Parameter Dependent Extremal Norms for Linear Parameter Varying Systems

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Abstract

We study families of time-varying linear systems with restrictions on the derivative of the parameter variation. This includes the systems usually considered in the area of linear parameter varying (LPV) systems. We show that it is possible to construct exact parameterized Lyapunov norms for a wide class of such systems. This may be used to derive (locally Lipschitz) continuous dependence of the exponential growth rate on the systems data. Furthermore, it is shown that the exponential growth rate may be approximated by exponential growth rates of periodic parameter variations.

1 Introduction

The control and robustness analysis of linear parameter-varying systems have been actively investigated during the last decade. In particular, parameter dependent quadratic Lyapunov functions for such systems are discussed in the literature and many results have been obtained in the framework of linear matrix inequalities, see [1, 2, 3, 4, 7, 8, 10, 11].

Despite this activity some basic questions have remained unanswered, at least to the best of the knowledge of the author. These relate to the continuity properties of the exponential growth rate as a function of the system data as well as to the question whether periodic parameter variations are sufficient to approximate the exponential growth rate of the system.

In this paper we generalize results obtained in [12] on the exponential growth rate of families of time-varying systems with measurable parameter variations to linear parameter varying systems with bounds on the derivative of the parameter variations. In particular, a procedure for the construction of exact parameter dependent Lyapunov norms is presented. Using these norms we can show that the exponential growth rate of an LPV system depends continuously on the data and is even locally Lipschitz continuous on an open and dense set in the space of systems. Furthermore, an analogue of the Gelfand formula holds, which states that the exponential growth rate can be approximated by periodic parameter variations.

We proceed as follows. In the ensuing Section 2 we define the class of LPV systems under consideration and in Section 3 the exponential growth of a linear parameter varying system is defined. This is the quantity of interest in this paper. In Section 4 it is shown under the assumption of irreducibility how to construct parameter dependent Lyapunov norms that exactly characterize the exponential growth rate. This result is then used in Section 5 to show local Lipschitz continuity of the exponential growth rate on the set of irreducible LPV systems. Finally, in Section 6 the Gelfand formula and continuity is proved. This note is a preliminary version of a research article in preparation. For reasons of space some proofs are only sketched or omitted. Full details which will appear elsewhere.

2 Problem formulation

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In this paper we study linear parameter-varying systems of the form

$$\dot{x}\,(t) = A(\theta(t))x(t)\,, \quad t \geq 0\,. \tag{2.1}$$

Here the map $A : \Theta \to \mathbb{K}^{n\times n}$ is assumed to be continuous and should be interpreted as a map on a parameter space $\Theta \subset \mathbb{K}^m$. The admissible parameter variations $\theta(\cdot)$ are specified by two sets Θ, Θ_1 as follows. We assume that Θ, Θ_1 are compact convex sets. The parameter variations $\theta : \mathbb{R} \to \Theta$ are assumed to be Lipschitz continuous with $\dot{\theta}(t) \in \Theta_1$, almost everywhere. It is then natural to assume that $0 \in \Theta_1$ and $\operatorname{span}\Theta_1 \subset \operatorname{span}\Theta - \theta_0$, $\theta_0 \in \Theta$. Note that $\theta(t) \in \Theta$ for all $t \geq 0$ and Lipschitz continuity implies that $\dot{\theta}(t)$ is contained in $\operatorname{span}\Theta - \theta_0$ a.e. From now on $\mathcal{U}$ always denotes the set of Lipschitz continuous functions that are described in this manner by the sets Θ, Θ_1. Given any $u \in \mathcal{U}$ the corresponding evolution operator defined through (2.1) is denoted by $\Phi_u(t,s)\,, t \geq s \geq 0$. In the following we will identify the system (2.1) with the triple (Θ, Θ_1, A).

