A linearization principle for robustness with respect to time-varying perturbations *

Fabian Wirth[1]

Zentrum für Technomathematik, Universität Bremen, 28334 Bremen, Germany,
fabian@math.uni-bremen.de

Abstract. We study nonlinear systems with an asymptotically stable fixed point subject to time-varying perturbations that do not perturb the fixed point. Based on linearization theory we show that in discrete time the linearization completely determines the local robustness properties at exponentially stable fixed points of nonlinear systems. In the continuous time case we present a counterexample for the corresponding statement. Sufficient conditions for the equality of the stability radii of nonlinear respective linear systems are given. We conjecture that they hold on an open and dense set.

1 Introduction

A natural question in perturbation or robustness theory of nonlinear systems concerns the information that the linearization of a nonlinear system at a singular point contains with respect to local robustness properties. This question has been treated for time-invariant perturbations in [8] for continuous time, (see the references therein for the discrete time case). The result obtained in these papers was that generically the linearization determines the local robustness of the nonlinear system, where genericity is to be understood in the sense of semi-algebraic geometry (on the set of linearizations).

Specifically, the objects under consideration are the local stability radius of the nonlinear system and the stability radius of the linear system, where as usual the stability radius of a system is the infimum of the norms of destabilizing perturbations in a prescribed class. The question is then, whether these two quantities are equal or more precisely when this is case, see also [4, Chapter 11].

In this paper we treat this problem for nonlinear systems subject to time-varying perturbations. Our analysis is based on recent results on the generalized spectral radius of linear inclusions. In particular, we see a surprising difference between continuous and discrete time. While the linearization always determines the robustness of the nonlinear system if the nominal system is exponentially stable this fails to be true for continuous time. On the other hand we are able to give a sufficient condition which guarantees equality between linear and nonlinear stability radius on an open set of systems. As it is

* Research supported by the European Nonlinear Control Network.

known from [9] that the Lebesgue measure of those linearizations for which it is possible that the nonlinear stability radius is different from the linear is zero it seems therefore natural to conjecture that the set of systems where these two quantities coincide is open and dense.

We proceed as follows. In Section 2 we recall the definition of the stability radius for nonlinear systems with time varying perturbations and state some relevant results from the theory of linear inclusions. In particular, we recall upper and lower bounds of the stability radius of the nonlinear system in terms of the stability radius and the strong stability radius of the linearization. In Section 3 we develop a local robustness theory based on the linearization of the system for the discrete time case. It is shown that the two linear stability radii coincide under weak conditions, demonstrating that one need only consider the linearization in order to determine the local nonlinear robustness properties of a system. The continuous time case is treated in Section 4. We first present a counterexample showing that analogous statements to the discrete time case cannot be expected in continuous time. We then present a sufficient condition for the equality of the two linear stability radii on an open set. Concluding remarks are found in Section 5.

2 Preliminaries

Consider nominal discrete and continuous time nonlinear systems of the form

$$x(t+1) = f_0(x(t))\,, \quad t \in \mathbb{N}, \tag{1a}$$
$$\dot{x}(t) = f_0(x(t))\,, \quad t \in \mathbb{R}_+\,, \tag{1b}$$

which are exponentially stable at a fixed point which we take to be 0. By this we mean that there exists a neighborhood U of 0 and constants $c > 1, \beta < 0$ such that the solutions $\varphi(t;x,0)$ of (1a),(1b) satisfy $\|\varphi(t;x,0)\| \leq ce^{\beta t}\|x\|$ for all $x \in U$.

As the concepts we will discuss do not differ in continuous and discrete time we will summarize our notation by writing $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ for the time-scale and $x^+(t) := \dot{x}(t), x(t+1)$ according to the time-scale we are working on.

