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A globalization procedure for locally stabilizing controllers

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Abstract

For a nonlinear system with a singular point that is locally asymptotically nullcontrollable we present a class of feedbacks that globally asymptotically stabilizes the system on the domain of asymptotic nullcontrollability.

The design procedure is twofold. In a neighborhood of the singular point we use linearization arguments to construct a sampled (or discrete) feedback that yields a feedback invariant neighborhood of the singular point and locally exponentially stabilizes without the need for vanishing sampling rate as the trajectory approaches the equilibrium. On the remainder of the domain of controllability we construct a piecewise constant patchy feedback that guarantees that all Carathéodory solutions of the closed loop system reach the previously constructed neighborhood.

1 Introduction

It is the aim of this paper to present a procedure to combine local stabilization procedures with global ones to obtain a globally defined feedback with desirable properties near the fixed point that are designed using inherently local arguments. Specifically, we use linearization techniques at a singular point to find control Lyapunov functions that yield sampled feedbacks that exponentially stabilize with non-vanishing sampling rate. Globally, we use ideas for the construction of piecewise constant feedback in order to control the system to the neighborhood, where the sampled feedback is in force.

It is known that asymptotic nullcontrollability is equivalent to the existence of a control Lyapunov function, see [20], and in recent years numerous papers have appeared on the question on how to construct stabilizing feedbacks from such functions, see e.g. [6, 7, 11, 12, 19]. A fundamental question in this area is precisely the question of the underlying solution concept for which the constructed feedback should be interpreted. Often sampling concepts are considered, which have the advantage to avoid the topological pitfalls associated as well with continuous feedbacks, see [3, 19] and [5], as with discontinuous feedback in the sense of Filippov solutions [18]. Unless added structure like homogeneity is used [11], the sampled feedbacks often require vanishing sampling intervals as the trajectory approaches the origin. From a practical point of view, this appears to be undesirable.

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For our local considerations we will rely on the work of Grüne, who has shown for linear and more generally homogeneous systems how to construct discrete feedbacks that stabilize the origin where the sampling rate can be chosen to be positive [11, 12]. In fact, in [10] Grüne also shows that this procedure works locally for general nonlinear systems if the linearization is asymptotically null-controllable. Unfortunately, the results in that paper do not provide feedback invariant sets, which we will need in order to construct well defined feedbacks. Here, we will employ ideas from nonsmooth analysis already used in [6, 11] to regularize a known control Lyapunov function.

The local method will be combined with ideas that are inherently global in nature and that all depend in one way or another on the construction of piecewise constant feedbacks. This line of thought can be found in [15, 4, 16, 1]. We follow the line of reasoning of the paper of Ancona and Bessan [1] as it has the advantage of guaranteeing properties of all Carathéodory solutions generated by the discontinuous feedback. A question that is unfortunately neglected in the other references on this subject, which makes some statements in [15], [4, Chapter 12], [16] somewhat imprecise, as it is unclear, to what solution set statements on "all solutions" refer. One has to interpret these constructions carefully, by just considering the solutions that were intended, while constructing the piecewise constant feedback, which leaves a less than clear picture of the value of such constructions. Note, that also in the seemingly easier problem of feedback controlling the system to a set we cannot always rely on controllers in the Filippov or Krasovskii sense, as then extensions of the arguments in Brockett's work provide topological obstructions to the existence of such controllers, see [17].

The theoretical interest in this is that one obtains in this way a construction of a feedback that globally stabilizes the system without the need for increasingly faster sampling near the origin to ensure convergence. Also the local construction is computable as discussed in [11]. The piecewise constant approach has been implemented by Lai [15], and we expect that the necessary refinements due to the more careful analysis necessary pose no major problems. This remark is speculative, as we are not aware of any existing implementation. We do not discuss perturbation problems in this paper. Outer perturbations for the local and global procedures are discussed by Grüne [10], respectively Ancona and Bessan [1]. The problem of measurement noise, however, persists, see the discussion in [19].

In the following Section 2 we define the system class and make precise what we mean by a feedback in the Carathéodory and in the sampled sense. The ensuing Section 3 details a special class of Carathéodory feedbacks, that are defined by patches in a piecewise constant manner. The key in this definition is an "inward pointing" assumption which guarantees that there are no solutions with behavior other than the one intended in the construction of the patches. We show that we can always construct a patchy feedback controlling to an open set B from its domain of attraction. If we want furthermore, that there are no solutions existing for all positive times in the domain of attraction we need a further inward pointing assumption on the boundary of the open set B. In Section 4 we make the further assumption that the system is C^2 in the state and has a singular fixed point. Under the assumption that the linearization is asymptotically nullcontrollable we construct a feedback in the sense of sampling for the nonlinear system, which renders a neighborhood of the fixed point feedback invariant and guarantees local exponential stability for the closed loop system. In the final Section 5 we show that the previous constructions can be used in a complementary fashion. In particular, the inward pointing condition is satisfied on a suitable sublevel set of a control Lyapunov function, so that we can apply the results of Sections 3 and 4 to obtain a global feedback strategy, that is "hybrid" in the sense that we employ different notions of feedback in different regions of the state space.

2 Preliminaries

We study systems of the form

$$\dot{x} = f(x,u)\,, \tag{1}$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous and locally Lipschitz continuous in x uniformly in u. Here $U \subset \mathbb{R}^m$ is a compact convex set with nonvoid interior. The unique trajectory corresponding to an initial condition $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{U} := \{u : \mathbb{R} \to U \mid u \text{ measurable}\}$ is denoted by $x(\cdot; x_0, u)$. A feedback for system (1) is a map $F : U \to \mathbb{R}^n$. If F is not continuous (as will be the case in the feedbacks we aim to construct) this immediately raises the question what solution concepts are appropriate for the solution of the discontinuous differential equation $\dot{x} = f(x, F(x))$. A number of concepts have been put forward to deal with these problems. We therefore propose to always regard feedbacks along with the notion of solution that is considered.

Definition 2.1 (Carathéodory closed loop system) *Consider a feedback law $F : \mathbb{R}^n \to U$. For an interval $J \subset \mathbb{R}$ a* Carathéodory solution $\gamma : J \to \mathbb{R}^n$ *of*

$$\dot{x} = f(x, F(x)) \tag{2}$$

is an absolutely continuous functions $x : J \to \mathbb{R}^n$ such that $\dot{x} = f(x, F(x))$ almost everywhere. The map F is called a **C**-feedback, *if we consider Carathéodory solutions of* (2).

