\mathbb{K}}$ and a control sequence $u \in U^{\mathbb{N}}(\xi_0)$ is denoted by $\xi(\cdot;\xi_0,u)$.

Let U_{inv} be the set $\{u \in U;\ \det A(u) \neq 0\}$, which is clearly the complement of a set defined by analytic equations in U. The following general assumption will be made throughout the remainder of this article.

(i) $U_{inv} \neq \emptyset, U \subset \operatorname{cl}\operatorname{int} U \subset \tilde{U}$.

(ii) $\operatorname{int} U$ is connected, U is bounded.

We are interested in the exponential growth rates that system (1) can exhibit. Let $\lambda(x_0, u)$ denote the Lyapunov exponent corresponding to the initial value $x_0 \in \mathbb{K}^n$ and the sequence $u \in U^{\mathbb{N}}$, i.e.

$$\lambda(x_0, u) = \limsup_{t \to \infty} \frac{1}{t} \log \|\Phi_u(t,0)x_0\|.$$

As a special case Floquet exponents are defined to be the Lyapunov exponents corresponding to periodic sequences $u \in U^{\mathbb{N}}$. It is easy to see that they are of the form $\frac{1}{t}\log|\lambda|$ for some $\lambda \in \sigma(\Phi_u(t,0))$ where $t \in \mathbb{N}, u \in U^t$. For a system of form (1) determined by the map A and the set of admissible controls U the Floquet and Lyapunov spectrum are the sets of all corresponding exponents, i.e.

$$\begin{aligned} \Sigma_{Fl}(A,U) &:= \bigcup_{t \geq 1, u \in U^t} \sigma_{Fl}(u)\,, \\ \Sigma_{Ly}(A,U) &:= \{\lambda(x_0,u)\,;\, x_0 \in \mathbb{K}^n \backslash \{0\}, u \in U^{\mathbb{N}}\}\,. \end{aligned}$$

3 Control Sets

It is known that the structure of the set of Floquet exponents is determined by the controllability properties of system (3), see [12]. We briefly quote the appropriate details. For $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ and a time $t \in \mathbb{N}$ we define the *forward orbit a time t* by $\mathcal{O}^+_t(\xi) := \{\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}; \exists\, u \in U^t(\xi) \text{ with } \eta = \xi(t;\xi,u)\}$. The forward orbit of ξ is then defined by $\mathcal{O}^+(\xi) := \bigcup_{t\in\mathbb{N}} \mathcal{O}^+_t(\xi)$. The backward orbit of ξ at time t is given by $\mathcal{O}^-_t(\xi) := \{\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}; \exists\, u \in U^t(\eta) \text{ with } \xi = \xi(t;\eta,u)\}$ and again $\mathcal{O}^-(\xi) := \bigcup_{t\in\mathbb{N}} \mathcal{O}^-_t(\xi)$. Recall that system (3) is called *forward accessible from* ξ if $\operatorname{int} \mathcal{O}^+(\xi) \neq \emptyset$ and *forward accessible* if it is forward accessible from all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$. A related concept is that of regularity. Consider the map

$$F_t(\xi,\cdot) : U^t(\xi) \to \mathbb{P}^{n-1}_{\mathbb{K}}\,, \quad F_t(\xi,u) := \xi(t;\xi,u)\,. \tag{4}$$

We call a pair $(\xi, u) \in \mathbb{P}^{n-1}_{\mathbb{K}} \times \operatorname{int} U^t$ *regular*, if $u \in \operatorname{int} U^t(\xi)$ and the Jacobian of $F_t(\xi,\cdot)$ has full rank in u. A control $u \in \operatorname{int} U^t$ is called *universally regular*, if (ξ,u) is a regular pair for all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$.

A very useful result in this context is that under the assumption of forward accessibility the universally regular sequences are generic provided the length of the sequence is sufficiently long, see [11]. We denote regular forward orbits that is the set of points reachable from ξ such that ξ and the applied control sequence form a regular pair by $\hat{\mathcal{O}}^+_t(\xi)$, the notations $\hat{\mathcal{O}}^+(\xi), \hat{\mathcal{O}}^-_t(\xi), \hat{\mathcal{O}}^-(\xi)$ are to be understood in the same sense. The regions of approximate controllability are now defined as follows.