The following assumptions are important in the sequel. Recall that a set of matrices $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is called irreducible, if only the trivial subspaces 0 and $\mathbb{K}^n$ are invariant under all $A \in \mathcal{M}$.

$$\text{For all } \theta_0 \in \Theta \text{ it holds that } \operatorname{span}\Theta_1 = \operatorname{span}\{\Theta - \theta_0\}\,. \tag{A1}$$

$$\text{The point 0 is contained in the interior of } \Theta_1 \text{ relative to its span.} \tag{A2}$$

$$\text{The set } A(\Theta) \subset \mathbb{K}^{n\times n} \text{ is irreducible.} \tag{A3}$$

Note that (A1) and (A2) guarantee that there is an $h > 0$ such that for any pair $\theta, \eta \in \Theta$ and all $t > \|\theta - \eta\|/h$ there is a $u \in \mathcal{U}$ with $u(0) = \theta, u(t) = \eta$. IN particular, as Θ is compact there is an $\bar{h} > 0$ such that $\bar{h} \geq \|\theta - \eta\|/h$ for all $\theta, \eta \in \Theta$.

Remark 2.1. *(i) We explicitly exclude the case where the parameter variations $\theta(\cdot)$ are arbitrary measurable functions taking values in Θ. The reason being that for this case the results analogous to those obtained in this note are already available in the literature, see [6, 5, 12, 13].*

(ii) In a large number of papers it is assumed that the parameter variations $\theta(\cdot)$ are continuously differentiable and that the derivative satisfies certain constraints. However, it can be shown that the exponential growth rates defined by the sets

$$\{\theta : \mathbb{R} \to \Theta \mid \theta \text{ Lipschitz continuous and } \dot{\theta}(t) \in \Theta_1\,,\ \text{a. e.}\ \}$$

and

$$\{\theta : \mathbb{R} \to \Theta \mid \theta \text{ continuously differentiable and } \dot{\theta}(t) \in \Theta_1, \ \forall t \in \mathbb{R}\}$$

are the same. So that our setup encompasses this standard case. We just find the set of Lipschitz continuous parameter variations easier to handle for several analytic reasons.

(iii) The results presented in this paper are a discussion of a special case which may be subsumed under the following more general framework, see [13]. Consider systems of the form

$$\begin{aligned} \dot{x}\,(t) \;&=\; A(\theta\,(t))x(t)\,, \quad t \geq 0\,, \\ \dot{\theta}(t) &\in \mathcal{F}\,(\theta\,(t))\,, \quad a.e.\ t \geq 0 \end{aligned} \tag{2.2}$$

where $A : \Theta \to \mathbb{K}^{n\times n}$ is a given continuous map, $\Theta \subset \mathbb{K}^m$ is a compact, pathwise connected set, and $\mathcal{F} : \Theta \to \mathbb{K}^m$ is a set-valued map with compact values that defines a complete dynamical system on Θ. Under controllability assumptions for the parameter variations a number the basic results of the present paper hold.

(iv) Under the convexity assumption on Θ, Θ_1 the set $\mathcal{U}$ is convex and weak*-compact in $L^\infty(\mathbb{R}_+, \Theta)$. Thus we may associate to (2.1) a linear flow ϕ on the vector bundle $\mathbb{K}^n \times \mathcal{U}$ defined by

$$\phi(t;(x,u)) = (\Phi_u(t,0)x, u(t+\cdot))\,. \tag{2.3}$$

This setup is studied in some detail in [5].

3 Exponential growth rates

We now define the object of interest in this paper which is the (maximal) exponential growth rate associated to the system (2.1). Given the map A and the set of admissible parameter variations $\mathcal{U}$ define the sets of finite time evolution operators

$$\mathcal{S}_t(A,\mathcal{U}) := \{\, \Phi_u(t,0) \mid u \in \mathcal{U}\}\,, \quad \mathcal{S}(A,\mathcal{U}) := \bigcup_{t\geq 0} \mathcal{S}_t(A,\mathcal{U}).$$

Remark 3.1. *The main technical problem of LPV systems is the fact that $\mathcal{S}(A,\mathcal{U})$ does not naturally carry the structure of a semigroup. Indeed, by requiring that the elements of $\mathcal{U}$ are Lipschitz continuous it follows for $u_1, u_2 \in \mathcal{U}$ that the concatenation of $u_{1|(-\infty,t]}$ and $u_{2|(t,\infty)}$ is an admissible parameter variation if and only if $u_1(t) = u_2(t)$. This complicates matters compared to the case of linear inclusions of the form*

$$\dot{x} \in \{Ax \mid A \in \mathcal{M}\}\,,$$

where $\mathcal{M} \subset \mathbb{K}^{n\times n}$ is compact as studied in [6, 12] and references therein.

