Dynamics of time-varying discrete-time linear systems: Spectral theory and the projected system

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Abstract

We study structural properties of linear time-varying discrete-time systems. At first an associated system on projective space is introduced as a basic tool to understand the linear dynamics. We study controllability properties of this system, and characterize in particular the control sets and their cores. Sufficient conditions for an upper bound on the number of control sets with nonempty interior are given. Furthermore exponential growth rates of the linear system are studied. Using finite time controllability properties in the cores of control sets the Floquet spectrum of the linear system may be described. In particular, the closure of the Floquet spectrum is contained in the Lyapunov spectrum.

1 Introduction

In recent years spectral theory for time-varying linear systems has attracted renewed interest. While the foundations of the theory have been laid by Floquet [25], Lyapunov [40] and Bohl [16] the introduction of the problems and considerations of control posed new questions to which different approaches have been proposed. Here we present an approach to the spectral theory of families of discrete-time time-varying linear systems of the form

$$x(t+1) = A(u(t))x(t) \quad t \in \mathbb{N},$$

where the entries of A depend analytically on the time-varying parameter u, which takes values in a prescribed set. In order to gain an insight in the dynamics of this system the system that is obtained by projecting on projective space is analyzed. This approach leads to two generalizations of objects well understood for time-invariant systems. The concept

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of eigenspace is extended to what is called control set on projective space that is a set that is characterized by certain controllability properties. Eigenvalues find natural and well understood generalizations in Floquet, Lyapunov, and Bohl exponents. We examine these different exponential growth rates and how control sets may be employed to characterize them.

Exponential stability is characterized by the Bohl exponent of a time-varying linear system [16], see also [24]. Przyłuski and Rolewicz studied Bohl exponents (or generalized spectral radii in their terminology) for discrete time systems in [46] with further work appearing in [43] - [45]. On the other hand Lyapunov exponents characterize exponential growth along trajectories. Properties of Lyapunov exponents were studied by Barabanov in [8]-[11], where sufficient conditions so that Lyapunov exponents characterize exponential stability for families of time-varying systems are shown. Berger and Wang [15], Lagarias and Wang [38] and Gurvits [27] study the joint and the generalized spectral radius given by a discrete inclusion (not to be confused with the notion of generalized spectral radius due to Przyłuski and Rolewicz). The works cited so far are concerned mainly with the largest exponents characterizing stability. In this article we are interested in the complete spectrum of exponential growth rates associated with the system. Also we will briefly discuss the relation between the different notions appearing in the literature.

The basic idea of our approach is to study a system on projective space that can be constructed from the linear system by Bogolyubov's projection introduced by Has'minskii [28]. The study of this projection in connection with control theory has found numerous applications for continuous time systems in the analysis of Lyapunov spectrum. For deterministic systems the work of Colonius and Kliemann [20], [21] and [22] presents a full picture of what is known. In particular the relation to exponential dichotomies and the dynamical spectrum as studied by Sacker and Sell [47] and Johnson, Palmer and Sell [35] is analyzed in these references.

Interest in the complete spectrum of the linear system stems from diverse lines of research. One of these is the question of robust stability. Let $A(u_0)$ be a Hurwitz stable matrix, i.e. the spectrum of $A(u_0)$ consists of values with negative real part and interpret U as a set determining the structure of possible perturbations to the time-invariant system given by $\dot{x} = A(u_0)x$. The problem of robust stability is to determine whether the perturbed system is exponentially stable under all possible perturbations $u : \mathbb{R} \to U$, that are e.g. piecewise continuous, see Hinrichsen and Pritchard [29], [30] and Colonius and Kliemann [19]. The discrete-time problem has been treated by the author and Hinrichsen in [57], [55].

Interpreting u as a control term knowledge about the set of exponential growth rates or Lyapunov exponents can be employed in the stabilization of such systems, see Colonius, Kliemann and Krull [23], and Grüne [26].

If $u(t)$ is given a stochastic interpretation we are in the realm of stochastic systems. This problem was treated for continuous time systems by Has'minskii [28], Arnold, Kliemann and Oeljeklaus [6], Arnold and Kliemann [5] and Arnold and San Martin [7]. The discrete-time case was studied by Homblé in [32], [33] and Baxendale and Has'minskii [14], however, with the restriction that the discrete-time system is invertible.

In this article we wish to lay the foundation for the theory and treat some of the difficulties inherent to the discrete-time case. It is explained how the problem of non-invertibility can be partially overcome while retaining the possibility of obtaining a reasonable system on projective space. We study asymptotic properties of the projected system, show the existence of controls with universal properties and examine the controllability structure of the projected system. This supplies the tools we need for an analysis of the different spectra.

We proceed as follows. Section 2 contains the problem statement along with the assumptions we make. In Section 3 we study accessibility, transitivity and regularity of discrete-time systems. Orbits and regular orbits are introduced and it is explained why forward accessibility can be characterized by the rank of a Jacobian. This has been noted by several other authors [41], [32]. What is particularly useful in the case of the projected system is that by Proposition 3.6 it is not necessary to check this in local coordinates on the projective space $\mathbb{P}_{\mathbb{K}}^{n-1}$.

In Section 4 we exhibit some asymptotic properties of the system on projective space. The study of ω-limit sets follows the approach of Colonius and Kliemann [20], and is standard if the projections of linear systems on projective space are studied. Using the regularity arguments from Section 3 we obtain sufficient conditions for the generalized eigenspace of a transition matrix to project to a region of exact controllability.

In Section 5 we state a result on universally regular controls and a controllability property that can be proved using the existence of universally regular controls. In spite of the activity in the study of accessibility of discrete-time systems, the existence of universal controls has only recently been investigated [54], [50]. In [49] Sontag shows the existence of universally regular (universal nonsingular in his terminology) controls for analytic, strongly accessible continuous-time systems. Related, and at first glance more interesting, is the existence of universally distinguishing controls which has been studied by Sussmann [51], and Sontag and Wang [48]. It cannot be overemphasized, however, that without the existence of universally regular controls, the following results would lose a considerable amount of strength. The main result of this section is that forward accessibility on projective space implies that a whole linear subspace may be steered so as to simultaneously avoid a complementary linear subspace. An analogue of this statement (Proposition 5.3) has to our knowledge not been studied in continuous time.

A starting point in the study of nonlinear control systems are questions of controllability of a system. Unlike the linear case where controllability is a global property in the state space, nonlinear systems may possess several regions of controllability. An important conceptual tool is to study sets, where it is possible to steer arbitrarily close from any one point to any other. These are the so-called *control sets*, which are introduced in Section 6.

Kliemann [37] studied properties of control sets of locally accessible systems on smooth manifolds in continuous time. For the projected system obtained in the continuous time case an upper bound on the number of control sets with non-void interior has been obtained in [20]. An improved version of this result has been given by Bragas and San Martin [12], where smaller upper bounds than the dimension of the state space have been given depending on the group that is acting on projective space. In the discrete-time case control sets have been studied by Albertini and Sontag [3], [4], [2] who also introduced the concept of the core of a control set which is a strictly discrete-time concept. Introducing a further assumption we define regular cores which can be shown to enjoy the same properties one would expect for cores, in fact for the class of systems studied in [3] the definition of core and regular core coincide. We give an example of a system where the interior of a control set and its regular core do not coincide.

What is surprising is that neither in the continuous nor in the discrete-time case an effort has been undertaken to study control sets for *complex* systems, although it has been known for some time that even for real systems it is useful to study complex perturbations by the results of Hinrichsen and Pritchard [31].

A first observation for our system on projective space is that the generalized eigenspaces corresponding to universally regular controls project to the cores of appropriate control sets. Using this property we show in Section 7 that under weak assumptions there exist a unique invariant control set and a unique open control set on projective space. These are maximal, respectively minimal in the control order on the control sets. Here is the first time where the importance of the universally regular controls becomes clear, as their existence yields an easy proof for the existence of the maximal and the minimal control set. This is also the point where we have to depart from lines of proof available in the literature that are based on properties of Lie groups, if we do not want to restrict ourselves to the invertible case.

In the subsequent Section 8 further results on control sets with nonempty interior are presented. For these it is of importance, what the minimal possible rank drop on a path connecting two admissible invertible matrices is. Depending on this singularity index, we show that the eigenspaces of universally regular controls corresponding to an eigenvalue whose modulus has index greater than the singularity index, project to a control set uniquely determined by the index of the modulus. Control sets with this property are called main control sets. This leads to a sufficient condition in terms of the singularity index guaranteeing that there exist a most n control sets with nonempty interior, which are all main control sets. It is briefly explained in what sense control sets may be viewed as a generalization of generalized eigenspaces.

In Section 9 we begin our discussion of spectral theory by introducing the different exponents we want to study. Our definition of Floquet and Lyapunov spectra follows Colonius and Kliemann [20], [22] with the exception, that in these references the collection of the i-th Floquet exponents are not introduced.

In Section 10 the Floquet spectrum of the discrete time system is analyzed. We study Floquet spectra corresponding to control sets with non-empty core. To each such control set an associated set of Floquet exponents is defined. The idea of the proof that the closure of such a set is an interval follows the continuous-time case. The key is here a finite time controllability property in the cores of control sets. In Section 11 we study Lyapunov and Bohl spectra and their relation to the Floquet spectrum. Using an idea already developed in [18] we show under which conditions it is possible to approximate Lyapunov exponents by periodic controls. Furthermore, it is shown that without any further assumptions the closure of a Floquet spectrum of a control set actually consists of Lyapunov exponents corresponding to trajectories that remain in that control set. This is the statement of Theorem 11.1 (ii). It follows that the closure of the Floquet spectrum is contained in the Lyapunov spectrum. It has been shown by Berger and Wang [15] that the joint and the generalized spectral radius of a discrete inclusion given by a bounded set of matrices are equal. For our systems this implies the equality of the suprema of Bohl, Floquet and Lyapunov spectra. We show that the infima of Floquet and Lyapunov spectra coincide as well.

To indicate a further line of research let us point out that an extension to the theory of control sets is given by the so-called chain control sets, which have been introduced by Colonius and Kliemann [20], [22]. The idea is not to consider trajectories of the system but (ε, T)-chains to define chain-orbits and using these to define chain control sets. For discrete-time systems this has been studied by Albertini and Sontag in [4]. The extension of these concepts to the kind of systems we have studied will be an interesting direction for further research, as with chain control sets it is possible to describe the Morse spectrum of the discrete time system, which is an outer approximation of the set of Lyapunov exponents.

2 Problem statement

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $\tilde{U} \subset \mathbb{K}^m$ be open and connected. For an analytic map

$$A : \tilde{U} \to \mathbb{K}^{n\times n}\,, \tag{1}$$

we consider a family of time-varying linear system of the form

$$\begin{aligned} x(t+1) &= A(u(t))x(t)\,, \quad t \in \mathbb{N} & (2)\\ x(0) &= x_0 \quad \in \mathbb{K}^n\,, & (3) \end{aligned}$$

where $u : \mathbb{N} \to U \subset \tilde{U}$. The set-up we have chosen contains in particular systems affine in u and positive systems as subclasses. Also it naturally extends to periodic systems.

For $t \in \mathbb{N}$ U^t denotes the set of admissible finite control sequences $u = (u(0), \ldots, u(t-1))$, while $U^{\mathbb{N}}$ is the set of infinite control sequences $u = (u(0), u(1), \ldots)$. It will always be clear from the context whether u denotes an element of U, U^t or $U^{\mathbb{N}}$.

For two finite control sequences $u_1 \in U^{t_1}$, $u_2 \in U^{t_2}$ we define the concatenation (u_1, u_2) to be the sequence in $U^{t_1+t_2}$ given by $(u_1, u_2) = (u_1(0), \ldots, u_1(t_1 - 1), u_2(0), \ldots, u_2(t_2 - 1))$. The k-times repeated concatenation of $u \in U^t$ is denoted by $(u)^k \in U^{tk}$. For infinite control sequences $u \in U^{\mathbb{N}}$ we consider for $t \in \mathbb{N}$, $u_{[0,t-1]} := (u(0), \ldots, u(t-1)) \in U^t$ the "first part" of the control sequence u. The evolution operator generated by a control sequence $u \in U^{\mathbb{N}}$ is defined by

$$\Phi_u(s,s) = I\,, \quad \Phi_u(t+1,s) = A(u(t))\Phi_u(t,s)\,, \quad t \geq s \in \mathbb{N}\,. \tag{4}$$

With this notation $\Phi_u(t,0)x_0$ is the solution of (2) corresponding to the initial value x_0 and the control u at time t.

We denote by U_{inv} the set $\{u \in U;\ \det A(u) \neq 0\}$, which is clearly the complement of a set defined by analytic equations in U. Thus U_{inv} is either ω-generic in U or empty, where we call a set ω-generic if its complement is contained in a proper analytic subset of $\tilde{U}$. The term generic will be used for sets whose complements are contained in closed subanalytic sets of dimension strictly less than the manifold considered. For details on the theory of analytic and subanalytic sets we refer the reader to [42], [36] and [52]. In the sequel we will have to make use of the existence of invertible matrices $A(u)$, so that we have to assume that $U_{inv} \neq \emptyset$.

The following general assumption will be made throughout the remainder of this article. Note, however, that the first one is just for convenience and without loss of generality.

Assumption 2.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (2). We assume that the map A in (1) and the sets $U \subset \tilde{U} \subset \mathbb{K}^m$ are such that:*

(i) $0 \in U$.

(ii) The set U_{inv} is ω-generic in U.

(iii) $\operatorname{int} U$ *is connected.*

(iv) $U \subset \operatorname{cl}\operatorname{int} U \subset \tilde{U}$.

(v) U is bounded.

One tool for the study of Lyapunov exponents has been the projection onto the projective space, known as Bogolyubov's projection. It is based on the fact that in continuous time the angular component of the system may be decoupled from the radial and studied independently.

In our discrete–time system we do not exclude the possibility that the origin may be reached from non–zero states. If this is regarded from the point of view of stability or robust stability this poses no problem for once system (2) is at zero it remains there, as it is totally uncontrollable at zero. However, this means that system (2) as such may not be projected onto projective space. First the maximal subsystem that can be projected has to be identified.

To consider the discrete time analogue of Bogolyubov's projection, we define for $x \in \mathbb{K}^n$

$$U(x) := \{u \in U; A(u)x \neq 0\},$$

and with a slight abuse of notation the analogous sets for finite and infinite control sequences are denoted by $U^t(x)$ and $U^{\mathbb{N}}(x)$.

As $U_{inv} \subset U(x)$ and $U^t_{inv} := (U_{inv})^t \subset U^t(x)$ for all $x \in \mathbb{K}^n \backslash \{0\}$ it follows that for $x \neq 0$ the sets $U(x)$ and $U^t(x)$ are ω-generic in U resp. U^t. In the sequel $\mathbb{P}^{n-1}_{\mathbb{K}}$ denotes the $n-1$ dimensional projective space, and for $W \subset \mathbb{K}^n$, $\mathbb{P}W$ denotes the natural projection of $W \backslash \{0\}$ onto the projective space $\mathbb{P}^{n-1}_{\mathbb{K}}$.

For $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ we define the admissible control values for ξ by

$$U(\xi) := U(x) \quad \text{iff } \xi = \mathbb{P}x,$$

and in an analogous fashion $U^t(\xi), U^{\mathbb{N}}(\xi)$. This is well defined as $\operatorname{Ker} A(u)$ is a linear subspace. With this notation the projected system corresponding to our linear system (2) is given by

$$\begin{aligned} \xi(t+1) &= \mathbb{P}A(u(t))\xi(t), \quad t \in \mathbb{N} &(5)\\ \xi(0) &= \xi_0 \in \mathbb{P}^{n-1}_{\mathbb{K}} &(6)\\ u &\in U^{\mathbb{N}}(\xi_0). &(7) \end{aligned}$$

We denote the solution of (5) corresponding to an initial value ξ_0 and a control sequence $u \in U^{\mathbb{N}}(\xi_0)$ by $\xi(\cdot;\xi_0,u)$. For a subset $V \subset \mathbb{P}^{n-1}_{\mathbb{K}}$, $t \in \mathbb{N}$, $u \in U^t$ the notation $\xi(t;V,u) := \{\xi(t;\eta,u); \eta \in V \text{ such that } u \in U^t(\eta)\}$ will be used.

3 Accessibility, Transitivity and Regularity

Let us now study the projected system (5) from a control point of view. The variable "u" will be treated as if it were available for control of the system. A basic question in control theory is that of accessibility. We begin with the following basic definitions.

Definition 3.1 (Orbits) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Consider system (5). The forward orbit of ξ at time t is defined as*

$$\mathcal{O}^+_t(\xi) := \{\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}; \exists\, u \in U^t(\xi) \text{ with } \eta = \xi(t;\xi,u)\}.$$

The forward orbit of ξ *is then defined by* $\mathcal{O}^+(\xi) := \bigcup_{t\in\mathbb{N}} \mathcal{O}_t^+(\xi)$. *The backward orbit of* ξ *at time* t *is given by*

$$\mathcal{O}_t^-(\xi) := \{\eta \in \mathbb{P}_{\mathbb{K}}^{n-1}; \exists\, u \in U^t(\eta) \text{ with } \xi = \xi(t;\eta,u)\}\,.$$

which leads to a definition of $\mathcal{O}^-(\xi)$ *analogous to that of the positive forward orbit. Let*

$$\mathcal{O}_0(\xi) := \{\xi\} \quad \mathcal{O}_{t+1}(\xi) := \bigcup_{\eta\in\mathcal{O}_t(\xi)} \mathcal{O}^+(\eta) \cup \mathcal{O}^-(\eta)\,, \quad t\in\mathbb{N}\,.$$

The orbit of ξ *is then defined by*

$$\mathcal{O}(\xi) = \bigcup_{t\in\mathbb{N}} \mathcal{O}_t(\xi)\,. \tag{8}$$

Definition 3.2 (Accessibility) *The system (5) is called* forward accessible from ξ *if* $\operatorname{int}\mathcal{O}^+(\xi) \neq \emptyset$*,* backward accessible from ξ *if* $\operatorname{int}\mathcal{O}^-(\xi) \neq \emptyset$*,* forward *(respectively* backward*)* accessible *if it is forward (backward) accessible from all* $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$*.*
System (5) is called transitive*, if* $\operatorname{int}\mathcal{O}(\xi) \neq \emptyset$ *for all* $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$*.*

We note the following properties of the forward orbit.