Definition 3.1 (Control set) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ Consider system (3). A set $\emptyset \neq D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ is called a* control set, *if*

(i) $D \subset \operatorname{cl} \mathcal{O}^+(\xi)$, $\forall \xi \in D$.

(ii) For every $\xi \in D$ there exists a $u \in U^{\mathbb{N}}(\xi)$ such that $\xi(t;\xi,u) \in D$ for all $t \in \mathbb{N}$.

(iii) D is a maximal set with respect to inclusion satisfying (i).

A control set C is called invariant control set, *if $\operatorname{cl} C = \operatorname{cl} \mathcal{O}^+(\xi)$, $\forall \xi \in C$.*

If $D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ is a control set with $\operatorname{int} D \neq \emptyset$, the (regular) core *of D is defined by*

$$\operatorname{core}(D) := \{\xi \in D; \hat{\mathcal{O}}^+(\xi) \cap D \neq \emptyset \text{ and } \hat{\mathcal{O}}^-(\xi) \cap D \neq \emptyset\}\,. \tag{5}$$

See [1], [12] for a discussion of further properties of the core. We now begin to recall the relation between the control sets with nonempty interior in the projective space $\mathbb{P}^{n-1}_{\mathbb{K}}$ and the Floquet spectrum of the system (1). For this we introduce the following notation, given $u \in U^t$ and $\lambda \in \sigma(\Phi_u(t,0))$ we denote by $GE(\lambda,u)$ the generalized eigenspace of $\Phi_u(t,0)$ corresponding to λ. The projection of this eigenspace $\mathbb{P}GE(\lambda,u)$ is called *regular*, if for any $\xi \in \mathbb{P}GE(\lambda,u)$ the pair (ξ,u) is regular. Note, that in particular $\mathbb{P}GE(\lambda,u)$ is always regular, if u is universally regular. Regular generalized eigenspaces are of interest, as they always project to the core of a control set, as the following result states.

Proposition 3.2 *[12] Consider system (1) and its projection (3). If for $u \in U^t, 0 \neq \lambda \in \sigma(\Phi_u(t,0))$, the projection $\mathbb{P}GE(\lambda,u)$ is regular, then there exists a control set D such that*

$$\mathbb{P}GE(\lambda,u) \subset \operatorname{core}(D)\,.$$

Given a control set D with nonempty core the associated set of Floquet exponents is now defined as follows.

$$\begin{aligned} \Sigma_{Fl}(D) := \bigcup_{t\geq 1, u \in U^t} \{&\frac{1}{t}\log|\lambda|;\, \lambda \in \sigma(\Phi_u(t,0)), \\ &\mathbb{P}GE(\lambda,u) \subset \operatorname{core}(D) \text{ and} \\ &\mathbb{P}GE(\lambda,u) \text{ is regular } \}. \end{aligned}$$

It is known that the closure of $\Sigma_{Fl}(D)$ is an interval [12]. Under the assumption of forward accessibility, there is exactly one invariant control set $C \subset \mathbb{P}^{n-1}_{\mathbb{K}}$. The corresponding Floquet spectral interval is given by $\operatorname{cl} \Sigma_{Fl}(C)$. It is of special interest as the top Lyapunov exponent of (1) is the maximal element of $\operatorname{cl} \Sigma_{Fl}(C)$, and this quantity is also the "generalized spectral radius" studied in [3, 6] or the "Lyapunov indicator" studied in [2].

