ASYMPTOTICS OF VALUE FUNCTIONS OF DISCRETE-TIME DISCOUNTED OPTIMAL CONTROL

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Abstract. We study deterministic discounted optimal control problems associated with discrete-time systems. It is shown that for small discount rates controllability properties of the underlying system can guarantee the convergence of the discounted value function to the value function of the average yield. An application in the theory of exponential growth rates of discrete inclusions is presented. This application motivates the analysis of the infinite horizon optimal control problems with running yields that are unbounded from below.

1. Introduction

This paper investigates the problem when the value functions of discrete-time discounted optimal control problems converge to the value function of average yield (or 0-discount) optimal control problems as the discount rate tends to 0. This topic has attracted the interest of various researchers in recent years, where in the discrete time-case the studies are motivated by problems in Markov decision chains, while in the continuous-time case results were obtained motivated by the analysis of Lyapunov exponents, i.e. exponential growth rates of families of time-varying systems.

In this paper we wish to present results for deterministic discrete-time optimal control problems. In the theory of Markov decision chains the behavior of optimal policies and of the value functions as the discount rate δ tends to zero has attracted considerable attention, see Cavazos-Cadena and Lasserre [4] and Laušmanová [13] for early references. In [7] Dutta proved convergence of the discounted value function to the average yield for stochastic Markov decision chains in a set-up encompassing deterministic systems. The assumptions in this paper, however, guarantee that the value function of the average yield optimal control problem is constant on the state space. This excludes many applications of the deterministic theory. Yushkevich studied convergence in [20], [21] for stochastic systems satisfying a simultaneous Doeblin-Doob condition which essentially states that there exists a subset D of the state space with positive measure such that for any measurable set $D' \subset D$ with positive measure the following holds: given any control value and any point in the state space there is positive probability that at the next time step the system state is in D'. This assumption excludes all but the most simplistic deterministic systems. A maximum principle for discounted optimal control was presented by Sorger in [15]. In Stern [16] several criteria of optimality in the infinite horizon case are discussed.

In this paper we follow an approach that was outlined in Colonius [5], Wirth [17] and Grüne [10], [11] for the continuous-time case. It has been shown in [17], that in general it is not possible to approximate average yield optimal control problems by discounted ones. This example easily translates to the discrete-time case. Thus it is necessary to find conditions that ensure convergence. The basic idea of the present approach is to use the dynamics of the discrete-time system in the construction of approximately optimal periodic controls. This is done by introducing control sets, that is sets where approximate controllability holds, and use controllability properties in their respective interiors. We present results on uniform convergence of the discounted value functions on compact subsets of cores of control sets, which largely resemble results obtained in continuous time, though we use different ideas of proof and obtain results with less stringent conditions. Further results on uniform convergence on compact subsets of backward orbits

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of control sets, are essentially new in the sense, that their continuous time analogues are not available.

The paper is organized as follows. In Section 2 we present the class of optimal control problems we wish to consider. In particular we will study the minimization as well as the maximization problem. Motivated by problems in discrete inclusions the running yield will only be bounded from above. This leads to problems not considered in [5], [17], [10], [11] so that we have to develop different methods from those used in the continuous time case. Section 3 presents preliminary convergence results along trajectories and their implications for the value functions. The results of this section are independent of controllability properties. In Section 4 control sets are introduced and it is shown how their properties may be used to obtain uniform convergence in compact subsets of their interiors respectively their backward orbits. In particular the result on uniform convergence on the backward orbit of control sets for the maximization problem is new even in continuous time. As an application of the previously obtained results we discuss Lyapunov exponents of discrete inclusions in Section 5. This topic has been of particular interest in recent years, see Barabanov [2], Berger and Wang [3] and Wirth [19].

2. Problem Statement

Let M be a real, connected, paracompact, Riemannian, C^ω-manifold of dimension n, and $\tilde{U} \subset \mathbb{R}^m$ be an open set. Assume we are given an analytic set $X \subset M \times \tilde{U}$ and a real-analytic map $f : W \to M$, where $W := (M \times \tilde{U}) \setminus X$. We denote $f_0 = f$ and define recursively $f_{t+1}(x, u_0, \dots, u_t) := f(f_t(x, u_0, \dots, u_{t-1}), u_t)$. The set of admissible control values for a given point $x \in M$ is denoted by $U(x)$ which is determined by $\{x\} \times U(x) = (\{x\} \times U) \setminus X$. The sets of admissible control sequences of length t, respectively of infinite length is denoted by $U^t(x), U^{\mathbb{N}}(x)$. Assume that the set of admissible control values U and the map f satisfy

(i) For all $x \in M$ it holds that $\{x\} \times U \not\subset X$.
(ii) For all $t \in \mathbb{N}$ and all $x \in M$ $f_t(x, \cdot)$ is nontrivial with respect to u, i.e. if $\partial f_t(x, \cdot)/\partial u_0 \dots \partial u_{t-1}$ has full rank in some point $u \in U^t(x)$, then in every connected component of $U^t(x)$ there exists a point where this rank condition is satisfied.

We consider the discrete-time system

$$x(t+1) = f(x(t), u(t)) \quad , t \in \mathbb{N} \tag{1}$$

$$x(0) = x_0 \quad \in M ,$$

$$u \in U^{\mathbb{N}}(x_0).$$

The solution corresponding to an initial value x_0 and an admissible control sequence $u \in U^{\mathbb{N}}(x_0)$ is denoted by $\phi(\cdot; x_0, u)$. To define an optimal control problem assume we are given a continuous *running yield* function

$$g : M \times \operatorname{cl} U \to \mathbb{R} \cup \{-\infty\},$$

satisfying

$$\begin{aligned} g(x,u) \le G_1 \in \mathbb{R} \text{ for all } (x,u) \in M \times \operatorname{cl} U \,, \\ g(x,u) = -\infty \implies (x,u) \in X \,. \end{aligned} \tag{2}$$

The continuity of g is to be understood with respect to the topology on $\mathbb{R} \cup \{-\infty\}$ generated by the standard topology on $\mathbb{R}$ together with the sets $\{-\infty\} \cup (-\infty, c), c \in \mathbb{R}$ as a neighborhood basis for the point $-\infty$. Furthermore, we assume the existence of a constant $G_2 \in \mathbb{R}$ such that for every $x \in M$ there exists a $u(x) \in U$ with

$$G_2 \le g(x, u(x)). \tag{3}$$

Without loss of generality we will assume that $0 < G := G_1 = -G_2$. The following values are associated with a trajectory of (1). For $\delta > 0$ we consider the δ-discounted yield

$$J_\delta(x,u) := \sum_{t=0}^{\infty} e^{-\delta t} g(\phi(t;x,u),u(t)), \quad u \in U^{\mathbb{N}}(x), \tag{4}$$

and the average yield functionals

$$\overline{J_0}(x,u) := \limsup_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} g(\phi(s;x,u),u(s)), \quad u \in U^{\mathbb{N}}(x), \tag{5}$$

$$\underline{J_0}(x,u) := \liminf_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} g(\phi(s;x,u),u(s)), \quad u \in U^{\mathbb{N}}(x). \tag{6}$$

The associated optimal value functions are given by

$$V_\delta(x) := \sup_{u\in U^{\mathbb{N}}(x)} J_\delta(x,u), \quad v_\delta(x) := \inf_{u\in U^{\mathbb{N}}(x)} J_\delta(x,u), \tag{7}$$

$$\overline{V_0}(x) := \sup_{u\in U^{\mathbb{N}}(x)} \overline{J_0}(x,u), \quad \overline{v_0}(x) := \inf_{u\in U^{\mathbb{N}}(x)} \overline{J_0}(x,u), \tag{8}$$

$$\underline{V_0}(x) := \sup_{u\in U^{\mathbb{N}}(x)} \underline{J_0}(x,u), \quad \underline{v_0}(x) := \inf_{u\in U^{\mathbb{N}}(x)} \underline{J_0}(x,u). \tag{9}$$

Remark 2.1. *Note that for every $u \in U^{\mathbb{N}}(x)$ the expression for $J_\delta(x,u)$ is well defined. In fact, it holds that the infinite sum is either absolutely convergent, or the partial sums tend to $-\infty$. This may be seen as follows. Define $f_+(t) := \max\{0, e^{-\delta t} g(\phi(t;x,u),u(t))\}$ and $f_-(t) := \min\{0, e^{-\delta t} g(\phi(t;x,u),u(t))\}$. Then*

$$\sum_{s=0}^{t} |e^{-\delta s} g(\phi(s;x,u),u(s))| = \sum_{s=0}^{t} f_+(s) + |f_-(s)|.$$

Clearly the infinite sum over $f_+(s)$ exists, so that $\sum_{s=0}^{t} f_+(s) + f_-(s)$ converges absolutely iff $\lim_{t\to\infty} \sum_{s=0}^{t} f_-(s)$ is a real number. If this is not the case then for every $c > 0$ there exists a $T \in \mathbb{N}$ such that for all $t \geq T$ it holds that $\sum_{s=0}^{t} e^{-\delta s} g(\phi(s;x,u),u(s)) < -c$.