Note, that the definition of **C**-feedbacks does not require or guarantee that there exist any solutions to (2) or that there should be uniqueness. In the spirit of the remarks by Hájek [13], we simply refer the statement of necessary conditions for this to possible existence results. In a similar spirit one might define **K**- or **F**-feedbacks, where solutions are interpreted in the Krasovskii or Filippov sense. The second notion of solution we are interested in is the following.

Definition 2.2 (Sampled closed loop system) *Consider a feedback law $F : \mathbb{R}^n \to U$. An infinite sequence $\pi = (t_i)_{i\geq 0}$ with $0 = t_0 < \ldots < t_i < t_{i+1}$ and $t_i \to \infty$ is called a* sampling schedule *or* partition. *The values*

$$\Delta_i := t_{i+1} - t_i\,, \quad d(\pi) := \sup_{i\in\mathbb{N}} \Delta_i$$

are called intersampling times *and* sampling rate, *respectively. For any sampling schedule π the corresponding* sampled *or π-trajectory $x_\pi(t, x_0, F)$ with initial value $x_0 \in \mathbb{R}^n$ and initial time $t_0 = 0$ is defined recursively by solving*

$$\dot{x}(t) = f(x(t), F(x(t_i)))\,, \quad t \in [t_i, t_{i+1}]\,, \quad x(t_i) = x_\pi(t_i, x_0, F)\,. \tag{3}$$

The map F is called a sampled (or discrete) feedback *if we consider all sampled solutions of* (3) *and h-*sampled feedback *if we consider all solutions corresponding to sampling schedules π with $d(\pi) \leq h$.*

Note that the definition of sampled feedbacks guarantees existence and uniqueness of π-trajectories in forward time, on the respective interval of existence. On the other hand, nothing prevents trajectories from joining at some time instant.
A specific point of this paper is that we allow to switch between different solution concepts in different regions of the state space. In order to obtain a well defined global feedback we will require, that the region where a sampled feedback is defined remains invariant under the sampled solution in the following sense.

Definition 2.3 *A set $B \subset \mathbb{R}^n$ is called (forward)* feedback-invariant under h-sampling *for system (1) with respect to the feedback F, if for any initial condition $x_0 \in B$ and any sampling schedule π with $d(\pi) \leq h$ it holds that*

$$x_\pi(t; x_0, F) \in B \quad \textit{for all } t \geq 0.$$

Note that it is a peculiarity of sampled feedbacks, that the corresponding trajectories may for short times exist on regions of the state space where the sampled feedback is not defined, simply by leaving this area and returning before the next sampling instant. This is of course, somewhat undesirable and it is one of the aims of the paper to show how it can be prevented. The corresponding idea has already been used in [6], [11], [12].

3 Practical feedback stabilization with patchy feedbacks

In this section we study the system

$$\dot{x} = f(x, u)\,, \tag{4}$$

with the properties stated in Section 2. We are interested in applying the concept of patchy feedbacks that have recently been introduced in [1]. We slightly extend the definition to allow for less regularity. In particular, we replace the definition of *inward pointing* in terms of outer normals by the appropriate concept from nonsmooth analysis.
Let B be a closed subset of $\mathbb{R}^n$, if $x \notin B$, and if a point $y \in B$ is closest to x, i.e. $\operatorname{dist}(x, B) = \|x - y\|$, then $x - y$ is said to be a *proximal normal* to B in y. The cone generated by taking all positive multiples of these points is called *proximal normal cone* to B in y, denoted by $N_B^P(y)$, which is set to be $\{0\}$ if no proximal normal to y exists. One of the interests in this cone stems from the following characterization of strong invariance. Consider a locally Lipschitz continuous set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ with nonempty compact convex values. Here Lipschitz continuity means that for every compact set $K \subset \mathbb{R}^n$ there exists a constant L such that

$$F(x_1) \subset F(x_2) + L\|x_1 - x_2\| B(0,1)\,, \quad \forall x_1, x_2 \in K\,,$$

where as usual we denote $A + B := \{x + y \mid x \in A, y \in B\}$ for sets $A, B \subset \mathbb{R}^n$. The map F gives rise to a differential inclusion defined given by

$$\dot{x} \in F(x)\,,$$

where as usual solutions γ are absolutely continuous functions such that $\dot{\gamma}(t) \in F(\gamma(t))$ almost everywhere. Then a closed set B is *strongly invariant* (i.e. no trajectory leaves B) if and only if

$$\max\{\langle \zeta, w \rangle \mid \zeta \in N_B^P(x), w \in F(x)\} \leq 0\,, \tag{5}$$

see [8, Theorem 4.3.8].
We now introduce the notion of patchy vector fields slightly extending the notion of [1].

Definition 3.1 *Let $\Omega \subset \mathbb{R}^n$ be an open domain with boundary $\partial\Omega$ and let $D \subset \mathbb{R}^n$ be open. A Lipschitz continuous vector field g defined on a neighborhood of* $\operatorname{cl}\Omega$ *is called* inward pointing *on*

$(\partial\Omega)\setminus D$, if for every compact set K there exists a constant $c > 0$ such that for all $x \in K \cap \partial\Omega \setminus D$ it holds that

$$\max\{\langle \zeta, g(x)\rangle \mid \zeta \in N_\Omega^P(x)\} \leq -c\|\zeta\|\,,$$

where $n(x)$ denotes the outer normal of $\partial\Omega$ at x. The pair (Ω, g) is called a patch.

Definition 3.2 *Let $\Omega \subset \mathbb{R}^n$ be an open domain. We say that $g : \Omega \to \mathbb{R}^n$ is a* patchy vector field *if there exists a family of patches $\{(\Omega_\alpha, g_\alpha) \mid \alpha \in \mathcal{A}\}$ such that*

(i) $\mathcal{A}$ is a totally ordered index set,

(ii) the open sets Ω_α form a locally finite cover of Ω,

(iii) it holds that

$$g(x) = g_\alpha(x) \quad \text{if } x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta\,.$$

(iv) for every $\alpha \in \mathcal{A}$ the vector field g_α is inward pointing on $(\partial\Omega_\alpha) \setminus \bigcup_{\beta>\alpha} \Omega_\beta$.