Lemma 3.3 *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*. Consider system (5).*

(i) Let $\xi_1, \xi_2 \in \mathbb{P}_{\mathbb{K}}^{n-1}$*. If* $\xi_2 \in \operatorname{cl}\mathcal{O}^+(\xi_1)$ *then* $\operatorname{cl}\mathcal{O}^+(\xi_2) \subset \operatorname{cl}\mathcal{O}^+(\xi_1)$*.*

(ii) For all $t\in\mathbb{N}$*,* $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ *it holds that* $\operatorname{cl}\mathcal{O}_t^+(\xi)$ *is connected.*

Proof: (i) follows from a simple continuity argument. In order to prove (ii) we proceed by induction over $t\in\mathbb{N}$. Let $t = 1$ and $0 \neq x \in \mathbb{K}^n$. For an analytic path $\gamma : [0,1] \to \operatorname{int} U$ with $A(\gamma(\tau))x \not\equiv 0$, we will show that $\operatorname{cl}\mathbb{P}\{A(\gamma(\tau))x; \tau\in[0,1]\}$ is pathwise connected. Assume that $A(\gamma(\tau_0))x = 0$ (there are at most finitely many such τ). Let $k\in\mathbb{N}$ be the smallest integer such that $\frac{d^k}{d\tau^k}A(\gamma(\tau))x|_{\tau=\tau_0} \neq 0$ and without loss of generality assume that the first component of this vector is nonzero. In standard local coordinates around (1,0,...,0) we obtain a neighborhood of τ_0 where for $\tau\neq\tau_0$ it holds that

$$\mathbb{P}A(\gamma(\tau))x = \left(1, \frac{(A(\gamma(\tau))x)_2}{(A(\gamma(\tau))x)_1}, \ldots, \frac{(A(\gamma(\tau))x)_n}{(A(\gamma(\tau))x)_1}\right)\,. \tag{9}$$

Using the rule of de l'Hospital we obtain that $\lim_{\tau\to\tau_0}\mathbb{P}A(\gamma(\tau))x$ exists which shows our claim. As for every $u_1, u_2 \in \operatorname{int} U$ there exists a piecewise polynomial path connecting them and using Assumption 2.1 (iv), we see that $\operatorname{cl}\mathcal{O}_1^+(\xi)$ is connected.
Assume now that $\operatorname{cl}\mathcal{O}_t^+(\xi)$ is connected. Then for $u_0 \in U_{inv}$ it holds that $\mathbb{P}A(u_0)\operatorname{cl}\mathcal{O}_t^+(\xi)$ is connected as the continuous image of a connected set. Thus

$$\operatorname{cl}\mathcal{O}_{t+1}^+(\xi) = \operatorname{cl}\bigcup_{\eta\in\operatorname{cl}\mathcal{O}_t^+(\xi)} \operatorname{cl}\mathcal{O}_1^+(\eta)$$

is connected as each of the sets in the union is connected and each of the sets intersects the connected set $\mathbb{P}A(u_0)\operatorname{cl}\mathcal{O}_t^+(\xi)$. □

It has been shown that forward accessibility is intimately related to the rank of a certain mapping in the case of smooth invertible systems [4]. To carry this result over to our case let for every $t \in \mathbb{N}$

$$W_t := \left\{ (\xi, u) \in \mathbb{P}_{\mathbb{K}}^{n-1} \times \operatorname{int} U^t; \quad u \in U^t(\xi) \right\} , \tag{10}$$

and consider the map

$$F_t : W_t \to \mathbb{P}_{\mathbb{K}}^{n-1} , \quad F_t(\xi, u) := \xi(t; \xi, u) . \tag{11}$$

For fixed $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ and $u_0 \in \operatorname{int} U^t(\xi)$ we consider the rank of the linearization of $F_t(\xi, \cdot) : U^t(\xi) \to \mathbb{P}_{\mathbb{K}}^{n-1}$ at $u_0 \in U^t \subset \mathbb{K}^{mt}$ with respect to $u = (u(0)_1, \ldots, u(0)_m, u(1)_1, \ldots, u(t-1)_1, \ldots, u(t-1)_m)$. We define the following shorthand notation

$$\frac{\partial F_t}{\partial u}(\xi, u_0) = \left(\frac{\partial F_{t,i}}{\partial u(s)_j}(\xi, u_0) \right)_{i=1,\ldots,n-1; s=0,\ldots,t-1; j=1,\ldots,m} ,$$

where the $F_{t,i}$ are the i-th components of the map $F_t(\xi, \cdot)$ with respect to some coordinate chart around $F_t(\xi, u_0)$. The important detail for us is the rank of this Jacobian which is denoted by

$$r(t; \xi, u_0) := \operatorname{rk} \frac{\partial F_t}{\partial u}(\xi, u_0). \tag{12}$$

Definition 3.4 (Regularity) *A pair $(\xi, u) \in \mathbb{P}_{\mathbb{K}}^{n-1} \times \operatorname{int} U^t$ is called* regular*, if $u \in \operatorname{int} U^t(\xi)$ and $r(t; \xi, u) = n - 1$. A control $u \in \operatorname{int} U^t$ is called* universally regular*, if (ξ, u) is a regular pair for all $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$.*

The following lemma summarizes some easy properties in connection with regularity.

Lemma 3.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $u_0 \in \operatorname{int} U^t$, $v_0 \in \operatorname{int} U^s$. For $\xi_0 \in \mathbb{P}_{\mathbb{K}}^{n-1}$ let $F_{t+s}(\xi_0, (u_0, v_0))$ be defined, then*

(i) $r(t + s; \xi_0, (u_0, v_0)) \geq r(s; \xi(t; \xi_0, u_0), v_0)$.

(ii) If $v_0 \in \operatorname{int} U^s_{inv}$ then $r(t + s; \xi_0, (u_0, v_0)) \geq r(t; \xi_0, u_0)$.

Proof: Both assertions follow from an application of the chain rule. □

Proposition 3.6 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, consider system (5). For all $x \in \mathbb{K}^n \backslash \{0\}, t \in \mathbb{N}, u \in \operatorname{int} U^t$ the following statements are equivalent:*

(i) $(\mathbb{P}x, u)$ is a regular pair.

(ii) $\Phi_u(t, 0)x \neq 0$ and the following rank condition holds:

$$\operatorname{rk} G_t(x, u) := \operatorname{rk} \left[\Phi_u(t, 0)x \,\vdots\, \frac{\partial}{\partial u} \Phi_u(t, 0)x \right] = n . \tag{13}$$

Proof: It is clear that $\Phi_u(t, 0)x \neq 0$ is necessary for regularity. An application of the chain rule and a simple calculation in local coordinates yields the desired result. □

The preceding criterion will be frequently used, as it is easily handled in lower dimensions, where all our examples will be situated. Of course, if the dimension is high, or the structure of the map A is complicated, this criterion is much too involved to yield a feasible procedure for checking whether a system is forward accessible.

Definition 3.7 (Regular orbit) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (5). For $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ we define the* regular forward orbit *and* regular backward orbit *by*

$$\hat{\mathcal{O}}_t^+(\xi) := \{\eta;\ \exists u \in \operatorname{int} U^t(\xi) \ \text{s. t.}\ (\xi, u) \ \text{is regular and}\ \eta = \xi(t; \xi, u)\}, \tag{14}$$

$$\hat{\mathcal{O}}_t^-(\xi) := \{\eta;\ \exists u \in \operatorname{int} U^t(\eta) \ \text{s. t.}\ (\eta, u) \ \text{is regular and}\ \xi = \xi(t; \eta, u)\}. \tag{15}$$

The definitions of $\hat{\mathcal{O}}^+(x)$ and $\hat{\mathcal{O}}^-(x)$ are then analogous to Definition 3.1.

For subsets $V \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ we will use the notations $\hat{\mathcal{O}}^+(V) := \bigcup_{\xi \in V} \hat{\mathcal{O}}^+(\xi)$ etc. The following results exhibit some properties of the regular forward orbits. Items (iii) and (v) are shown in [4] for analytic invertible systems and similar arguments are applicable here.

Lemma 3.8 *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ consider system (5). Let $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$, then*

(i) $\hat{\mathcal{O}}_t^+(\xi)$ *is open in* $\mathbb{P}_{\mathbb{K}}^{n-1}$.

(ii) $\hat{\mathcal{O}}_t^-(\xi)$ *is open in* $\mathbb{P}_{\mathbb{K}}^{n-1}$.

(iii) $\operatorname{int} \mathcal{O}_t^+(\xi) \neq \emptyset$ *iff* $\hat{\mathcal{O}}_t^+(\xi) \neq \emptyset$.

(iv) If, for $t \in \mathbb{N}$, $\hat{\mathcal{O}}_t^+(\xi) \neq \emptyset$, then $\hat{\mathcal{O}}_s^+(\xi) \neq \emptyset$ for all $s \geq t$.

(v) $\operatorname{int} \mathcal{O}_t^+(\xi) \neq \emptyset \Rightarrow \operatorname{cl} \mathcal{O}_t^+(\xi) = \operatorname{cl} \hat{\mathcal{O}}_t^+(\xi)$.

In the case when ξ is a fixed point under a control u such that (ξ, u) is a regular pair, the following property is immediately obtained.

Proposition 3.9 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ there exist $u_\xi \in \operatorname{int} U^t, t \in \mathbb{N}$ such that (ξ, u_ξ) is a regular pair and*

$$\xi = \xi(t; \xi, u_\xi) \tag{16}$$

if and only if there exists an open neighborhood V of ξ such that $V \subset \hat{\mathcal{O}}_t^+(\xi) \cap \hat{\mathcal{O}}_t^-(\xi)$.

Proof: "$\Rightarrow$": This follows as $\xi \in \hat{\mathcal{O}}_t^+(\xi) \cap \hat{\mathcal{O}}_t^-(\xi)$ and the fact that both $\hat{\mathcal{O}}_t^+(\xi)$ and $\hat{\mathcal{O}}_t^-(\xi)$ are open by Lemma 3.8. "$\Leftarrow$": This is obvious as $\xi \in \hat{\mathcal{O}}_t^+(\xi)$. □

Let us now extend this property to connected sets.

Lemma 3.10 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. If $\Gamma \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ is a connected set such that for every $\xi \in \Gamma$ the assumption of Proposition 3.9 holds for some $t(\xi) \in \mathbb{N}$ then there exists a connected open set V such that*

$$\Gamma \subset V \subset \bigcap_{\xi \in \Gamma} \hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\xi). \tag{17}$$

Proof: Let $\xi \in \Gamma$ and consider the set $\hat{\mathcal{O}}^+(\xi) \cap \Gamma$, which is clearly open in Γ. Let $\eta \in \Gamma \cap \mathrm{cl}\,\hat{\mathcal{O}}^+(\xi)$. As $\eta \in \hat{\mathcal{O}}^-(\eta)$, which is open, it follows that $\hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\eta) \neq \emptyset$ and hence $\eta \in \hat{\mathcal{O}}^+(\xi)$. Thus $\hat{\mathcal{O}}^+(\xi) \cap \Gamma$ is open and closed in Γ and nonempty. As Γ is connected it follows that $\Gamma \subset \hat{\mathcal{O}}^+(\xi)$ and as $\xi \in \Gamma$ was arbitrary it holds for all $\xi_1, \xi_2 \in \Gamma$ that $\xi_1 \in \hat{\mathcal{O}}^+(\xi_2)$ and thus $\hat{\mathcal{O}}^+(\xi_1) \subset \hat{\mathcal{O}}^+(\xi_2)$ by Lemma 3.5 (i). By symmetry we obtain $\hat{\mathcal{O}}^+(\xi_1) = \hat{\mathcal{O}}^+(\xi_2)$. Furthermore it follows for every $\eta \in \Gamma$ that $\Gamma \subset \hat{\mathcal{O}}^-(\eta)$ and again for all $\xi_1, \xi_2 \in \Gamma$ it holds that $\hat{\mathcal{O}}^-(\xi_1) = \hat{\mathcal{O}}^-(\xi_2)$. As Γ is connected we can thus choose V to be the connected component of $\hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\xi)$ containing Γ for some $\xi \in \Gamma$. □

4 Asymptotic Properties on Projective Space

A first step in the study of the discrete-time system on projective space is the study of the ω-limit sets defined by constant matrices in Jordan block form, where we follow the argumentation from [20] and extend the arguments used there so that we may treat cases not considered in that reference. The following notation is used from now on.

Let $B \in \mathbb{K}^{n\times n}$. For an eigenvalue $\lambda \in \sigma(B) \cap \mathbb{K}$ $E(\lambda)$ denotes the eigenspace and $GE(\lambda)$ denotes the generalized eigenspace corresponding to λ. If $B \in \mathbb{R}^{n\times n}$ and $\lambda \in \sigma(B)$ is complex then $E(\lambda)$ denotes the real part of the sum of the eigenspaces corresponding to the eigenvalues $\lambda, \overline{\lambda}$. $GE(\lambda)$ denotes the appropriate generalized eigenspaces.

It will also be convenient to consider the set of absolute values of the eigenvalues defined by $|\sigma(B)| := \{|\lambda|\,;\, \lambda \in \sigma(B)\}$. For $1 \leq i \leq n$ let $r_i(B)$ be equal to the i-th entry of the ordered sequence $|\lambda_1| \leq \ldots \leq |\lambda_n|$, where each element of the spectrum of B appears according to its algebraic multiplicity. For $r \in |\sigma(B)|$ we denote

$$E(r) = \bigoplus_{\substack{\lambda\in\sigma(B)\\ |\lambda|=r}} E(\lambda)\,, \quad GE(r) = \bigoplus_{\substack{\lambda\in\sigma(B)\\ |\lambda|=r}} GE(\lambda)\,. \tag{18}$$

In the sequel we will be concerned with eigenspaces of $\Phi_u(t,0)$ generated by some finite control sequence $u \in U^t$. To make the dependence on u explicit we write $E(\lambda,u), E(r,u)$ etc. The projection of generalized eigenspaces is particularly important if regularity arguments can be applied.

Definition 4.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in U^t$, $r \in |\sigma(\Phi_u(t,0))|$. If $r > 0$ we call $\mathbb{P}GE(r,u)$* regular *if u can be partitioned as $u = (u_1, u_2)$ with $u_1 \in U^{t_1}$, $u_2 \in \mathrm{int}\, U^{t_2}$ and $t = t_1 + t_2$ and it holds that*

$$(\xi, u_2) \textit{ is a regular pair for every } \xi \in \mathbb{P}\Phi_{u_1}(t_1,0)GE(r,u)\,. \tag{19}$$

Definition 4.2 (Limit sets) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $u \in U^{\mathbb{N}}$, $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$. The positive ω-limit set is defined by*

$$\omega^+(\xi,u) := \left\{\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}; \exists\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, \lim_{k\to\infty} t_k = \infty \textit{ such that } \eta = \lim_{k\to\infty} \xi(t_k;\xi,u)\right\}\,. \tag{20}$$

The negative ω-limit set is defined by

$$\begin{aligned}\omega^-(\xi,u) \; := \; & \Big\{\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}; \exists\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}, \lim_{k\to\infty} t_k = \infty,\; \exists\{\eta_k\} \subset \mathbb{P}^{n-1}_{\mathbb{K}}, \\ & \xi = \xi(t_k;\eta_k,u) \textit{ such that } \eta = \lim_{k\to\infty} \eta_k\Big\}\,.\end{aligned} \tag{21}$$

For $t \in \mathbb{N}$, $u \in U^t$, $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$ $\omega^+(\xi, u)$ $(\omega^-(\xi, u))$ denotes the positive (resp. negative) ω-limit set that is obtained by applying the t-periodic continuation of u.

Note that with this definition we do not exclude the possibility that ω-limit sets may be empty, e.g. if $u \notin U^{\mathbb{N}}(\xi)$. For a discussion of the concept of ω-limit sets we refer the reader to [1], Chapter 1. In the following lemma we collect some simple properties of limit sets pertinent to our problem. The proof is left to the reader.

Lemma 4.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}, u \in U^t$, $\xi \in \mathbb{P}_{\mathbb{K}}^{n-1}$.*

(i) $\omega^+(\xi, u)$, $\omega^-(\xi, u)$ are closed.

(ii) $\Phi_u(t, 0)\omega^+(\xi, u) = \omega^+(\xi, u)$.

(iii) If $\xi = \mathbb{P}x = \mathbb{P}\sum_{j=1}^{l} x_j$ with $x_j \in GE(r_j, u)$ is the spectral decomposition of ξ and $r_1 < r_2 < \ldots < r_l$ then

$$\omega^+(\xi, u) \subset \mathbb{P}GE(r_l)\,. \tag{22}$$

If $r_1 = 0$ then $\omega^-(\xi, u) = \emptyset$, otherwise

$$\omega^-(\xi, u) \subset \mathbb{P}GE(r_1)\,. \tag{23}$$

(iv) If $r > 0$ then $\xi \in \mathbb{P}E(r, u) \Rightarrow \xi \in \omega^+(\xi, u) = \omega^-(\xi, u) \subset \mathbb{P}E(r, u)$.

The following lemma states the fundamental asymptotic property of the projected system.

Lemma 4.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.*

(i) Let $J_n(\lambda)$ denote a $n \times n$ Jordan block to an eigenvalue $\lambda \in \mathbb{K}\backslash\{0\}$. Then for any $x \in \mathbb{K}^n\backslash\{0\}$

$$\lim_{t \to \pm\infty} \mathbb{P}J_n(\lambda)^t x = \mathbb{P}[1, 0, \ldots, 0]'\,. \tag{24}$$

(ii) Let $\mathbb{K} = \mathbb{R}$ and let $J_n(\lambda, \bar{\lambda})$ denote a $2n \times 2n$ Jordan block to a complex pair of eigenvalues $\lambda, \bar{\lambda}$. Then for any Riemannian metric d on $\mathbb{P}_{\mathbb{R}}^{2n-1}$ and any $x \in \mathbb{R}^{2n}\backslash\{0\}$ it holds that

$$\lim_{t \to \pm\infty} d(\mathbb{P}J_n(\lambda, \bar{\lambda})^t x, \mathbb{P}\mathrm{span}\{[1, 0, \ldots, 0]', [0, 1, 0, \ldots, 0]'\}) = 0\,. \tag{25}$$

Proof: (i) For $\lambda \in \mathbb{K}\backslash\{0\}$, $t > n$ it holds that

$$J_n(\lambda)^t = \begin{bmatrix} \lambda^t & t\lambda^{t-1} & \ldots & \ldots & \binom{t}{t-(n-1)}\lambda^{t-(n-1)} \\ 0 & \lambda^t & t\lambda^{t-1} & \ldots & \binom{t}{t-(n-2)}\lambda^{t-(n-2)} \\ 0 & 0 & \lambda^t & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \ldots & 0 & \lambda^t \end{bmatrix}. \tag{26}$$

For $i > 1$ it follows immediately that

$$\lim_{t\to\infty} \left| \frac{(J_n(\lambda)^t e_j)_i}{(J_n(\lambda)^t e_j)_1} \right| = 0\,, \tag{27}$$

which proves the assertion in the limit $t \to +\infty$. The assertion for $t \to -\infty$ follows upon noting that $J_n(\lambda)^{-t}$ is similar to $J_n(\frac{1}{\lambda})^t$, where the vector e_1 is fixed under the similarity transformation.

(ii) The proof for the complex pair of eigenvalues follows the same pattern and is omitted.

□

Corollary 4.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in U^t$. If for $r \in |\sigma(\Phi_u(t,0))|$, $r > 0$ the generalized eigenspace $\mathbb{P}GE(r,u)$ is regular then there exists an open set V such that*

$$\mathbb{P}E(r,u) \subset V \subset \bigcap_{\xi \in \mathbb{P}E(r,u)} \hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\xi)\,. \tag{28}$$

Proof: Let $u = (u_1, u_2)$ be partitioned in accordance with Definition 4.1. If $\xi_0 \in \mathbb{P}E(r,u)$, then there exists a $\xi_2 \in \mathbb{P}\Phi_{u_1}(t_1,0)E(r,u)$ such that $\xi_0 = \xi(t_2;\xi_2,u_2)$ and (ξ_2,u_2) is regular. Furthermore it holds by Lemma 4.3 (iv) that $\xi_2 \in \mathbb{P}\Phi_{u_1}(t_1,0)\omega^+(\xi_0,u)$ and so $\mathcal{O}^+(\xi_0) \cap \hat{\mathcal{O}}^-(\xi_0) \neq \emptyset$ as the regular backward orbit is open by Lemma 3.8 (ii). Using Lemma 3.8 (v) it follows that $\xi_0 \in \hat{\mathcal{O}}^+(\xi_0)$.