We now introduce finite time growth constants given by

$$\widehat{\rho}_t(A,\mathcal{U}) := \sup\left\{\frac{1}{t}\log\|S\| \mid S \in \mathcal{S}_t(A,\mathcal{U})\right\}.$$

It is easy to see that the function $t \longmapsto t\widehat{\rho}_t(A,\mathcal{U})$ is subadditive so that the following limit exists

$$\widehat{\rho}(A,\mathcal{U}) := \lim_{t\to\infty} \widehat{\rho}_t(A,\mathcal{U}).$$

As for $A \in A(\Theta)$ and $t \geq 0$ we have $e^{At} \in \mathcal{S}_t(A,\mathcal{U})$ it is clear that $\widehat{\rho}(A,\mathcal{U}) > -\infty$. The quantity $\hat{\rho}(A,\mathcal{U})$ is called *exponential growth rate* of system (2.1). A trajectory-wise definition of exponential growth would be to define the Lyapunov exponent corresponding to an initial condition $x_0 \in \mathbb{K}^n$ and $u \in \mathcal{U}$ by

$$\lambda(x_0, u) := \limsup_{t\to\infty} \frac{1}{t}\log\|\Phi_u(t,0)x_0\|,$$

and to define as exponential growth rate $\kappa(A,\mathcal{U}) := \sup\{\lambda(x,u) \mid 0 \neq x \in \mathbb{K}^n, u \in \mathcal{U}\}$. However, by appealing to the equivalent reformulation in the context of linear flows on vector bundles as in (2.3) it follows using Fenichel's uniformity lemma that $\kappa(A,\mathcal{U}) = \hat{\rho}(A,\mathcal{U})$, see [5, Prop. 5.4.15].

One might now be tempted to look for norms that characterize the quantity $\hat{\rho}(A,\mathcal{U})$ as can be done for the case of linear differential inclusions, see [12]. However, the following lemma shows that this is not a very fruitful enterprise.

Lemma 3.1. *[13] Consider system (2.1). If there is a norm v on $\mathbb{K}^n$ such that for all $x \in \mathbb{K}^n$, $u \in \mathcal{U}$ and the corresponding evolution operator $\Phi_u(t,s)$ it holds that*

$$v(\Phi_u(t,0)x) \leq e^{\hat{\rho}(A,\mathcal{U})t}\, v(x), \quad \forall t \geq 0, \tag{3.4}$$

then $\hat{\rho}(A,\mathcal{U}) = \max\{\lambda(x,B) \mid 0 \neq x \in \mathbb{K}^n, B : \mathbb{R}_+ \to A(\Theta)$ measurable$\}$.

The previous lemma states that a norm satisfying (3.4) can only exists for system (2.1) if the parameter varying system realizes the exponential growth that would be possible by allowing all measurable functions with values in $A(\Theta)$, that is by studying (2.1) with $\mathcal{U} = L^\infty(\mathbb{R},\Theta)$. One of the reasons to study LPV systems lies, of course, in the fact that this situation is rarely encountered. For this reason we use another approach that introduces a family of norms with an extremal property. This is the topic of the following section.

We will see that the idea that has been proposed by several authors, (see e.g. [1, 8]) to use parameter dependent Lyapunov functions can in fact be made exact. That is, we show how to construct a family of parameter dependent Lyapunov norms that are extremal in the sense that the exponential growth rate of system (2.1) is the incremental growth rate with respect to this family. Note that this implies that we cannot restrict our attention to quadratic norms.

4 Parametrized families of Lyapunov functions

In this section we assume the LPV system (Θ, Θ_1, A) to be given. For ease of notation we will therefore suppress the dependence on these data of $\hat{\rho}(A, \mathcal{U})$, $\mathcal{S}_t(A, \mathcal{U})$ and other objects we intend to define.