Assume that (1a),(1b) are subject to perturbations of the form

$$x^+(t) = f_0(x(t)) + \sum_{i=1}^{m} d_i(t) f_i(x(t)) =: F(x(t), d(t))\,, \tag{2}$$

where the perturbation functions f_i leave the fixed point invariant, i.e. $f_i(0) = 0, i = 0, 1, \ldots, m$. We assume that the f_i are continuously differentiable in 0 (and locally Lipschitz in the case $\mathbb{T} = \mathbb{R}_+$). The *unknown* perturbation function d is assumed to take values in $\alpha D \subset \mathbb{R}^m$,

$$d : \mathbb{T} \to \alpha D\,,$$

where in the case $\mathbb{T} = \mathbb{R}_+$ we impose that d is measurable. Here $\alpha > 0$ describes the perturbation intensity, which we intend to vary in the sequel, while the perturbation set D is fixed. Thus structural information about the perturbations one wants to consider can be included in the functions $f_i, i = 1, \ldots, m$ and in the set D. For the perturbation set $D \subset \mathbb{R}^m$ we assume that it is compact, convex, with nonempty interior, and $0 \in \operatorname{int} D$. Solutions to the initial value problem (2) with $x(0) = x_0$ for a particular time-varying perturbation d will be denoted $\varphi(t; x_0, d)$.

The question we are interested in concerns the critical perturbation intensity at which the system (2) becomes unstable. The *stability radius* is thus defined as

$$r_{nl}(f_0, (f_i)) := \inf\{\alpha > 0 \mid \exists d^* : \mathbb{T} \to \alpha D : x^+(t) = F(x(t), d^*(t)) \\ \text{is not asymptotically stable at } 0\}\,. \tag{3}$$

By linearizing the perturbed system in (2) we are led to the system

$$x^+(t) = \left(A_0 + \sum_{i=1}^m d_i(t) A_i \right) x(t)\,, \quad t \in \mathbb{T}\,. \tag{4}$$

This is a (discrete or differential) linear inclusion, which is in principle determined by the set

$$\mathcal{M}(A_0, \ldots, A_m, \alpha) := \left\{ A_0 + \sum_{i=1}^m d_i A_i \;\middle|\; \|d\| \leq \alpha \right\}\,.$$

If the matrices A_i are fixed we will denote this set by $\mathcal{M}(\alpha)$ for the sake of succinctness.

The inclusion (4) is called exponentially stable, if there are constants $M \geq 1, \beta < 0$ such that

$$\|\psi(t)\| \leq M e^{\beta t} \|\psi(0)\|\,, \quad \forall t \in \mathbb{T}$$

for all solutions ψ of (4).

Exponential stability is characterized by the number

$$\rho(\mathcal{M}(A_0, \ldots, A_m, \alpha)) := \sup_{\psi} \limsup_{t \to \infty} \|\psi(t)\|^{1/t}\,,$$

where the supremum is taken over all solutions of (4). Namely, (4) is exponentially stable iff $\rho(\mathcal{M}(A_0, \ldots, A_m, \alpha)) < 1$. Again we will write $\rho(\alpha)$ if there is no fear of confusion.

In the discrete time case the number ρ is known as the joint or the generalized spectral radius. We refer to [2,10] for further characterizations of this number and for further references. In the continuous time case it is more customary to consider the quantity $\kappa(\alpha) := \log \rho(\alpha)$, which is known under the name of maximal Lyapunov exponent, see [4] and references therein.

As in the nonlinear case we now define stability radii by

$$r_{Ly}(A_0, (A_i)) := \inf\{\alpha \geq 0 \mid \rho(\alpha) \geq 1\},$$
$$\bar{r}_{Ly}(A_0, (A_i)) := \inf\{\alpha \geq 0 \mid \rho(\alpha) > 1\}.$$

The relation between the linear and the nonlinear stability radii is indicated by the following result which is contained in [3] for the continuous and in [7] for the discrete time case.