As $\mathbb{P}E(r,u)$ is connected the assertion follows from Lemma 3.10. □

Corollary 4.6 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in U^t$. If for $r \in |\sigma(\Phi_u(t,0))|$, $r > 0$ the generalized eigenspace $\mathbb{P}GE(r,u)$ is regular then there exists an open set W such that*

$$\mathbb{P}GE(r,u) \subset W \subset \bigcap_{\xi \in \mathbb{P}GE(r,u)} \hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\xi)\,. \tag{29}$$

Proof: Let $u = (u_1, u_2)$ be partitioned in accordance with Definition 4.1 and $\xi \in \mathbb{P}GE(r,u)$. By Lemma 4.4 and Corollary 4.5 there exists $\eta \in \mathcal{O}^+(\xi) \cap \hat{\mathcal{O}}^-(\mathbb{P}E(r,u))$ and it follows that $\mathbb{P}E(r,u) \subset \hat{\mathcal{O}}^+(\xi)$. On the other hand $\omega^-(\xi,u) \subset \mathbb{P}E(r,u)$ and so by Corollary 4.5 there exits an $\eta \in \hat{\mathcal{O}}^+(\mathbb{P}E(r,u))$ and a $k \in \mathbb{N}$ such that $\xi = \xi(kt;\eta,(u)^k)$. By regularity of the pair $(\xi((k-1)t + t_1;\eta,((u)^{k-1},u_1)),u_2)$ and using the fact that $\hat{\mathcal{O}}^-_{t_2}(\xi)$ is open we see that $\eta \in \hat{\mathcal{O}}^-_{kt}(\xi)$. Hence $\mathbb{P}E(r,u) \subset \hat{\mathcal{O}}^-(\xi)$. It follows that $\xi \in \hat{\mathcal{O}}^+(\xi)$ and an application of Lemma 3.10 completes the proof. □

Now that we have seen that for generalized eigenspaces in projective space certain controllability properties hold if a regularity condition is satisfied, it is reasonable to ask, if we can for certain controls guarantee that this condition holds. This is discussed in the next section.

5 Universally Regular Controls

A crucial point in the development of the theory is the construction of universally regular controls and the proof of their genericity in U^t for t large enough. The following result is largely a restatement of results shown in [54] and [50].

Proposition 5.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For the projected system (5) the following statements are equivalent.*

(i) System (5) is forward accessible.

(ii) There exist $t \in \mathbb{N}$, $u^* \in \operatorname{int} U^t$ such that u^* is universally regular.

(iii) There exists a $t^* \in \mathbb{N}$ such that for all $t > t^*$ the set of universally regular control sequences is generic in $\operatorname{int} U^t$.

(iv) There exists a $t \in \mathbb{N}$, $u \in \operatorname{int} U^t$ such that for every $r \in |\sigma(\Phi_u(t,0))|$ the generalized eigenspace $\mathbb{P}GE(r,u)$ is regular.

Proof: The equivalence of (i),(ii) and (iii) follows from Corollaries 3.2 and 3.3 in [50]. For this note in particular that by Proposition 3.6 the set of non-regular pairs in $\mathbb{P}^{n-1}_{\mathbb{K}} \times U^t$ is analytic. To complete the proof note that "(ii) $\Rightarrow$ (iv)" is obvious. For the converse direction let u be such that (iv) is satisfied. For any $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ Lemma 4.4 implies that $\omega^+(\xi,u) \subset \mathbb{P}E(r,u)$ for some $r \in |\sigma(\Phi_u(t,0))|$. Corollary 4.6 implies that $\hat{\mathcal{O}}^+(\xi) \neq \emptyset$, so that (i) holds. □

The set of universally regular $u \in U^t$ will be denoted by U^t_{reg}, while t^* denotes the smallest $t \in \mathbb{N}$ such that $U^t_{reg} \neq \emptyset$. It follows from the results in [50] that if $\operatorname{int} \mathcal{O}^+_t(\xi) \neq \emptyset$ for all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ then $t^* \leq tn$. Note that U^t_{reg} is open for all $t \in \mathbb{N}$.

Remark 5.2 Let us point out that we use the term *generic* for sets that are the complement of closed subanalytic sets of lower dimension in the real case or proper analytic subsets in the complex case. The reason that we work with analytically defined sets lies in the analytic dependence of A on u. In particular we use in the proof of Proposition 8.1 that if the complement of a set Z is generic then from every $x \in Z$ there exists a path that starts in $x \in Z$ and leaves Z immediately. This is due to the fact that subanalytic sets can be represented as a locally finite union of embedded analytic submanifolds, see [52]. □

Proposition 5.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible, then for linear subspaces $X, Y \subset \mathbb{K}^n$ such that*

$$\dim X + \dim Y \leq n \tag{30}$$

the set $\{u \in U^t_{reg}; \Phi_u(t,0)X \cap Y = \{0\}\}$ is generic in $\operatorname{int} U^t$ for all $t \geq t^*$.

Proof: For $X = \{0\}$ there is nothing to show, so assume $\dim X \geq 1$. Note that the set $\{(\xi,u) \in \mathbb{P}X \times \operatorname{int} U^t; u \notin U^t(\xi) \text{ or } [u \in U^t(\xi) \text{ and } \xi(t;\xi,u) \in \mathbb{P}Y]\}$ is analytic in $\mathbb{P}X \times \operatorname{int} U^t$. From Remmert's proper mapping theorem ([36] Theorem 45.17), respectively the definition of subanalytic sets ([52] Section 8) it follows that the projection of this set given by

$$\{u \in \operatorname{int} U^t; \exists \xi \in \mathbb{P}X \text{ such that } [u \notin U^t(\xi) \text{ or } \xi(t;\xi,u) \in \mathbb{P}Y]\} \tag{31}$$

is analytic in $\operatorname{int} U^t$ for $\mathbb{K} = \mathbb{C}$ or subanalytic in $\operatorname{int} U^t$ for $\mathbb{K} = \mathbb{R}$. As the set is clearly closed and the intersection of two generic sets is generic the assertion is thus proved in the real and the complex case, if the following statement is shown:

$$\begin{gathered}\text{If } t \geq t^* \text{ and } u \in \operatorname{int} U^t \text{ then in any open neighborhood of } u \\ \text{there exists a } v \in U^t_{reg} \text{ such that } \Phi_v(t,0)X \cap Y = \{0\}\,. \end{gathered} \tag{32}$$

We prove (32) by induction over $\dim X$. Let $\dim X = 1$. Due to $u \in \operatorname{cl} U^t_{reg}$ it holds that $\xi(t;\xi,u) \in \operatorname{cl} \hat{\mathcal{O}}^+_t(\xi)$ for $\xi = \mathbb{P}X$, so (32) follows immediately. Assume that (32) is shown for $\dim X = k < n-1$ and let $X = \operatorname{span}\{x_1,\ldots,x_{k+1}\}$ for a linearly independent set of vectors $x_i \in \mathbb{K}^n$, $i = 1,\ldots,k+1$. Without loss of generality let $Y \subset \operatorname{span}\{e_{k+2},\ldots,e_n\}$. Denote $X' = \operatorname{span}\{x_1,\ldots,x_k\}$. Fix $u \in \operatorname{int} U^t$ and an open neighborhood $V \subset \operatorname{int} U^t$ of u. Thus there exists $v \in V \cap U^t_{reg}$ such that

$$\Phi_v(t,0)X' \cap \operatorname{span}\{e_{k+2},\ldots,e_n\} = \{0\}\,. \tag{33}$$

Due to forward accessibility v may be chosen such that

$$\Phi_v(t,0)x_{k+1} \notin \operatorname{span}\{e_{k+2},\ldots,e_n\}. \tag{34}$$

Let $W \subset V \cap U^t_{reg}$ be a neighborhood of v such that (33) and (34) are satisfied for all $v' \in W$. Let $P \in \mathbb{K}^{k+1\times n}$ be defined by

$$P = \begin{bmatrix} 1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \\ 0 & & 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{35}$$

then

$$\operatorname{rk} P\Phi_v(t,0)[x_1 \vdots \cdots \vdots x_{k+1}] \geq k\,. \tag{36}$$

If the rank is equal to $k+1$, then indeed

$$\Phi_v(t,0)X \cap \operatorname{span}\{e_{k+2},\ldots,e_n\} = \{0\}\,. \tag{37}$$

Let $u' \in \mathbb{K}^{mt}$ and consider the mappings

$$h_i : (-\varepsilon,\varepsilon) \to \mathbb{K}^{k+1} \tag{38}$$

$$h_i(\tau) = P\Phi_{v+\tau u'}(t,0)x_i \tag{39}$$

for $i = 1,\ldots,k+1$, where ε is small enough such that $v + \tau u' \in W$ for $|\tau| < \varepsilon$. We claim that there exist $u' \in \mathbb{K}^{mt}$ such that (37) holds for $\Phi_{v+\tau u'}(t,0)$ for some $|\tau| < \varepsilon$. Assume this is not the case, then $h_{k+1}(\tau) \in \operatorname{span}\{h_i(\tau)\}_{i=1,\ldots,k}$ for all $|\tau| < \varepsilon$. Hence there exist continuously differentiable functions

$$\mu_i : (-\varepsilon,\varepsilon) \to \mathbb{K} \quad, i = 1,\ldots,k \tag{40}$$

such that

$$h_{k+1}(\tau) = \sum_{i=1}^{k} \mu_i(\tau)h_i(\tau), \tag{41}$$

where the differentiability follows from the differentiability of the h_i and the fact that the $h_i(\tau)$, $i = 1, \ldots, k$ are linearly independent. Hence if we differentiate with respect to τ at $\tau = 0$

$$h'_{k+1}(0) = \sum_{i=1}^{k} \mu'_i(0) h_i(0) + \mu_i(0) h'_i(0) \tag{42}$$

or equivalently using the chain rule

$$P \frac{\partial \Phi_v(t,0) x_{k+1}}{\partial u} \cdot u' = \sum_{i=1}^{k} \mu'_i(0) h_i(0) + \mu_i(0) P \frac{\partial \Phi_v(t,0) x_i}{\partial u} \cdot u' \,. \tag{43}$$

Let

$$B_i := P \frac{\partial \Phi_v(t,0) x_i}{\partial u} \in \mathbb{K}^{k+1 \times mt} \tag{44}$$

be our shorthand notation, then we obtain that if u' is such that $B_i u' \in \operatorname{span}\{h_1(0), \ldots, h_k(0)\}$ for $i = 1, \ldots, k$ and $B_{k+1} u' \notin \operatorname{span}\{h_1(0), \ldots, h_k(0)\}$ then (43) cannot be solved and there exist τ arbitrarily small such that

$$\operatorname{rk} P \Phi_{u+\tau u'}(t,0)[x_1 \vdots \cdots \vdots x_{k+1}] = \operatorname{rk} \Phi_{u+\tau u'}(t,0)[x_1 \vdots \cdots \vdots x_{k+1}] = k+1 \tag{45}$$

and hence (32) holds.

Assume there is no u' satisfying these properties, i.e. for all $u \in \mathbb{K}^{mt}$ it holds that

$$B_i u \in \operatorname{span}\{h_1(0), \ldots, h_k(0)\}, i = 1, \ldots, k \Rightarrow B_{k+1} u \in \operatorname{span}\{h_1(0), \ldots, h_k(0)\} \,.$$

By (33) we obtain $\dim\{u \in \mathbb{K}^{mt};\, B_i u \in \operatorname{span}\{h_1(0), \ldots, h_k(0)\}\} \geq mt - 1$ and thus

$$\begin{aligned} & \dim\{u \in \mathbb{K}^{mt};\, B_i u \in \operatorname{span}\{h_1(0), \ldots, h_k(0)\} \text{ for } i = 1, \ldots, k\} && (46) \\ = \;& \dim\{u \in \mathbb{K}^{mt};\, B_i u \in \operatorname{span}\{h_1(0), \ldots, h_k(0)\} \text{ for } i = 1, \ldots, k+1\} && (47) \\ \geq \;& mt - k \,. && (48) \end{aligned}$$

Hence in suitable coordinates the matrices B_i, $i = 1, \ldots, k+1$ are of the form

$$\begin{array}{c c} & \begin{array}{cc} mt-k & k \end{array} \\ \begin{array}{c} k \\ \\ \\ 1 \end{array} & \left[\begin{array}{ccc ccc} * & \dots & * & * & \dots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ * & \dots & * & * & \dots & * \\ 0 & \dots & 0 & * & \dots & * \end{array} \right] \end{array} \tag{49}$$

and there are scalars $\nu_i \in \mathbb{K}$, $i = 1, \ldots, k+1$ not all zero such that

$$\operatorname{Im} \sum_{i=1}^{k+1} \nu_i B_i \subseteq \operatorname{span}\{h_1(0), \ldots, h_k(0)\} \,. \tag{50}$$

Now for the vector $\overline{x} = \sum_{i=1}^{k+1} \nu_i x_i \neq 0$ and with G_t as defined in (13) it follows that

$$\operatorname{rk} P G_t(\overline{x}, v) = \operatorname{rk} \left[\sum_{i=1}^{k+1} \nu_i h_i(0) \vdots \sum_{i=1}^{k+1} \nu_i B_i \right] \leq k \tag{51}$$

which contradicts the universal regularity of v by Proposition 3.6. □

In [34], [2], [4] accessibility and transitivity properties of analytic, invertible systems have been studied. In particular Lie algebraic characterizations of these properties were obtained. Also it was obtained that on compact manifolds transitivity, forward and backward accessibility are all equivalent. Since forward accessibility of system (5) implies the existence of a universally regular control, we can state the following

Proposition 5.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume system (5) is forward accessible, then it is backward accessible and transitive. Furthermore, it holds for every $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, that $\hat{\mathcal{O}}^{-}(\xi) \neq \emptyset$.*

Proof: This is clear from the existence of a universally regular control. $\square$

This proposition shows in particular that the criteria for accessibility developed in [34],[4] can be brought to use in our case, even though the full requirements of the theorems stated in these references are not met. This is due to the fact that system (5) is forward accessible iff there is an analytic invertible subsystem that is forward accessible. Where we call a system a subsystem if the map A is the same but the set of control values is restricted. Thus we could choose an open subset U' of U_{inv} that is relatively compact in U_{inv}. For the system with control values in U' it is clear that its forward accessibility implies forward accessibility of the original system. But also the converse is true as forward accessibility implies the generic existence of universally regular controls. Which implies that there exists a universally regular control in $U'^{t^*}_{reg}$, where t^* is the constant of the original system.
The converse of the statement in Proposition 5.4 does not hold as shown by the following example.

Example 5.5 Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $U = \mathbb{K}$. Define

$$A(u) := \begin{bmatrix} 1+2u & 0 \\ 0 & 1+u \end{bmatrix}.$$

Then the system

$$\xi(t+1) = \mathbb{P}A(u(t))\xi(t), \quad t \in \mathbb{N}$$

is clearly not forward accessible, as $\mathcal{O}^{+}(\mathbb{P}[1,0]') = \{\mathbb{P}[1,0]'\}$, $\mathcal{O}^{+}(\mathbb{P}[0,1]') = \{\mathbb{P}[0,1]'\}$. However, an open set can be steered to $\mathbb{P}[1,0]'$ by applying the constant control given by $\lambda = -1$ (respectively $\mathbb{P}[0,1]'$ and $\lambda = -1/2$). It is then easy to see that $\operatorname{int} \mathcal{O}^{-}_{1}(\xi) \neq \emptyset$ for all $\xi \in \mathbb{P}^{1}_{\mathbb{K}}$. So that the system is backward accessible. $\square$

6 Control Sets

Let us now give a precise meaning to the words "sets where it is possible to steer arbitrarily close from one point to another". Control sets are defined as maximal sets where a controllability property holds. Precontrol sets satisfy the same controllability properties without being maximal. We note that different control sets are disjoint, and that to every precontrol set there exists a unique control set containing it. Furthermore to every point in a control set there exists a control sequence such that the corresponding trajectory stays in that control set for all times, and the closures of the forward orbits of two points contained in the same control set coincide.

Definition 6.1 (Control set) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Consider system (5). A set $\emptyset \neq D \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ is called a* precontrol set*, if*

(i) $D \subset \operatorname{cl} \mathcal{O}^+(\xi)$, $\forall \xi \in D$.

(ii) For every $\xi \in D$ there exists a $u \in U^{\mathbb{N}}(\xi)$ and an increasing sequence $(t_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $\xi(t_k; \xi, u) \in D$ for all $k \in \mathbb{N}$.

A precontrol set D is called control set*, if furthermore*

(iii) D is a maximal set with respect to inclusion satisfying (i).

A control set C is called invariant control set*, if*

$$\operatorname{cl} C = \operatorname{cl} \mathcal{O}^+(\xi), \ \forall \xi \in C\,. \tag{52}$$

With this definition we obtain the following basic results.

Proposition 6.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (5).*

(i) For two control sets D_1, $D_2 \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ it holds that either $D_1 = D_2$ or $D_1 \cap D_2 = \emptyset$.

(ii) To every precontrol set $D' \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ exists a unique control set D such that $D' \subset D$.

(iii) If $\xi_1, \xi_2 \in D$ for some control set $D \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ and for some $u \in U^t$ it holds that

$$\xi_2 = \xi(t; \xi_1, u) \tag{53}$$

then

$$\xi(s; \xi_1, u) \in D \text{ for } s = 0, \dots, t\,. \tag{54}$$

(iv) For a control set D it holds that

$$\operatorname{cl} \mathcal{O}^+(\xi_1) = \operatorname{cl} \mathcal{O}^+(\xi_2)\,, \quad \forall \xi_1, \xi_2 \in D\,. \tag{55}$$

(v) Let D be a control set. For every $\xi \in D$ there exists a control $u \in U^{\mathbb{N}}(\xi)$ such that

$$\xi(t; \xi, u) \in D\,, \quad \forall t \in \mathbb{N}\,. \tag{56}$$

(vi) Let D be a control set. For every $\xi \in D$ and every $T \in \mathbb{N}$ it holds that

$$\operatorname{cl} \mathcal{O}^+(\xi) = \operatorname{cl} \bigcup_{t=T}^{\infty} \mathcal{O}_t^+(\xi)\,. \tag{57}$$

In the forward accessible case invariant control sets enjoy further useful properties.

Proposition 6.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible. A control set C is invariant iff it is closed and satisfies* $\operatorname{int} C \neq \emptyset$.