For each $\theta \in \Theta$ and $t \geq 0$ we define the set of evolution operators "starting in θ" by

$$\mathcal{S}_t(\theta) := \{\, \Phi_u(t,0) \mid u \in \mathcal{U} \text{ with } u(0) = \theta\}. \tag{4.5}$$

Similarly, we define for $\theta, \eta \in \Theta$ and for $t \geq 0$ the sets of evolution operators "starting in θ and ending at η" by $\mathcal{R}_t(\theta, \eta) := \{\Phi_u(t,0) \mid u \in \mathcal{U} \text{ with } u(0) = \theta, u(t) = \eta\}$. Then we define

$$\begin{aligned} \mathcal{S}_{\leq T}(\theta) \;&:=\; \bigcup_{0 \leq t \leq T} \mathcal{S}_t(\theta) \;\text{ and }\; \mathcal{S}(\theta) := \bigcup_{t \geq 0} \mathcal{S}_t(\theta)\,, \quad \text{respectively} \\ \mathcal{R}_{\leq T}(\theta, \eta) \;&:=\; \bigcup_{0 \leq t \leq T} \mathcal{R}_t(\theta, \eta) \;\text{ and }\; \mathcal{R}(\theta, \eta) := \bigcup_{t \geq 0} \mathcal{R}_t(\theta, \eta)\,. \end{aligned}$$

Note that the definition entails that for every $\theta \in \Theta$ the set $\mathcal{R}(\theta, \theta)$ is a semigroup. Furthermore, we note the following properties which are essential in the proofs of the ensuing results.

Proposition 4.1. *Consider system (2.1) with (A1)-(A3). For all $\theta, \eta \in \Theta$ and all $s < t \in \mathbb{R}_+$ we have*

(i) if $\mathcal{R}_s(\theta, \eta) \neq \emptyset$ then the set $\cup_{s \leq \tau \leq t} \mathcal{R}_\tau(\theta, \eta)$ is irreducible,

(ii) the set $\cup_{s \leq \tau \leq t} \mathcal{S}_\tau(\theta)$ is irreducible.

If we want to describe the exponential growth rate within the subsets of evolution operators with given initial and end condition, this leads to the definitions

$$\widehat{\rho}_t(\theta) := \max\left\{\frac{1}{t}\log\|S\| \mid S \in \mathcal{S}_t(\theta)\right\}, \quad \widehat{\rho}_t(\theta, \eta) := \max\left\{\frac{1}{t}\log\|S\| \mid S \in \mathcal{R}_t(\theta, \eta)\right\},$$

which has the problem that the functions $t \longmapsto t\widehat{\rho}_t(\theta)$, and $t \longmapsto t\widehat{\rho}_t(\theta, \eta)$ are no longer submultiplicative, so that is useful to point out the following.

Lemma 4.1. *Consider system (2.1) with (A1)-(A3). There is a constant $C > 0$ such that for every $t \geq 0$ and for all $\theta, \eta \in \Theta$ there is an $R \in \mathcal{R}_{t+2\bar{h}}(\theta, \eta)$ such that*

$$\|R\| \geq C e^{\hat{\rho} t}\,.$$

This result may now be used to prove the following statement.

Lemma 4.2. *Consider the system (2.1) with (A1)-(A3). For every $\theta, \eta \in \Theta$ we have that*

$$\widehat{\rho} = \lim_{t \to \infty} \widehat{\rho}_t(\theta, \eta), \quad \widehat{\rho} = \lim_{t \to \infty} \widehat{\rho}_t(\theta). \tag{4.6}$$

Proof: Fix $\theta, \eta \in \Theta$. Clearly, for all $t \geq 0$ we have $\widehat{\rho}_t(\theta,\eta) \leq \widehat{\rho}_t(\theta) \leq \widehat{\rho}_t$, so that in order to show (4.6) it is sufficient to show that $\widehat{\rho} \leq \liminf_{t\to\infty} \widehat{\rho}_t(\theta,\eta)$. This however, is an immediate consequence of Lemma 4.1, as we have for every $t \geq 0$, that $\widehat{\rho}_t(\theta,\eta) \geq 1/t(\log C + (t - 2\bar{h})\hat{\rho})$.