Lemma 1. *Let $\mathbb{T} = \mathbb{N}, \mathbb{R}_+$ and consider system* (2) *and its linearization* (4), *then*

$$r^{\mathbb{T}}_{Ly}(A_0, (A_i)) \leq r^{\mathbb{T}}_{nl}(f_0, (f_i)) \leq \bar{r}^{\mathbb{T}}_{Ly}(A_0, (A_i)).$$

It is the aim of this paper to obtain further results on the information the linear stability radii contain for the nonlinear system.

The following set of matrix sets will play a vital role in our analysis. Recall that a set of matrices $\mathcal{M}$ is called irreducible if only the trivial subspaces of $\mathbb{R}^n$ are invariant under all $A \in \mathcal{M}$.

We define

$$I(\mathbb{R}^{n\times n}) := \{\mathcal{M} \subset \mathbb{R}^{n\times n} \mid \mathcal{M} \text{ compact and irreducible}\}.$$

Note that this set is open and dense in the set of compact subsets of $\mathbb{R}^{n\times n}$ endowed with the usual Hausdorff metric.

The proof of the following statements can be found in [10]. They are the foundation for our analysis of linearization principles.

Theorem 1. *(i) The generalized spectral radius is locally Lipschitz continuous on $I(\mathbb{R}^{n\times n})$.*
(ii) The maximal Lyapunov exponent is locally Lipschitz continuous on $I(\mathbb{R}^{n\times n})$.

Furthermore in the discrete time case a strict monotonicity property can be shown to hold, under the assumption that the following condition can be satisfied. Given $A \in \mathbb{R}^{n\times n}$ we denote by P_A the reducing projection corresponding to the eigenvalues $\lambda \in \sigma(A)$ with $|\lambda| = r(A)$.

Property 1. The set $\mathcal{M} \subset \mathcal{K}(\mathbb{R}^{n\times n})$ is said to have Property 1 if $n = 1, 2$ or if there exists an $A \in \mathcal{M}$ such

$$r(A) < \rho(\mathcal{M}), \quad \text{or} \quad \operatorname{rank} P_A \neq 2, \quad \text{or} \quad \sigma((I - P_A)A) \neq \{0\}.$$

In the following statement we denote the affine subspace generated by a set $\mathcal{M} \subset \mathbb{R}^{n\times n}$ by $\operatorname{aff} \mathcal{M}$ while $\operatorname{int}_{\operatorname{aff} \mathcal{M}}$ denotes the interior with respect to this affine subspace.

Proposition 1. *Let* $\mathcal{M}_1, \mathcal{M}_2 \in I(\mathbb{R}^{n\times n})$ *satisfy* $\mathcal{M}_1 \neq \mathcal{M}_2$ *and*

$$\mathcal{M}_1 \subset \operatorname{int}_{\operatorname{aff} \mathcal{M}_2} \operatorname{conv} \mathcal{M}_2 \,. \tag{5}$$

Assume that $\mathcal{M}_1$ *has Property 1 then*

$$\rho(\mathcal{M}_1) < \rho(\mathcal{M}_2) \,.$$

3 The discrete time case

In discrete time the situation turns out to be particularly simple. In fact, if Property 1 holds then we can immediately conclude the following linearization principle.

Theorem 2. *Let* $\mathbb{T} = \mathbb{N}$ *and consider the discrete-time system* (1b) *and the perturbed system* (2) *along with its linearization* (4). *If for some* $\alpha^* < r_{Ly}(A_0, (A_i))$ *the set* $\mathcal{M}(\alpha^*)$ *is irreducible and satisfies Property* (1), *then*

$$r_{Ly}(A_0, (A_i)) = r_{nl}(f_0, (f_i)) = \bar{r}_{Ly}(A_0, (A_i)) \,.$$

Proof. The assumptions guarantee that the map $\alpha \mapsto \rho(\alpha)$ is strictly increasing on $[\alpha^*, \infty)$. This implies $r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i))$. The assertion now follows from Lemma 1.