Proof: "$\Rightarrow$": If $C = \mathbb{P}_{\mathbb{K}}^{n-1}$ there is nothing to show. Assume that $\xi \in \operatorname{cl} C \setminus C$. This implies that $\xi \in \operatorname{cl} \mathcal{O}^+(\eta)$ for all $\eta \in C$. As $\xi \notin C$ it follows that $\mathcal{O}^+(\xi) \cap C = \emptyset$ for otherwise $C \subset \operatorname{cl} \mathcal{O}^+(\xi)$ and this would imply $\xi \in C$. By assumption there exist $t \in \mathbb{N}$, $u \in U^t(\xi)$ such that $\xi(t; \xi, u) \in \operatorname{int} \mathcal{O}^+(\xi)$. By continuity there exists a neighborhood V of ξ that is steered to $\operatorname{int} \mathcal{O}^+(\xi)$ and therefore there exists $\eta \in C$ such that $\mathcal{O}^+(\eta) \cap \operatorname{int} \mathcal{O}^+(\xi) \neq \emptyset$. But $\mathcal{O}^+(\eta) \subset \operatorname{cl} C$, a contradiction. Hence C is closed, and $C = \operatorname{cl} \mathcal{O}^+(\xi)$ for $\xi \in C$. As $\operatorname{int} \mathcal{O}^+(\xi) \neq \emptyset$ it follows that $\operatorname{int} C \neq \emptyset$.
"$\Leftarrow$": Let C be a closed control set with $\operatorname{int} C \neq \emptyset$. If $C = \mathbb{P}_{\mathbb{K}}^{n-1}$ there is nothing to show. Otherwise we have to show for every $\xi \in C$ that $\operatorname{cl} \mathcal{O}^+(\xi) \subset C$, or equivalently as C is closed $\mathcal{O}^+(\xi) \subset C$. For every $\eta \in C$ there exists $t \in \mathbb{N}$, $u \in U^t(\eta)$ such that $\xi(t; \eta, u) \in \operatorname{int} C$. By continuous dependence on the initial values there exists an open neighborhood $V(\eta)$ of η such that $\xi(t; V(\eta), u) \subset \operatorname{int} C$. Hence there exists an open set $V \supset C$ such that $\mathcal{O}^+(\xi) \cap \operatorname{int} C \neq \emptyset$ and therefore $C \subset \operatorname{cl} \mathcal{O}^+(\xi)$ for every $\xi \in V$.
Assume now that there exists a $\xi \in C$ and a $u \in U(\xi)$ such that $\xi(1; \xi, u) \notin C$. As $C \subset \operatorname{cl} \mathcal{O}^+(\xi)$ there exists an $\eta \in \mathcal{O}^+(\xi) \cap C$ and Proposition 6.2 (iii) guarantees that there exists a $v \in U(\xi)$ such that $\xi(1; \xi, v) \in C$. Now $\operatorname{cl} \mathcal{O}_1^+(\xi) \cap C \neq \emptyset$ but also $\operatorname{cl} \mathcal{O}_1^+(\xi) \not\subset C$. Since $\operatorname{cl} \mathcal{O}_1^+(\xi)$ is connected, it follows that there exists a $\zeta \in \mathcal{O}_1^+(\xi) \cap (V \setminus C)$. But then $\zeta \in \operatorname{cl} \mathcal{O}^+(\eta)$ for all $\eta \in C$ and $C \subset \operatorname{cl} \mathcal{O}^+(\zeta)$ and thus $\zeta \in C$, which is a contradiction. □

Cores of control sets, a strictly discrete time concept, have been introduced in [4]. We give a definition of core that slightly differs from the original definition in that we require a regularity condition to hold. So to contrast it it might be called *regular core* of a control set. It should, however, be noted that for the systems studied in [4] core and regular core of a control set coincide.

Definition 6.4 (Regular core) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $D \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ be a control set with $\operatorname{int} D \neq \emptyset$. The* (regular) core *of D is defined as*

$$\operatorname{core}(D) := \{\xi \in D; \hat{\mathcal{O}}^+(\xi) \cap D \neq \emptyset \text{ and } \hat{\mathcal{O}}^-(\xi) \cap D \neq \emptyset\}. \tag{58}$$

Proposition 6.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (5). It holds that $\xi \in \hat{\mathcal{O}}^+(\xi)$, iff there exists a control set D such that $\xi \in \operatorname{core}(D)$.*

Proof: "$\Rightarrow$": This follows from Proposition 3.9.

"$\Leftarrow$": Let $\eta \in \hat{\mathcal{O}}^-(\xi) \cap D$. By the implicit function theorem there exists a neighborhood V of η with $V \subset \hat{\mathcal{O}}^-(\xi)$. As $\eta \in D$ it follows that $V \cap \mathcal{O}^+(\xi) \neq \emptyset$. Therefore $\hat{\mathcal{O}}^-(\xi) \cap \mathcal{O}^+(\xi) \neq \emptyset$ and so $\xi \in \hat{\mathcal{O}}^+(\xi)$. □

Proposition 6.6 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (5). Let $D \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ be a control set with $\operatorname{int} D \neq \emptyset$. If system (5) is forward accessible from every $\xi \in D$, then*

(i) $\operatorname{core}(D)$ *is open in* $\mathbb{P}_{\mathbb{K}}^{n-1}$.

(ii) $\operatorname{cl} \operatorname{core}(D) = \operatorname{cl} \operatorname{int}(D) = \operatorname{cl} D$.

(iii) If $\xi \in \operatorname{core}(D)$ then $\operatorname{core}(D) \subset \hat{\mathcal{O}}^+(\xi)$ and $D \subset \hat{\mathcal{O}}^-(\xi)$.

(iv) If $\xi \in \operatorname{core}(D)$, $t \in \mathbb{N}$, $u \in \operatorname{int} U_{inv}^t$ and $\xi(t; \xi, u) \in D$ then $\xi(s; \xi, u) \in \operatorname{core}(D)$ for $s = 0, \ldots, t$.

Proof: (i) If $\xi \in \mathrm{core}(D)$, then by Proposition 6.5 $\xi \in \hat{\mathcal{O}}^+(\xi)$. Thus the assertion follows from Proposition 3.9, as there exists an open neighborhood V of ξ satisfying $V \subset \hat{\mathcal{O}}^+(\xi) \cap \hat{\mathcal{O}}^-(\xi)$. V is a precontrol set satisfying the rank condition in (58), and thus contained in $\mathrm{core}(D)$.
(ii) Clearly $\mathrm{cl}\,\mathrm{core}(D) \subset \mathrm{cl}\,\mathrm{int}\,D \subset \mathrm{cl}\,D$. Let $\xi \in \mathrm{cl}\,D$ and V be any open neighborhood of ξ. Let $\eta \in \mathrm{int}\,D$. By Lemma 3.8 (v) and Proposition 6.2 (vi) we have $D \subset \mathrm{cl}\,\hat{\mathcal{O}}^+(\eta)$. Thus we may choose $\zeta \in D \cap V \cap \hat{\mathcal{O}}^+(\eta)$ and it follows that $\hat{\mathcal{O}}^-(\zeta) \cap \mathrm{int}\,D \neq \emptyset$. As $\zeta \in D$ we have as before that $\mathrm{int}\,D \subset D \subset \hat{\mathcal{O}}^+(\zeta)$ and so also $\hat{\mathcal{O}}^+(\zeta) \cap \mathrm{int}\,D \neq \emptyset$. Thus $\zeta \in \mathrm{core}(D) \cap V$.
(iii) If $\xi \in \mathrm{core}(D)$ then $\xi \in \mathrm{cl}\,\mathcal{O}^+(\eta)$ for every $\eta \in D$. By Proposition 3.9, $\xi \in \hat{\mathcal{O}}^-(\xi)$ and so $\mathcal{O}^+(\eta) \cap \hat{\mathcal{O}}^-(\xi) \neq \emptyset$ and hence $\eta \in \hat{\mathcal{O}}^-(\xi)$. This shows that $D \subset \hat{\mathcal{O}}^-(\xi)$. As $\xi \in \mathrm{core}(D)$ was arbitrary this implies also that $\mathrm{core}(D) \subset \hat{\mathcal{O}}^+(\xi)$ for every $\xi \in \mathrm{core}(D)$.
(iv) This is clear as $D \subset \hat{\mathcal{O}}^-(\xi) \subset \hat{\mathcal{O}}^-(\xi(s;\xi,u))$ for $s = 0,\ldots,t$ by Lemma 3.5, and $\mathrm{core}(D) \subset \hat{\mathcal{O}}^+(\xi(t;\xi,u)) \subset \hat{\mathcal{O}}^+(\xi(s;\xi,u))$. □

From now on control sets of the system on projective space are studied, using the underlying linear structure which allows more precise statements. We begin by considering projected generalized eigenspaces that satisfy a regularity condition.

Proposition 6.7 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$, $u \in \mathrm{int}\,U^t$. Assume that for $r \in |\sigma(\Phi_u(t,0))|$, $r > 0$ the generalized eigenspace $\mathbb{P}GE(r,u)$ is regular, then there exists a control set D such that*

$$\mathbb{P}GE(r,u) \subset \mathrm{core}(D)\,. \tag{59}$$

Proof: This follows from Corollary 4.6 (i), and the fact that to every precontrol set there is a control set containing it. □

Proposition 6.8 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $t \in \mathbb{N}$. Assume that $\gamma : [0,1] \to U^t$ is a continuous path with an associated continuous path $\gamma_2 : [0,1] \to \mathbb{R}$ such that for every $\tau \in [0,1]$*

$$0 \neq \gamma_2(\tau) \in \left|\sigma(\Phi_{\gamma(\tau)}(t,0))\right| \tag{60}$$

and $\mathbb{P}GE(\gamma_2(\tau),\gamma(\tau))$ is regular, then there exists a connected open precontrol set D contained in the core of a control set with

$$\bigcup_{\tau\in[0,1]} \mathbb{P}GE(\gamma_2(\tau),\gamma(\tau)) \subset D\,. \tag{61}$$

Proof: By Proposition 6.7 for every $\tau \in [0,1]$ there exists an open precontrol set $V(\tau) \supset \mathbb{P}GE(\gamma_2(\tau),\gamma(\tau))$ which we may assume without loss of generality to be connected and contained in the core of a control set. By the continuity properties of the eigenprojections, see [13] Chapter II.8 for every $\tau \in [0,1]$ there exists an $\varepsilon(\tau) > 0$ such that $\mathbb{P}GE(\gamma_2(\tau'),\gamma(\tau')) \subset V(\tau)$ if $|\tau - \tau'| < \varepsilon(\tau)$. This shows that

$$D := \bigcup_{\tau\in[0,1]} V(\tau) \tag{62}$$

is connected, and by Lemma 3.10 a precontrol set with the desired properties. □

For the system (5) the core of a control set corresponds to regular pairs (ξ, u), where ξ is an eigenvector of $\Phi_u(t,0)$ by Proposition 6.5. For a forward accessible system even more is true. For any control set D with nonempty core we may find universally regular controls u that generate an eigenspace whose projection lies in any prescribed open subset of $\operatorname{core}(D)$.

Proposition 6.9 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (5) is forward accessible. For every control set $D \subset \mathbb{P}_{\mathbb{K}}^{n-1}$ with $\operatorname{core}(D) \neq \emptyset$ and every open set $\emptyset \neq V \subset \operatorname{core}(D)$ there exist $t \in \mathbb{N}$, $u \in U_{reg}^t$ such that for some $r \in |\sigma(\Phi_u(t,0))|$*

$$\mathbb{P}E(r,u) \cap V \neq \emptyset\,, \text{ and } \mathbb{P}GE(r,u) \subset \operatorname{core}(D)\,. \tag{63}$$

Proof: Let $\xi \in V$. By Proposition 6.5 $\xi \in \hat{\mathcal{O}}^+(\xi)$, and we can choose $t \in \mathbb{N}$, $u \in \operatorname{int} U^t$ such that $r(t;\xi,u) = n-1$ and

$$\xi = \xi(t;\xi,u)\,. \tag{64}$$

Without loss of generality let $t > t^*$. As the set of universally regular controls is generic in $\operatorname{int} U^t$ and by Proposition 6.5 we can choose $u_1 \in U_{reg}^t$ such that $\eta_1 := \xi(t;\xi,u_1) \in \hat{\mathcal{O}}_t^+(\xi) \cap \hat{\mathcal{O}}_t^-(\xi) \cap V$. Using the universal regularity of u_1 and applying the implicit function theorem it may be concluded that there exists an open neighborhood $V(\xi) \subset V$ such that for every $\eta \in V(\xi)$ there exists a universally regular $u(\eta) \in U_{reg}^t$ with $\eta_1 = \xi(t;\eta,u(\eta))$. Furthermore as $\eta_1 \in \hat{\mathcal{O}}_t^-(\xi)$ we may choose $u_2 \in \operatorname{int} U_{inv}^t$ such that $\eta_2 := \xi(t;\eta_1,u_2) \in V(\xi)$. Hence

$$\eta_1 = \xi\left(2t;\eta_1,(u_2,u(\eta_2))\right) \tag{65}$$

and as $u_2 \in \operatorname{int} U_{inv}^t$ and $u(\eta_2) \in U_{reg}^t$ it follows by Lemma 3.5 that $(u_2, u(\eta_2))$ is universally regular. Now η_1 is the projection of an eigenvector of $\Phi_{(u_2,u(\eta_2))}(2t,0)$, which proves the first half of (63). To complete the proof note that by Proposition 6.7 there exists a control set $D_2 \supset \mathbb{P}GE(r,u)$. But then $D \cap D_2 \neq \emptyset$ and hence $D = D_2$ by Proposition 6.2 (i). $\square$

It should be noted that elements of the cores of control sets need not be eigenvectors corresponding to eigenvalues for *universally regular* controls even though it holds that $\xi \in \hat{\mathcal{O}}^+(\xi) \Leftrightarrow \xi \in \operatorname{core}(D)$ for some control set D. This phenomenon will be exhibited in the following example. The small sidestep necessary in the proof of the previous Proposition 6.9 is thus explained.

Example 6.10 Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and consider the map

$$A : \mathbb{K}^2 \to \mathbb{K}^{2\times 2}\,, \quad A(a,b) = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}\,. \tag{66}$$

Define $U := \{[a\,,\,b]' \in \mathbb{K}^2;\, |a| < 1,\, |b| < \frac{1}{4}\}$. The system (5) given with these data is forward accessible, which is most easily seen using the rank criterion of Proposition 3.6. Define

$$V := \left\{[x_1,x_2]' \in \mathbb{K}^2; x_1 \neq 0,\, \frac{|x_2|}{|x_1|} < \frac{1}{2}\right\}\,. \tag{67}$$

It is easy to show that $\mathbb{P}V$ is an invariant subset of $\mathbb{P}_{\mathbb{K}}^1$. Also for the point $\xi_0 := \mathbb{P}[1,0]' \in V$ and the control $u_0 = (0,0)$ it may be seen that $\xi_0 = \mathbb{P}A(u_0)\xi_0$ and (ξ_0, u_0) is a regular pair. Thus by Proposition 6.5 there exists a control set D satisfying $\xi_0 \in \operatorname{core}(D)$ and by invariance of $\mathbb{P}V$ it holds that $D \subset \mathrm{cl}\,\mathbb{P}V$. (In fact D is the unique invariant control set, but this will be shown later.) However, ξ_0 does not belong to the projection of a generalized eigenspace of a universally regular control. Note that there is no generalized eigenspace of dimension 2 corresponding to a universally regular control as otherwise $\mathbb{P}^1_{\mathbb{K}}$ would be contained in the core of a control set (by Proposition 6.7), which contradicts the invariance of V. It is easy to see that if $\det A(u) \neq 0$ and $\xi_0 = \mathbb{P}A(u)\eta_0$, then $\eta_0 = \mathbb{P}[0,1]' \notin \mathrm{cl}\,\mathbb{P}V$. As universal regularity implies invertibility it follows that if $\xi_0 = \xi(t;\xi_0,u)$ for some universally regular control u then $\xi(t-1;\xi_0,u) = \eta_0$, contradicting the invariance of $\mathbb{P}V$. □

This difference between the projected eigenspaces of universally regular controls and the regions of complete controllability is unique for discrete systems and does not occur in continuous time. Compare [20] Proposition 3.8. The reason appears to be the non-invertibility possible in discrete-time.

Proposition 6.11 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (5) is forward accessible. If $U = U_{inv}$, then*

$$\xi \in \mathrm{core}(D) \Leftrightarrow \exists t \in \mathbb{N},\ u \in U^t_{reg} \text{ such that } \xi = \xi(t;\xi,u)\,, \tag{68}$$

where D is some control set.

Proof: "$\Leftarrow$" This is clear from Proposition 6.5.

"$\Rightarrow$" As $\xi \in \mathrm{core}(D)$ by Proposition 6.6 (iii) it follows that $\mathrm{core}(D) \subset \hat{\mathcal{O}}^+(\xi)$. Hence there exist $t \in \mathbb{N}$, $u \in U^t_{reg}$ such that $\xi(t;\xi,u) \in \mathrm{core}(D)$. As $\mathrm{core}(D) \subset \hat{\mathcal{O}}^-(\xi)$ there exist $s \in \mathbb{N}$, $v \in \mathrm{int}\,U^s = \mathrm{int}\,U^s_{inv}$ such that $\xi = \xi(t+s;\xi,(u,v))$. By Lemma 3.5 $(u,v) \in \mathrm{int}\,U^{t+s}$ is universally regular. □

A slight modification of the previous Example 6.10 will show that there exist indeed cases where $\mathrm{core}(D) \neq \mathrm{int}\,D$ for control sets D.

Example 6.12 Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

$$A : \mathbb{K}^2 \to \mathbb{K}^{2\times 2}\,, \quad A(a,b) = \begin{bmatrix} 1 & a^3 \\ b^3 & 0 \end{bmatrix}. \tag{69}$$

Let $U := \{[a\,,\,b]' \in \mathbb{K}^2;\ |a| < 1,\ |b| < (\frac{1}{4})^{\frac{1}{3}}\}$. Note that with this definition the system defined by (69) behaves no different from the system in Example 6.10, in the sense that for every point ξ the forward and backward orbits of the two systems coincide, which is clear from the definitions of A and U. Hence there exists the same control set D as in Example 6.10. But still the point ξ_0 that was critical in the previous example now does not even belong to the core of D. For this we show that $\hat{\mathcal{O}}^-(\mathbb{P}[1,0]') = \hat{\mathcal{O}}^-(\mathbb{P}[0,1]')$. Let $t \in \mathbb{N}$, $u = (u(0),\ldots,u(t-1)) \in U^t$ with $u(t-1) = [a,0]'$ and $x \notin \mathrm{Ker}\,\Phi_u(t,0)$ then $\xi(t;\mathbb{P}x,u) = \mathbb{P}[1,0]' = \xi_0$ but

$$G_t(x,u) = \left[\begin{array}{c:c} * & \begin{bmatrix} 1 & a^3 \\ 0 & 0 \end{bmatrix} \cdot \dfrac{\partial \Phi_u(t-1,0)x}{\partial u} \quad \begin{matrix} * & 0 \\ 0 & 0 \end{matrix} \\ 0 & \end{array}\right] \tag{70}$$

and hence $\mathrm{rk}\,G_t(x,u) = 1$, and $(\mathbb{P}x,u)$ is not a regular pair. If $b \neq 0$ and $\xi_0 = \mathbb{P}A(a,b)x$ it follows that $A(a,b)$ is invertible. As we have seen in the previous Example 6.10 that if $\det A(u) \neq 0$ then any trajectory going to the point ξ_0 must first go through $\eta_0 = \mathbb{P}[0,1]' \notin \mathrm{cl}\,\mathcal{O}^+(\xi)$ for all $\xi \in \mathbb{P}V$. So $\xi_0 \notin \hat{\mathcal{O}}^+(\xi_0)$ and hence $\xi_0 \in \mathrm{int}\,D \backslash \mathrm{core}(D)$. □

It should also be noted that it cannot be concluded that the projection of an arbitrary eigenspace corresponding to any control is contained in the closure of a control set with nonempty interior. In fact, in the following example we show that any point of the projective space may be a precontrol set, but the control sets with nonempty interior do not cover the whole projective space. Note that the following example is given here as it fits well in our discussion of control sets and generalized eigenspaces. We do, however, use a fact from the next section namely the existence of a unique open and and a unique invariant control set.