The second assertion follows from similar argument. □

As the exponential growth in $\mathcal{S}$ and in the subsets $\mathcal{S}(\theta)$, $\mathcal{R}(\theta,\eta)$ is essentially the same it makes sense to pursue ideas of the construction in [12] and to define limit sets as follows.

$$\mathcal{S}_\infty(\theta) \;:\; = \{\, S \in \mathbb{K}^{n\times n} \mid \exists\, t_k \to \infty\,, S_k \in \mathcal{S}_{t_k}(\theta) : \, e^{-\hat{\rho}t_k} S_k \to S \,\}. \tag{4.7}$$

$$\mathcal{R}_\infty(\theta,\eta) \;:\; = \{\, S \in \mathbb{K}^{n\times n} \mid \exists\, t_k \to \infty\,, S_k \in \mathcal{R}_{t_k}(\theta,\eta) : \, e^{-\hat{\rho}t_k} S_k \to S \,\}. \tag{4.8}$$

We note the following properties of $\mathcal{S}_\infty(\theta)$ and $\mathcal{R}_\infty(\theta,\eta)$.

Lemma 4.3. *Consider the system (2.1) with (A1)-(A3). For all $\theta, \eta \in \Theta$ it holds that*

(i) $\mathcal{R}_\infty(\theta,\eta)$ is a compact, nonempty set not equal to $\{0\}$,

(ii) $\mathcal{S}_\infty(\theta)$ is a compact, nonempty set not equal to $\{0\}$, and $\cup_{\theta\in\Theta}\mathcal{S}_\infty(\theta)$ is bounded,

(iii) for every $R \in \mathcal{R}_t(\theta,\eta)$ and every $S \in \mathcal{S}_\infty(\eta)$ we have $e^{-\hat{\rho}t} SR \in \mathcal{S}_\infty(\theta)$,

(iv) for every $S \in \mathcal{S}_\infty(\theta)$ and every $t \in \mathbb{R}_+$ there exist $\eta \in \Theta$, $R \in \mathcal{R}_t(\theta,\eta)$, and $T \in \mathcal{S}_\infty(\eta)$ such that $S = e^{-\hat{\rho}t} TR$,

(v) $\mathcal{R}_\infty(\theta,\eta)$, $\mathcal{S}_\infty(\theta)$ are irreducible.

Proof: Without loss of generality we may assume that $\hat{\rho} = 0$ in this proof.

(i) A standard argument shows that $\mathcal{R}_\infty(\theta,\eta)$ is closed. Lemma 4.1 shows that there exists a constant $C > 0$ and sequences $t_k \to \infty\,, S_k \in \mathcal{R}_{t_k}(\theta,\eta)$ with $\|S_k\| \geq C$ for all $k \in \mathbb{N}$. So if the sequence $\{\|S_k\|\}_{k\in\mathbb{N}}$ has a convergent subsequence then its limit must be different from zero. Thus to conclude the proof it is sufficient to show that there is a bound $M \geq \|S\|$, for all t large enough and all $S \in \mathcal{R}_t(\theta,\eta)$. This shows that $\mathcal{R}_\infty(\theta,\eta)$ is bounded and that the sequence constructed above indeed does have a convergent subsequence. So assume that the constant M does not exist, so that there are sequences $t_k \to \infty$, $S_k \in \mathcal{R}_{t_k}(\theta,\eta)$ with $\|S_k\| \to \infty$. By a small variation of [12, Lemma 3.1] there are constants $1 \geq \varepsilon > 0$ and $T > 0$ only depending on the set $\mathcal{R}(\eta,\theta)$ such that for all $x \in \mathbb{K}^n$ and all $B \in \mathbb{K}^{n\times n}$ there is an $R_{\leq T} \in \mathcal{R}(\eta,\theta)$ with $\|BRx\| \geq \varepsilon\|B\|\|x\|$. Choose k large enough such that

$$\|S_k\| > 4/\varepsilon.$$

Pick an arbitrary $x_0 \in \mathbb{K}^n$, such that $\|S_k x_0\| \geq \|S_k\|\,\varepsilon/2$. Then we can choose $R_1 \in \mathcal{R}_{\leq T}(\eta,\theta)$ such that

$$\|S_k R_1 S_k x_0\| \geq \left(\|S_k\|\,\frac{\varepsilon}{2}\right)^2 .$$

Note that by construction $S_k R_1 S_k \in \mathcal{R}(\theta, \eta)$. Applying the same arguments again we can choose $R_2 \in \mathcal{R}_{\le T}(\eta, \theta)$ such that

$$\|S_k R_2 S_k R_1 S_k x_0\| \ge \left(\|S_k\| \frac{\varepsilon}{2}\right)^3 .$$