The discounted optimal control problem is far easier to analyze, which is why one tries to gain a relation between it and the average yield problem. The following theorem summarizes some known properties of the value functions. The proof is omitted as we only slightly depart from the standard assumptions in that g is merely bounded from above. The standard arguments, however, can be applied to obtain a proof. For details we refer to [8] Chapters 1 and 2.

Theorem 2.2. *Consider a discounted optimal control problem given by (1)–(7). The following properties hold*

(i) (Bellman's principle of optimality)
For all $t \in \mathbb{N}$ it holds that

$$V_\delta(x_0) = \sup_{u\in U^t(x)} \left[\sum_{s=0}^{t-1} e^{-\delta s} g(\phi(s;x_0,u)) + e^{-\delta t} V_\delta(\phi(t;x_0,u)) \right], \tag{10}$$

$$v_\delta(x_0) = \inf_{u\in U^t(x)} \left[\sum_{s=0}^{t-1} e^{-\delta s} g(\phi(s;x_0,u)) + e^{-\delta t} v_\delta(\phi(t;x_0,u)) \right]. \tag{11}$$

(ii) V_δ is bounded and continuous. v_δ is bounded from above and lower semi-continuous.

(iii) (Bellman's principle of optimality II)
For all $t \in \mathbb{N}$ it holds that

$$\overline{V_0}(x) = \sup_{u \in U^t(x)} \overline{V_0}(\phi(t;x,u))\,, \quad \overline{v_0}(x) = \inf_{u \in U^t(x)} \overline{v_0}(\phi(t;x,u))\,, \tag{12}$$

$$\underline{V_0}(x) = \sup_{u \in U^t(x)} \underline{V_0}(\phi(t;x,u))\,, \quad \underline{v_0}(x) = \inf_{u \in U^t(x)} \underline{v_0}(\phi(t;x,u))\,. \tag{13}$$

3. Convergence of the Value Functions

We consider a system of the form (1) and present results on the values of the different optimal control problems along trajectories. First properties on the convergence of the discounted problems to the average yield problems may thus be obtained. To several of the following statements the continuous time analogue is to our knowledge not available in the literature. Note that the analytic structure of the system is not used in this section. For a sequence $a : \mathbb{N} \to \mathbb{R}$, define

$$\underline{J_0}(a) := \liminf_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t-1} a(s)\,, \quad \overline{J_0}(a) := \limsup_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t-1} a(s) \tag{14}$$

and if $\underline{J_0}(a) \in \mathbb{R}$, respectively $\overline{J_0}(a) \in \mathbb{R}$, let

$$\underline{r_a}(t) := \inf_{T \geq t} \frac{1}{T+1} \sum_{s=0}^{T} a(s) - \underline{J_0}(a)\,, \quad \overline{r_a}(t) := \sup_{T \geq t} \frac{1}{T+1} \sum_{s=0}^{T} a(s) - \overline{J_0}(a)\,.$$

Note that for any a satisfying $\underline{J_0}(a) \in \mathbb{R}$ we have that $\underline{r_a} \leq 0$ and also $\underline{r_a}(t)$ converges monotonically to 0 as t goes to infinity. Converse statements hold for $\overline{r_a}$.

Proposition 3.1. *Let $a : \mathbb{N} \to \mathbb{R}$.*

(i) If $\underline{J_0}(a) \in \mathbb{R}$ then for any $\delta > 0$ we have

$$\underline{J_0}(a) + (1 - e^{-\delta})^2 \sum_{s=0}^{\infty} e^{-\delta s}(s+1)\underline{r_a}(s) \leq (1 - e^{-\delta}) \liminf_{t \to \infty} \sum_{s=0}^{t} e^{-\delta s} a(s)\,.$$

(ii) If $\overline{J_0}(a) \in \mathbb{R}$ then for any $\delta > 0$ we have

$$\overline{J_0}(a) + (1 - e^{-\delta})^2 \sum_{s=0}^{\infty} e^{-\delta s}(s+1)\overline{r_a}(s) \geq (1 - e^{-\delta}) \limsup_{t \to \infty} \sum_{s=0}^{t} e^{-\delta s} a(s)\,.$$

Proof. (i) Note that because of the convergence of $\underline{r_a}$ to 0 the series on the left hand side converges absolutely and we may freely rearrange the sum. Thus to prove the assertion it is sufficient to show that for all $t \in \mathbb{N}$

$$\sum_{s=0}^{t} e^{-\delta s} \underline{J_0}(a) + (1 - e^{-\delta}) \sum_{s=0}^{t} e^{-\delta s}(s+1)\underline{r_a}(s) + e^{-\delta(t+1)}(t+1)\underline{r_a}(t) =$$

$$\sum_{s=0}^{t} e^{-\delta s} \left(\underline{J_0}(a) + (s+1)\underline{r_a}(s) - s\underline{r_a}(s-1) \right) \leq \sum_{s=0}^{t} e^{-\delta s} a(s)\,, \tag{15}$$

where we set $\underline{r_a}(-1) := 0$. We proceed by induction. For $t = 0$ equation (15) is immediate from the definition of $\underline{r_a}(0)$. Assume that (15) is shown for $t-1$ and consider

$$\sum_{s=0}^{t} e^{-\delta s} a(s) = e^{-\delta t} a(t) + \sum_{s=0}^{t-1} e^{-\delta s} a(s)\,. \tag{16}$$

By definition we have that

$$\sum_{s=0}^{t-1} a(s) \geq t(\underline{J_0}(a) + \underline{r_a}(t-1))\,, \quad \sum_{s=0}^{t} a(s) \geq (t+1)(\underline{J_0}(a) + \underline{r_a}(t))\,.$$

Note that the factor of $a(t)$ in (16) is the smallest. Thus in order to minimize (16) we may assume $\sum_{s=0}^{t-1} a(s)$ to be as small as possible, i.e. equal to $t(\underline{J_0}(a) + \underline{r_a}(t-1))$. It follows that we may assume that

$$a(t) \geq \underline{J_0}(a) + (t+1)\underline{r_a}(t) - t\underline{r_a}(t-1) .$$

This combined with the induction hypothesis implies (15).
(ii) follows from (i) considering $-a$. □

Remark 3.2. *Note that by monotonicity we have for all $t \in \mathbb{N}$*

$$(t+1)\underline{r_a}(t) - t\underline{r_a}(t-1) \geq \underline{r_a}(t) ,$$

and conversely for $\overline{r_a}$. Thus by rearranging the series it follows immediately under the assumptions of Proposition 3.1 that

$$\underline{J_0}(a) + (1-e^{-\delta}) \sum_{s=0}^{\infty} e^{-\delta s} \underline{r_a}(s) \leq (1-e^{-\delta}) \liminf_{t\to\infty} \sum_{s=0}^{t} e^{-\delta s} a(s) \tag{17}$$

$$\overline{J_0}(a) + (1-e^{-\delta}) \sum_{s=0}^{\infty} e^{-\delta s} \overline{r_a}(s) \geq (1-e^{-\delta}) \limsup_{t\to\infty} \sum_{s=0}^{t} e^{-\delta s} a(s) . \tag{18}$$

From this property we obtain a number of corollaries. In particular, the right hand side in the following corollary is the analogue of the statement of Theorem 2.1 in [10], which has been proven in a different manner in that reference. In the following we do not assume that the the discounted sums converge.