Example 6.13 Let $\mathbb{K} = \mathbb{R}$,

$$A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}\,, \quad A(a,b) = \begin{bmatrix} 1 & ab \\ a & 1 \end{bmatrix}. \tag{71}$$

Define $U = \left\{[a,b]' \in \mathbb{R}^2;\ 0 \le a \le \frac{1}{2},\ 2 \le b \le 4\right\}$. Then clearly choosing $a = 0$ leads to a transition matrix for which every $\xi \in \mathbb{P}^1_{\mathbb{R}}$ is a fixed point. Furthermore b may be chosen such that the rank condition (13) is satisfied. However, the controls for which this is possible are not in the interior of U, and hence the statements made up til now do not infer that the system (5) is completely controllable on $\mathbb{P}^1_{\mathbb{R}}$. In fact, for the set

$$V := \{[x_1, x_2]' \in \mathbb{R}^2;\ 0 < x_2 < x_1\}\,, \tag{72}$$

$\mathbb{P}V$ is an invariant set of system (5), and thus the invariant control set satisfies $C \subset \mathrm{cl}\,\mathbb{P}V$. On the other hand we have that the open control set satisfies

$$C^- \subset \mathbb{P}\{[x_1, x_2]' \in \mathbb{R}^2;\ x_1 x_2 \le 0\}\,. \tag{73}$$

as for every $t \in \mathbb{N}$, $u \in \mathrm{int}\,U^t$ the matrix $\Phi_u(t,0)$ has only strictly positive entries. Thus by the Perron-Frobenius theory for positive matrices $\Phi_u(t,0)$ does not have two linearly independent nonnegative eigenvectors, and the eigenvalue corresponding to the nonnegative eigendirection has algebraic multiplicity 1 (see [39] Chapter 15.3 Theorem 1 and Exercise 11). This implies that for any $u \in U^t_{reg}$ the evolution operator $\Phi_u(t,0)$ has an eigenvector $x = [x_1, x_2]'$ satisfying $x_1 x_2 \le 0$ corresponding to an eigenvalue of algebraic multiplicity 1, while the eigenvector corresponding to the other eigenvalue of algebraic multiplicity 1 projects to $\mathbb{P}V$. As to every control set D with nonempty interior there exists a universally regular control u such that $\mathbb{P}GE(r,u) \subset D$ for a suitable value r by Proposition 6.9, it follows that the set $\mathbb{P}\{[x_1, x_2]' \in \mathbb{R}^2;\ 0 < x_1 < x_2\}$ does not intersect a control set with nonempty interior, although every point in this set is a precontrol set. □

7 The Maximal and the Minimal Control Set

It is now shown that there exists a unique invariant and a unique open control set. These two can be described in a particularly easy fashion: they are the intersection of the closures of forward orbits, respectively in the interior of the intersection of closures of backward orbits. We call these control sets the maximal respectively minimal control sets. This terminology is justified as we may introduce a natural order on the set of all control sets on $\mathbb{P}^{n-1}_{\mathbb{K}}$, in which the maximal control set is the invariant one and the minimal is open.

Theorem 7.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (5) is forward accessible, then* *(i) There exists a unique invariant control set $C \subset \mathbb{P}^{n-1}_{\mathbb{K}}$. It is given by*

$$C := \bigcap_{\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}} \operatorname{cl} \mathcal{O}^+(\xi)\,. \tag{74}$$

(ii) There exists a unique open control set $C^- \subset \mathbb{P}^{n-1}_{\mathbb{K}}$. It satisfies

$$\operatorname{cl} C^- = C^* := \bigcap_{\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}} \operatorname{cl} \mathcal{O}^-(\xi)\,. \tag{75}$$

Moreover is holds that $\operatorname{core}(C^-) = C^-$.

Proof: (i) To begin with it has to be shown that C as defined by (74) is not empty. Let $u \in U^{t^*}_{reg}$ and $|\sigma(\Phi_u(t,0))| = \{r_1, \ldots, r_\nu\}$ with $r_1 < \ldots < r_\nu$. By Proposition 6.7 there exists a control set D such that $\mathbb{P}GE(r_\nu, u) \subset \operatorname{core}(D)$. By Lemma 4.3 it holds for all $\xi \notin \mathbb{P} \bigoplus_{j=1}^{\nu-1} GE(r_j, u)$ that

$$\omega^+(\xi, u) \subset \mathbb{P}GE(r_\nu, u)\,. \tag{76}$$

Note that the set of ξ for which (76) holds is generic in $\mathbb{P}^{n-1}_{\mathbb{K}}$. By forward accessibility we may steer from any point into that generic set, as the interior of each forward orbit is open, and it follows that $\mathbb{P}GE(r_\nu, u) \cap \operatorname{cl} \mathcal{O}^+(\xi) \neq \emptyset$ for all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$. However, we know that $\mathbb{P}GE(r_\nu, u) \subset \operatorname{core}(D)$ so that $\mathcal{O}^+(\xi) \cap \operatorname{core}(D) \neq \emptyset$ and therefore $\operatorname{core}(D) \subset \mathcal{O}^+(\xi)$. In all we have obtained that $\operatorname{core}(D) \subset C$. By definition of C it follows furthermore that $D = C$, for if $\xi \in C$, then $\operatorname{core}(D) \subset \operatorname{cl} \mathcal{O}^+(\xi)$, and also $\xi \in \operatorname{cl} \mathcal{O}^+(\eta)$ for all $\eta \in D$, so that $\xi \in D$. C is therefore a closed control set with nonempty interior and invariant by Proposition 6.3. As $C \subset \operatorname{cl} \mathcal{O}^+(\eta)$ for every $\eta \in \mathbb{P}^{n-1}_{\mathbb{K}}$ there can be no other invariant control set.
(ii) Let D be the control set with $\mathbb{P}GE(r_1, u) \subset \operatorname{core}(D)$. Recall that by Proposition 5.4 $\hat{\mathcal{O}}^-(\xi) \neq \emptyset$ for all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$. Hence for all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, we may choose a control $v \in \operatorname{int} U^{t^*}$ and $\xi_2 \in \mathbb{P}^{n-1}_{\mathbb{K}}$ such that (ξ_2, v) is a regular pair, $\xi_2 \notin \mathbb{P} \bigoplus_{j=2}^{\nu} GE(r_j, u)$ and $\xi = \xi(t^*; \xi_2, v)$. By Lemma 4.3 (iii) it follows that $\omega^-(\xi_2, u) \subset \mathbb{P}GE(r_1, u)$. Thus there exists a $\xi_3 \in \operatorname{core}(D)$ such that $\xi_3 \in \hat{\mathcal{O}}^-(\xi)$. As by Proposition 6.6 $\operatorname{core}(D) \subset \hat{\mathcal{O}}^-(\xi)$ for $\xi \in \operatorname{core}(D)$, it follows that $\operatorname{core}(D) \subset \hat{\mathcal{O}}^-(\xi)$ for all $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ and thus $\operatorname{core}(D) \subset C^*$.
In particular for $\eta \in D$ it is obtained that $\operatorname{core}(D) \subset \hat{\mathcal{O}}^-(\eta)$ and thus $\eta \in \operatorname{core}(D)$. This implies that D is an open control set by Proposition 6.6 (i).
Finally, it has to be shown that $\operatorname{cl} D = C^*$. Let $\eta \in C^* \setminus \operatorname{cl} D$. As $\eta \in C^*$ it follows that $\eta \in \operatorname{cl} \mathcal{O}^-(\xi)$ for all $\xi \in D$. Hence in every neighborhood of η there exists a ζ such that $D \subset \mathcal{O}^+(\zeta)$. On the other hand $D \subset \hat{\mathcal{O}}^-(\zeta)$ and thus $\zeta \in D$, by maximality. This however implies that $\eta \in \operatorname{cl} D$, a contradiction. Thus $\operatorname{cl} D = C^*$ and hence $C^- = D$ is the only open control set contained in C^*.
It remains to show that there is no other open control set in $\mathbb{P}^{n-1}_{\mathbb{K}}$. If D is a control set with $\operatorname{core}(D) \neq \emptyset$, then by Proposition 6.5 there exists $\xi \in \operatorname{core}(D)$, $t \in \mathbb{N}$, $u \in U^t$. such that (ξ, u) is regular and $\xi = \xi(t; \xi, u)$. By Proposition 6.9 we may assume that u is universally regular. Let $|\sigma(\Phi_u(t,0))| = \{r_1, \ldots, r_\nu\}$, $r_1 < \ldots < r_\nu$. Thus $\xi \in \mathbb{P}GE(r_i, u)$ for some $i > 1$, for otherwise $\xi \in C^-$ which may be seen using the previous arguments. Now for $\eta \in \mathbb{P}(GE(r_i, u) \oplus GE(r_1, u)) \setminus \mathbb{P}GE(r_1, u)$ it holds that $\omega^+(\eta, u) \subset \mathbb{P}GE(r_i, u)$ by Lemma 4.3 (iii). Thus $\partial D \cap \mathbb{P}(GE(r_i, u) \oplus GE(r_1, u)) \subset D$ and D is not open. □

Corollary 7.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (5) is forward accessible. If there exists exactly one control set D in $\mathbb{P}^{n-1}_{\mathbb{K}}$, then* $D = \operatorname{core}(D) = \mathbb{P}^{n-1}_{\mathbb{K}}$.

Proof: By the previous Theorem 7.1 it follows that $D = C = C^- = \mathrm{core}(C^-)$. Thus D is open and closed and not empty, which shows that $D = \mathrm{core}(D) = \mathbb{P}_{\mathbb{K}}^{n-1}$. $\square$

Using the fact that the invariant control set is closed we may prove the following result on its connectedness.

Proposition 7.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible. Then the invariant control set C is connected.*

Proof: For each connected component Y of C and $u \in U_{inv}$ the image $\mathbb{P}A(u)Y$ is connected as the continuous image of a connected set. As also $\mathcal{O}_1^+(\xi)$ is connected for all $\xi \in Y$ and $\mathrm{cl}\,\mathcal{O}_1^+(Y) \subset C$ it follows that there exists a connected component Y' of C such that $\mathrm{cl}\,\mathcal{O}_1^+(Y) \subset Y'$. Let $u \in U_{reg}^{t^*}$. For the connected component of C satisfying $\mathbb{P}GE(r_n, u) \subset Y$, which exists as $\mathbb{P}GE(r_n, u)$ is connected, it follows that $\mathcal{O}_{t^*}^+(Y) \subset Y$, but then $C = \mathrm{cl}\,\mathcal{O}^+(Y) \subset \bigcup_{s=1}^{t^*} \mathrm{cl}\,\mathcal{O}_s^+(Y) \subset C$, so that there are $k \le t^*$ connected components of C. Hence we may assume that the connected components of C are ordered in such a way that

$$\mathrm{cl}\,\mathcal{O}_1^+(Y_i) \subset Y_{i+1} \quad , i = 1, \dots, k-1 \,,$$

and

$$\mathrm{cl}\,\mathcal{O}_1^+(Y_k) \subset Y_1 \,.$$

Let $v \in U_{reg}^{kt^*+1}$. Then by universal regularity of v

$$\mathbb{P}GE(r_n, v) \subset \mathrm{core}(C) \,,$$

and for every $i = 1, \dots, k$ it holds that

$$\xi \in Y_i \Rightarrow \xi(kt^* + 1; \xi, v) \in Y_{i \bmod k+1} \,.$$

But if $\xi \in \mathbb{P}GE(r_n, v)$ then clearly $\xi(kt^*+1; \xi, v) \in \mathbb{P}GE(r_n, v)$ and $\mathbb{P}GE(r_n, v)$ is connected. So that $i = i \bmod k + 1$ and thus $k = 1$. $\square$

Remark 7.4 (i) The uniqueness of the invariant control set system (5) has been shown in [32] for the case that all system matrices $A(u)$ are invertible. The proof relies, however, on a theorem in [7], where it has to be assumed that the group generated by $\{A(u); u \in U\}$ is a Lie group. We have shown that in our case these assumptions are not necessary.
(ii) From the proof of Theorem 7.1 it follows that for all $t \ge t^*$, $u \in U_{reg}^t$ we have

$$\mathbb{P}GE(r_1(\Phi_u(t,0)), u) \subset C^- \,, \tag{77}$$

$$\mathbb{P}GE(r_n(\Phi_u(t,0)), u) \subset C \,. \tag{78}$$

$\square$

The last argument in the proof of Theorem 7.1 contains the fundamental idea on what order is in a sense natural on the set of control sets.

Let D_1, D_2 be control sets in $\mathbb{P}_{\mathbb{K}}^{n-1}$ for the system (5). We define

$$D_1 \le D_2 :\Leftrightarrow \text{ There exist } \xi \in D_1,\ t \in \mathbb{N},\ u \in U^t \text{ such that } \xi(t; \xi, u) \in D_2 \,. \tag{79}$$

A priori this defines only a partial order on the control sets. What is however evident at this point is the following.

Proposition 7.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that system (5) is forward accessible.*

(i) C is the unique maximal control set with respect to the order "$\leq$" on the control sets.

(ii) C^- is the unique minimal control set with respect to the order "$\leq$" on the control sets.

Proof: (i) is immediate from (74), while (ii) follows from (75). □

8 Main Control Sets

In this section we give sufficient conditions for which it is possible to recover exactly those results that are known in the continuous-time case. Namely, the number of control sets with non-void interior is bounded by n, the dimension of the state space, the control sets are completely ordered with respect to the order defined in the previous section, and to each control set an index may be assigned as the sum of the algebraic multiplicities of all the eigenvalues corresponding to a universally regular u, whose generalized eigenspace is projected into the core of that control set. Furthermore in the complex or real invertible case the control sets are connected.

We begin with the following definition. For every $t \in \mathbb{N}$, $u \in U^t$, we will from now on consider the set $\{r_1, \ldots, r_n\}$, where $r_i \in |\sigma(\Phi_u(t, 0))|$, $r_1 \leq \ldots \leq r_n$ and each r_i occurs as often as the sum of the algebraic multiplicities of those $\lambda \in \sigma(\Phi_u(t, 0))$ with $r_i = |\lambda|$. We define for $i = 1, \ldots, n$

$$Q_i(t) := \bigcup_{u \in U^t_{reg}} \mathbb{P}GE(r_i, u), \quad Q_i := \bigcup_{t=1}^{\infty} Q_i(t). \tag{80}$$

Furthermore for a map $A : \tilde{U} \to \mathbb{R}^{n \times n}$ we introduce the following index which is a measure of what sets of rank deficient matrices separate $A(\operatorname{int} U)$. Define the sets

$$U_i := \{u \in U; \dim \operatorname{Ker} A(u) \leq i\}, \tag{81}$$

and the singularity index

$$\bar{i}(A, U) := \min\{i; \operatorname{int} U_i \text{ is pathwise connected}\}. \tag{82}$$

Note that all the sets U_i are generic in U, as $U_i \supset U_{inv} \neq \emptyset$. Moreover, $\mathbb{K} = \mathbb{C}$ implies that $\bar{i}(A, U) = 0$ as proper analytic subsets are nowhere separating in the complex case, see [36] Proposition 7.4. The significance of the indices $i > \bar{i}(A, U)$ is explained in the following proposition.

Proposition 8.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible. If $i > \bar{i}(A, U)$ then Q_i is contained in a precontrol set.*

Proof: Let $u, v \in U^t_{reg}$, where we assume without loss of generality that the length of the sequences is the same, and that $t \geq t^* + 1$. Denote $u = (u(0), u')$ and $v = (v(0), v')$ where $u(0), v(0) \in \operatorname{int} U_{inv}$ and $u', v' \in \operatorname{int} U^{t-1}_{inv}$. Let $\gamma_1 : [0, 1] \to \operatorname{int} U$ be a continuous path connecting $u(0)$ and $v(0)$. γ_1 can be chosen piecewise analytic. Hence we may assume there is a finite number of points τ_j, $j = 1, \ldots, k$ such that $\det(A(\gamma_1(\tau_j))) = 0$. By definition we may assume that $\dim \operatorname{Ker} A(\gamma_1(\tau_j)) \leq \bar{i}(A, U)$ for $j = 1, \ldots, k$. By Proposition 5.3, the set

$$Z := \left\{ u \in U_{reg}^{t-1};\ \Phi_u(t-1, 0) \operatorname{Im} A(\gamma_1(\tau_j)) \cap \operatorname{Ker} A(\gamma_1(\tau_j)) = \{0\} \text{ for } j = 1, \ldots, k \right\} \tag{83}$$

is generic in $\operatorname{int} U^{t-1}$ as it is the finite intersection of generic sets. We may therefore choose a continuous path $\gamma_2 : [0, 1] \to \operatorname{int} U^{t-1}$ such that $\gamma_2(0) = u'$ and $\gamma_2(\tau) \in Z$ for all $\tau \in (0, 1]$. Let $\tilde{u}' := \gamma_2(1)$.
Now consider the path:

$$\gamma_3 : [0, 2] \;\; \to \;\; \operatorname{int} U^t \tag{84}$$

$$\gamma_3(\tau) \;\; = \;\; \begin{cases} (u(0), \gamma_2(\tau)) & 0 \leq \tau \leq 1 \\ (\gamma_1(\tau - 1), \tilde{u}') & 1 \leq \tau \leq 2 \end{cases} . \tag{85}$$

For $0 \leq \tau \leq 1$, $\gamma_3(\tau)$ is universally regular as $\gamma_3(0) = u$ and $\gamma_2(\tau) \in U_{reg}^{t-1}$ for $\tau \in (0, 1]$. Furthermore we obtain for $1 \leq \tau \leq 2$ and $i > \bar{i}(A, U)$ that $r_i(\Phi_{\gamma_3(\tau)}(t, 0)) > 0$. This is clear if $\det(A(\gamma_1(\tau - 1))) \neq 0$. For $\tau = 1 + \tau_j$, $j = 1, \ldots, k$ we have that

$$\Phi_{\tilde{u}'}(t-1, 0) \operatorname{Im} A(\gamma_1(\tau_j)) \cap \operatorname{Ker} A(\gamma_1(\tau_j)) = \{0\}$$

and hence for the eigenvalue 0 of $\Phi_{\tilde{u}'}(t-1, 0) A(\gamma_1(\tau_j))$ algebraic and geometric multiplicity coincide.
In all we have constructed a continuous path from $u = (u(0), u')$ to $(v(0), \tilde{u}')$ such that $r_i > 0$ along this path if $i > \bar{i}(A, U)$ and furthermore $\tilde{u}' \in Z$ can be chosen arbitrarily close to u'. We wish to continue this procedure in an inductive manner. Assume that for some $0 < j < t-1$ we have constructed a continuous path from u to $(v'(0), \ldots, v'(j-1), v(j), w(j+1), \ldots, w(t-1)) \in U_{reg}^t$, where $(v'(0), \ldots, v'(j-1))$ is arbitrarily close to $(v(0), \ldots, v(j-1))$ and $(w(j+1), \ldots, w(t-1))$ is arbitrarily close to $(u(j+1), \ldots, u(t-1))$. Furthermore along this path the i-th entry in the ordered spectrum is never 0 if $i > \bar{i}(A, U)$.