4 Floquet and Lyapunov spectrum

The following statement summarizes some known results on the relation of Floquet spectral intervals of a control set and the set of Lyapunov equations. We need the following notation. For $u \in U^{\mathbb{N}}$, $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, the positive ω-limit set is defined by

$$\omega^+(\xi,u) := \Big\{ \eta \in \mathbb{P}^{n-1}_{\mathbb{K}} ; \exists \{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, \lim_{k\to\infty} t_k = \infty \quad (6)$$
$$\text{such that } \eta = \lim_{k\to\infty} \xi(t_k;\xi,u) \Big\} .$$

Theorem 4.1 *[12, Theorem 11.1] Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (3) is forward accessible.*

(i) Let D be a control set, with $\operatorname{core}(D) \neq \emptyset$. Assume that $(\xi_0, u) \in \mathbb{P}^{n-1}_{\mathbb{K}} \times U^{\mathbb{N}}(\xi_0)$ are given with $\omega^+(\xi_0,u) \subset D$. If there exists a $t_0 \in \mathbb{N}$ with $\xi(t_0;\xi_0,u) \in \operatorname{core}(D)$ then $\lambda(\xi_0,u) \in \operatorname{cl}\Sigma_{Fl}(D)$.

(ii) Let D be a control set, with $\operatorname{core}(D) \neq \emptyset$, then $\operatorname{cl}\Sigma_{Fl}(D) \subset \Sigma_{Ly}(A,U)$.

The previous theorem leaves a gap in that it does not resolve the question whether there may be trajectories whose ω-limit set is contained in a control set, but with a Lyapunov exponent that is not contained in the closure of the corresponding Floquet interval. The following example shows that this may indeed be the case.

Example 4.2 Let $\mathbb{K} = \mathbb{R}$, $n = 2$, and consider the map

$$A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}, \quad A(a,b) = \begin{bmatrix} 1 & a \\ b & b \end{bmatrix} . \tag{7}$$

Define $U := \{[a\,,\,b]' \in \mathbb{R}^2;\ 0 \le a \le 1,\ 0 \le b \le 1\}$. It is easy to see that the projected system given by these data is forward accessible. As all matrices $A(u)$ have only positive entries the positive orthant $\mathbb{R}^2_{\ge 0}$ is invariant under all $A(u)$ and so is the projection of the positive orthant for system (3). It follows by [12, Theorem 7.1 (i)] that $C \subset \mathbb{P}\mathbb{R}^2_{\ge 0}$ as C is contained in the forward orbit of every $\xi \in \mathbb{P}^1_{\mathbb{R}}$. We claim that indeed $C = \mathbb{P}\mathbb{R}^2_{>0}$. This may be seen by noting that the vector $[1/b - 1, 1]'$ is the eigenvector to the eigenvalue 1 of $A(0,b)$ and hence of $A(0,b)^t$. For t large enough there exists a sequence of universally regular $u_n \in U^t$ such that $\Phi_{u_n}(t,0) \to A(0,b)^t$. As the spectrum of $A(0,b)^t$ is simple and $\Phi_{u_n}(t,0)$ is positive the (projection of the) eigenspaces corresponding to the maximal eigenvalue of $\Phi_{u_n}(t,0)$ converge to $\mathbb{P}[1/b - 1, 1]'$. By universal regularity and Proposition 3.2 these eigenspaces project to the core of some control set, which can only be C by the maximality of the eigenvalues [12, Remark 7.4 (ii)].

Clearly, we have for $\xi = \mathbb{P}[0,1]'$ and the constant control $u_0 := (0,0)$ that $\lambda(\xi,u_0) = -\infty$, so on the boundary of C the Lyapunov exponent $-\infty$ may be realized. We intend to show that on the other hand $\inf \Sigma_{Fl}(C) > -\infty$. Choose any $\xi_0 \in \operatorname{int} C$, $t \in \mathbb{N}$, $u \in U^t$ such that $\xi_0 = \xi(t;\xi_0,u)$. Let $x(0), x(1), \ldots, x(t) = \lambda x(0)$ be a corresponding trajectory in $\mathbb{R}^2$ satisfying $x(s+1) = A(u(s))x(s)$, $s = 0,\ldots,t-1$. As for any $u \in U$ it holds that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge 0\,, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A(u) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 \le y_1\,.$$