Corollary 1. *Let* $\mathbb{T} = \mathbb{N}$ *and consider the discrete-time system* (1b) *and the perturbed system* (2) *along with its linearization* (4). *If the point* $x^* = 0$ *is exponentially stable for the unperturbed system*

$$x(t+1) = f_0(x(t))$$

then

$$r_{Ly}(A_0, (A_i)) = r_{nl}(f_0, (f_i)) = \bar{r}_{Ly}(A_0, (A_i)) \,.$$

Proof. There exists a similarity transformation T such that all $A_i, i = 0, \dots, m$ are similar to matrices of the form

$$TA_iT^{-1} = \begin{bmatrix} A^i_{11} & A^i_{12} & \dots & \dots & A^i_{1d} \\ 0 & A^i_{22} & A^i_{23} & \dots & A^i_{2d} \\ 0 & 0 & A^i_{33} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \dots & 0 & A^i_{dd} \end{bmatrix},$$

where each of the sets $M_j := \{A^i_{jj} \mid i = 0, \dots, m\}, j = 1, \dots, d$ is irreducible. It holds that $\rho(\alpha) = \max_{j=1,\dots,d} \rho(M_j(\alpha))$.

Thus it is sufficient to consider the blocks individually to determine r_{Ly}, resp. $\bar{r}_{Ly}$. Under the assumption of exponential stability we have $r(A_0) < 1$. Hence for each j we have $r(A^0_{jj}) < 1$ and the set $\mathcal{M}_j(\alpha)$ has Property 1 for all $\alpha > 0$ such that $\rho(\mathcal{M}_j(\alpha)) > r(A_0)$. Now the result follows from Theorem 2.

Corollary 2. *Let $\mathbb{T} = \mathbb{N}$. The stability radius of linear systems with respect to time-varying perturbations r_{Ly} is continuous on the set*

$$\{(A_0, \dots, A_m) \in (\mathbb{R}^{n\times n})^{m+1} \mid r(A_0) \neq 1\}.$$

Furthermore, the set

$$\{(A_0, \dots, A_m) \in (\mathbb{R}^{n\times n})^{m+1} \mid r_{Ly}(A_0, \dots, A_m) \neq \bar{r}_{Ly}(A_0, \dots, A_m)\}$$

is contained in a lower dimensional algebraic set.

Proof. It was shown in [7] that $r_{Ly}, \bar{r}_{Ly}$ are upper respectively lower semicontinuous on $(\mathbb{R}^{n\times n})^{m+1}$. The preceding Corollary 1 shows that these two functions coincide if $r(A_0) < 1$, which shows continuity in this case. If $r(A_0) > 1$ the statement is obvious as both functions are equal to 0.

The second statement now follows because a necessary condition for the condition $r_{Ly}(A_0, \dots, A_m) \neq \bar{r}_{Ly}(A_0, \dots, A_m)$ is $r(A_0) = 1$. The latter condition defines a lower dimensional algebraic set.

The result for the linear stability radii extends to the case of nonlinear systems as follows. First, denote by $C^1(\mathbb{R}^n, \mathbb{R}^n, 0)$ the set of continuously differentiable maps from $\mathbb{R}^n$ to itself satisfying $f(0) = 0$. This space may be endowed with the C^1 topology inherited from the topologies on the space $C^1(\mathbb{R}^n, \mathbb{R}^n)$, (see [6, Chapter 17]).

Corollary 3. *Given $n, m \in \mathbb{N}$, the set $\mathcal{W}$ of functions $(f_0, f_1, \dots, f_m) \in C^1(\mathbb{R}^n, \mathbb{R}^n, 0)^{m+1}$ for which*

$$r_{nl}(f_0, (f_i)) = r_{Ly}(A_0, (A_i)) \tag{6}$$

contains an open and dense subset of $C^1(\mathbb{R}^n, \mathbb{R}^n, 0)^{m+1}$ with respect to both the coarse and the fine C^1 topology.

Proof. This is immediate from the definition of the C^1 topology.