As for all $w_1' \in U^{t-j-2}, w_2' \in U, w_3' \in U_{inv}^j$ the Jordan structures of

$$\Phi_{w_1'}(t - j - 2, 0) A(w_2') \Phi_{w_3'}(j + 1, 0)$$

and

$$\Phi_{w_3'}(j + 1, 0) \Phi_{w_1'}(t - j - 2, 0) A(w_2')$$

coincide by similarity, we may work as in the first part to construct a path with the desired properties from $(w(j+1), w(j+2), \ldots, w(t-1), v'(0), \ldots, v'(j-1), v(j))$ to $(v(j+1), \tilde{w})$, where $\tilde{w}$ may be chosen arbitrarily close to $(w(j+2), \ldots, w(t-1), v'(0), \ldots, v'(j-1), v(j))$. Note that this rearrangement does not destroy universal regularity by Lemma 3.5. By rearranging the sequence to the original order, we obtain the desired path in the j-th step.
Continuing this procedure we obtain a continuous path γ_4 from u to $\tilde{v}$, where $\tilde{v}$ may be chosen arbitrarily close to v. As $v \in U_{reg}^t$, the path may be assumed to go from u to v.
By construction $r_i(\Phi_{\gamma_4(\tau)}(t, 0)) > 0$ along this path if $i > \bar{i}(A, U)$. Now consider the continuous paths

$$\gamma_5, \gamma_6 : [0, 1] \;\; \to \;\; \operatorname{int} U^t \tag{86}$$

$$\gamma_5(\tau) \;\; = \;\; (\gamma_4(\tau), u) \tag{87}$$

$$\gamma_6(\tau) \;\; = \;\; (\gamma_4(1 - \tau), v) \tag{88}$$

connecting (u,u) with (v,u) and (u,v) with (v,v), respectively. As u and v are universally regular, we have that for $i > \bar{i}(A,U)$ and all $\tau \in [0,1]$ the sets

$$\mathbb{P}GE(r_i(\tau), \gamma_5(\tau)), \;\; \mathbb{P}GE(r_i(\tau), \gamma_6(\tau)) \tag{89}$$

are regular. Hence each of the sets

$$\bigcup_{\tau \in [0,1]} \mathbb{P}GE(r_i(\tau), \gamma_5(\tau))\,, \tag{90}$$

$$\bigcup_{\tau \in [0,1]} \mathbb{P}GE(r_i(\tau), \gamma_6(\tau)) \tag{91}$$

is contained in an open precontrol set by Proposition 6.8. Furthermore it holds that

$$\mathbb{P}GE(r_i, (v,u)) = \mathbb{P}\Phi_v(t,0)GE(r_i,(u,v))\,, \tag{92}$$

which is clear by the relation $\Phi_{(u,v)}(2t,0) = \Phi_v(t,0)\Phi_{(v,u)}(2t,0)\Phi_v(t,0)^{-1}$. By symmetry we obtain furthermore that

$$\mathbb{P}GE(r_i, (u,v)) = \mathbb{P}\Phi_u(t,0)GE(r_i,(v,u))\,, \tag{93}$$

Thus it may be concluded that

$$\bigcup_{\tau \in [0,1]} \mathbb{P}GE(r_i(\tau), \gamma_5(\tau)) \cup \bigcup_{\tau \in [0,1]} \mathbb{P}GE(r_i(\tau), \gamma_6(\tau)) \tag{94}$$

is contained in a precontrol set. The proof is completed by fixing one universally regular control and noting that we may apply the procedure of this proof for a path to any other universally regular control. □

Remark 8.2 In the preceding theorem we did not make a statement about connectedness. In Example 6.12 in the case $\mathbb{K} = \mathbb{R}$ we have seen a system, where indeed the core of the invariant control set C is not connected. On the other hand we know by Proposition 6.9 and by the fact that Q_1 is contained in the open control set C^- that for every connected component W of $\mathrm{core}(C)$ it holds that $Q_2 \cap W \neq \emptyset$. Note that in this example the index $\bar{i}(A,U) = 1$ as $\mathrm{rk}\, A(u) \geq 1$ for all $u \in U$ and the controls $(1/2, -\varepsilon)$ $(1/2, \varepsilon)$ can only be connected through a point of the form $(a, 0)$ which leads to a rank drop. □

The following statement includes in particular the case of real invertible and complex systems.

Proposition 8.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible. If $\bar{i}(A,U) = 0$ then for each $i = 1, \dots, n$ the set Q_i is contained in a connected component of* $\mathrm{core}(D)$ *for some control set D.*

Proof: Fix $u_1, u_2 \in U^t_{reg}$ (where again without loss of generality the length of the control sequences is the same) and let $\gamma : [0,1] \to \mathrm{int}\, U^t_{inv}$ be a continuous connecting path. Such a path exists as $\mathrm{int}\, U^t_{inv}$ is connected, but there may be $\tau \in [0,1]$ such that $\gamma(\tau)$ is not universally regular. Then the path

$$\gamma_2 : [0,2] \;\; \to \;\; \mathrm{int}\, U^{2t} \tag{95}$$

$$\gamma_2(\tau) \;\; = \;\; \begin{cases} (u_1, \gamma(\tau)) & , \; 0 \leq \tau \leq 1 \\ (\gamma(\tau - 1), u_2) & , \; 1 \leq \tau \leq 2 \end{cases} \tag{96}$$

is a continuous path connecting (u_1,u_1) and (u_2,u_2) in $\mathrm{int}\, U^{2t}$. By Lemma 3.5, the invertibility of $A(\gamma(\tau))$ and the universal regularity of u_1, u_2 it follows furthermore that $\gamma_2(\tau)$ is universally regular for all $\tau \in [0,2]$. The assertion now follows due to Proposition 6.8. □

Theorem 8.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible and that $\bar{\imath}(A, U) \leq 1$, then the following statements hold:*

(i) The number κ of control sets $D_1, \dots, D_\kappa$ with nonempty interior satisfies

$$1 \leq \kappa \leq n\,. \tag{97}$$

(ii) For every $t > 0$, $u \in U^t_{reg}$, $r \in |\sigma(\Phi_u(t,0))|$ there exists a control set D_i $1 \leq i \leq \kappa$ such that

$$\mathbb{P}GE(r,u) \subset \mathrm{core}(D_i)\,. \tag{98}$$

(iii) The core of the control sets $D_1, \dots, D_\kappa$ consists of exactly those elements $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$ which are eigenvectors to a nonzero eigenvalue of some $\Phi_u(t,0)$ where (ξ, u) is a regular pair. If $U = U_{inv}$ the control may be chosen to be universally regular.

(iv) For every $t > 0$, $u \in U^t$, $r \in |\sigma(\Phi_u(t,0))|$ there exists an $j \in \{1, \dots, \kappa\}$ with $\mathbb{P}GE(r,u) \cap \mathrm{cl}\, D_j \neq \emptyset$. Also for every $t \in \mathbb{N}$, $u \in U^t$ and every $j = 1, \dots, \kappa$ there exists an $r \in |\sigma(\Phi_u(t,0))|$ with $\mathbb{P}GE(r,u) \cap \mathrm{cl}\, D_j \neq \emptyset$.

Proof: (i) Let D be a control set with $\mathrm{core}(D) \neq \emptyset$. By Proposition 6.9 there exists an $i \in \{1, \dots, n\}$ such that $Q_i \cap D \neq \emptyset$. If $i = 1$ then Q_i is contained in a control set by Remark 7.4 (ii). Using Proposition 8.1 it follows that $Q_i \subset D$. Thus the number of control sets with nonempty interior is bounded by n, the number of the sets Q_i.

(ii) This follows from Corollary 4.6 and (i).

(iii) This follows from Propositions 6.5 and 6.11.

(iv) The statement is clear for $u \in U^t_{reg}$. If $t < t^*$ choose l such that $lt \geq t^*$ and consider the control $(u)^l$. If $t \geq t^*$ and $u \notin U^t_{reg}$ by genericity of the universally regular controls there exists a sequence $(u_k)_{k \in \mathbb{N}} \subset U^t_{reg}$ with $\lim_{k \to \infty} u_k = u$. Using again the continuity properties of the eigenprojections (Chapter II.8 in [13]) it follows that for $r_i \in |\sigma(\Phi_u(t,0))|$ it holds that $\mathbb{P}GE(r_i, u) \cap \mathrm{cl}\, Q_i \neq \emptyset$. This implies the assertion.

□

It has been shown that under the assumption of the previous theorem for every $i \in \{1, \dots, n\}$ there exists a control set D such that $Q_i \subset D$. From now on the following terminology is used.

Definition 8.5 (Main control set) *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible. A control set D is called* main control set *if for every index $1 \leq i \leq n$ it holds that*

$$Q_i \cap D \neq \emptyset \;\Rightarrow\; Q_i \subset D\,.$$

The result of the previous theorem may then be paraphrased by saying that in the case where $\bar{\imath}(A, U) \leq 1$, i.e. in particular in complex or real invertible case the only control sets with nonempty core are main control sets. Let us now examine further properties of main control sets. Recall that $n(\lambda, u)$ denotes the dimension of the generalized eigenspace of the eigenvalue λ of $\Phi_u(t,0)$.

Theorem 8.6 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Assume that (5) is forward accessible, then the following holds.*

(i) If $\bar{i}(A, U) = 0$ then the core of every main control set is connected.

(ii) The main control sets are completely ordered with respect to the order "$\leq$".

(iii) For each main control set D the number

$$m(D) = \sum_{\mathbb{P}GE(\lambda,u) \subset \mathrm{core}(D)} n(\lambda, u) \tag{99}$$

is independent of $u \in U^t_{reg}$ and $t \in \mathbb{N}$.

Proof: (i) Let D be a main control set. For any open set $W \subset \mathrm{core}(D)$ there exists an i such that $W \cap Q_i \neq \emptyset$ by Proposition 6.9. As the sets Q_i are contained in connected components of the core by Proposition 8.3, it is sufficient to show the following: If there exists $i, j \in \{1, \ldots, n\}$ $i \neq j$ such that $Q_i, Q_j \subset D$, then there exists a $1 \leq k \leq n$ such that $Q_k \subset D$, $Q_i \cap Q_k \neq \emptyset$ and $Q_j \cap Q_k \neq \emptyset$.
Let $\xi \in Q_i$, $\eta \in Q_j$. Hence there exist $t, s \in \mathbb{N}$, $u \in U^t_{reg}$, $v \in U^s_{reg}$ such that

$$\eta = \xi(t; \xi, u)\,, \tag{100}$$
$$\xi = \xi(s; \eta, v)\,. \tag{101}$$

(Indeed if $\xi \in \mathbb{P}GE(r_i, u')$ for $u' \in U^{t'}_{reg}$ and $\xi = \xi(t'; \xi', u')$ then by the implicit function theorem there is an open neighborhood of ξ' that can be steered to ξ with universally regular controls. Into this neighborhood we can steer from η using an invertible control. A concatenation yields the desired control.)
Now $(v, u), (u, v) \in U^{t+s}_{reg}$. Furthermore as $\sigma(\Phi_v(s, 0)\, \Phi_u(t, 0)) = \sigma(\Phi_u(t, 0)\, \Phi_v(s, 0))$ it follows that there exists a $\lambda \in \mathbb{C}^*$ such that

$$\xi \in \mathbb{P}GE(\lambda, (v, u))\,, \tag{102}$$

and

$$\eta \in \mathbb{P}GE(\lambda, (u, v))\,. \tag{103}$$

If $|\lambda| = r_k(\Phi_{(u,v)}(s+t, 0)) = r_k(\Phi_{(v,u)}(s+t, 0))$ it follows that $\xi, \eta \in Q_k$. Hence $Q_k \subset D_i$ and $Q_i \cup Q_j \cup Q_k$ is contained in a connected component of the core of D.

(ii) Let D_1, D_2 be two main control sets. Then there exists $Q_i \subset D_1$, $Q_j \subset D_2$. Let us assume that $i \leq j$ then we claim that $D_1 \leq D_2$. Indeed let $u \in U^{t^*}_{reg}$ and $\xi \in \mathbb{P}(GE(r_i, u) \oplus GE(r_j, u))$. As $r_i \leq r_j$ it follows that

$$\omega^+(\xi, u) \subset \begin{cases} \mathbb{P}GE(r_i, u) & \text{if } \xi \in \mathbb{P}GE(r_i, u) \\ \mathbb{P}GE(r_j, u) & \text{otherwise} \end{cases}\,. \tag{104}$$

As $\mathbb{P}GE(r_i, u) \subset \mathrm{core}(D_1)$ there exists $\eta \in D_1$ such that $\omega^+(\eta, u) \subset \mathrm{core}(D_2)$. This proves the assertion.

(iii) It is clear that

$$m(D) = \#\{1 \leq i \leq n;\, Q_i \subset D\}\,, \tag{105}$$

which is independent of $u \in U^t_{reg}$, $t \in \mathbb{N}$.

□

As a result of the preceding Theorem 8.6 the following definition is straightforward.

Definition 8.7 (Index of a main control set) *Assume that (5) is forward accessible. For a main control set $D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ the number $m(D)$ is called the* index of the control set D.

It remains to analyze the case where $\bar{\imath}(A,U) > 1$. By the discussion up to this point it is clear that for $i = 1, n$ and $i > \bar{\imath}(A,U)$ there exists a main control set D_i such that $Q_i \subset D_i$. For the remainder of the indices the question of whether there exists a unique control set with this property must for the moment be left unresolved.

To summarize we have obtained the following picture of the control structure of the system on projective space. For a map A and a set of admissible controls U such that the system on $\mathbb{P}^{n-1}_{\mathbb{K}}$ is forward accessible and $\bar{\imath}(A,U) \leq 1$, there exists a sequence of indices $i_1, \ldots, i_\kappa$, with $\sum_{j=1}^{\kappa} i_j = n$.
To each index i_j there exists a control set D_j such that $m(D_j) = i_j$. More specifically it is shown in [56] that if we write

$$\mu_j = \sum_{l=1}^{j} i_l$$

for $j = 1, \ldots, \kappa$ then

$$\bigcup_{i=\mu j-1+1}^{\mu_j} Q_i \subseteqq \mathrm{core}(D_j)\,.$$

where equality holds if $U = U_{inv}$. So the numbers from 1 to n are partitioned into κ non-interlacing subsequences which represent the indices i such that $Q_i \subset \mathrm{core}(D_j)$:

$$\underbrace{1, \ldots, \mu_1}_{D_1}, \underbrace{\mu_1 + 1, \ldots, \mu_2}_{D_2}, \underbrace{\mu_2 + 1, \ldots,}_{\cdots\cdots} \ldots, \underbrace{\ldots, \mu_{\kappa-1}}_{\cdots\cdots}, \underbrace{\mu_{\kappa-1} + 1, \ldots, n}_{D_\kappa}\,.$$

The order between the main control sets is simply reflected in the order of the subsequences. In case there are control sets with nonempty core that are not main control sets this can be extended in a natural way by considering indices that do not correspond to main control sets, but to control set clusters, see [53].

With this notation we may formulate the the following invariance principle which also motivates the interpretation of control sets and their indices as an extension of eigenspaces and their dimension. For a proof we refer to [53] or [56].

Theorem 8.8 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (5) is forward accessible. For $u \in U^{\mathbb{N}}$ define $d(u) := \max_{t\in\mathbb{N}} \dim\ker \Phi_u(t,0)$. Let $\mu_1, \ldots, \mu_\kappa$ be the indices for the control set structure as described above.*

(i) For every main control set D_j with $\mu_{j-1} > d(u)$ there exists a linear subspace $X_j(u)$ satisfying

$$\dim X_j(u) = m(D_j) = \mu_j - \mu_{j-1}\,,$$

$$\textit{for all } t \in \mathbb{N} \textit{ it holds that } \mathbb{P}\Phi_u(t,0)X_j(u) \subset \mathrm{cl}\, D_j\,.$$

(ii) If $d(u) > 0$ and a main control set D_j exists such that $\mu_{j-1} < d(u) < \mu_j$ then there exists a linear subspace $X_j(u)$ satisfying

$$\dim X_j(u) = \mu_j - d(u)\,,$$

$$\textit{for all } t \in \mathbb{N} \textit{ it holds that } \mathbb{P}\Phi_u(t,0)X_j(u) \subset \mathrm{cl}\, D_j\,.$$

9 Characteristic exponents

Up to now we have described the control structure of a system on projective space. With the insight that has been gained let us now discuss properties of the set of characteristic exponents that may be deduced from our knowledge about the control sets.

For systems of the form (2) let $\lambda(x_0, u)$ denote the Lyapunov exponent corresponding to the initial value $(0, x_0) \in \mathbb{N} \times \mathbb{K}^n$ and the sequence $A(u(\cdot)) \in \ell^\infty(\mathbb{N}, \mathbb{K}^{n\times n})$ determined by $u \in U^{\mathbb{N}}$, i.e. the exponential growth rate of the corresponding solution:

$$\lambda(x_0, u) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_u(t, 0)x_0\|,$$

while $\beta(u)$ denotes the Bohl exponent determined by $u \in U^{\mathbb{N}}$:

$$\beta(u) = \limsup_{s,t-s\to\infty} \frac{1}{t-s} \log \|\Phi_u(t, s)\| .$$

Note that it is sufficient to study Lyapunov exponents corresponding to the initial time 0, as control sequences may be shifted, i.e. the Lyapunov exponent to the initial value (t, x_t) and the control sequence $u \in U^{\mathbb{N}}$ may be recaptured by studying the initial value $(0, x_t)$ and the control sequence $v \in U^{\mathbb{N}}$ defined by $v(s) = u(s + t)$. It is known that in general $\max_{x_0\neq 0} \lambda(x_0, u) \leq \beta(u)$ where strict inequality is possible, see [24].

Floquet exponents are the Lyapunov exponents corresponding to periodic sequences $u \in U^{\mathbb{N}}$. For $t \in \mathbb{N}$, $u \in U^t$ it is easy to see that the set of Floquet exponents determined by the t–periodic continuation of u is given by

$$\sigma_{Fl}(u) := \{\frac{1}{t} \log r;\ r \in |\sigma(\Phi_u(t, 0))|\} , \tag{106}$$

where we continue to use the convention $\log 0 = -\infty$. For a system of form (2) determined by the map A and the set of admissible controls U the Lyapunov spectrum is defined as the union

$$\Sigma_{Ly}(A, U) := \{\lambda(x_0, u)\, ;\ x_0 \in \mathbb{K}^n\backslash\{0\}, u \in U^{\mathbb{N}}\} . \tag{107}$$

The Floquet spectrum of (2) is defined by

$$\Sigma_{Fl}(A, U) := \bigcup_{t\geq 1, u\in U^t} \sigma_{Fl}(u). \tag{108}$$

Furthermore we define

$$\Sigma_{Fl,i}(A, U) := \{\frac{1}{t} \log r_i(\Phi_u(t, 0)); t \geq 1\ ,\ u \in U^t\}. \tag{109}$$

Recall that $\mathbb{P}GE(r, u)$ is called *regular*, if $u = (u_1, u_2)$ and (ξ, u_2) is a regular pair for all $\xi \in \mathbb{P}\Phi_{u_1}(t_1, 0)GE(r, u)$. For a control set D with nonempty core we define the Floquet spectrum of D to be

$$\Sigma_{Fl}(D) := \bigcup_{t\geq 1, u\in U^t} \{\frac{1}{t} \log r;\ r \in |\sigma(\Phi_u(t, 0))|,\ \mathbb{P}GE(r, u) \subset \mathrm{core}(D) \text{ and } \mathbb{P}GE(r, u) \text{ is regular } \}. \tag{110}$$

Finally, we consider the Bohl spectrum of (2) defined as the set of all Bohl exponents the system can generate

$$\Sigma_{Bo}(A, U) := \{\beta(u); u \in U^{\mathbb{N}}\}. \tag{111}$$

Let us begin by explaining how to obtain the Lyapunov exponent $\lambda(x_0, u)$ from the trajectory $\xi(\cdot; \mathbb{P}x_0, u)$ of the projected system. For $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, $u \in U(\xi)$ define

$$q(\xi, u) := \log \frac{\|A(u)x\|}{\|x\|}, \quad \text{where} \quad x \neq 0,\ \mathbb{P}x = \xi. \tag{112}$$

This is well defined as multiplication of x with a non-zero scalar does not alter the value of $q(\xi, u)$. For $\xi \in \mathbb{P}^{n-1}_{\mathbb{K}}$, $t \in \mathbb{N}$, $u \in U^t(\xi)$ define

$$J(t; \xi, u) = \sum_{s=0}^{t-1} q(\xi(s; \xi, u), u(s)). \tag{113}$$

Then we obtain the following expression for Lyapunov exponents:

Lemma 9.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For $x_0 \in \mathbb{K}^n \setminus \{0\}$, $u \in U^{\mathbb{N}}$ it holds that*

$$\lambda(x_0, u) = \begin{cases} \limsup_{t\to\infty} \frac{1}{t} J(t; \mathbb{P}x_0, u), & \text{if} \quad u \in U^{\mathbb{N}}(x_0). \\ -\infty, & \text{otherwise.} \end{cases} \tag{114}$$

Proof: This can be shown by a straightforward calculation. □

The previous lemma shows that we may speak of the Lyapunov exponent corresponding to $(\xi_0, u) \in \mathbb{P}^{n-1}_{\mathbb{K}} \times U^{\mathbb{N}}$ which we denote by $\lambda(\xi_0, u)$.