Corollary 3.3. *Let $a : \mathbb{N} \to \mathbb{R}$, with $\underline{J_0}(a) < \infty$, $\overline{J_0}(a) > -\infty$ then*

$$\underline{J_0}(a) \leq \liminf_{\delta\to 0} (1-e^{-\delta}) \liminf_{t\to\infty} \sum_{s=0}^{t} e^{-\delta s} a(s) \leq \limsup_{\delta\to 0} (1-e^{-\delta}) \limsup_{t\to\infty} \sum_{s=0}^{t} e^{-\delta s} a(s) \leq \overline{J_0}(a) .$$

Proof. To consider the first inequality note that if the limit inferior over the averaged sums is $-\infty$ there is nothing to show. If $\underline{J_0}(a) \in \mathbb{R}$ the assertion follows from (17) and the fact that because of the convergence of $\underline{r_a}$ we have

$$\lim_{\delta\to 0} (1-e^{-\delta}) \sum_{s=0}^{\infty} e^{-\delta s} \underline{r_a}(s) = 0 ,$$

which may be seen by a straightforward calculation. The middle inequality in the assertion is obvious and the statement on the RHS can be shown by considering $-a$ and using the first part of the proof. □

Remark 3.4. *In particular it follows that if the limit of $1/t \sum_{s=0}^{t-1} a(s)$ exists than also the limit of the discounted values exists. Note that for a periodic there is a particularly easy way to calculate the average yield. Namely, if a has period p, then*

$$\liminf_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} a(s) = \limsup_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} a(s) = \frac{1}{p} \sum_{s=0}^{p-1} a(s) .$$

It is worth noting an immediate consequence of the preceding proposition with respect to the value function.

Corollary 3.5. *Consider system (1). For all $x \in M$ it holds that*

$$\liminf_{\delta\to 0} (1-e^{-\delta}) V_\delta(x) \geq \underline{V_0}(x) ,$$

$$\limsup_{\delta\to 0} (1-e^{-\delta}) v_\delta(x) \leq \overline{v_0}(x) .$$

Proof. Let $\{u_k\} \subset U^{\mathbb{N}}(x)$ be a sequence such that

$$\lim_{k\to\infty} \underline{J_0}(x,u_k) = \underline{V_0}(x)\,.$$

By Corollary 3.3 for all $k \in \mathbb{N}$ it holds that

$$\liminf_{\delta\to 0}(1-e^{-\delta})V_\delta(x) \geq \liminf_{\delta\to 0}(1-e^{-\delta})J_\delta(x,u_k) \geq \underline{J_0}(x,u_k)\,.$$

This shows the first assertion and the proof of the second is analogous. □

For a converse statement we introduce the following definition.

Definition 3.6. *Consider system* (1). *The average yield* $\overline{V_0}(x)$ *is called* uniformly approximable *if there exists a function* $r : \mathbb{N} \to \mathbb{R}_+$ *with* $\lim_{t\to\infty} r(t) = 0$ *such that for all* $t \in \mathbb{N}$

$$\sup_{u\in U^{\mathbb{N}}(x)} \frac{1}{t}\sum_{s=0}^{t-1} g(\phi(s;x,u),u(s)) \leq \overline{V_0}(x) + r(t)\,.$$

$\underline{v_0}(x)$ *is called* uniformly approximable *if for all* $t \in \mathbb{N}$

$$\inf_{u\in U^{\mathbb{N}}(x)} \frac{1}{t}\sum_{s=0}^{t-1} g(\phi(s;x,u),u(s)) \geq \underline{v_0}(x) - r(t)\,,$$

for a suitable positive function r *converging to* 0.

Corollary 3.7. *Consider system* (1) *and fix* $x \in M$.

(i) If $\overline{V_0}(x)$ *is uniformly approximable then*

$$\underline{V_0}(x) \leq \liminf_{\delta\to 0}(1-e^{-\delta})V_\delta(x) \leq \limsup_{\delta\to 0}(1-e^{-\delta})V_\delta(x) \leq \overline{V_0}(x)\,.$$

(ii) If $\underline{v_0}(x)$ *is uniformly approximable then*

$$\underline{v_0}(x) \leq \liminf_{\delta\to 0}(1-e^{-\delta})v_\delta(x) \leq \limsup_{\delta\to 0}(1-e^{-\delta})v_\delta(x) \leq \overline{v_0}(x)\,.$$

Proof. (i) By Corollary 3.5 we only have to show the right hand side. Proposition 3.1 (i) and Remark 3.2 show that for all $\delta > 0$ and all $u \in U^{\mathbb{N}}(x)$ we have

$$(1-e^{-\delta})J_\delta(x,u) \leq \overline{J_0}(x,u) + (1-e^{-\delta})^2\sum_{s=0}^{\infty} e^{-\delta s} r(s)\,.$$

This shows that

$$(1-e^{-\delta})V_\delta(x) \leq \overline{V_0}(x) + (1-e^{-\delta})^2\sum_{s=0}^{\infty} e^{-\delta s} r(s)\,,$$

and the assertion follows.

(ii) If $\underline{v_0}(x) = -\infty$ there is nothing to show. Otherwise we proceed as in part (i). □

Proposition 3.1 may be strengthened if we consider shifted bounded functions for a fixed discount rate. In Theorem 2.2 in [10] the inequality (20) has been shown and here we use the same idea of proof.

Proposition 3.8. *Let* $\delta > 0$ *be a fixed discount rate and* $a : \mathbb{N} \to \mathbb{R}$ *be bounded, then*

$$\liminf_{t\to\infty}(1-e^{-\delta})\sum_{s=0}^{\infty} e^{-\delta s} a(s+t) \leq \liminf_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1} a(s) \leq \tag{19}$$

$$\limsup_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1} a(s) \leq \limsup_{t\to\infty}(1-e^{-\delta})\sum_{s=0}^{\infty} e^{-\delta s} a(s+t)\,. \tag{20}$$

Proof. (i) We begin by proving (19). Let

$$C := \liminf_{t\to\infty}(1-e^{-\delta})\sum_{s=0}^{\infty} e^{-\delta s}a(s+t)\,.$$

and fix $\varepsilon > 0$. An easy calculation shows that if we replace a by $\tilde{a} := a - (C - 2\varepsilon)$ then

$$\liminf_{t\to\infty}(1-e^{-\delta})\sum_{s=0}^{\infty} e^{-\delta s}\tilde{a}(s+t) = 2\varepsilon\,. \tag{21}$$

If it is shown that

$$\liminf_{t\to\infty}\frac{1}{t}\sum_{s=0}^{t-1}\tilde{a}(s) \geq 0\,,$$

then the assertion for a follows as $\varepsilon > 0$ is arbitrary. In the following assume that a already satisfies (21). Let $t_0 \in \mathbb{N}$ be such that for all $t \geq t_0$

$$(1-e^{-\delta})\sum_{s=0}^{\infty} e^{-\delta s}a(s+t) \geq \frac{3\varepsilon}{2}\,.$$

First we show that for all $t \geq t_0$ there is a $\tilde{t}(t) \geq t$ such that

$$\sum_{s=t}^{\tilde{t}(t)} a(s) \geq \frac{\varepsilon}{1-e^{-\delta}}\,. \tag{22}$$

To this end fix $t \geq t_0$. By assumption there is a unique $\tilde{t}(t) \geq t$ such that for all $t \leq t' < \tilde{t}(t)$

$$\sum_{s=t}^{t'} e^{-\delta(s-t)}a(s) < \frac{\varepsilon}{1-e^{-\delta}} \quad \text{and} \quad \sum_{s=t}^{\tilde{t}(t)} e^{-\delta(s-t)}a(s) \geq \frac{\varepsilon}{1-e^{-\delta}}\,.$$

Thus it follows that $f(\tilde{t}(t)) > 0$ and therefore

$$\sum_{s=t}^{\tilde{t}(t)-1} e^{-\delta(s-t)}a(s) + e^{\delta}e^{-\delta(\tilde{t}(t)-t)}f(\tilde{t}(t)) > \sum_{s=t}^{\tilde{t}(t)} e^{-\delta(s-t)}a(s) \geq \frac{\varepsilon}{1-e^{-\delta}}\,. \tag{23}$$

As

$$\sum_{s=t}^{\tilde{t}(t)-2} e^{-\delta(s-t)}a(s) < \frac{\varepsilon}{1-e^{-\delta}}\,,$$

equation (23) implies that $f(\tilde{t}(t)-1) + e^{\delta}f(\tilde{t}(t)) > 0$ and therefore

$$\sum_{s=t}^{\tilde{t}(t)-2} a(s) + e^{\delta}e^{-\delta(\tilde{t}(t)-1-t)}f(\tilde{t}(t)-1) + e^{2\delta}e^{-\delta(\tilde{t}(t)-t)}f(\tilde{t}(t)) > \sum_{s=t}^{\tilde{t}(t)} e^{-\delta(s-t)}a(s) \geq \frac{\varepsilon}{1-e^{-\delta}}\,. \tag{24}$$

Continuing this process we obtain

$$\sum_{s=t}^{\tilde{t}(t)} a(s) = \sum_{s=t}^{\tilde{t}(t)} e^{\delta(s-t)}e^{-\delta(s-t)}a(s) \geq \frac{\varepsilon}{1-e^{-\delta}}\,. \tag{25}$$