It follows by induction that $x_1(t) \ge x_1(0)$ and hence $\lambda \ge 1$ and $\inf \Sigma_{Fl}(C) \ge 0$. As $r(A(u)) = 1$ for all $u = [0,b]' \in U$ we obtain $\min \Sigma_{Fl}(C) = 0$. ■

This example is extreme in the sense that it exhibits a large difference between the two quantities considered. It should be noted however that the difference between the lower boundary of the Floquet spectrum of a control set and the minimal Lyapunov exponent realizable in this control set does not depend on the non-invertibility used in the previous example. In fact, consider the same map A as in (7) and the set of admissible control values

$$U_\varepsilon := \{[a\,,\,b]' \in \mathbb{R}^2;\ 0 \le a \le 1/2,\ \varepsilon \le b \le 1\}\,,$$

for some $0 < \varepsilon < 1$. In this case $\det(A(u)) \ge \varepsilon/2$. It is easy to see that again $\min \Sigma_{Fl}(C_\varepsilon) = 0$, while $\lambda(\mathbb{P}[0,1]', (\varepsilon,0)) = \log(\varepsilon) < 0$ and $\mathbb{P}[0,1]' \in \partial C_\varepsilon$.

5 Strict monotonicity of the top Lyapunov exponent

We now turn to the examination of parameter dependent systems. That is we consider a family $U_\rho, \rho \in \mathbb{R}^+$, satisfying $0 \in U_\rho \subset \tilde{U}$ and $\operatorname{cl} U_{\rho_1} \subset \operatorname{int} U_{\rho_2}$ for $\rho_1 < \rho_2$. We denote the top Lyapunov exponent

$$\kappa(\rho) := \max \Sigma_{Ly}(A, U_\rho)\,.$$

The following result shows a strict monotonicity property of this quantity. The argumentation makes use of a remarkable result due to Barabanov [2], that associates a norm, that can also be interpreted as an "optimal" Lyapunov function to system (1), under the assumption that the set $A(U)$ is irreducible. Recall, that a set of matrices $M \subset \mathbb{K}^{n\times n}$ is called irreducible, if only the trivial subspaces $\{0\}, \mathbb{K}^n$ are invariant under all $A \in M$.

As it is well possible that our sets of matrices are contained in some subspaces $V \subset \mathbb{R}^{n\times n}$, we wish to formulate properties with respect to these subspaces. We denote $V_\rho := \operatorname{span} A(U_\rho) \subset \mathbb{R}^{n\times n}$. The interior of a set $M \subset V_\rho$ with respect to the relative topology on V_ρ will be denoted by $\operatorname{int}_\rho M$.

Theorem 5.1 *Let $\operatorname{conv} A(U_\rho) \subset \operatorname{int}_\rho \operatorname{conv} A(U_\theta)$ for all $0 < \rho < \theta$. If $A(U_\rho)$ is irreducible for all $\rho > 0$ then there exists a constant $\bar\rho \in [0,\infty)$ such that $\kappa(\cdot)$ is constant on $[0,\bar\rho]$ and strictly increasing on $[\bar\rho,\infty)$.*

Proof. Let $\rho > 0$. We may assume that $\kappa := \kappa(\rho) = 0$, by considering the map $e^{-\kappa}A$. By [2] there exists a norm v on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ we have $\max\{v(A(u)x) \mid u \in U_\rho\} = v(x)$. Let S denote the sphere with respect to the norm v. We consider two cases:

1. There exist $x \in S$ and $u \in U_0^{\mathbb{N}}$ such that $x(t;x,u) \in S$ for all $t \ge 0$ then clearly $\kappa(0) \ge \lambda(x,u) = 0$. On the other hand by monotonicity $\kappa(0) \le \kappa = 0$, so that $\kappa(\theta) = 0$ for all $\theta \in [0,\rho]$.