10 The Floquet Spectrum

The Floquet spectrum is closely related to the structure of the control sets examined up to now. In order to explore this relationship we need a controllability property in the cores of control sets. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (5) on $\mathbb{P}^{n-1}_{\mathbb{K}}$. Consider the function

$$h : \mathbb{P}^{n-1}_{\mathbb{K}} \times \mathbb{P}^{n-1}_{\mathbb{K}} \to \mathbb{N} \cup \{\infty\} \tag{115}$$

$$h(\xi, \eta) := \min\{t \in \mathbb{N};\ \text{there is a } u \in U^t \text{ such that } \xi(t; \xi, u) = \eta\},$$

where $\min \emptyset = \infty$.
The previous definition is the discrete–time analogue of the *first–time hitting map*, as defined for instance in [17], [18]. As we treat non–invertible systems as well it is important for us to obtain information not only on the time that elapses to steer from ξ to η, but also on the "cost" incurred in doing so. For the projected system (5) and the function q interpreted as a cost $|q(\xi, u)|$ may be arbitrarily large if u is chosen such that $A(u)$ is almost singular. In analogy to the first time hitting map, we define the *minimal absolute cost map* by

$$H : M \times M \to \mathbb{R}_+ \cup \{\infty\} \tag{116}$$

$$H(\xi, \eta) := \inf\{\max_{1\leq s\leq t} |J(s; \xi, u)|; \quad t \in \mathbb{N};\ u \in U^t \text{ such that } \xi(t; \xi, u) = \eta\},$$

where $\inf \emptyset = \infty$. The essential point is that both these values may be simultaneously bounded if one tries to reach a compact subset of the core of a control set.

Lemma 10.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that system (5) is forward accessible. Let $D \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ be a control set. Assume there are two non-void compact sets K_1, K_2 with $K_1 \subset \mathcal{O}^-(D)$ and $K_2 \subset \operatorname{core}(D)$, then the following statements hold:*

(i) There are constants $\overline{h} \in \mathbb{N}, \overline{H} \in \mathbb{R}_+$ such that

$$h(\xi, \eta) \leq \overline{h} \text{ for all } \xi \in K_1,\, \eta \in K_2\,, \tag{117}$$

$$H(\xi, \eta) \leq \overline{H} \text{ for all } \xi \in K_1,\, \eta \in K_2\,. \tag{118}$$

(ii) If $K_2 = \mathbb{P}GE(r, u)$ for some $t \in \mathbb{N}$, $u \in U^t_{reg}$ and $r \in |\sigma(\Phi_u(t, 0))|$, then $\overline{h}$, $\overline{H}$ may be chosen such that for all $\xi \in K_1$, $\eta \in K_2$ there exists $v \in U^t_{reg}$ with

$$\eta = \xi(t; \xi, v), \tag{119}$$

$$t \leq \overline{h}, \tag{120}$$

$$\max_{1 \leq s \leq t} |J(s; \xi, v)| \leq \overline{H}. \tag{121}$$

Proof: (i) Let $\xi \in K_1$, $\eta \in K_2$. Choose any point $\zeta \in \operatorname{core}(D) \cap \hat{\mathcal{O}}^+(\xi)$, which is possible by Lemma 3.8 (i) and Proposition 6.6 (iii). Thus there exist $u_1 \in \operatorname{int} U^{t_1}(\xi)$ such that $\zeta = \xi(t_1; \xi, u_1)$ and (ξ, u_1) is a regular pair. By the implicit function theorem there exist open neighborhoods V_1 of ξ, W_1 of u_1 and a continuous function $w : V_1 \to W_1$ such that $\zeta = \xi(t_1, \xi', w(\xi'))$ for every $\xi' \in V_1$. This shows that $h(\xi', \zeta) \leq t_1$ for all $\xi' \in V_1$. Furthermore, by continuous dependence of $J(s; \xi', w(\xi'))$ on ξ' it may be also obtained that $H(\xi', \zeta) \leq H_1$ for some suitable constant $H_1 \in \mathbb{R}$ and all $\xi' \in V_1$, where possibly V_1 has to be chosen to be smaller than the original choice.
On the other hand there exist $t_2 \in \mathbb{N}$, $u_2 \in \operatorname{int} U^{t_2}(\zeta)$ such that $\eta = \xi(t_2; \zeta, u_2)$ and (ζ, u_2) is a regular pair. By regularity for any open neighborhood W_2 of u_2 the set $\{\xi(t_2; \zeta, u');\ u' \in W_2\}$ contains an open neighborhood V_2 of η. Choosing W_2 small enough so that $\operatorname{cl} W_2 \subset \operatorname{int} U^{t_2}(\zeta)$ we see that $h(\zeta, \eta') \leq t_2$ for all $\eta' \in V_2$ and also $H(\zeta, \eta') \leq H_2$ for all $\eta' \in V_2$ and some suitable constant H_2.
In all we have obtained that

$$h(\xi', \eta') \leq t_1 + t_2 \quad \text{for all} \quad \xi' \in V_1, \eta' \in V_2\,,$$

and

$$H(\xi', \eta') \leq H_1 + H_2 \quad \text{for all} \quad \xi' \in V_1, \eta' \in V_2.$$

The assertion now follows because we may choose a finite sub-cover of the open cover

$$\{V_1(\xi) \times V_2(\eta); \quad \xi \in K_1, \eta \in K_2\}$$

of the compact set $K_1 \times K_2$.

(ii) Let $\xi \in K_1$, $\eta \in K_2$. Choose ζ' such that $\xi(t; \zeta', u) = \eta$. As u is universally regular there exists an open neighborhood V of ζ', $V \subset \operatorname{core}(D)$, such that for every $\zeta'' \in V$ there exists $u(\zeta'') \in U^t_{reg}$ with $\eta = \xi(t; \zeta'', u(\zeta''))$. As $\zeta' \in \operatorname{core}(D)$ there exists $t_1 \in \mathbb{N}$, $u_1 \in \operatorname{int} U^{t_1}_{inv}$ such that $\zeta := \xi(t_1; \xi, u_1) \in V$. Let $t_2 = t$, $u_2 = u(\zeta)$, then $\eta = \xi(t_1 + t_2; \xi, (u_1, u_2))$, (u_1, u_2) is universally regular and we may proceed as in the proof of part (i) by genericity of $U^{t_1}_{inv}$ and $U^{t_2}_{reg}$. □

With this result in hand we may start to examine the structure of the set of Floquet exponents.

Proposition 10.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The set $\Sigma_{Fl,i}(A, U)$ is an interval.*

Proof: Consider the function

$$\begin{array}{rccc} \lambda_{i,t}: & U^t & \to & \mathbb{R} \cup \{-\infty\} \\ & u & \mapsto & \frac{1}{t} \log r_i(\Phi_u(t,0)). \end{array}$$

By Chapter II.8 in [13] $\lambda_{i,t}$ is continuous and therefore $\lambda_{i,t}(U^t)$ is connected as the continuous image of a connected set, and thus an interval. Now

$$\Sigma_{Fl,i}(A, U) = \bigcup_{t=1}^{\infty} \lambda_{i,t}(U^t)$$

and furthermore for $u \in U$ and all $t \geq 1$

$$\log |r_i(A(u))| \in \lambda_{i,t}(U^t),$$

as we may simply consider the sequence $(u)^t$. Thus the assertion follows. □

Thus from the connectedness of the set of admissible controls it is immediately obtained, that the Floquet spectrum is the union of at most n intervals. However, a weak point of this statement is that it totally ignores the dynamics of the system. The interplay between Floquet spectrum and dynamical behavior is studied from now on.

Proposition 10.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (5) is forward accessible. For a control set D with $\mathrm{core}(D) \neq \emptyset$ the set $\mathrm{cl}\, \Sigma_{Fl}(D)$ is an interval.*

Proof: By Proposition 6.9 there exist $t \geq t^*$, $u_1 \in U^t_{reg}$ and $\lambda_1 \in \sigma(\Phi_{u_1}(t, 0))$ such that $\mathbb{P}GE(\lambda_1, u_1) \subset \mathrm{core}(D)$. It is sufficient to show that for any $\lambda \in \Sigma_{Fl}(D)$ the Floquet exponents of D are dense in the interval determined by λ and $\frac{1}{t} \log |\lambda_1|$.

Let $t_2 \in \mathbb{N}$ $u_2 \in \mathrm{int}\, U^{t_2}$ be such that for some $\lambda_2 \in \sigma(\Phi_{u_2}(t_2, 0))$ we have that $\mathbb{P}GE(\lambda_2, u_2) \subset \mathrm{core}(D)$ and the eigenspace is regular. Without loss of generality we may assume that $t = t_2$ and $|\lambda_1| \leq |\lambda_2|$.

By Lemma 10.1 there exist constants $\overline{h}$, $\overline{H}$ such that for any $\xi, \eta \in \mathbb{P}E(\lambda_1, u_1) \cup \mathbb{P}E(\lambda_2, u_2)$ it holds that

$$h(\xi, \eta) \leq \overline{h},$$
$$H(\xi, \eta) \leq \overline{H},$$

where furthermore the corresponding control steering from ξ to η may be chosen to be universally regular if $\eta \in \mathbb{P}E(\lambda_1, u_1)$. Choose $\xi_j \in \mathbb{P}E(\lambda_j, u_j)$, $j = 1, 2$. Clearly it holds for $j = 1, 2$

$$\lambda(\xi_j, u_j) = \frac{1}{t} \log |\lambda_j|.$$

We wish to construct controls such that the corresponding Floquet exponents are dense in the interval $[\frac{1}{t} \log |\lambda_1|, \frac{1}{t} \log |\lambda_2|]$. To this end define the control $u_{k,l,m}$, $k, l, m \in \mathbb{N}$ by

$$u_{k,l,m} := \left((u_1)^{mk}, v_{1,k,m}, (u_2)^{ml}, v_{2,m,l}\right),$$

where $s_{1,k,m}, s_{2,m,l} \leq \overline{h}$ and $v_{1,k,m} \in \operatorname{int} U^{s_{1,k,m}}$ is chosen such that $\xi(s_{1,k,m}; \xi(mkt; \xi_1, (u_1)^{mk}), v_{1,k,m}) = \xi_2$ and analogously $\xi(s_{2,l,m}; \xi(mlt; \xi_2, (u_2)^{ml}), v_{2,l,m}) = \xi_1$ for a universally regular control $v_{2,l,m}$, which is possible by Lemma 10.1(ii). We obtain in all that $\xi_1 = \xi(m(k+l)t + s_{1,k,m} + s_{2,k,m}; \xi_1, u_{k,l,m})$. Thus for some $r \in \mathbb{R}$ it holds that $\xi_1 \in \mathbb{P}GE(r, u_{k,l,m})$. This projected sum of generalized eigenspaces is regular by the universal regularity of $v_{2,l,m}$. The corresponding Floquet exponent is given by

$$\lambda(\xi_1, u_{k,l,m}) = \frac{1}{m(k+l)t + h(k,l,m)}(J(mkt; \xi_1, (u_1)^{mk}) + J(mlt; \xi_2, (u_2)^{ml}) + H(k,l,m))$$

where $h(k,l,m) \leq 2\overline{h}$ and $|H(k,l,m)| \leq 2\overline{H}$ for all $k,l,m \in \mathbb{N}$. Thus for $k,l \geq 1$ it may be seen that

$$\begin{aligned}\lim_{m\to\infty} \lambda(\xi_1, u_{k,l,m}) &= \lim_{m\to\infty} \frac{1}{m(k+l)t}(J(mkt; \xi_1, (u_1)^{mk}) + J(mlt; \xi_2, (u_2)^{ml})) \\ &= \frac{k\lambda(\xi_1, u_1) + l\lambda(\xi_2, u_2)}{k+l} \in \operatorname{cl} \Sigma_{Fl}(D).\end{aligned}$$

Clearly the set of points that may be obtained by choosing different $k,l \in \mathbb{N}$ is dense in $[\lambda(\xi_1, u_1), \lambda(\xi_2, u_2)]$. □

Corollary 10.4 *Assume that (5) is forward accessible.*

(i) If $\mathbb{K} = \mathbb{R}, \mathbb{C}$ then for every control set D with $\operatorname{core}(D) \neq \emptyset$ it holds that

$$\operatorname{cl} \Sigma_{Fl}(D) = \operatorname{cl} \bigcup_{t\in\mathbb{N}, u\in U^t_{reg}} \{\frac{1}{t}\log|\lambda|;\ \lambda \in \sigma(\Phi_u(t,0)),\ \mathbb{P}GE(\lambda, u) \subset \operatorname{core}(D)\}\,. \tag{122}$$

(ii) If $\mathbb{K} = \mathbb{R}$ then for every control set D with nonempty core

$$\operatorname{cl} \Sigma_{Fl}(D) = \operatorname{cl} \bigcup_{t\in\mathbb{N}, u\in U^t_{reg}} \{\frac{1}{t}\log|\lambda| \in \Sigma_{Fl}(D);\ \lambda \in \sigma(\Phi_u(t,0)) \cap \mathbb{R}\}\,. \tag{123}$$

Proof: (i) If for some $u \in U^t$ and $r \in |\sigma(\Phi_u(t,0))|$ it holds that $\mathbb{P}GE(r,u) \subset \operatorname{core}(D)$ then by the genericity of the universally regular controls and the continuity of the eigenvalues and eigenprojections we may choose universally regular controls whose eigenspaces project to the core of D and whose corresponding Floquet exponents approximate the Floquet exponent $\frac{1}{t}\log r$ arbitrarily close. This shows the assertion.

(ii) As the intermediate values $\lambda(\xi_1, u_{k,l,m})$ constructed in the previous proof are in fact Floquet exponents corresponding to a real eigenvalue of $\Phi_{u_{k,l,m}}(m(k+l)t + s_{1,k,m} + s_{2,l,m}, 0)$ it follows that it is sufficient to consider real eigenvalues. Now we may argue as in part (i).

□

Theorem 10.5 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (5) is forward accessible. Let κ be equal to the number of main control sets.*

(i) For each main control set D_j $j = 1, \dots, \kappa$ the closed Floquet spectrum is an interval. We define

$$\mathrm{cl}\, \Sigma_{FL}(D_j) =: [\alpha_j, \beta_j], \quad \alpha_j \leq \beta_j \,. \tag{124}$$

(ii) If all control sets with nonempty interior are main control sets then

$$\mathrm{cl}\, \Sigma_{FL}(A, U) = \bigcup_{j=1}^{\kappa} [\alpha_j, \beta_j] \,. \tag{125}$$

(iii) If there exist control sets with nonempty interior that are not main control sets then there exists a constant $\bar{\beta} \in \mathbb{R}$ such that

$$\mathrm{cl}\, \Sigma_{FL}(A, U) = \bigcup_{j=1}^{\kappa} [\alpha_j, \beta_j] \cup [-\infty, \bar{\beta}] \,. \tag{126}$$

(iv) If for two main control sets $D_{j_1} < D_{j_2}$ then

$$\alpha_{j_1} \leq \alpha_{j_2} \,, \tag{127}$$

$$\beta_{j_1} \leq \beta_{j_2} \,. \tag{128}$$

(v) For $j = 1, \dots, \kappa$ it holds that

$$\# \, \mathrm{cl}\, \Sigma_{FL}(D_j) \setminus \Sigma_{FL}(D_j) \leq m(D_j) + 1 \,. \tag{129}$$

Proof: (i) This is clear by Proposition 10.3.

(ii) Let $t \in \mathbb{N}$, $u \in U^t$ and consider $\sigma_{Fl}(u)$. As the Floquet spectrum of u does not change if we consider $(u)^l$ for some $l \geq 1$ we may assume that $t \geq t^*$. Hence, we may choose a sequence $\{u_k\}_{k \in \mathbb{N}} \subset U^t_{reg}$ converging to u for k tending to infinity. By the continuity of the spectrum it follows that $\sigma_{Fl}(u) \subset \bigcup_{j=1}^{\kappa} [\alpha_j, \beta_j]$.

(iii) For a control set D with nonempty interior that is not a main control set it holds by Proposition 8.1 that $\inf \Sigma_{Fl}(D) = -\infty$. The assertion thus follows from Proposition 10.3 and the argumentation of (ii).

(iv) If for two main control sets $D_{j_1} \leq D_{j_2}$ then $Q_i \subset D_{j_1}$ and $Q_j \subset D_{j_2}$ implies that $i < j$. Thus the assertion follows from the obvious inequalities $\inf \Sigma_{Fl,i}(A, U) \leq \inf \Sigma_{Fl,j}(A, U)$ and $\sup \Sigma_{Fl,i}(A, U) \leq \sup \Sigma_{Fl,j}(A, U)$ if $i < j$.

(v) Let $t \in \mathbb{N}$ and $u, v \in U^t_{reg}$. Consider a continuous path $\gamma : [0, 1] \to \mathrm{int}\, U^t$ with $\gamma(0) = u$ and $\gamma(1) = v$. Now consider the control $(\gamma(\tau), u)$. For every $\tau \in [0, 1]$ and every $i \in \{1, \dots, n\}$ it holds by the universal regularity of u that $r_i(\tau) := r_i(\Phi_{(\gamma(\tau),u)}(2t, 0)) > 0$ iff $\mathbb{P}GE(r_i(\tau), (\gamma(\tau), u))$ is regular. Thus it follows for every $i \in \{1, \dots, n\}$ that the interval $[r_i(\Phi_{(u,u)}(2t, 0)), r_i(\Phi_{(v,u)}(2t, 0))]$ is contained in $\Sigma_{Fl}(D)$ for some control set D by Proposition 6.8. As the sets $\mathrm{int}\, \Sigma_{Fl,i}(A, U)$ are intervals and by $\mathrm{cl}\, \Sigma_{Fl}(D_j) = \bigcup_{Q_i \subset D_j} \mathrm{cl}\, \Sigma_{Fl,i}(A, U)$, it follows that the only points where the Floquet spectrum of a main control set and its closure may differ are the endpoints of the intervals $\Sigma_{Fl,i}(A, U)$. Of these there are at most $m(D_j) + 1$, which shows the assertion.

□

It should be noted, that the spectral intervals corresponding to different main control sets may overlap, i.e. that the statement $\alpha_i \leq \alpha_j$, $\beta_i \leq \beta_j$ in Theorem 10.5 does in no way exclude the possibility that $\beta_i > \alpha_j$. In fact, it is even possible that $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$ for $i \neq j$. To illustrate this phenomenon consider the following example.