Patchy vector fields g and solutions to the differential equation $\dot{x} = g(x)$ are discussed in detail in [1] for the case that the boundaries of Ω_α are C^1. The arguments, however, carry over to our definition. For Carathéodory solutions it can be shown that to each initial condition there exists at least one forward and at most one backward solution. Furthermore, along each Carathéodory solution x it holds that $t \mapsto \max\{\alpha \in A \mid x(t) \in \Omega_\alpha\}$ is nondecreasing and left continuous (with respect to the discrete topology on A). In an example [1, p. 457 ff.] the differences to other solution concepts are explained[1].
Note furthermore, that by [13, Lemma 2.8] the set of Krasovskii and Filippov solutions for patchy vector fields coincide, but we will not dwell on this issue, as in this article only Carathéodory solutions will be considered and will be henceforth called *solutions* for the sake of brevity.
We will now turn to the problem of practical feedback stabilization based on the concept of patchy feedbacks. Assume we are given a totally order index $\mathcal{A}$, open sets $\Omega_\alpha, \alpha \in \mathcal{A}$ and functions $F_\alpha : W_\alpha \to U, \alpha \in \mathcal{A}$, where W_α is an open neighborhood of Ω_α. We say that F is a patchy **C**-feedback for system (4), if

(i) $F(x) = F_\alpha(x)$ if $x \in \Omega_\alpha \setminus \bigcup_{\beta>\alpha} \Omega_\beta$,

(ii) $f(x, F(x))$ is a patchy vector field on $\Omega := \cup\,\Omega_\alpha$.

Given a set $Q \subset \mathbb{R}^n$ and an open set $B \subset Q$ we define the backward orbit of B relative to Q by

$$\mathcal{O}^-(B)_Q := \{y \in \mathbb{R}^n \mid \exists t \geq 0, u \in \mathcal{U} : x(t; y, u) \in B \text{ and } x(s; y, u) \in Q, s \in [0, t]\}\,. \tag{6}$$

Note that it is obvious by definition that $\mathcal{O}^-(B)_Q \subset Q$. Furthermore, it is an easy consequence of continuous dependence on the initial value that $\mathcal{O}^-_Q(B)$ is open, if Q is open.

[1] Incidentally, note that in this example **not all** maximal Filippov solutions are given, contrary to what is claimed. The ones that remain in $(0, 0)$ for some interval $[1, r]$ and then follow the parabola to the right are missing.

Definition 3.3 *System* (4) *is called* practically **C**-feedback controllable *if for every closed set* $Q \subset \mathbb{R}^n$ *and every open set* $B \subset Q$ *there is* **C**-*feedback* $F_{B,Q} : \mathcal{O}^-(B)_{\operatorname{int} Q} \setminus \operatorname{cl} B \to \mathbb{R}^n$ *so that*

(i) *for every* $x \in \mathcal{O}^-(B)_{\operatorname{int} Q} \setminus \operatorname{cl} B$ *there exists a Carathéodory solution* γ *with* $\gamma(0) = x$,

(ii) *for every Carathéodory solution* γ *starting in* $x \in \mathcal{O}^-(B)_{\operatorname{int} Q} \setminus \operatorname{cl} B$ *of*

$$\dot{x} = f(x, F_{B,Q}(x)),$$

there is a time T *such that* $\gamma(T) \in \partial B$.

Note that we do not require anything with respect to controllability on the boundary of Q. The reason will become clear in the construction of the proof, where we will approximate measurable controls by piecewise constant ones. The following result shows that the foregoing notion of practical feedback controllability is always fulfilled.

Theorem 3.4 *System* (4) *is practically* **C**-*feedback controllable using patchy* **C**-*feedbacks.*

The proof of this statement follows the ideas explained in [1], with the necessary modifications for our case.

Proof: Let Q be closed and $B \subset Q$ be open. Let $x \in \mathcal{O}^-(B)_{\operatorname{int} Q} \setminus \operatorname{cl} B$, if the set is empty there is nothing to show. We will first construct a patchy feedback in a neighborhood of a trajectory from x to B. In a later step we will construct the global patchy feedback from these smaller entities.

Since B is open and the set of piecewise constant controls is dense in $\mathcal{U}$ there exist $T > 0$ and a piecewise constant control u such that $x(T; x, u) \in B$ and $x(t; x, u) \in \operatorname{int} Q$ for all $t \in [0, T]$. We may assume that $\gamma_0(\cdot) := x(\cdot; x, u)$ is injective on $[0, T]$, as we may otherwise cut away "loops". We denote the points of discontinuity of u by $t_0 = 0 < t_1 < \ldots < t_k = T$ and let $v_j := u(s), s \in (t_{j-1}, t_j), j = 1, \ldots, k$.

The essential idea due to Ancona and Bessan now consists in enriching the systems dynamics around our trajectory of interest. Choosing $\rho_0, \chi_0 > 0$ small enough, it may be seen that for any $0 \le \rho \le \rho^*, 0 \le \chi \le \chi^*$ any solution γ of the differential inclusion

$$\dot{x}(t) \in f(x(t), u(t)) + \overline{B}(0, \chi) \tag{7}$$

with initial condition $\gamma(\tau) = y$, $\tau \in [0, T], y \in \overline{B}(\gamma_0(\tau), \rho)$ exists on the interval $[\tau, T]$. Furthermore, a constant c_0 may be chosen such that it holds that

$$\sup_{t \in [\tau, T+\rho]} \|\gamma(t) - \gamma_0(t)\| < c_0(\rho + \chi), \quad 0 \le \rho \le \rho^*, 0 \le \chi \le \chi^*.$$

From this we conclude inductively for $j = 1, \ldots, k-1$ as follows. If

$$\bar{x} \in B\left(\gamma_0(t_j), \sum_{l=1}^{j-1} c_0^l \chi + c_0^j(\rho + \chi)\right)$$

and γ is a solution of (7) with $\gamma(t_j) = \bar{x}$, then it follows that

$$\sup_{t \in [t_j, T+\rho]} \|\gamma(t) - \gamma_0(t)\| < \sum_{l=1}^{j} c_0^l \chi + c_0^{j+1}(\rho + \chi), \tag{8}$$

provided that ρ, χ are chosen small enough so that all sums are bounded by χ^*. We now choose $\rho_0 > 0, \chi_0 > 0$ such that additionally

$$\sum_{l=1}^{k-1} c_0^l \chi + c_0^k(\rho + \chi) < \operatorname{dist}(\gamma_0(T), \partial B)/2\,,$$

and define

$$\rho_1 := \rho_0\,, \quad \rho_j = \sum_{l=1}^{j-2} c_0^l 2\chi_0 + c_0^{j-1}(\rho_0 + 2\chi_0)\,, \quad j = 2, \dots, k\,.$$