2. Assume the contrary, so that for any $x \in S$ there exists a control $u \in U_0^{\mathbb{N}}$ such that $x(t;x,u) \in S$ for $0 \le t \le t_x$ and for $y = x(t_x; x, u)$ it holds that $\max_{u \in U_0} v(A(u)y) < 1$. Now by construction of v there exists a $u_1 \in U_\rho$ such that $v(A(u_1)y) = 1$. Now let $u \in U_0$ then for $\mu \in [0,1]$ we have

$$A(u_0) + \mu(A(u_1) - A(u_0)) \in \operatorname{conv} A(\rho) .$$

Fix $\varepsilon > 0$ and define

$$B_\varepsilon := A(u_0) + (1+\varepsilon)(A(u_1) - A(u_0)) \in \operatorname{conv} A(\rho_\varepsilon) ,$$

where $\rho_\varepsilon > \rho$. As $v(A(u_0)y) < 1$ and $v(A(u_1)y) = 1$ it follows that $v(B_\varepsilon y) > 1$. Consequently, it follows that $v(B_\varepsilon \Phi(t_x, u_x)x) > 1$ and by continuity there exists a neighborhood V_x of x in S such that $v(B\Phi(t_x, u_x)z) > 1$ for all $z \in V_x$. Using a standard compactness argument it follows that there exists a $T > 0$, $\rho' > \rho$ such that for any $x \in S$ there exist matrices $B_1, \ldots, B_T \in \operatorname{conv} A(\rho')$ with $v(B_T \ldots B_1 x) > 1$. By concatenation this implies that there exists an unbounded solution to the discrete inclusion

$$x(t+1) \in \{Ax(t) \mid A \in \operatorname{conv} A(\rho')\} .$$

Now Corollaries 2 and 3 of [2] imply that $\kappa(\rho') = \kappa(\operatorname{conv} A(\rho')) > 0$.

By the assumption on the monotonic growth of $\operatorname{conv} A(U_\rho)$ the value ρ' can be chosen arbitrarily close to ρ by choosing ε small enough. Thus under the assumption of 2. $\kappa(\theta) > \kappa(\rho)$ for all $\theta > \kappa$. ■

Let us note the following immediate corollary.

Corollary 5.2 *Let* $\operatorname{conv} A(U_\rho) \subset \operatorname{int}_\rho \operatorname{conv} A(U_\theta)$ *for all* $0 < \rho < \theta$. *There exists a constant* $\bar\rho \in [0, \infty)$ *such that* $\kappa(\cdot)$ *is constant on* $[0, \bar\rho]$ *and strictly increasing on* $[\bar\rho, \infty)$.

Proof. Given any set $M \subset \mathbb{R}^{n \times n}$ we can find a similarity transformation T such that any $A \in M$ has the form

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & \ldots & A_{1d} \\ 0 & A_{22} & A_{23} & \ldots & A_{2d} \\ 0 & 0 & A_{33} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \ldots & 0 & A_{dd} \end{bmatrix} ,$$

where each of the sets $M_{ii} := \{A_{ii}; A \in M\}, i = 1, \ldots, d$ is irreducible. It holds that $\kappa(M) = \max_{i=1,\ldots,d} \kappa(M_i)$.

Thus in our case there exist $l \in \{0, \ldots, n-1\}$ and $0 = t_0 < t_1 < \ldots < t_l < t_{l+1} < \infty$ such that on the intervals $(t_i, t_{i+1}], i = 0, \ldots, l$ the number of irreducible blocks of $A(U_\rho)$ is constant. Fix one interval $[t_i, t_{i+1})$. For each irreducible block Theorem 5.1 immediately applies. Taking the maximum of the finite number of blocks clearly preserves the desired property.