Arguing inductively we construct times $lt_k \le \tau_l \le l(t_k + T)$ and matrices $T_l \in \mathcal{R}_{\tau_l}(\theta, \eta)$ with

$$\frac{1}{\tau_l} \log \|T_l\| \ge \frac{l}{\tau_l} \log \left(\|S_k\| \frac{\varepsilon}{2}\right) \ge \frac{l}{\tau_l} \log 2 \ge \frac{1}{t_k + T} \log 2.$$

This contradicts the assumption that $\limsup_{l \to \infty} \frac{1}{\tau_l} \log \|T_l\| \le 1$.

(ii) As $\mathcal{R}_\infty(\theta, \eta) \subset \mathcal{S}_\infty(\theta)$ it is clear from (i) that $\mathcal{S}_\infty(\theta)$ is nonempty and not equal to $\{0\}$. Closedness is immediate from the definition. It remains to show that $\cup_{\theta \in \Theta} \mathcal{S}_\infty(\theta)$ is bounded. If this is not the case then there are $t_k \to \infty$, $S_k \in \mathcal{R}_{t_k}(\theta_k, \eta_k)$ with $\|S_k\| \to \infty$. This is brought to a contradiction similar to the proof of (i).

(iii) This is an easy exercise.

(iv) Let $t_k \to \infty$, $\Phi_k(t_k, 0) \in \mathcal{S}_{t_k}(\theta)$ be sequences such that $\Phi_k(t_k, 0) \to S \in \mathcal{S}_\infty(\theta)$. Fix $t \ge 0$. Let $u_k \in \mathcal{U}$ be the generators for Φ_k. As the family $\{u_k\}$ is bounded and equicontinuous (due to the compactness of Θ and Θ_1) we may apply the Arzela-Ascoli theorem and assume without loss of generality that $u_k \to u \in \mathcal{U}$ uniformly on $[0, t]$. Denote $\eta = u(t)$ and let $R \in \mathcal{R}(\theta, \eta)$ be the transition matrix generated by u. By construction there are nonnegative times $s_k \to 0$ and $S_k \in \mathcal{R}_{s_k}(\eta, u_k(t))$. Then we have

$$\Phi_k(t_k, t) S_k R \in \mathcal{S}_{t_k + s_k}(\theta) .$$

Defining $T_k := \Phi_k(t_k, t) S_k$ we may by (ii) assume without loss of generality that $T_k \to T \in \mathcal{S}_\infty(\eta)$. Furthermore, as $\Phi_k(t, 0) \to R$ it follows that $TR = S$. This shows the assertion.

(v) Fix $\theta \in \Theta$. As we have noted the set $\mathcal{R}(\theta, \theta)$ is a semigroup that is irreducible by Proposition 4.1. By (iii) it is easy to see that if $S \in \mathcal{R}(\theta, \theta) \cup \mathcal{R}_\infty(\theta, \theta)$, $T \in \mathcal{R}_\infty(\theta, \theta)$ then $ST, TS \in \mathcal{R}_\infty(\theta, \theta)$. Thus $\mathcal{R}_\infty(\theta, \theta)$ is a nonzero semigroup ideal of the irreducible semigroup

$$\mathcal{R}_\infty(\theta, \theta) \cup \mathcal{R}(\theta, \theta) .$$

By [9, Lemma 1] this shows irreducibility of $\mathcal{R}_\infty(\theta, \theta)$. The second assertion follows from $\mathcal{R}_\infty(\theta, \theta) \subset \mathcal{S}_\infty(\theta)$.