Now consider the differential inclusion

$$\dot{x} \in f(x, v_j) + \overline{B}(0, \chi_0)\,. \tag{9}$$

Given $x \in \mathbb{R}^n$ the reachable set at time t is defined by

$$\mathcal{R}_t(x) := \{y \in \mathbb{R}^n \mid \exists \gamma \text{ solving (9) such that } \gamma(0) = x, \gamma(t) = y\}\,.$$

Then in an obvious way for a set $W \subset \mathbb{R}^n$ and an interval $[0,t]$ we denote $\mathcal{R}_{[0,t]}(W) := \cup \mathcal{R}_s(x)$, where the union is taken over $s \in [0,t], x \in W$. We abbreviate $\mathcal{R}_j := \mathcal{R}_{[0,t_j - t_{j-1}]}(B(\gamma_0(t_{j-1}), \rho_j))$, and define artificially $\mathcal{R}_{k+1} := B$, to handle the final set.
Now consider $y \in (\partial \mathcal{R}_j) \setminus \mathcal{R}_{j+1}, j = 1, \dots, k$. By the construction of the constants ρ_j and χ_0 this implies that $y = \gamma(\tau)$ for some solution of (9) with $\gamma(0) \in B(\gamma_0(t_{j-1}), \rho_j)$ and $\tau < t_j - t_{j-1}$. By (5), and using the fact that by definition no solution of (9) can leave $\mathcal{R}_j$ at z we have that

$$\langle \zeta, w \rangle \leq 0\,, \quad \text{for all } \zeta \in N^P_{\mathcal{R}_j}(y), w \in f(y, v_j) + \overline{B}(0, \chi_0)\,,$$

which using the particular choice $w_\zeta := f(y, v_j) + \chi_0 \zeta / \|\zeta\|$ implies immediately that

$$0 \geq \langle \zeta, w_\zeta \rangle = \langle \zeta, f(y, v_j) \rangle + \chi_0 \|\zeta\|\,.$$

This implies that for all $y \in (\partial \mathcal{R}_j) \setminus \mathcal{R}_{j+1}$, $\zeta \in N^P_{\mathcal{R}_j}(y)$ we have

$$\langle \zeta, f(y, v_j) \rangle \leq -\chi_0 \|\zeta\|\,,$$

and hence $f(\cdot, v_j)$ is inward pointing on $(\partial \mathcal{R}_j) \setminus \mathcal{R}_{j+1}$. Now, we use the usual order on the index set $\mathcal{A}_x := \{1, \dots, k\}$ and define for some $\eta > 0$, the feedback $F_j(x) = v_j, x \in \mathcal{R}_j + \eta B(0,1), j = 1, \dots, k$. Then the patchy **C**-feedback F_x defined on $\Omega_x := \cup_{j=1}^k \mathcal{R}_j$ has the property that all solutions starting in a $\Omega_x \setminus B$ reach ∂B at a time $T_\gamma < T + \rho$, using (8).
If we now take a compact subset $K \subset \mathcal{O}^-(B)_{\operatorname{int} Q}$, then we obtain an open cover of K by performing the construction of Ω_x for every $x \in K$. We may choose a finite subcover $\Omega_{x_1}, \dots, \Omega_{x_l}$ and define the overall index set $\mathcal{A}_K := \{(i,j) \mid i = 1, \dots, l; j = 1, \dots, k_{x_i}\}$, where k_{x_i} denotes the number of patches composing Ω_{x_i}. This index set may be ordered by $(i_1, j_1) < (i_2, j_2)$ if $i_1 < i_2$ or in the case of equality $i = i_1 = i_2$ by using the original order among the index set $\mathcal{A}_{x_i}$ for the j-component. The result now follows by taking an increasing family of compact subset $K_i, i \in \mathbb{N}$, $K_i \subset K_{i+1}, i \in \mathbb{N}$ with $\cup_{i \in \mathbb{N}} K_i = \mathcal{O}^-(B)_{\operatorname{int} Q} \setminus \operatorname{cl} B$ and ordering the index set accordingly. $\square$

Note that we have not shown, that there is no solution γ with $\gamma(0) \in \mathcal{O}^-(B)_{\mathrm{int}\,Q} \setminus \mathrm{cl}\, B$ such that $\gamma(t_1), \gamma(t_2) \in \mathrm{cl}\, B$ for some $0 < t_1 < t_2$. For this we need the added requirement that the entry into the set is "non-tangential" in the sense of proximal normals. For our goal of stabilization it is of course essential that such "double" touching is not possible, for otherwise we cannot exclude the possibility that some trajectories evolve in $\mathcal{O}^-(B)_{\mathrm{int}\,Q} \setminus B$ for all times under the patchy **C**-feedback, we have constructed.

Corollary 3.5 *Let $Q \subset \mathbb{R}^n$ be closed and $B \subset \mathrm{int}\, Q$ be open and bounded. Assume that for every $x \in \partial B$ there exists a $u \in U$ such that the inward pointing condition*

$$\max\{\langle \zeta, f(y,u)\rangle \mid \zeta \in N_B^P(y)\} \leq -c\|\zeta\|\,, \tag{10}$$

is satisfied for all $y \in \partial B$ in a neighborhood of x, then there is a piecewise constant patchy **C***-feedback F so that for every trajectory γ starting in $x \in \mathcal{O}^-(B)_{\mathrm{int}\,Q}$ of*

$$\dot{x} = f(x, F(x))\,,$$

there is a time T such that $\gamma(T) \in \partial B$. Furthermore, F can be chosen so that there exists no trajectory γ on an interval J such that $\gamma(t_1), \gamma(t_2) \in \partial B$ for $t_1 < t_2 \in J$ and $\gamma(t) \notin B$ else.