For $N := \sup_{t\in\mathbb{N}}|a(t)|$ let $T \in \mathbb{N}$ be such that

$$\sum_{s=T}^{\infty} e^{-\delta s}N < \frac{\varepsilon}{4(1-e^{-\delta})}\,.$$

Thus it follows for all $t \in \mathbb{N}$ that $\tilde{t}(t) - t < T$, for otherwise we obtain a contradiction between (21) and the definition of $\tilde{t}(t)$. Denote $b := \max_{t\in\mathbb{N}}(\tilde{t}(t)-t)$. For $t \geq t_0$ there is a unique sequence $t_0, t_{k+1} = \tilde{t}(t_k) + 1, t_l = t$, for $0 \le k \le l$, where $t_l \le \tilde{t}(t_{l-1}) + 1$. It is readily obtained that

$$\sum_{s=0}^{t} a(s) = \sum_{s=0}^{t_0-1} a(s) + \sum_{k=0}^{l-2} \sum_{s=t_k}^{t_{k+1}-1} a(s) + \sum_{s=t_{l-1}}^{t} a(s) \tag{26}$$

$$\ge (l-1)\frac{\varepsilon}{1-e^{-\delta}} - (t_0 + t - t_{l-1})N \ge \frac{(t-t_0)}{b}\frac{\varepsilon}{1-e^{-\delta}} - (b+t_0)N\,.$$

Taking t to infinity it follows that

$$\liminf_{t\to\infty} \frac{1}{t}\sum_{s=0}^{t} a(s) \ge \liminf_{t\to\infty} \frac{\varepsilon}{b(1-e^{-\delta})} - \frac{(b+t_0)N}{t} > 0\,.$$

(ii) The middle inequality is obvious and (20) follows by using (i) for $-a$. □

Remark 3.9. *It has to be pointed out that part of the statement of the previous proposition remains true if we only assume that f is bounded from above (or below). More specifically* (19) *remains true if f is bounded from above, because in this case if* $\liminf_{t\to\infty} f(t) = -\infty$ *it is easy to see that also*

$$\liminf_{t\to\infty} (1-e^{-\delta}) \sum_{s=0}^{\infty} e^{-\delta s} a(s+t) = -\infty\,.$$

The converse statement holds for (20).

Again the previous proposition entails an immediate corollary. First we need the following definition.

Definition 3.10 (Orbits)**.** *Consider system* (1). *The forward orbit of x at time t is defined as*

$$\mathcal{O}_t^+(x) := \{y \in M; \exists\, u \in U^t(x) \text{ with } y = \phi(t;x,u)\}\,.$$

The forward orbit of x is then defined by

$$\mathcal{O}^+(x) := \bigcup_{t\in\mathbb{N}} \mathcal{O}_t^+(x)\,. \tag{27}$$

The backward orbit of x at time t is given by

$$\mathcal{O}_t^-(x) := \{y \in M; \exists\, u \in U^t(y) \text{ with } x = \phi(t;y,u)\}\,.$$

which leads to a definition of $\mathcal{O}^-(x)$ analogous to (27).

Corollary 3.11. *Let $\delta > 0$ be fixed. For all $x \in M$ it holds that*

(i)

$$\inf_{y\in\mathcal{O}^+(x)} (1-e^{-\delta})V_\delta(y) \le \underline{V_0}(x)\,.$$

(ii) If there exists a sequence $\{u_n\} \subset U^{\mathbb{N}}(x)$ such that
(a) *For every $n \in \mathbb{N}$ the set $\{g(\phi(t;x,u_n), u_n(t)) \mid t \in \mathbb{N}\}$ is bounded,*
(b) $\lim_{n\to\infty} J_\delta(\phi(t;x_0,u_{n,\delta}), u_{n,\delta}(t+\cdot)) = v_\delta(\phi(t;x_0,u_{n,\delta}))$ *uniformly in $t \in \mathbb{N}$,*
then

$$\sup_{y\in\mathcal{O}^+(x)} (1-e^{-\delta})v_\delta(y) \ge \overline{v_0}(x)\,.$$

Proof. (i) Fix $\varepsilon > 0$ and choose $u_\varepsilon \in U^{\mathbb{N}}(x)$ such that for all $t > 0$ it holds that $V_\delta(\phi(t;x,u_\varepsilon)) - \varepsilon < J_\delta(\phi(t;x,u_\varepsilon), u_\varepsilon(t+\cdot))$. Note that by assumption on g the set $\{g(\phi(t;x,u_\varepsilon), u_\varepsilon(t)) \mid t \in \mathbb{N}\}$ is bounded from above. By Proposition 3.8 and Remark 3.9 we have for every $\varepsilon > 0$

$$\underline{V_0}(x) \ge \underline{J_0}(x,u_\varepsilon) \ge \liminf_{t\to\infty}(1-e^{-\delta})J_\delta(\phi(t;x,u_\varepsilon), u_\varepsilon(t+\cdot)) \ge$$

$$\liminf_{t\to\infty}(1-e^{-\delta})\,(V_\delta(\phi(t;x,u_\varepsilon))-\varepsilon) \geq \inf_{y\in\mathcal{O}^+(x)}(1-e^{-\delta})\,(V_\delta(y)-\varepsilon)\ ,$$

which shows the assertion as $\varepsilon > 0$ was arbitrary.
(ii) This may be seen just as (i). □

Thus on the set of points that can be reached from x the infimum of the discounted value function $(1-e^{-\delta})V_\delta$ is less than the average yield while in x itself we have the converse. This suggests that convergence results can be proven using controllability properties, which relate the forward orbit of x with x. This is the topic of the next section.

4. Convergence in Control Sets

We begin with the following definitions are vital for our analysis of the optimal control problem.

Definition 4.1 (Accessibility). *System* (1) *is called* forward accessible from x *if* $\operatorname{int}\mathcal{O}^+(x)\neq\emptyset$ *and* forward accessible *if it is forward accessible from all* $x\in M$.

It has been shown that forward accessibility is intimately related to the rank of the map f_t in the case of smooth invertible systems [1]. In our case we consider the rank of the linearization of $f_t(x,\cdot):U^t(x)\to M$ at $u_0\in\operatorname{int}U^t\subset\mathbb{R}^{mt}$ with respect to the control variables and denote it by $r(t;x,u_0)$.

Definition 4.2 (Regularity). *A pair* $(x,u)\in M\times\operatorname{int}U^t$ *is called* regular, *if* $u\in\operatorname{int}U^t(x)$ *and* $r(t;x,u)=n$.

By Sard's theorem forward accessibility implies the existence of regular pairs for every $x\in M$. We denote by $\hat{\mathcal{O}}_t^+(x)$ the regular forward orbit of x at time t, which is defined as the set of points reachable with a control sequence $u\in U^t(x)$ such that the pair (x,u) is regular. In an obvious way analogous definitions hold for $\hat{\mathcal{O}}^+(x)$, $\hat{\mathcal{O}}_t^-(x)$ and $\hat{\mathcal{O}}^-(x)$. Using local surjectivity and the implicit function theorem it is easy to see that all these objects are open sets. The nontriviality of f_t guarantees that $\operatorname{cl}\hat{\mathcal{O}}_t^+(x)=\operatorname{cl}\mathcal{O}_t^+(x)$ if and only if $\operatorname{int}\mathcal{O}_t^+(x)\neq\emptyset$. Further properties in connection with regularity are shown in [14].

Definition 4.3 (Control set). *Consider system* (1). *A set* $\emptyset\neq D\subset M$ *is called a* control set, *if*

(i) $D\subset\operatorname{cl}\mathcal{O}^+(x)$, $\forall x\in D$.
(ii) For every $x\in D$ *there exists a* $u\in U^{\mathbb{N}}(x)$ *such that* $\phi(t;x,u)\in D$ *for all* $t\in\mathbb{N}$.
(iii) D *is a maximal set with respect to inclusion satisfying (i).*

A control set C *is called an* invariant control set, *if*

$$\operatorname{cl}C=\operatorname{cl}\mathcal{O}^+(x),\ \forall x\in C\,. \tag{28}$$

In particular it holds that if (1) is forward accessible and C is an invariant control set, then C is closed and has nonempty interior. This may be seen as in [19]. The importance of control sets lies in the ability to construct periodic orbits. Contrary to the continuous time case, in discrete time it may not be possible to construct periodic orbits through any point in the interior of a control set. This has been noted in [1], where the concept of a core of a control set has been introduced. We slightly modify the definition for our possibly non-invertible case in introducing the *regular core*. It should, however, be noted that for the systems studied in [1] core and regular core of a control set coincide.