□

We note the following corollary with respect to the maps $\theta \longmapsto \mathcal{S}_\infty(\theta)$, $(\theta, \eta) \longmapsto \mathcal{R}_\infty(\theta, \eta)$.

Corollary 4.1. *Consider the system (2.1) with (A1)-(A3). The set-valued maps*

$$\theta \longmapsto \mathcal{S}_\infty(\theta)\,, \tag{4.9}$$
$$(\theta,\eta) \rightarrow \mathcal{R}_\infty(\theta,\eta) \tag{4.10}$$

are Lipschitz continuous on Θ, respectively $\Theta \times \Theta$, with respect to the Hausdorff topology.

Proof: Let $\theta, \eta \in \Theta$ then for $t := \|\theta - \eta\|/h$ there exists a transition matrix $R \in \mathcal{R}_t(\theta, \eta)$. Defining $m := \max\{\|A - \hat{\rho} I\| \mid A \in A(\Theta)\}$ we have by an application of Gronwall's lemma that

$$\left\|I - e^{-\hat{\rho} t} R\right\| \leq \exp(tm) - 1\,. \tag{4.11}$$

Now for any $S \in \mathcal{S}_\infty(\eta)$ we have by Lemma 4.3 that $e^{-\hat{\rho} t} SR \in \mathcal{S}_\infty(\theta)$. And so by (4.11)

$$\left\|S - e^{-\hat{\rho} t} SR\right\| \leq \|S\| \left\|I - e^{-\hat{\rho} t} R\right\| \leq \max\{\|S\| \mid S \in \cup_{\theta\in\Theta} \mathcal{S}_\infty(\theta)\} \left(\exp(mt) - 1\right)\,.$$

This shows that $\max\{\operatorname{dist}(S, \mathcal{S}_\infty(\theta)) \mid S \in \mathcal{S}_\infty(\eta)\} \leq C\|\theta - \eta\|$ and by symmetry the same holds for the Hausdorff distance $H(\mathcal{S}_\infty(\theta), \mathcal{S}_\infty(\eta))$. This completes the proof of the first statement. The second statement is shown in exactly the same manner.

$\square$

We now define a family of norms by setting for $\theta \in \Theta$

$$v_\theta(x) := \max\,\{\|Sx\| \mid S \in \mathcal{S}_\infty(\theta)\}\,. \tag{4.12}$$

Using Lemma 4.3 (ii) it is easy to see that for every $\theta \in \Theta$ the function defined in (4.12) is a norm on $\mathbb{K}^n$. The following result shows that in this manner we have defined a family of parameterized Lyapunov functions for our LPV system that

Proposition 4.2. *Consider the system (2.1) with (A1)-(A3). Then*

(i) For all $u \in \mathcal{U}$ and all $x \in \mathbb{K}^n$ it holds that

$$v_{u(t)}(\Phi_u(t,0)x) \leq e^{\hat{\rho} t}\, v_{u(0)}(x)\,,$$

(ii) for every $x \in \mathbb{K}^n$, every $t \geq 0$, and every $\theta \in \Theta$ there exists an $u \in \mathcal{U}$ such that $u(0) = \theta$ and such that

$$v_{u(t)}(\Phi_u(t,0)x) = e^{\hat{\rho} t}\, v_\theta(x)\,.$$

Proof: Without loss of generality we may assume that $\hat{\rho} = 0$.

(i) Assume that $v_{u(t)}(Sx) > v_{u(0)}(x)$. Then $\|TSx\| > v_{u(0)}(x)$ for some $T \in \mathcal{S}_\infty(u(t))$. Now Lemma 4.3 (iii) shows that $TS \in \mathcal{S}_\infty(u(0))$. This contradicts the definition of $v_{u(0)}$.