Proof: In view of Proposition 3.4 we only have to construct suitable patches at the boundary ∂B, as we have already shown the existence of the desired patchy **C**-feedback away from $\mathrm{cl}\, B$. Let $x \in \partial B$ and $u \in U$ be such that (10) is satisfied. Now this assumption guarantees that in a sufficiently small neighborhood W_x of x any solution $x(\cdot; y, u)$ of $\dot{x} = f(x,u)$ with $y \in W_x$ satisfies

$$\mathrm{dist}\,(x(t;y,u), \mathrm{cl}\, B) \leq \max\{\mathrm{dist}\,(y, \mathrm{cl}\, B) - \eta t, 0\}$$

as long as $x(t;y,u) \notin \mathrm{cl}\, B$, for a suitable $\eta > 0$, see [8, p. 220]. We may therefore construct a patch Ω_x, u as a neighborhood of x contained in W_x as in the proof of Proposition 3.4. We perform this for every $x \in \partial B$ and invoking compactness choose a finite cover $\Omega_{x_1}, \ldots, \Omega_{x_k}$ of ∂B. Let $\bar{\eta} = \min\{\eta_{x_1}, \ldots, \eta_{x_k}\}$. Then in a neighborhood of $\mathrm{cl}\, B$ we have that any solution of the Carathéodory closed loop system approaches $\mathrm{cl}\, B$ linearly in t. Now, it is immediate that no solution γ can exist such that $\gamma(t) \in \mathrm{cl}\, B$ at two different time instances. $\square$

In [15] the author proves the above result and imposes two further conditions: that the system be affine in the controls and locally accessible. We have seen that this is indeed not necessary. However, the aim in [15] is different, as there the feedback is defined using just extremal values of the set U thereby reducing the complexity of actually designing such a patchy feedback (in particular if U is a polyhedron). We note the following corollary, in which we see that local accessibility is unnecessary in any case. We denote the set of extremal points of U by $\mathrm{ext}\, U$. Recall that the set of extremal points of a compact convex set need not be compact.

Corollary 3.6 *Consider the system*

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x) \tag{11}$$

System (11) *is practically feedback controllable using patchy feedbacks, with values in* $\mathrm{ext}\, U$.

Proof: By convexity of U and using the affine structure of system (11) all trajectories can be approximated uniformly on compact intervals by trajectories generated by controls with values in ext U. Thus the control u at the beginning of the proof of Theorem 3.4 can be chosen in $L^\infty(\mathbb{R}, \text{ext } U)$. Now the result follows by repeating the remainder of the proof of Theorem 3.4. □

4 A sufficient condition for local sampled feedback stabilization with positive sampling rate

In this section we give a brief review of the result of Grüne for local stabilization at singular points. For us there remains one detail to supply, namely that the control Lyapunov functions constructed in [11] remain control Lyapunov functions for the nonlinear system.

$$\dot{x} = f(x, u)\,, \tag{12}$$

where we assume in addition to the assumptions previously stated that $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is twice differentiable in x and, furthermore, that 0 is a singular point for (12), i.e. $f(0, u) = 0$ for all $u \in U$.

We are interested in constructing a locally stabilizing feedback $F : \mathbb{R}^n \to U$ that stabilizes in the sense of sampling, see Definition 2.2. The local design procedure we will investigate relies on linearization techniques. We thus introduce the linearization of (12) in 0 with respect to x given by

$$\dot{z}(t) = A(u(t))z(t)\,, \tag{13}$$

where $A(u)$ denotes the Jacobian of $f(\cdot, u)$ in $x = 0$. We assume that $A(\cdot) : U \to \mathbb{R}^{n\times n}$ is Lipschitz continuous. The trajectories of (13) are denoted by $z(\cdot; z_0, u)$. It is known that asymptotic nullcontrollability of system (13), i.e. the property that for every $z \in \mathbb{R}^n$ there exists a $u \in \mathcal{U}$ such that $z(t; z, u) \to 0$, can be characterized via the use of Lyapunov exponents. These are defined for initial conditions $z \neq 0$ and $u \in \mathcal{U}$ by

$$\lambda(z, u) := \limsup_{t\to\infty} \frac{1}{t} \log \|z(t; z, u)\|\,.$$

Indeed by the results in [10] asymptotic nullcontrollability of (13) is equivalent to the requirement

$$\overline{\lambda} := \sup_{z\neq 0} \inf_{u\in\mathcal{U}} \lambda(z, u) < 0\,. \tag{14}$$

An approximation scheme that has been investigated in proves to be successful. The linear system (13) can be projected to the sphere, where it satisfies $\dot{s} = A(u)s - \langle s, A(u)s\rangle s$. Denoting the radial component $q(s, u) := \langle s, A(u)s\rangle$ we obtain that the Lyapunov exponent for $z \in \mathbb{R}^n \setminus \{0\}$ is given by

$$\lambda(z, u) = \lambda(z/\|z\|, u) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t q(s(\tau, z/\|z\|, u), u(\tau))d\tau\,.$$

Instead of considering this infinite horizon averaged integral we approximate by

$$J_\delta(z, u) := \int_0^\infty e^{-\delta\tau} q(s(\tau), u(\tau))d\tau\,, \text{ and } v_\delta(y) := \inf_{u\in\mathcal{U}} J_\delta(y, u)\,. \tag{15}$$

Then it can be shown that (14) is equivalent to the existence of a $\delta_0 > 0$ small enough such that

$$\max_{y\in\mathbb{R}^n} v_\delta(y) < 0\,, \quad \text{for all } 0 < \delta < \delta_0\,. \tag{16}$$

In [11] it is shown that $v_\delta(y)$ gives rise to a control Lyapunov function for the linearized system (13). We briefly recall the relevant notions. Given a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ and $v \in \mathbb{R}^n$ the *lower directional derivative* of V in x in direction v is defined by

$$DV(x;v) := \liminf_{t\searrow 0,\, v'\to v} \frac{1}{t}\,(V(x+tv') - V(x)).$$

Two continuous functions $V, W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are said to be a *control Lyapunov pair* for system (12), if they satisfy the following requirements:

(i) **(positive definiteness)** $V(z) > 0$ and $W(z) > 0$ for all $z \in \mathbb{R}^n\backslash\{0\}$, and $V(0) = 0$;

(ii) **(properness)** The set $\{z \,|\, V(z) \leq \beta\}$ is bounded for every $\beta \geq 0$;

(iii) **(infinitesimal decrease)** For all $z \in \mathbb{R}^n$ it holds that

$$\min_{v\in \mathrm{co} f(z,U)} DV(z;v) \leq -W(z). \tag{17}$$

The function V is called *control Lyapunov function*, if there is a W such that (V,W) is a control Lyapunov pair. The function v_δ gives rise to a control Lyapunov function for (13) as follows.