4 Continuous time

A natural question is if statements similar to those of Theorem 2 and Corollary 1 hold in continuous time. The fundamental tool for this results is the monotonicity property given by Proposition 1. This statement is unfortunately in general false in continuous time, as any subset $\mathcal{M}_1$ of the skew-symmetric matrices generates a linear inclusion whose system semigroup is a subset of the orthogonal group and for which the maximal Lyapunov exponent is therefore equal to 0. Taking a set $\mathcal{M}_2$ which contains $\mathcal{M}_1$ in its interior (with respect to the skew-symmetric matrices) does not yield a Lyapunov exponent larger than one, so that the strict monotonicity property fails to hold. This example leaves still some hope that maybe a statement corresponding to Corollary 1 remains true in continuous time. The following example shows that even such expectations are unfounded.

Example 1. Consider the matrices

$$A(d) := \begin{bmatrix} 0 & d \\ -d & -2+d \end{bmatrix}.$$

It is easy to see that $A^*(d) + A(d) \leq 0$ for all $d \in (-\infty, 2)$. Hence for $D \subset (-\infty, 2)$ it is immediate that $\kappa(D) \leq 0$ as the Euclidean unit ball is forward invariant under the associated time-varying linear system. On the other hand while $\gamma(A(0)) = 0$, we have $\gamma(A(d)) < 0$ for all $d \in (0, 2)$, see Figure 1.

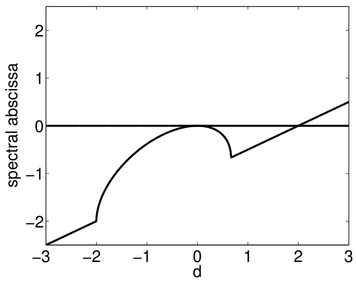

Fig. 1. The spectral abscissa of $A(d)$ in dependence of d.

The consequence of this is the following. If we define $A_0 = A(1/2)$ and

$$A_1 := \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},$$

then

$$0 < r_{Ly}(A_0, A_1) \leq \frac{1}{2} < \frac{3}{2} = \bar{r}_{Ly}(A_0, A_1),$$

because at least $A_0 - 1/2A_1 = A(0)$ is not asymptotically stable. While on the other hand for $\alpha < 3/2$ the perturbation set is a strict subset of $(-\infty, 2)$ and $\gamma(A_0) = -(3 + \sqrt{5})/4 \approx -0.191$ so that the unperturbed system is exponentially stable.

While this example shows that we cannot expect a continuous time counterpart to the discrete-time results of Section 3 we are able to show that the property that the stability radius of the linearization determines the nonlinear stability radius is true on certain open sets. We even conjecture that it is true on an open and dense set, but this point remains open for the moment. The following theorem strengthens the result obtained in [9, Theorem 3.1 (i)]. Here the local Lipschitz continuity property of the maximal Lyapunov exponent will play a vital role, as it will allow the application of the implicit function theorem for Lipschitz continuous functions. To this end we will need the Clarke subdifferential of a function g, which we denote by $\partial_{\mathrm{Cl}} g(x)$. Here we will not need the most general definition. For our purposes it is sufficient to know that if we assume that $g : \mathbb{R}^p \to \mathbb{R}$ is locally Lipschitz continuous then

$$\partial_{\mathrm{Cl}} g(x) = \operatorname{conv} \left\{ c \in \mathbb{R}^p \;\middle|\; \exists x_k \to x : c = \lim_{k\to\infty} \nabla g(x_k) \right\} , \tag{7}$$

see [5, Theorem II.1.2], where we tacitly assume that the gradient ∇g exists in x_k if we write $\nabla g(x_k)$. Note that Lipschitz continuity of g implies that it is differentiable almost everywhere by Rademacher's theorem. For further details we refer to [5].

The following lemma ensures that the theory of the Clarke generalized gradient is applicable in our case.

Lemma 2. *The map*

$$(A_0, \dots , A_m, \alpha) \mapsto \kappa(A_0, \dots , A_m, \alpha) := \kappa \left(\left\{ A_0 + \sum_{i=1}^{m} d_i A_i \;\middle|\; \|d\| \le \alpha \right\} \right)$$

is locally Lipschitz continuous on the set $I(\mathbb{R}^{n\times n}) \times \mathbb{R}_{>0}$.