Example 10.6 Let $\mathbb{K} = \mathbb{R}$. Define

$$A: \mathbb{R}^4 \longrightarrow \mathbb{R}^{2\times 2}$$
$$A(a,b,c,d) := \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let $\mathbb{R}^n_{\geq 0}$ denote the set of vectors with nonnegative real entries. Define

$$U := \{[a\ b\ c\ d] \in \mathbb{R}^4_{\geq 0}; \quad a + c \leq 1;\ b + d \leq 1\}.$$

Then $A(U)$ is exactly the set of nonnegative matrices in $\mathbb{R}^{2\times 2}$ with 1–norm less or equal to 1. As the set of nonnegative vectors in $\mathbb{R}^2$ is invariant under $A(u)$ for any $u \in U$, i.e. $A(u)\mathbb{R}^2_{\geq 0} \subset \mathbb{R}^2_{\geq 0}$, it follows that the invariant control set $D_2 = C \subset \mathbb{P}\mathbb{R}^2_{\geq 0}$. Hence there exists also a minimal control set $D_1 = C^-$ and no other control set D with $\operatorname{core}(D) \neq \emptyset$.
Clearly $\alpha_1 = \alpha_2 = -\infty$ as $0 \in A(U)$. Let us show that also $\beta_1 = \beta_2 = 0$. For any $t \geq 1$, $u \in U^t$, it holds that

$$r(\Phi_u(t,0)) \leq \|\Phi_u(t,0)\|_1 \leq \|A(u(t-1))\|_1 \cdot \ldots \cdot \|A(u(0))\|_1 \leq 1.$$

Hence $\beta_1, \beta_2 \leq \log 1 = 0$. On the other hand $I \in A(U)$ and so $0 \in \operatorname{cl}\Sigma_{FL}(C)$, $0 \in \operatorname{cl}\Sigma_{FL}(C^-)$ and $\beta_1, \beta_2 \geq 0$.
In order to construct a two-dimensional example with identical spectral intervals and $U = U_{inv}$ it is sufficient to replace the set U of the previous example by $U' := \{u \in U\ ;\ \det(A(u)) > 0\}$. If we require that $\operatorname{cl} A(U)$ consists of invertible matrices then it is still possible to make upper or lower boundaries of spectral intervals equal, e.g. if the map A is replaced by $u \mapsto \exp(A(u))$. Note that also $\exp(A(U))$ consists of nonnegative matrices. Then similarly to the preceding discussion it is possible to obtain that for this modified example $\beta_1 = \beta_2 = 1$. However, this comes with the price that $\alpha_1 = -1 \neq \alpha_2 = 0$. It is not known whether identical spectral intervals to different main control sets are possible if it is assumed that $\det(A(u)) \neq 0$ for all $u \in \operatorname{cl} U$.

11 The Lyapunov and the Bohl Spectrum

Let us now discuss how the results on the Floquet spectrum can be related to the other spectra of characteristic exponents. We begin by showing that the Lyapunov exponents corresponding to trajectories that evolve in a specific way in the core of control sets, are contained in the closure of the associated Floquet interval. On the other hand to every element of the closure of the Floquet interval of a control set there exists a control sequence that realizes this number as a Lyapunov exponent.

Theorem 11.1 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume that (5) is forward accessible.*

(i) Let D be a control set, with $\operatorname{core}(D) \neq \emptyset$. Assume that $(\xi_0, u) \in \mathbb{P}^{n-1}_{\mathbb{K}} \times U^{\mathbb{N}}(\xi_0)$ are given with $\omega^+(\xi_0, u) \subset D$. If there exists a $t_0 \in \mathbb{N}$ with $\xi(t_0; \xi_0, u) \in \operatorname{core}(D)$ then $\lambda(\xi_0, u) \in \operatorname{cl}\Sigma_{Fl}(D)$.

(ii) Let D be a control set, with $\operatorname{core}(D) \neq \emptyset$*, then*

$$\operatorname{cl} \Sigma_{Fl}(D) \subset \Sigma_{Ly}(A, U). \tag{130}$$

Proof: (i) Without loss of generality we may assume that $t_0 = 0$ as the Lyapunov exponents satisfy $\lambda(\xi_0, u) = \lambda(\xi(t_0; \xi_0, u), u(t_0 + \cdot))$ where $u(t_0 + \cdot) = (u(t_0), u(t_0 + 1), \ldots)$ is the shifted control. Let $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ be an increasing sequence such that

$$\lim_{k \to \infty} \frac{1}{t_k} J(t_k; \xi_0, u) = \lambda(\xi_0, u)\,. \tag{131}$$

Taking a subsequence we may assume that

$$\lim_{k \to \infty} \xi(t_k; \xi_0, u) =: \eta \in \omega^+(\xi_0, u) \subset D\,. \tag{132}$$

As $\xi_0 \in \operatorname{core}(D)$ it follows that $\eta \in \hat{\mathcal{O}}^-(\xi_0)$ and hence there is a $t \in \mathbb{N}$ and a neighborhood $V(\eta)$ such that $V(\eta) \subset \hat{\mathcal{O}}^-_t(\xi_0)$. For k large enough it holds that $\xi(t_k; \xi_0, u) \in V(\eta)$. By continuous dependence of $\xi(t_k; \xi_0, u)$ on u, the continuous dependence of $J(t_k; \xi_0, u)$ on u and the genericity of $U^{t_k}_{reg}$ we may choose controls $u_k \in U^{t_k}_{reg}$ such that $\xi(t_k; \xi_0, u_k) \in V(\eta)$ for all k large enough and

$$\lim_{k \to \infty} \frac{1}{t_k} J(t_k; \xi_0, u_k) = \lambda(\xi_0, u)\,.$$

We can therefore find a controls $v_k \in \operatorname{int} U^t$ such that $\xi_0 = \xi(t; \xi(t_k; \xi_0, u_k), v_k)$. The Floquet exponent corresponding to the control (u_k, v_k) and ξ_0 is given by

$$\begin{aligned} \lambda(\xi_0, (u_k, v_k)) &= \frac{1}{t_k + t} \left(J(t_k; \xi_0, u_k) + J(t; \xi(t_k; \xi_0, u_k), v_k) \right) \\ &= \lambda(\xi(t_k; \xi_0, u_k), (v_k, u_k)) \in \Sigma_{Fl}(D)\,, \end{aligned} \tag{133}$$

by the universal regularity of u_k. Letting $k \to \infty$ the assertion follows after noting that Lemma 10.1 guarantees that the v_k can be chosen so that $|J(t; \xi(t_k; \xi_0, u_k), v_k)|$ is bounded independently of k.

(ii) Let $\lambda^* \in \operatorname{cl} \Sigma_{Fl}(D)$. Let $u_k \in U^{t_k}_{reg}$, $k \in \mathbb{N}$ be a sequence of controls such that

$$\lim_{k \to \infty} \frac{1}{t_k} \log |\lambda_k| = \lambda^* \tag{134}$$

where $\lambda_k \in \sigma(\Phi(t_k, u_k))$ and $\mathbb{P}E(\lambda_k, u_k) \subset \operatorname{core}(D)$. By Corollary 10.4 such a sequence exists and we may assume that $\lambda_k \in \mathbb{R}$, if $\mathbb{K} = \mathbb{R}$. For $k \in \mathbb{N}$ let $\xi_k \in \mathbb{P}E(\lambda_k, u_k)$. Therefore it holds for all $l, k \in \mathbb{N}$ that $\xi(lt_k; \xi_k, (u_k)^l) = \xi_k \in \operatorname{core}(D)$. For all $k \in \mathbb{N}$ there exists a control $v_k \in U^{s_k}$ such that $\xi_{k+1} = \xi(s_k; \xi_k, v_k)$. Let H_k be such that $|J(s; \xi_k, v_k)| < H_k$ for $0 \leq s \leq s_k$. We construct a control that generates the Lyapunov exponent λ^* as follows: Choose $m_1 \in \mathbb{N}$ such that

$$\left| \left(\frac{m_1 t_1}{m_1 t_1 + s_1 + t_2} - 1 \right) \frac{1}{t_1} \log |\lambda_1| \right| < \frac{1}{8}\,, \tag{135}$$

$$\frac{H_1}{m_1 t_1} < \frac{1}{8}\,, \tag{136}$$

$$\left|\frac{J(s;\xi_2,u_2)}{m_1t_1}\right| < \frac{1}{8}, \quad 0 \le s \le t_2\,, \tag{137}$$

Let $u_1^* := ((u_1)^{m_1}, v_1) \in U^{T_1}$ and $T_1 := m_1t_1 + s_1$. Using (135) and (136) it may be seen that for $0 \le s \le s_1$

$$\begin{aligned} &\left|\frac{1}{m_1t_1+s}J(m_1t_1+s;\xi_1,u_1^*) - \frac{1}{t_1}\log|\lambda_1|\right| \\ &\le \frac{1}{m_1t_1+s}|J(s;\xi_1,v_1)| + \left|\left(\frac{m_1t_1}{m_1t_1+s}-1\right)\frac{1}{t_1}\log|\lambda_1|\right| < \frac{1}{4}\,. \end{aligned} \tag{138}$$

Note also that by (137), we obtain as in (138) that for $0 \le s \le t_2$ and $v = (u_1^*, u_2)$

$$\begin{aligned} &\left|\frac{1}{T_1+s}J(T_1+s;\xi_1,v) - \frac{1}{t_1}\log|\lambda_1|\right| \\ &\le \left|\frac{1}{T_1+s}J(m_1t_1;\xi_1,v) - \frac{1}{t_1}\log|\lambda_1|\right| + \frac{H_1}{T_1+s} + \left|\frac{1}{T_1+s}J(s;\xi_2,u_2)\right| < \frac{1}{2}\,. \end{aligned} \tag{139}$$

For $k > 1$ assume that we have constructed u_{k-1}^*, m_{k-1} and T_{k-1} such that for $-s_{k-1} \le s \le t_k$ it holds that

$$\left|\frac{1}{(T_{k-1}+s)}J(T_{k-1}+s;\xi_1,(u_{k-1}^*,u_k)) - \frac{1}{t_{k-1}}\log|\lambda_{k-1}|\right| < 2^{-(k-1)}\,. \tag{140}$$

Choose $m_k \in \mathbb{N}$ such that

$$\left|\frac{1}{T_{k-1}+m_kt_k}J(T_{k-1};\xi_1,u_{k-1}^*)\right| < 2^{-(k+3)}\,, \tag{141}$$

$$\left|\left(\frac{m_kt_k}{T_{k-1}+m_kt_k+s_k+t_{k+1}}-1\right)\frac{1}{t_k}\log|\lambda_k|\right| < 2^{-(k+3)}\,, \tag{142}$$

$$\frac{H_k}{T_{k-1}+m_kt_k} < 2^{-(k+3)}\,, \tag{143}$$

$$\left|\frac{J(s;\xi_{k+1},u_{k+1})}{m_kt_k}\right| < 2^{-(k+2)}, \quad 0 \le s \le t_{k+1}\,, \tag{144}$$

Set $u_k^* := (u_{k-1}^*, (u_k)^{m_k}, v_k)$ and $T_k := T_{k-1} + m_kt_k + s_k$. For $T_{k-1} + m_kt_k \le t \le T_k$ we obtain with (141),(142) and (143) that

$$\begin{aligned} &\left|\frac{1}{t}J(t;\xi_1,u_k^*) - \frac{1}{t_k}\log|\lambda_k|\right| \\ &\le \left|\frac{1}{t}J(T_{k-1};\xi_1,u_{k-1}^*)\right| + \left|\left(\frac{m_kt_k}{t}-1\right)\frac{1}{t_k}\log|\lambda_k|\right| + \frac{1}{t}|J(t-T_{k-1}-m_kt_k;\xi_k,v_k)| \\ &< 2^{-(k+3)} + 2^{-(k+3)} + 2^{-(k+3)} < 2^{-(k+1)}\,. \end{aligned}$$

Analogously to (139) it may be seen from (142) and (144) that for $0 \le s \le t_{k+1}$ and $v = (u_k^*, u_{k+1})$

$$\left|\frac{1}{T_k+s}J(T_k+s;\xi_1,v) - \frac{1}{t_k}\log|\lambda_k|\right| < 2^{-k}\,. \tag{145}$$

For the control u^* that is recursively defined via $u^*_{[0,T_k]} = u^*_k$ we claim that

$$\lambda(\xi_1, u^*) = \lim_{k\to\infty} \frac{1}{t_k} \log|\lambda_k| = \lambda^* . \tag{146}$$

As we have shown that for $k > 1$ and $T_{k-1} + m_k t_k \leq t \leq T_k + t_{k+1}$ it holds that

$$\left| \frac{1}{t} J(t; \xi_1, u^*) - \frac{1}{t_k} \log|\lambda_k| \right| < 2^{-k} ,$$

our claim follows if we can show that for $t = T_{k-1}, \ldots, T_{k-1} + (m_k - 1)t_k$ the following relation holds

$$\left| \frac{1}{t} J(t; \xi_1, u^*_k) - \frac{1}{t_k} \log|\lambda_k| \right| \geq \left| \frac{1}{t + t_k} J(t + t_k; \xi_1, u^*_k) - \frac{1}{t_k} \log|\lambda_k| \right| , \tag{147}$$

because this means that the sequences behaves in a monotonic way, at least if viewed at every t_k-th step. For $l = 0, \ldots, m_k - 1$ and $t = T_{k-1} + l t_k$ this is clear by

$$\begin{aligned} \left| \frac{1}{t} J(t; \xi_1, u^*_k) - \frac{1}{t_k} \log|\lambda_k| \right| &= \left| \frac{1}{t} (J(T_{k-1}; \xi_1, u^*_k) + l \log|\lambda_k|) - \frac{1}{t_k} \log|\lambda_k| \right| \\ &= \frac{1}{t} \left| J(T_{k-1}; \xi_1, u^*_k) - \frac{T_{k-1}}{t_k} \log|\lambda_k| \right| . \end{aligned}$$

The other cases can be treated using the same argument, with the modification that the time from which periodicity is used is not T_{k-1} but $T_{k-1} + s$ for some $0 \leq s \leq t_k - 1$. This proves the assertion. □

A further question of interest, especially if stabilization and robust stability questions are considered, concerns the lower and upper bounds of the spectral sets that we have defined. For a general discrete inclusion given by a bounded set $\Sigma \subset \mathbb{K}^{n\times n}$ and

$$x(t+1) \in \{Ax(t) \ ; \ A \in \Sigma\} , \quad t \in \mathbb{N} \tag{148}$$

this has been studied in [8], [15], [38], [27]. In particular the latter three references study the relation between the generalized spectral radius

$$\bar{\rho}(\Sigma) := \limsup_{t\to\infty} \bar{\rho}_t(\Sigma)^{1/t} ,$$

where

$$\bar{\rho}_t(\Sigma) := \sup\{r(A_{t-1} \cdot \ldots \cdot A_0); A_s \in \Sigma, s = 0, \ldots, t-1\} .$$

and the joint spectral radius

$$\hat{\rho}(\Sigma) := \limsup_{t\to\infty} \hat{\rho}_t(\Sigma)^{1/t} ,$$

where

$$\hat{\rho}_t(\Sigma) := \sup\{\|A_{t-1} \cdot \ldots \cdot A_0\|; A_s \in \Sigma, s = 0, \ldots, t-1\} .$$

Theorem IV in [15] states that for every bounded set Σ we have $\bar{\rho}(\Sigma) = \hat{\rho}(\Sigma)$. Although Berger and Wang restrict themselves to the real case, it is clear that they also prove the complex case, which may be seen via identification of $\mathbb{C}^{n\times n}$ with $\mathbb{R}^{2n\times 2n}$. Note that these definitions correspond to our definitions but for the fact that we have introduced the logarithm thus it is easy to see that

$$\log(\bar{\rho}(A(U))) = \sup \Sigma_{Fl}(A, U) , \tag{149}$$

and

$$\log(\hat\rho(\Sigma)) = \limsup_{t\to\infty} \sup_{u\in U^{\mathbb{N}},\, \xi\in\mathbb{P}^{n-1}_{\mathbb{K}}} \frac{1}{t} J(t;\xi,u)\,. \tag{150}$$

We therefore immediately obtain the following corollaries where we do not have to make our usual forward accessibility assumption. In order to conform to our previously introduced notation we will still think of the discrete inclusion to be given by an analytic map A and a set U. Note, however, that if we drop Assumption 2.1 then any bounded set of matrices may be represented in this way.

Corollary 11.2 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (2). Assume that $A(U)$ is bounded then*

$$\sup \Sigma_{Fl}(A,U) = \sup \Sigma_{Ly}(A,U) = \sup \Sigma_{Bo}(A,U) = \limsup_{t\to\infty} \sup_{u\in U^{\mathbb{N}},\, \xi\in\mathbb{P}^{n-1}_{\mathbb{K}}} \frac{1}{t} J(t;\xi,u)\,.$$

Using this result we can also prove the following statements on the infima of the spectra.

Proposition 11.3 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and consider system (2). Assume that $A(U)$ is bounded, then*

$$\inf \Sigma_{Fl}(A,U) = \inf \Sigma_{Ly}(A,U) = \liminf_{t\to\infty} \inf_{u\in U^{\mathbb{N}},\, \xi\in\mathbb{P}^{n-1}_{\mathbb{K}}} \frac{1}{t} J(t;\xi,u). \tag{151}$$

Proof: Obviously, it holds that

$$\inf \Sigma_{FL}(A,U) \geq \inf \Sigma_{Ly}(A,U) \geq \liminf_{t\to\infty} \inf_{u\in U^{\mathbb{N}},\, \xi\in\mathbb{P}^{n-1}_{\mathbb{K}}} \frac{1}{t} J(t;\xi,u)\,.$$

If there exists a $u \in \operatorname{cl} U$ such that $\det(A(u)) = 0$ the claim is trivially true as both infima are given by $-\infty$. If this is not the case we may consider the time–reversed system

$$\begin{array}{rcl} x(t+1) &=& A(u(t))^{-1}x(t),\ t\in\mathbb{N} \\ x(0) &=& x_0 \in \mathbb{K}^n \\ u(t) &\in& U,\ t\in\mathbb{N}. \end{array} \tag{152}$$

Denote the Floquet spectrum of the time–reversed system by $\Sigma^-_{Fl}(A,U)$. It is immediate that $\sup \Sigma^-_{Fl}(A,U) = -\inf \Sigma_{Fl}(A,U)$. Note also that

$$\inf_{x\in\mathbb{K}^n, \|x\|=1} \log \|\Phi_u(t,0)x\| = - \sup_{x\in\mathbb{K}^n, \|x\|=1} \log \|\Phi_u(t,0)^{-1}x\|$$

and therefore

$$\liminf_{t\to\infty} \inf_{u\in U^{\mathbb{N}},\, \xi\in\mathbb{P}^{n-1}_{\mathbb{K}}} \frac{1}{t} J(t;\xi,u) = - \limsup_{t\to\infty} \sup_{u\in U^{\mathbb{N}}, \xi\in\mathbb{P}^{n-1}_{\mathbb{K}}} \frac{1}{t} J^-(t;\xi,u)$$

where $J^-(t;\xi,u) = \log \frac{\|\Phi_u(t,0)^{-1}x\|}{\|x\|}$ for $\xi = \mathbb{P}x$. The assertion now follows by applying Corollary 11.2. □

Barabanov [9] proved that to each discrete inclusion given by a bounded set of matrices there exists a trajectory that realizes the maximal Lyapunov exponent. The following statement brings this in relation to the control structure of system (5).

Proposition 11.4 *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, let Assumption 2.1 hold and assume that (5) is forward accessible, then*

(i) There exist $u \in U^{\mathbb{N}}$, $\xi \in C$ such that $\lambda(\xi, u) = \beta(u) = \sup \Sigma_{Ly}(A, U)$.

(ii) There exist $v \in U^{\mathbb{N}}$, $\eta \in C^-$ such that $\lambda(\eta, v) = \inf \Sigma_{Ly}(A, U)$.

Proof: (i) resp. (ii) follow from Corollary 11.2 resp. Proposition 11.3, Remark 7.4(ii) and Theorem 11.1 (ii). □

If the finiteness conjecture holds as discussed by Lagarias and Wang [38] then the previous result can be restated in terms of the Floquet spectrum, i.e. it would be possible to realize maximal and minimal Floquet exponent via some periodic control sequence u. This is the topic of ongoing research.

Let us also note that Gurvits [27] has shown that for discrete inclusions given by finitely many matrices the indices $\inf \Sigma_{Fl,n}(A, U)$ and $\inf \Sigma_{Bo}(A, U)$ coincide. It remains to be investigated how this result may be carried over to our case.

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