Definition 4.4 (Regular core). *Let* $D\subset M$ *be a control set with* $\operatorname{int}D\neq\emptyset$. *The* (regular) core *of* D *is defined as*

$$\operatorname{core}(D):=\{x\in D;\hat{\mathcal{O}}^+(x)\cap D\neq\emptyset \textit{ and } \hat{\mathcal{O}}^-(x)\cap D\neq\emptyset\}\,. \tag{29}$$

We note the following properties of the core of a control set [1].

Proposition 4.5. *Consider system* (1). *Let* $D \subset M$ *be a control set with* $\operatorname{int} D \neq \emptyset$. *If system* (1) *is forward accessible, then*

(i) $\operatorname{core}(D)$ *is open in* M.
(ii) $\operatorname{cl}\operatorname{core}(D) = \operatorname{cl}\operatorname{int}(D) = \operatorname{cl} D$.
(iii) If $x \in D$ *then* $\operatorname{core}(D) \subset \hat{\mathcal{O}}^+(x)$. *If* $x \in \operatorname{core}(D)$ *then* $D \subset \hat{\mathcal{O}}^-(x)$.

Let us now introduce the main technical tool for the construction of periodic orbits in control sets. Consider the function

$$h : M \times M \to \mathbb{N} \cup \{\infty\} \tag{30}$$

$$h(x,y) := \min\{t \in \mathbb{N};\ \text{there is a } u \in U^t \text{ such that } \phi(t;x,u) = y\}\,,$$

where $\min \emptyset = \infty$.
The previous definition is the discrete–time analogue of the *first–time hitting map*. As g may be unbounded it is important for us not only to obtain information on the time that elapses to steer from x to y but also on the cost that is necessary to do so. In analogy to the first time hitting map, we define the *minimal absolute cost map* by

$$H_g : M \times M \to \mathbb{R}_+ \cup \{\infty\} \tag{31}$$

$$\begin{aligned} H_g(x,y) \ &:= \ \inf\{\max_{1\le s\le t} |g(\phi(s;x,u),u(s))|;\quad t \in \mathbb{N}\,,\ u \in U^t(x) \\ &\qquad \text{satisfying } \phi(t;x,u) = y\}\,, \end{aligned}$$

where $\inf \emptyset = \infty$. The essential point is that both these values may be simultaneously bounded if one tries to reach a compact subset of the core of a control set. This will be vital in the construction of periodic trajectories.

Lemma 4.6. *Let system* (1) *be forward accessible. Let* $D \subset M$ *be a control set. Assume there are two non-void compact sets* K_1, K_2 *with* $K_1 \subset \mathcal{O}^-(D) := \bigcup_{x\in D} \mathcal{O}^-(x)$ *and* $K_2 \subset \operatorname{core}(D)$. *Then there are constants* $\overline{h} \in \mathbb{N}, \overline{H} \in \mathbb{R}_+$ *such that*

$$h(x,y) \le \overline{h} \ \text{ for all } x \in K_1,\ y \in K_2\,, \tag{32}$$

$$H_g(x,y) \le \overline{H} \ \text{ for all } x \in K_1,\ y \in K_2\,. \tag{33}$$

Proof. (i) Let $x \in K_1$, $y \in K_2$. Choose any point $z \in \operatorname{core}(D) \cap \hat{\mathcal{O}}^+(x)$, which is possible by Proposition 4.5 (iii). Thus there exist $u_1 \in \operatorname{int} U^{t_1}(x)$ such that $z = \phi(t_1;x,u_1)$ and (x,u_1) is a regular pair. By the implicit function theorem there exist open neighborhoods V_1 of x, W_1 of u_1 and a continuous function $w : V_1 \to W_1$ such that $z = \phi(t_1;x',w(x'))$ for every $x' \in V_1$. This shows that $h(x',z) \le t_1$ for all $x' \in V_1$. Furthermore, by continuous dependence of $g(s;x',w(x'))$ on x' it may be also obtained that $H_g(x',z) \le H_1$ for some suitable constant $H_1 \in \mathbb{R}$ and all $x' \in V_1$, where possibly V_1 has to be chosen to be smaller than the original choice.
On the other hand using again Proposition 4.5 (iii) there exist $t_2 \in \mathbb{N}$, $u_2 \in \operatorname{int} U^{t_2}(z)$ such that $y = \phi(t_2;z,u_2)$ and (z,u_2) is a regular pair. By the regularity for any open neighborhood W_2 of u_2 the set $\{\phi(t_2;z,u');\ u' \in W_2\}$ contains an open neighborhood V_2 of y. Choosing W_2 small enough so that $\operatorname{cl} W_2 \subset \operatorname{int} U^{t_2}(z)$ we see that $h(z,y') \le t_2$ for all $y' \in V_2$ and also $H_g(z,y') \le H_2$ for all $y' \in V_2$ and some suitable constant H_2.
In all we have obtained that

$$h(x',y') \le t_1 + t_2 \quad \text{for all} \quad x' \in V_1, y' \in V_2$$

and

$$H_g(x',y') \le H_1 + H_2 \quad \text{for all} \quad x' \in V_1, y' \in V_2.$$

The assertion now follows because we may choose a finite sub-cover of the open cover

$$\{V_1(x) \times V_2(y);\quad x \in K_1, y \in K_2\}$$

of the compact set $K_1 \times K_2$. $\square$

An immediate consequence of the preceding proposition and Theorem 2.2 (iii) is that $\overline{V_0}, \underline{V_0}, \overline{v_0}, \underline{v_0}$ are constant on the cores of control sets. The following proposition gives a sufficient condition for equality equality of the maximization, respectively minimization problems. We denote the ω-limit set of the trajectory $\{\phi(t;x,u)\}$, i.e. the set of limit points of this sequence, by $\omega(x,u)$. The idea for the following proof is contained in [6], where in particular approximation of average yields by periodic trajectories is studied.

Proposition 4.7. *Assume that system* (1) *is forward accessible. Let $x \in M$ and $D \subset M$ be a control set.*

(i) Assume there are sequences $\{u_n\} \subset U^{\mathbb{N}}(x)$ and $\{t_n\} \subset \mathbb{N}$ such that for every $n \in \mathbb{N}$

$$\phi(t_n; x, u_n) \in \mathrm{core}(D)\,, \quad \textit{and} \quad \omega(x,u_n) \subset D\,, \tag{34}$$

and

$$\lim_{n\to\infty} \overline{J_0}(x,u_n) = \overline{V_0}(x)\,, \tag{35}$$

then

$$\overline{V_0}(x) = \underline{V_0}(x)\,. \tag{36}$$

If, furthermore, $\overline{V_0}(x)$ is uniformly approximable then

$$\lim_{\delta\to 0}(1-e^{-\delta})V_\delta(x) = \overline{V_0}(x) = \underline{V_0}(x)\,. \tag{37}$$

(ii) Assume there are sequences $\{u_n\} \subset U^{\mathbb{N}}(x)$ and $\{t_n\} \subset \mathbb{N}$ such that for every $n \in \mathbb{N}$

$$\phi(t_n; x, u_n) \in \mathrm{core}(D)\,, \quad \textit{and} \quad \omega(x,u_n) \subset D\,, \tag{38}$$

and

$$\lim_{n\to\infty} \underline{J_0}(x,u_n) = \underline{v_0}(x)\,, \tag{39}$$

then

$$\overline{v_0}(x) = \underline{v_0}(x)\,. \tag{40}$$

If, furthermore, $\underline{v_0}(x)$ is uniformly approximable then

$$\lim_{\delta\to 0}(1-e^{-\delta})v_\delta(x) = \overline{v_0}(x) = \underline{v_0}(x)\,. \tag{41}$$

Proof. (i) Clearly it holds that $\overline{V_0}(x) \geq \underline{V_0}(x)$. By Remark 3.4 and Theorem 2.2 (iii) it is sufficient to show that there exist periodic controls v_n generating periodic trajectory through x such that $\lim_{n\to\infty} \overline{J_0}(x,v_n) = \overline{V_0}(x)$. Fix $n \in \mathbb{N}$. Without loss of generality we may assume that $t_n = 0$. Let $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be an increasing sequence such that

$$\lim_{k\to\infty} \frac{1}{t_k} \sum_{s=0}^{t_k-1} g(\phi(s;x,u_n), u_n(s)) = \overline{J_0}(x,u_n)\,.$$