Proof. Note that the map

$$(A_0, \dots , A_m, \alpha) \mapsto \left\{ A_0 + \sum_{i=1}^{m} d_i A_i \;\middle|\; \|d\| \le \alpha \right\}$$

is Lipschitz continuous. As the composition of Lipschitz continuous maps is again Lipschitz continuous the claim follows from Theorem 1 (ii).

Proposition 2. *Let* $n, m \in \mathbb{N}$. *Fix* $\{A_0^*, \dots , A_m^*\} \in I(\mathbb{R}^{n\times n})$ *and let*

$$r_{Ly}(A_0, (A_i)) < \infty .$$

Consider the map $\kappa : (A_0, \dots , A_m, \alpha) \mapsto \kappa(\mathcal{M}(\alpha))$ *and denote*

$$\partial_{\mathrm{Cl},\alpha} \kappa(z) := \left\{ c \in \mathbb{R} \mid \exists p' \in (\mathbb{R}^{n\times n})^{m+1} : (p', c) \in \partial_{\mathrm{Cl}} \kappa(z) \right\} .$$

If

$$\inf \partial_{\mathrm{Cl},\alpha} \kappa(A_0^*, \dots , A_m^*, r_{Ly}(A_0^*, (A_i^*))) > 0 , \tag{8}$$

then $r_{Ly} = \bar{r}_{Ly}$ *on a neighborhood of* $(A_0, \dots , A_m) \in (\mathbb{R}^{n\times n})^{m+1}$ *and on this neighborhood* r_{Ly} *is locally Lipschitz continuous.*

Proof. By Lemma 2 and (8) we may apply the implicit function theorem for Lipschitz continuous maps [5, Theorem VI.3.1] which states that for every $(B_0, \dots, B_m)$ in a suitable open neighborhood of $(A_0, \dots, A_m) \in (\mathbb{R}^{n\times n})^{m+1}$ the map

$$\alpha \mapsto \kappa(\mathcal{M}(B_0, \dots, B_m, \alpha))$$

has a unique root and this root is a locally Lipschitz continuous function of $(B_0, \dots, B_m)$. In other words, this means that on this neighborhood the functions r_{Ly} and $\bar{r}_{Ly}$ coincide and are locally Lipschitz continuous.

Conjecture 1. For fixed $m \geq 1$ the set $\mathcal{L} \subset (\mathbb{R}^{n\times n})^{m+1}$ given by

$$\{(A_0, \dots, A_m) \,|\, r_{Ly}(A_0, (A_i)) = \bar{r}_{Ly}(A_0, (A_i))\}$$

contains an open and dense set. Furthermore, the Lebesgue measure of the complement $\mathcal{L}^c$ is 0.

Remark 1. (i) The statement that the complement has measure zero is shown in [9, Theorem 3.1 (i)].

(ii) With the help of Proposition 2 it is easy to identify open sets on which $r_{Ly} = \bar{r}_{Ly}$ in the continuous time case. For instance, if $A_i = cI$ for some $i = 1, \dots, m$ this implies that condition (8) holds. The problem is whether this conditions holds generically.

5 Conclusion

In this paper it was shown that linearization at singular points can provide information about the stability radius of a nonlinear system with respect to time-varying perturbations. In discrete time this information is complete if the nominal system is exponentially stable, while this is false in continuous time.

The fundamental difference between discrete and continuous time lies in the fact that the perturbation in discrete time is on the level of the systems semigroup, whereas in continuous time the perturbations act on the level of the Lie algebra of the system. This at least gives an indication that some differences are to be expected.

We conjecture that also in continuous time the linearization provides sufficient information at least on an open and dense set of systems. If Conjecture 1 can be proved to hold it is clear how to formulate results for the continuous time case analogous to Corollaries 2,3.

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