Lemma 4.1 *[11, Lemma 4.1] For every $\rho \in (0,\overline{\lambda})$ there exist a $\delta_\rho > 0$ such that for every $\delta \in (0,\delta_\rho]$ the function $V_\delta : \mathbb{R}^n \to \mathbb{R}$ defined by $V_\delta(0) = 0$ and*

$$V_\delta(z) := e^{2v_\delta(z)}\|z\|^2\,, \qquad z \neq 0\,, \tag{18}$$

satisfies

$$\min_{v\in \mathrm{co} A(U)z} DV_\delta(z;v) \leq -2\rho V_\delta(z) \tag{19}$$

Note that by definition we have the homogeneity property $V_\delta(\alpha z) = \alpha^2 V_\delta(z), \alpha > 0$. In general, V_δ is only Hoelder continuous. We follow an approach originating in [6] and employed for our special case in [11] that obtains Lipschitz continuous Lyapunov functions from V_δ. This also yields Lyapunov functions that describe feedback invariant sets as already remarked in [6, Lemma IV.2]. We introduce the (quadratic) *inf-convolution* of V_δ given by

$$V_\beta(x) := \inf_{y\in\mathbb{R}^n}\left[V_\delta(y) + \frac{1}{2\beta^2}\|y-x\|^2\right]. \tag{20}$$

This function is Lipschitz for $\beta > 0$ and converges pointwise to V_δ as $\beta \to 0$. Furthermore, the homogeneity of V_δ implies that we also have $V_\beta(\alpha x) = \alpha^2 V_\beta(x)$. For $x \in \mathbb{R}^n$ we denote by $y_\beta(x)$ a minimizing vector in (20), which exists by continuity of V_δ, and introduce the vector

$$\zeta_\beta(x) := \frac{x - y_\beta(x)}{\beta^2}. \tag{21}$$

By [6, Lemma III.1] we have for all $x, v \in \mathbb{R}^n$ and all $\tau \geq 0$ that

$$V_\beta(x + \tau v) \leq V_\beta(x) + \tau \langle \zeta_\beta(x), v \rangle + \frac{\tau^2 \|v\|^2}{2\beta^2}. \tag{22}$$

which can be interpreted as a Taylor inequality for V_β in direction v. The following statement shows that we retain the property of being a control Lyapunov function if V_δ is replaced by V_β where $\delta > 0$ and $\beta > 0$ are small enough. The feedback we envisage is now given as a pointwise minimizer of $\langle \zeta_\beta(z), A(u)z \rangle$, that is $F_\beta(z)$ is chosen so that

$$\langle \zeta_\beta(z), A(F_\beta(z))z \rangle = \min_{u \in U} \langle \zeta_\beta(z), A(u)z \rangle. \tag{23}$$

This choice is of course not unique, nor can we expect that a regular choice is possible, we will however always assume that we have obtained a pointwise minimizer satisfying $F_\beta(\alpha z) = F_\beta(z)$, for $z \in \mathbb{R}^n$, $\alpha > 0$, which is easily seen to be possible.
We quote the following result by Grüne in a slightly extended manner that also states intermediate statements of the proof that we will need later on. We denote

$$A^t_{z_0} := \frac{1}{t} \int_0^t A(F_\beta(z_0)) z(\tau; z_0, F_\beta(z_0))\, d\tau\,,$$

and

$$M := \max \left\{ 2 \max_{u \in U} \|A(u)\|, \sup_{\|x\| \leq 2, u \in U} \|f(x,u)\| \right\}. \tag{24}$$

Proposition 4.2 *[11, Proposition 4.2] Assume that system (13) is asymptotically nullcontrollable and let $\rho \in (0, \overline{\lambda})$. Let $0 < \delta < \delta_\rho$, where δ_ρ is defined in Lemma 4.1. Then there exists a $\beta \in (0, 1]$, such that*

$$\min_{v \in \operatorname{co} A(U)z} DV_\beta(z; v) \leq -2\rho V_\beta(z) \tag{25}$$

for all $z \in \mathbb{R}^n$. Furthermore, there is a $\bar{t} > 0$ such that

$$V_\beta(z(t; z_0, F_\beta(z_0))) - V_\beta(z_0) \leq t \langle \zeta_\beta(z_0), A^t_{z_0} \rangle + t^2 \frac{M^2}{2\beta^2} \leq -\frac{3}{2} t \rho V_\beta(z_0)\,,$$

for all $z_0 \in \mathbb{R}^n$ and all $0 < t \leq \bar{t}$.

The previous result contains all the arguments necessary to prove that F_β is indeed a $\bar{t}$-sampled stabilizing feedback for the linearized system (13). We now wish to carry the result over to a local statement for the nonlinear system (12).
Given that V_β is a control Lyapunov function it is no real surprise that it is also one for the nonlinear system (12) in a neighborhood of the origin as we show now. To this end we need the following technical lemma. We denote

$$\underline{\sigma} := \inf_{z \in \mathbb{R}^n \setminus \{0\}} v_\delta(z) \quad \text{and} \quad \overline{\sigma} := \sup_{z \in \mathbb{R}^n \setminus \{0\}} v_\delta(z)\,.$$

Lemma 4.3 *For all $z \in \mathbb{R}^n$ and all $\beta > 0$ it holds that*

$$\|y_\beta(z) - z\| \leq \sqrt{2} e^{\overline{\sigma}} \|z\| \beta.$$

Proof: By definition we have that $V_\beta(z) \le V_\delta(z)$ for all $z \in \mathbb{R}^n$. Now the assertion follows from

$$\frac{1}{2\beta^2}\|y_\beta(z) - z\|^2 = V_\beta(z) - V_\delta(y_\beta(z)) \le V_\delta(z) - V_\delta(y_\beta(z)) \le V_\delta(z) \le e^{2\overline{\sigma}}\|z\|^2. \qquad \square$$

Theorem 4.4 *Consider the nonlinear system* (12) *and its linearization* (13). *Assume that* (13) *is asymptotically nullcontrollable. Let* $\rho \in (0, \overline{\lambda})$, $0 < \delta < \delta_\rho$ *and assume furthermore that* $\overline{t} > 0$ *and* $\beta \in (0, 1]$ *are chosen such that the assertion of Proposition 4.2 hold. Then there exists constants* $R > 0, \bar{t} > 0$, *such that*

$$\min_{v \in \mathrm{co} f(x,U)} DV_\beta(x; v) \le -\rho V_\beta(x) \tag{26}$$

for all $x \in \mathrm{cl}\, B_R(0)$ *and*

$$V_\beta(x(t; x_0, F_\beta(x_0))) - V_\beta(x_0) \le -t\rho V_\beta(x_0)\,, \tag{27}$$

for all $x_0 \in \mathrm{cl}\, B_R(0)$ *and all* $0 < t \le \overline{t}$.