Taking a subsequence we may assume that

$$\lim_{k\to\infty} \phi(t_k;x,u_n) =: y \in \omega(x,u_n) \subset D\,.$$

As $x \in \mathrm{core}(D)$ it follows that $y \in \hat{\mathcal{O}}^-(x)$ by Proposition 4.5 (iii) and hence there is a $t \in \mathbb{N}$ and a neighborhood $V(y)$ such that $V(y) \subset \hat{\mathcal{O}}_t^-(x)$. For all k large enough it holds that $\phi(t_k;x,u_n) \in V(y)$. We can therefore find a controls $v_k \in \mathrm{int}\, U^t$ such that $x = \phi(t;\phi(t_k;x,u_n),v_k)$. By Remark 3.4 the average yield corresponding to the (periodic continuation of the) concatenated control (u_n,v_k) and x is given by

$$\overline{J_0}(x,(u_n,v_k)) = \frac{1}{t_k+t}\left(\sum_{s=0}^{t_k-1} g(\phi(s;x,u_n),u_n(s)) + \sum_{s=0}^{t} g(\phi(s;\phi(t_k;x,u_n),v_k),v_k(s))\right)\,.$$

Letting $k \to \infty$ and noting that Lemma 4.6 guarantees that the v_k can be chosen so that $|g(\phi(s;\phi(t_k;x,u_n),v_k),v_k(s))|$ is bounded independently of $0 \le s \le t$ and k for k large enough, we see that $\lim_{k\to\infty} \overline{J_0}(x,(u_n,v_k)) = \overline{J_0}(x,v_n)$. Thus we may choose for each n a $k(n)$ such that $\lim_{n\to\infty} \overline{J_0}(x,(u_n,v_{k(n)})) = \overline{V_0}(x)$. This completes the proof of (36). The last assertion is immediate from Corollary 3.7.
(ii) This may be shown as in (i). □

For the convergence of the discounted yield we may formulate the following proposition. An equivalent statement to (ii) below appears in [10], but here we have to consider the case of unbounded yields, so that we have to argue in a slightly different manner.

Proposition 4.8. *Assume that system* (1) *is forward accessible. Let D be a control set, $x_0 \in$* core(D)*, and $K \subset D$ be a compact set.*

(i) If for every $\delta > 0$ there exists a sequence $\{u_{n,\delta}\} \subset U^{\mathbb{N}}(x_0)$ such that

$$\phi(t;x_0,u_{n,\delta}) \in K\,, \quad \forall t \in \mathbb{N}\,,\ \textit{and} \lim_{n\to\infty} J_\delta(x_0,u_{n,\delta}) = V_\delta(x_0)\,, \tag{42}$$

then

$$(1-e^{-\delta})V_\delta \to \underline{V_0}\,,$$

uniformly on compact subsets of core(D)*.*

(ii) If for every $\delta > 0$ there exists a sequence $\{u_{n,\delta}\} \subset U^{\mathbb{N}}(x_0)$ such that

$$\phi(t;x_0,u_{n,\delta}) \in K\,, \quad \forall t \in \mathbb{N}\,,\ \textit{and} \lim_{n\to\infty} J_\delta(x_0,u_{n,\delta}) = v_\delta(x_0)\,, \tag{43}$$

then

$$(1-e^{-\delta})v_\delta \to \overline{v_0}\,,$$

uniformly on compact subsets of core(D)*.*

Proof. (i) Note that $\underline{V_0}(x_0) > -\infty$ because of condition (3). Let $K_2 \subset \mathrm{core}(D)$ be compact. By Bellman's principle of optimality and finite-time controllability there exist constants $\bar{h} \in \mathbb{N}, \bar{H} > 0$ such that for any $x, y \in K_2$ it holds that

$$V_\delta(x) \ge e^{-\delta\bar{h}}V_\delta(y) - \bar{h}\bar{H}\,.$$

By symmetry we obtain that there are constants $c_1(\delta) > 0$ such that

$$(1-e^{-\delta})|V_\delta(x) - V_\delta(y)| < c_1(\delta)\,, \quad \forall x,y \in K_2\,,\ \delta > 0\,, \tag{44}$$

where $\lim_{\delta\to 0} c_1(\delta) = 0$. By the same argument applied to $y \in K$ and $x_0 \in \mathrm{core}(D)$ there is a constant $c_2(\delta) > 0$ with $\lim_{\delta\to 0} c_2(\delta) = 0$, such that

$$(1-e^{-\delta})V_\delta(y) \ge (1-e^{-\delta})V_\delta(x_0) - c_2(\delta)\,.$$

Assume without loss of generality that $x_0 \in K_2$. As $\underline{V_0}$ is constant on core(D) by Theorem 2.2 (iii), we only have to show that

$$\lim_{\delta\to 0}(1-e^{-\delta})V_\delta(x_0) = \underline{V_0}(x_0)\,,$$

then it follows from (44) that $(1-e^{-\delta})V_\delta$ converges uniformly on K_2.

Let $t_0 \ge 1$ be such that $h(y,x_0) \le t_0$ for all $y \in K$ which is possible by Lemma 4.6 and denote the corresponding bound on the yields by $\bar{H}$. As a first step we claim that for any control $u_{n,\delta}$ and any $t \in \mathbb{N}$ it holds that

$$\frac{1}{t}\sum_{s=0}^{t-1} g(\phi(s;x_0,u_{n,\delta}),u_{n,\delta}(s)) \le \underline{V_0}(x_0) + \frac{t_0}{t}(\bar{H} + \underline{V_0}(x_0))\,. \tag{45}$$

Otherwise, we may choose a control $u_{n,\delta}$ and a $T > 0$ such that (45) is not satisfied. Let v be a control that steers back from $\phi(T;x_0,u_{n,\delta})$ to x_0 in time $s_0 \le t_0$, i.e. we have

$$x_0 = \phi(s_0;\phi(T;x_0,u_{n,\delta}),v)\,.$$

Denote the periodic continuation of this control by $(u_{n,\delta}, v)$. Using Remark 3.4 and assuming that (45) does not hold we obtain

$$\underline{J_0}(x_0, (u_{n,\delta}, v)) > \frac{1}{T+s_0}\left(T\underline{V_0}(x_0) + t_0(\bar{H} + \underline{V_0}(x_0)) - s_0\bar{H}\right) \geq \underline{V_0}(x_0)\,,$$

a contradiction.

Using Proposition 3.1 (ii) we obtain that for any $\delta > 0$

$$(1-e^{-\delta})V_\delta(x_0) = \lim_{n\to\infty}(1-e^{-\delta})J_\delta(x_0, u_{n,\delta}) \leq \underline{V_0}(x_0) - c_2(\delta)\,, \tag{46}$$

where $c_2(\delta) \to 0$ as $\delta \to 0$ and thus $\underline{V_0}(x_0) \geq \limsup_{\delta\to 0}(1-e^{-\delta})V_\delta(x_0)$. Hence with Corollary 3.5 it follows that $\underline{V_0}(x_0) = \lim_{\delta\to 0}(1-e^{-\delta})V_\delta(x_0)$.
(ii) If $\overline{v_0} = -\infty$ on $\mathrm{core}(D)$ we can apply Corollary 3.5 and use finite-time controllability on $K_2 \subset \mathrm{core}(D)$ in conjunction with Bellman's principle of optimality as in (44) to obtain the assertion. If $\overline{v_0}$ is finite on $\mathrm{core}(D)$ the claim may be shown as in (i).

□

In particular, the preceding results yield the following statements for invariant control sets.

Theorem 4.9. *Consider system* (1). *Let $C \subset M$ be a compact invariant control set. Then for $\delta \to 0$ it holds that*

$$(1-e^{-\delta})V_\delta \to \overline{V_0} = \underline{V_0} \text{ uniformly on compact subsets of } \mathrm{core}(C)\,.$$

$$(1-e^{-\delta})v_\delta \to \overline{v_0} = \underline{v_0} \text{ uniformly on compact subsets of } \mathrm{core}(C)\,.$$

Proof. It suffices to show the statement for the maximization problem, as the proof follows exactly the same lines for the other case.
$\overline{V_0} = \underline{V_0}$ on $\mathrm{core}(C)$ holds as by the invariance of C the assumptions of Proposition 4.7 are automatically satisfied. For $x_0 \in \mathrm{core}(C)$ we may construct controls $u_{n,\delta}$ satisfying (42) by recursively choosing $u_{n,\delta}(t)$ such that

$$g(\phi(t;x_0,u_{n,\delta}), u_{n,\delta}(t)) + e^{-\delta t}V_\delta(\phi(t+1;x_0,u_{n,\delta})) > V_\delta(\phi(t;x_0,u_{n,\delta})) - \frac{1}{n}\,.$$

Then by Bellman‘s principle of optimality

$$V_\delta(x_0) < \sum_{s=0}^{\infty} e^{-\delta}\left(g(\phi(s;x_0,u_{n,\delta}), u_{n,\delta}(s)) + \frac{1}{n}\right)$$

$$= \sum_{s=0}^{\infty} e^{-\delta} g(\phi(s;x_0,u_{n,\delta}), u_{n,\delta}(s)) + \frac{1}{1-e^{-\delta}}\frac{1}{n}\,.$$

Choosing $K = C$ the assumptions of Proposition 4.8 are satisfied, and the assertion follows. □

The following theorem gives conditions for the convergence of the value functions on the backward orbits of control sets, under the assumption of pointwise convergence in the core of the control set and some uniform growth condition on the possible yields.