It thus remains to check, that it is impossible that $\kappa(\cdot)$ is strictly increasing on an interval $(a, t_i]$, $a < t_i$ and constant on (t_i, b), $b > t_i$, for some $i = 1, \ldots, l$. Let $V_i := \operatorname{span} A(U_{t_i})$. By assumption, $A(U_\rho) \not\subset V_i$ for all $\rho > t_i$, as otherwise the block structure would not change at t_i. The assumption on the growth of the sets $\operatorname{conv} A(U_\rho)$, however, implies in particular that

$$\operatorname{conv} A(U_{t_i}) \subset \operatorname{int}_{t_i} \operatorname{conv} V_i \cap A(U_\rho) , \quad \rho > t_i .$$

The assumption that $\kappa(\cdot)$ be strictly increasing on $(a, t_i]$ implies that

$$\kappa(t_i) < \kappa(V_i \cap A(U_\rho)) ,$$

for all $\rho > t_i$ by Theorem 5.1. By inclusion the right hand side, however, is less equal than $\kappa(\rho)$, which shows that $\kappa(\rho)$ is continuously increasing on (t_i, b). This concludes the proof. ■

6 Application: Nonlinear Stability radii

Using Theorem 5.1 we can strengthen the results on nonlinear stability radii that were obtained in [8]. We consider nonlinear systems of the form

$$x(t+1) = f_0(x(t)) , \quad t \in \mathbb{N} \tag{8}$$

with $f_0 \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ which have an exponentially stable fixed point x^*. Assume that (8) is subject to perturbations of the form

$$x(t+1) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)) , \quad t \in \mathbb{N} , \tag{9}$$

where the perturbation functions f_i leave the fixed point invariant, i.e. $f_i(x^*) = 0, f_i \in C^1(\mathbb{R}^n, \mathbb{R}^n), i = 1, \ldots, m$. We want to analyze the corresponding time-varying stability radius

$$r_{tv}(f_0, (f_i)) := \inf\{\|u\|_\infty \mid u : \mathbb{N} \to \mathbb{R}^m \text{ s. t. (9) is not asymptotically stable at } x^*\} .$$

Associated with the nonlinear system (9) we may study the linearization in x^* given by

$$y(t+1) = A_0 y(t) + \sum_{i=1}^{m} u_i(t) A_i y(t) , \quad t \in \mathbb{N} , \tag{10}$$

where A_i denotes the Jacobian of f_i in x^*.We abbreviate $A(u) = A_0 + \sum u_i A_i$ and denote $U_\rho := \{u \in \mathbb{R}^m \; ; \; \|u\| < \rho\}$. Define the linear stability radii

$$\begin{aligned} r_{Ly}(A_0, (A_i)) &= \inf\{\rho \mid \kappa(\rho) \ge 0\} , \\ \bar r_{Ly}(A_0, (A_i)) &= \inf\{\rho \mid \kappa(\rho) > 0\} . \end{aligned}$$

For a method for the calculation of $r_{Ly}(A_0, (A_i))$ we refer to [13]. The following result is a generalization of a result in [8].

Theorem 6.1 *If* $r_{Ly}(A_0, (A_i)) < 0$ *then*

$$r_{Ly}(A_0, (A_i)) r_{tv}(f_0, (f_i)) = \bar r_{Ly}(A_0, (A_i)) .$$

Proof. By [8] we already know that

$$r_{Ly}(A_0,(A_i)) \leq r_{tv}(f_0,(f_i)) \leq \bar{r}_{Ly}(A_0,(A_i)),$$

where equality holds generically. Now, note that $r_{Ly}(A_0,(A_i)) < \bar{r}_{Ly}(A_0,(A_i))$ can only occur, if there is an interval $[\rho_1,\rho_2]$ such that $\kappa(\rho_1) = \kappa(\rho_2)$. By Corollary 5.2 this implies that $\kappa(0) = 0$. This in turn implies $r_{Ly}(A_0,(A_i)) = 0$, which contradicts our assumption. ■

7 Conclusion

We have analyzed Lyapunov exponents of families of time-varying systems. In particular, we obtained a result on the strict monotonicity of the top Lyapunov exponent, which allows for an easy characterization of nonlinear stability radii. In particular, it follows from these results that the methods for the calculation of the linear stability radii in [13] are applicable for local nonlinear stability radii. This is surprising, as for the case of time-invariant perturbations such a statement is not always true, even though generically so [9, 10]. It is the topic of ongoing research if these results can be transferred to the continuous time case. A direct application of the discrete time ideas does not lead to an end here.

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