Proof: For the sake of abbreviation we denote $\tilde{f} := f - A$ and

$$f_{x_0}^t := \frac{1}{t}\int_0^t f(x(\tau; x_0, F_\beta(x_0)), F_\beta(x_0))\, d\tau.$$

Let M be as defined in (24). Then by decreasing $\bar{t}$ if necessary we have that $\|f_{x_0}^t\| \le M$ for all $x_0 \in \mathrm{cl}\, B_1(0)$ and all $0 < t \le \overline{t}$. By [4, Lemma 12.2.10 (iii)] there exists a constant $C > 0$, such that

$$\|x(t; x_0, F_\beta(x_0)) - z(t; x_0, F_\beta(x_0))\| \le Ct\|x_0\|^2 \tag{28}$$

for all $x_0 \in \mathrm{cl}\, B_1(0)$ and all $0 < t \le \overline{t}$. Now by Lemma 4.3, (28), and the definition of ζ_β we obtain for all $x_0 \in \mathrm{cl}\, B_1(0)$ and all $0 < t \le \overline{t}$ that

$$\begin{aligned}
\langle \zeta_\beta(x), f_{x_0}^t - A_{x_0}^t \rangle &\le \|\zeta_\beta(x)\|\|f_{x_0}^t - A_{x_0}^t\| = \frac{\|y_\beta(x_0) - x_0\|}{\beta^2}\|f_{x_0}^t - A_{x_0}^t\| \\
&= \frac{\|y_\beta(x_0) - x_0\|}{t\beta^2}\left\|\int_0^t \tilde{f}(x(\tau; x_0, F_\beta(x_0)), F_\beta(x_0))d\tau\right\| \\
&\le \frac{\sqrt{2}e^{\overline{\sigma}}\|x_0\|}{t\beta}\|x(t; x_0, F_\beta(x_0)) - z(t; x_0, F_\beta(x_0))\| \\
&\le \frac{\sqrt{2}e^{\overline{\sigma}}}{\beta}C\|x_0\|^3.
\end{aligned} \tag{29}$$

Using Proposition 4.2, (22), the definition of M and (29) this implies that

$$\begin{aligned}
V_\beta(x(t; x_0, F_\beta(x_0))) - V_\beta(x_0) &= V_\beta(x_0 + tf_{x_0}^t) - V_\beta(x_0) \\
&\le t\langle \zeta_\beta(x_0), f_{x_0}^t \rangle + t^2\frac{\|f_{x_0}^t\|^2}{2\beta^2} \\
&\le t\langle \zeta_\beta(x_0), A_{x_0}^t \rangle + t^2\frac{M^2}{2\beta^2} + t\langle \zeta_\beta(x_0), f_{x_0}^t - A_{x_0}^t \rangle \\
&\le -\frac{3}{2}t\rho V_\beta(x_0) + t\langle \zeta_\beta(x_0), f_{x_0}^t - A_{x_0}^t \rangle \\
&\le t\left(-\frac{3}{2}\rho V_\beta(x_0) + \frac{\sqrt{2}e^{\overline{\sigma}}C}{\beta}\|x_0\|^3\right)
\end{aligned}$$

for all $x_0 \in \operatorname{cl} B_1(0)$ and all $0 < t \leq \bar{t}$. Define

$$\chi_r := \sup_{\|x\|=r} \frac{\sqrt{2}e^{\overline{\sigma}}C\|x\|^3}{\beta V_\beta(x)} = \sup_{\|x\|=1} r\frac{\sqrt{2}e^{\overline{\sigma}}C}{\beta V_\beta(x)},$$

where we have used the homogeneity of V_β. Now choose $R > 0$, such that

$$\chi_R \leq \frac{1}{2}\rho,, \tag{30}$$

and it follows that

$$V_\beta(x(t; x_0, F_\beta(x_0))) - V_\beta(x_0) \leq t\left(-\frac{3}{2}\rho + \chi_R\right)V_\beta(x_0) \leq -t\rho V_\beta(x_0)$$

for all x_0 with $\|x_0\| \leq R$ and all $0 < t \leq \bar{t}$. This implies the assertion. □

The final result of this section shows that the sublevel sets of V_β describe sets that are feedback invariant under $\bar{t}$-sampling for system (12) with respect to the $\bar{t}$-sampled feedback F_β, at least close to zero. For $r > 0$ we denote the r^2-sublevel set by

$$G_r^\beta := \{x \mid V_\beta(x) \leq r^2\}.$$

Corollary 4.5 *Let the assumptions of Theorem 4.4 be satisfied. Choose $R > 0$ according to the assertions of that theorem. For any $r > 0$ such that $G_r^\beta \subset B(0, R)$ it holds that*

(i) The set G_r^β is feedback-invariant under $\bar{t}$-sampling for system (12) with respect to the feedback F_β,

(ii) there exists a constant $C > 0$ such that for any $x_0 \in G_r^\beta$ and any sampling schedule π with $d(\pi) \leq \bar{t}$ it holds for the π-trajectory defined by (3) that

$$\|x_\pi(t, x_0, F_\beta)\| \leq Ce^{-\rho t/2}\|x_0\|.$$

Proof: (i) This follows immediately from Theorem 4.4 via an inductive argument.
(ii) Fix a partition $\pi = (t_i)_{i\in\mathbb{N}_0}$ with $d(\pi) \leq \bar{t}$. By Theorem 4.4 it follows for $x_0 \in G_r^\beta \subset \operatorname{cl} B_R(0)$ and all $i \in \mathbb{N}_0$ that

$$V_\beta(x_{i+1}) \leq (1 - \rho(t_{i+1} - t_i))V_\beta(x_i) \leq e^{-\rho(t_{i+1}-t_i)}V_\beta(x_i),$$

that is we have the desired exponential bound in terms of V_β. By construction it is clear that there are constants $0 < a_1 < a_2$ such that

$$a_1\|x\|^2 \leq V_\beta(x) \leq a_2\|x\|^2.$$

Now the assertion follows by comparison arguments, see e.g. [14, Proof of Corollary 3.4]. □