Theorem 4.10. *Consider a forward accessible system of the form* (1) *on M. Let D be a control set and $K \subset \mathcal{O}^-(D)$ be compact. Assume there exists an $x_0 \in \mathrm{core}(D)$ such that*

$$\overline{V_0}(x_0) = \sup_{x\in K}\overline{V_0}(x) = \lim_{t\to\infty}\sup_{x\in K, u\in U^t(x)}\frac{1}{t}\sum_{s=0}^{t-1} g(\phi(s;x,u), u(s))\,.$$

If D is not invariant, assume furthermore that there exists a sequence $\{u_n\} \subset U^{\mathbb{N}}(x_0)$ such that

$$\overline{V_0}(x_0) = \lim_{n\to\infty}\overline{J_0}(x_0,u_n) \quad \text{and} \quad \omega(x_0,u_n) \subset D\,, \quad \forall n \in \mathbb{N}\,.$$

Then the following properties hold.

(i) $\overline{V_0} = \underline{V_0}$ is constant on K.

(ii) $(1-e^{-\delta})V_\delta \to \overline{V_0}$ *uniformly on* K *for* $\delta \to 0$.
(iii)

$$\lim_{\delta\to 0}\min_{x\in K}(1-e^{-\delta})V_\delta(x) = \lim_{\delta\to 0}\max_{x\in K}(1-e^{-\delta})V_\delta(x) = \overline{V_0}_{|K}. \tag{47}$$

Proof. (i) By Theorem 2.2 (iii) it holds for all $y \in K$ that

$$\overline{V_0}(y) \geq \overline{V_0}(x_0) = \max_{x\in K}\overline{V_0}(x) \geq \overline{V_0}(y)\,. \tag{48}$$

The equality $\overline{V_0} = \underline{V_0}$ follows by Theorem 4.9 in case D is invariant. Otherwise note that to every $x \in K$ there exist approximately optimal controls $u \in U^{\mathbb{N}}(x)$ such that $\phi(t;x,u) = x_0 \in \mathrm{core}(C)$ for some $t \in \mathbb{N}$ satisfying the conditions of Proposition 4.7.
(ii) Fix $\varepsilon > 0$. By assumption we may choose $T \in \mathbb{N}$ such that for all $t \geq T$ it holds that

$$L(\varepsilon) := \sup_{x\in K}\overline{V_0}(x) + \varepsilon \geq \sup_{x\in K, u\in U^t(x)} \frac{1}{t}\sum_{s=0}^{t-1} g(\phi(s;x,u),u(s))\,. \tag{49}$$

Denote

$$J(x,u,t) := \sum_{s=0}^{t-1} g(\phi(s;x,u),u(s))\,.$$

For every $x \in K$, $u \in U^{\mathbb{N}}(x)$, $\delta > 0$, such that $J_\delta(x,u) \in \mathbb{R}$ it holds that

$$J_\delta(x,u) = \lim_{t\to\infty}\sum_{s=0}^{t} e^{-\delta s} g(\phi(s;x,u),u(s))$$

$$= \lim_{t\to\infty}\left[e^{-\delta t}J(x,u,t) + \sum_{s=0}^{t-1}(1-e^{-\delta})e^{-\delta s}J(x,u,s)\right]$$

$$\leq \lim_{t\to\infty, t\geq T}\left[e^{-\delta t}tL(\varepsilon) + \sum_{s=T}^{t-1}(1-e^{-\delta})e^{-\delta s}sL(\varepsilon) + \sum_{s=0}^{T-1}(1-e^{-\delta})e^{-\delta s}sG\right]$$

$$= \left[\lim_{t\to\infty, t\geq T}\sum_{s=T}^{t} e^{-\delta s}L(\varepsilon) + (T-1)e^{-\delta T}L(\varepsilon) + \sum_{s=0}^{T-1}(1-e^{-\delta})e^{-\delta s}sG\right].$$

The last two terms are bounded by a constant $L_2(\varepsilon)$ independently of x, u and δ. Thus after multiplying with $(1-e^{-\delta})$ we obtain

$$\max_{x\in K}(1-e^{-\delta})V_\delta(x) \leq e^{-\delta T}L(\varepsilon) + (1-e^{-\delta})L_2(\varepsilon).$$

As the factor of $L(\varepsilon)$ tends to 1 for $\delta \to 0$ it follows by (49) that for every $\varepsilon > 0$ there exists δ_ε such that for $0 < \delta < \delta_\varepsilon$

$$\max_{x\in K}(1-e^{-\delta})V_\delta(x) \leq \sup_{x\in K}\overline{V_0}(x) + 2\varepsilon = \overline{V_0}(x_0) + 2\varepsilon\,. \tag{50}$$

By Lemma 4.6 there exist constants $\overline{h}$, $\overline{H}$, such that $h(x,x_0) < \overline{h}$ for all $x \in K$ and the absolute value of the yield along the trajectory is bounded by $\overline{H}$. Thus by Bellman's principle of optimality for every $x \in K$ and every $\delta > 0$ it holds that

$$(1-e^{-\delta})V_\delta(x) \geq e^{-\delta\overline{h}}(1-e^{-\delta})V_\delta(x_0) - (1-e^{-\delta})\overline{h}\,\overline{H}.$$

By assumption $\overline{V_0}(x_0)$ is uniformly approximable and by (i) and Proposition 4.7 we have $\lim_{\delta\to 0}(1-e^{-\delta})V_\delta(x_0) = \overline{V_0}(x_0)$. It follows that $\liminf_{\delta\to 0}(1-e^{-\delta})V_\delta \geq \overline{V_0}$ uniformly in $x \in K$. Combining this with (50) the assertion follows.

(iii) This is obvious by (i) and (ii).

□

A converse statement holds for the minimization problems. The proof is as in the previous case and is left to the reader.

Theorem 4.11. *Consider a forward accessible system of the form* (1) *on* M. *Let* D *be a control set and* $K \subset \mathcal{O}^-(D)$ *be compact. Assume there exists an* $x_0 \in \operatorname{core}(D)$ *such that*

$$\underline{v_0}(x_0) = \inf_{x \in K} \underline{v_0}(x) = \lim_{t\to\infty} \inf_{x \in K, u \in U^t(x)} \frac{1}{t} \sum_{s=0}^{t-1} g(\phi(s; x, u), u(s)) .$$

If D *is not invariant, assume furthermore that there exists a sequence* $\{u_n\} \subset U^{\mathbb{N}}(x_0)$ *such that*

$$\underline{v_0}(x_0) = \lim_{n\to\infty} \underline{J_0}(x_0, u_n) \quad \textit{and} \quad \omega(x_0, u_n) \subset D\,, \quad \forall n \in \mathbb{N}\,.$$

Then the following properties hold.

(i) $\overline{v_0} = \underline{v_0}$ *is constant on* K.

(ii) $(1-e^{-\delta})v_\delta \to \overline{v_0}$ *uniformly on* K *for* $\delta \to 0$.

(iii)

$$\lim_{\delta\to 0} \min_{x \in K} (1-e^{-\delta})v_\delta(x) = \lim_{\delta\to 0} \max_{x\in K}(1-e^{-\delta})v_\delta(x) = \overline{v_0}_{|K}. \tag{51}$$

5. Application: Lyapunov Exponents of Discrete Inclusions

Consider a time-varying linear system of the form

$$\begin{aligned} x(t+1) &= A(u(t))x(t) := \left(A_0 + \sum_{i=1}^m u_i(t) A_i\right) x(t)\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{R}^n, \end{aligned} \tag{52}$$

where $u(t) \in U \subset \mathbb{R}^m$ and U is a bounded set with connected interior satisfying $U \subset \operatorname{cl} \operatorname{int} U$. Let us assume without loss of generality that $0 \in U$. The evolution operator generated by a control sequence $u \in U^{\mathbb{N}}$ is defined by

$$\Phi_u(s,s) = I\,, \quad \Phi_u(t+1,s) = A(u(t))\Phi_u(t,s)\,, \quad t \geq s \in \mathbb{N}\,. \tag{53}$$

With this notation $\Phi_u(t,0)x_0$ is the solution of (52) corresponding to the initial value x_0 and the control u at time t. We are interested in the exponential growth rates of trajectories of the linear system which are given as follows.

Definition 5.1 (Lyapunov exponent). *Given a sequence* $u \in U^{\mathbb{N}}$ *and an initial condition* $x_0 \in \mathbb{R}^n \setminus \{0\}$ *the* Lyapunov exponent *corresponding to* (x_0, u) *is defined by*

$$\lambda(x_0, u) = \limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_u(t,0)x_0\|. \tag{54}$$

As we will see value functions $\overline{V_0}, \overline{v_0}$ may be defined which correspond to the problem of determining the supremal, respectively the infimal Lyapunov exponent that can be realized from a particular initial condition x_0. This corresponds to the following two problems in control theory.

If we assume that the time-invariant system

$$\begin{aligned} x(t+1) &= A(0)x(t)\,, \quad t \in \mathbb{N} \\ x(0) &= x_0 \quad \in \mathbb{R}^n \end{aligned}$$

is stable, then system (52) may be interpreted as a model for the time-varying uncertainty to which the nominal system is subjected. The set U and the matrices $A_1, \ldots, A_m$ determine the structure of the uncertainty. A question of interest is then to determine whether the perturbed system is still stable, i.e. if $\sup_{x\in\mathbb{R}^n} \overline{V_0}(x) < 0$, see [18].

A converse problem is to determine if (55) is open-loop stabilizable, that is for every initial condition a control sequence may be found such that the corresponding trajectory is exponentially decaying. In terms of the value functions this means $\sup_{x\in\mathbb{R}^n} \overline{v_0}(x) < 0$, see [9], [10].

Using formula (54) it is easy to see that the Lyapunov exponent is invariant with respect to (non-zero) scaling. This is why the study of Lyapunov exponents may be undertaken on the projective space $\mathbb{P}^{n-1}$. In our discrete–time system we do not exclude the possibility that the origin may be reached from non–zero states. If this is regarded from the point of view of stability or robust stability this poses no problem for once system (52) is at zero it remains there. However, this means that system (52) as such may not be projected onto projective space. To avoid degenerate situations we assume that $L := \bigcap_{u\in U} \ker A(u) = \{0\}$, otherwise system (52) may be studied on the quotient space $\mathbb{R}^n/L$.

On projective space the exceptional set X is naturally given by $\{(\xi, u) \in \mathbb{P}^{n-1} \times U \;;\; \xi \subset \operatorname{Ker} A(u)\}$. For $\xi \in \mathbb{P}^{n-1}$ and $u \in U(\xi)$ we define the transition map

$$\mathbb{P}A(u)\xi := \mathbb{P}A(u)x \quad \text{iff } \xi = \mathbb{P}x \,,$$

where $\mathbb{P}$ denotes the projection onto $\mathbb{P}^{n-1}$. With this notation the projected system corresponding to our linear system (52) is given by

$$\begin{aligned} \xi(t+1) &= \mathbb{P}A(u(t))\xi(t)\,, \quad t \in \mathbb{N} \\ \xi(0) &= \xi_0 \in \mathbb{P}^{n-1} \\ u &\in U^{\mathbb{N}}(\xi_0)\,, \end{aligned} \tag{55}$$

and it is easy to see that all the assumptions for system (1) are satisfied. Let us now explain how to obtain the Lyapunov exponent $\lambda(x_0, u)$ from the trajectory $\xi(\cdot; \mathbb{P}x_0, u)$ of the projected system. For $\xi \in \mathbb{P}^{n-1}$, $u \in U(\xi)$ define

$$g(\xi, u) := \log \frac{\|A(u)x\|}{\|x\|}, \quad \text{where} \quad x \neq 0,\ \mathbb{P}x = \xi. \tag{56}$$

This is well defined as multiplication of x with a non-zero scalar does not alter the value of $g(\xi, u)$. For $\xi \in \mathbb{P}^{n-1}$, $t \in \mathbb{N}$, $u \in U^t(\xi)$ define

$$J(t; \xi, u) = \sum_{s=0}^{t-1} g(\xi(s; \xi, u), u(s)). \tag{57}$$

Then we obtain the following expression for Lyapunov exponents:

Lemma 5.2. *For $x_0 \in \mathbb{R}^n \setminus \{0\}$, $u \in U^{\mathbb{N}}$ it holds that*

$$\lambda(x_0, u) = \begin{cases} \limsup_{t\to\infty} \frac{1}{t} J(t; \mathbb{P}x_0, u), & \text{if} \quad u \in U^{\mathbb{N}}(x_0). \\ -\infty, & \text{otherwise.} \end{cases} \tag{58}$$

Proof. This may be seen by a straightforward calculation. □

Thus we can use g as a running cost that satisfies the assumptions of the general optimal control problem considered in the previous sections. If system (55) is forward accessible, then there exists a unique invariant control set $C \subset \mathbb{P}^{n-1}$, as the projective space is compact. Existence may be seen via an argument using Zorn's lemma on the set

$$\{\operatorname{cl} \mathcal{O}^+(\xi) \;;\; \xi \in \mathbb{P}^{n-1}\}\,,$$

that is partially ordered by inclusion, see [12]. For uniqueness of C we refer to [19] where it is also shown that C is closed and has nonempty interior. Furthermore, for $\xi \in C$ it holds that $\overline{V_0}(\xi) = \sup_{\eta\in\mathbb{P}^{n-1}} \overline{V_0}(\eta)$ by Theorem 2 in [2], which shows that $\overline{V_0}$ is constant on $\mathbb{P}^{n-1}$.

So we have to check just one more item to fulfill the conditions of Theorem 4.10 for the invariant control set C. The following statement is a consequence of Theorem 4 in [3].

Proposition 5.3.

$$\sup\{\lambda(x,u)\ ;\ 0 \neq x \in \mathbb{R}^n, u \in U^{\mathbb{N}}\} = \limsup_{t\to\infty} \sup_{u\in U^{\mathbb{N}},\, \xi\in\mathbb{P}^{n-1}} \frac{1}{t} J(t;\xi,u), \tag{59}$$

Thus we obtain the following as a corollary to Theorem 4.10.

Corollary 5.4. *Let system* (55) *be forward accessible and consider the running yield g defined in* (56)*, then*

$$(1-e^{-\delta})V_\delta \to \overline{V_0} = \underline{V_0} = \sup\{\lambda(x,u)\ ;\ 0 \neq x \in \mathbb{R}^n, u \in U^{\mathbb{N}}\} \quad \textit{uniformly on } \mathbb{P}^{n-1}.$$

If we assume furthermore that there exists a $u \in U$ such that $A(u)$ is invertible, and that the set

$$\{u \in U\ ;\ \dim \operatorname{Ker} A(u) \geq 2\}$$

does not separate $\operatorname{int} U$ then by the results in [19] there are $D_1, \dots, D_\kappa$ different control sets with nonempty interior for (55) on $\mathbb{P}^{n-1}$, where $1 \leq \kappa \leq n$. These control sets may be linearly ordered such that $D_i \subset \mathcal{O}^-(D_{i+1})$, $i = 1, \dots, \kappa - 1$. It follows that $\overline{v_0}_{|\operatorname{core}(D_i)} \leq \overline{v_0}_{|\operatorname{core}(D_{i+1})}$. It is an open problem whether the conditions of Theorem 4.11 can be satisfied via a statement analogous to Proposition 5.3. We conjecture that the following statement may be obtained as a corollary to Theorem 4.11.

Conjecture 5.5. *Let system* (55) *be forward accessible and consider the running yield g defined in* (56)*. Assume there are $D_1, \dots, D_\kappa$ linearly ordered control sets with nonempty interior on $\mathbb{P}^{n-1}$. Let $1 \leq i \leq \kappa$ and $K \subset (\mathcal{O}^-(D_i) \setminus \operatorname{cl} \mathcal{O}^-(D_{i-1}))$ be compact then*

$$(1-e^{-\delta})v_\delta \to \overline{v_0} = \underline{v_0} = \inf\{\lambda(\xi,u)\ ;\ 0 \neq \xi \in D_i, u \in U^{\mathbb{N}}\} \quad \textit{uniformly on } K.$$

If further conditions on so-called chain control sets of the system are satisfied, and the running yield is bounded the preceding conjecture is shown in the continuous time case in [11, Theorem 8.4].

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