Remark 4.6 If we assume in addition to the assumptions of the previous Corollary 4.5 that $\beta < \frac{1}{\sqrt{2}}e^{-\overline{\sigma}}$, we can make possible choices for $r > 0$ and a_1, a_2 more concrete. Denoting $\kappa := 1 - \sqrt{2}e^{\overline{\sigma}}\beta > 0$, we obtain with Lemma 4.3 that

$$\|y_\beta(x)\| \geq \|x\| - \|y_\beta(x) - x\| \geq (1 - \sqrt{2}e^{\overline{\sigma}}\beta)\|x\| = \kappa\|x\| .$$

for all $x \in \mathbb{R}^n$. This implies that

$$\kappa^2 e^{2\underline{\sigma}}\|x\|^2 \leq V_\delta(y_\beta(x)) \leq V_\beta(x) \leq V_\delta(x) \leq e^{2\overline{\sigma}}\|x\|^2.$$

In particular, if $r < \kappa e^{\underline{\sigma}} R$ and $x \in G_r^\beta$ we have

$$\kappa^2 e^{2\underline{\sigma}}\|x\|^2 \leq V_\beta(x) \leq r^2 \leq \kappa^2 e^{2\underline{\sigma}} R^2 , \tag{31}$$

so that $x \in B(0, R)$.

5 Global feedback stabilization with positive sampling rate near the origin

In this section the two ingredients of the final feedback design will be put together. We continue to consider system (12) with the additional assumptions stated at the beginning of Section 4 and the associated linearization (13) in 0.

Definition 5.1 *System* (12) *is called* globally asymptotically feedback stabilizable, *if for every connected compact set $Q \subset \mathbb{R}^n$ with $0 \in \operatorname{int} Q$ and every open ball $B(0, r) \subset Q$ there exists a compact connected set $D \subset B(0, r)$ containing the origin, such that the following conditions are satisfied:*

(i) There exists a patchy **C**-*feedback F_1 on $\mathcal{O}^-(D)_{\operatorname{int} Q} \backslash D$, such that*

(a) for every $x \in \mathcal{O}^-(D)_{\operatorname{int} Q} \backslash D$ there exists a Carathéodory solution of

$$\dot{x} = f(x, F_1(x)) \tag{32}$$

with $\gamma(0) = x$ on some interval $[0, t_\gamma)$

(b) for every solution $\gamma(\cdot)$ of (32) *with $\gamma(0) = x \in \mathcal{O}^-(D)_{\operatorname{int} Q} \backslash D$ there exists a $T_\gamma > 0$, with*

$$\gamma(T_\gamma) \in \partial D ,$$

(c) there is no solution γ of (32) *with $\gamma(0) = x \in \mathcal{O}^-(D)_{\operatorname{int} Q} \backslash D$ such that $\gamma(t_1) \in \partial D$, $\gamma(t_2) \notin D$ for some $t_1 < t_2$.*

(ii) There exists a sampling bound $\overline{t} > 0$ and a $\overline{t}$-sampled feedback F_2 on D, such that D is feedback invariant under $\overline{t}$-sampling for system (12) with respect to the feedback F_2, and such that

$$\lim_{t \to \infty} x_\pi(t; x, F_2) = 0 ,$$

for every $x \in D$ and every π trajectory for sampling schedules π with $d(\pi) \leq \overline{t}$.

Theorem 5.2 *The system* (12) *is globally asymptotically feedback stabilizable in the sense of Definition 5.1 if its linearization* (13) *is asymptotically nullcontrollable.*

Proof: Let $Q \subset \mathbb{R}^n$ be closed with $0 \in \operatorname{int} Q$. Let $\bar{t}$ and $\beta \in (0,1]$ be such that $F_2 = F_\beta$ is an $\bar{t}$-sampled exponentially stabilizing feedback on $D' = G^\beta_{r'}$. We choose an $0 < r < r'$, then the statement is obviously also true for $D = G^\beta_r$. The linear decrease statement (27) from Theorem 4.4 guarantees that we can satisfy the inward pointing condition (10) for ∂G^β_r. Now by Theorem 3.4 and Corollary 3.5 we have the existence of a patchy $\mathbf{C}$-feedback F with ordered index $\mathcal{A}$ on $\mathcal{O}^-(G^\beta_{r'})_Q$, stabilizing to $\partial G^\beta_{r'}$. This concludes the proof. □

Remark 5.3 It is of course quite unrealistic from several points of view, to demand switching between the two controllers and solution concepts if the boundary of a set D is reached, especially as level sets of V_β do not lend themselves easily to computation. We can, however, relax the requirements here a bit. Take two values $0 < r_1 < r_2$ such that for a suitable β the sampled feedback F_β renders $G^\beta_{r_2}$ $\bar{t}$-sampled feedback invariant and exponentially stabilizes to 0. As in the proof of Corollary 3.5 we may construct a finite number of patches such that the corresponding patchy $\mathbf{C}$-feedback controls to $\partial G^\beta_{r_1}$.
Now we may just require that the switch between the Carathéodory and the sampled feedback is made somewhere in $G^\beta_{r_2} \setminus G^\beta_{r_1}$, say at a point $x \in G^\beta_r$. As for all $r_1 < r < r_2$ the set G^β_r is also $\bar{t}$-sampled feedback invariant we still have that the following π-trajectories remain in G^β_r and converge exponentially to zero. This makes the decision of switching less delicate for the price that we have concurring definitions for the feedback in a certain region of the state space, so that we need a further variable, a "switch", to remember which feedback strategy is applicable.

6 Conclusions

In this article we have presented a method to unite a local exponentially stabilizing sampled feedback with a global piecewise constant feedback interpreted in the sense of Carathéodory solutions. The key tools in this approach were methods from nonsmooth analysis in particular proximal normals and inf-convolution. In general, this approach is not restricted to the feedback types we have considered here, but can be performed for any feedback concepts that allow for the completion of the key step in our design. This consists in the construction of feedback invariant sets for the feedback F_2 that can be entered from the outside under the feedback F_1, and that have the additional property that no solutions under F_1 can move away (locally) from the feedback invariant set.

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