(ii) Let $S \in \mathcal{S}_\infty(\theta)$ be such that $\|Sx\| = v_\theta(x)$. By Lemma 4.3 (iv) there exist matrices $R \in \mathcal{S}_t(\theta,\eta)$, $T \in \mathcal{S}_\infty(\eta)$ such that $S = TR$. Let $u \in \mathcal{U}$ be generating for R. Then we have by part (i)

$$v_\theta(x) = \|TRx\| \le v_{u(t)}(Rx) \le v_\theta(x)\,.$$

This concludes the proof. □

In order to state a continuity result we need a notion of distance between norms. To this end we introduce the space of positively homogeneous functions on $\mathbb{K}^n$ defined by

$$\mathrm{Hom}\,(\mathbb{K}^n,\mathbb{R}) := \{f : \mathbb{K}^n \to \mathbb{R} \,|\, \forall \alpha \ge 0 : f(\alpha x) = \alpha f(x) \text{ and } f \text{ is continuous on } \mathbb{K}^n\}\,.$$

Clearly, all norms on $\mathbb{K}^n$ are elements of $\mathrm{Hom}\,(\mathbb{K}^n,\mathbb{R})$. This space becomes a Banach space if equipped with the norm

$$\|f\|_{\infty,\mathrm{hom}} := \max\left\{|f\,(x)| \;|\; \|x\|_2 = 1\right\}\,.$$

Proposition 4.3. *Consider the system (2.1) with (A1)-(A3). Then the map*

$$\theta \longmapsto v_\theta\,, \tag{4.13}$$

is Lipschitz continuous from Θ *to* $\mathrm{Hom}\,(\mathbb{K}^n,\mathbb{R})$.

Proof: By definition we have

$$\|v_\theta - v_\eta\|_{\infty,\mathrm{hom}} = \max_{\|x\|_2=1} \left|\max\,\{\|Sx\| \,|\, S \in \mathcal{S}_\infty(\theta)\} - \max\,\{\|Sx\| \,|\, S \in \mathcal{S}_\infty(\eta)\}\right|\,.$$

Assume that $v_\theta(x) = \|\tilde{S}x\|$ for some $\tilde{S} \in \mathcal{S}_\infty(\theta)$ then there is a $T \in \mathcal{S}_\infty(\eta)$ such that $\|\tilde{S} - T\| \le H(\mathcal{S}_\infty(\theta),\mathcal{S}_\infty(\eta))$ and we obtain

$$v_\theta(x) - v_\eta(x) \le \|\tilde{S}x\| - \|Tx\| \le \|\tilde{S} - T\|\|x\| \le CH(\mathcal{S}_\infty(\theta),\mathcal{S}_\infty(\eta))\|x\|_2\,,$$

where C is a constant such that $\|x\| \le C\|x\|_2$. This shows that

$$\|v_\theta - v_\eta\|_{\infty,\mathrm{hom}} \le CH(\mathcal{S}_\infty(\theta),\mathcal{S}_\infty(\eta))\,.$$

Now the assertion follows from Corollary 4.1. □

5 Lipschitz Continuity

Using the existence of the Lyapunov norms v_θ it is possible to prove results on Lipschitz continuity of $\hat{\rho}$ under the variation of the data. To this end we introduce the space of linear parameter varying systems $\mathcal{LPV}$ as the space of triples (Θ,Θ_1,A) where $\Theta,\Theta_1 \subset \mathbb{K}^m$ are compact and convex, $0 \in \Theta_1$, $\operatorname{span}\Theta_1 \subset \operatorname{span}\Theta - \theta_0$ for $\theta_0 \in \Theta$, and $A : \mathbb{K}^m \to \mathbb{K}^{n\times n}$ is continuous. This space becomes a complete metric space if endowed with the metric

$$d((\Theta,\Theta_1,A),(\Theta',\Theta_1',A')) := H(\Theta,\Theta') + H(\Theta_1,\Theta_1') + \|A - A'\|_{\infty,\Theta\cup\Theta'}\,,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff distance on the compact sets in $\mathbb{K}^{n\times n}$ and $\|A-A'\|_{\infty,\Theta\cup\Theta'}$ is the supremum norm of $A - A'$ on the set $\Theta\cup\Theta'$.

We are especially interested in the subset

$$\mathcal{I} := \{(\Theta,\Theta_1,A) \in \mathcal{LPV} \mid (\Theta,\Theta_1,A) \text{ satisfy conditions (A1) - (A3) }\}\,.$$

Note that with respect to the metric defined above $\mathcal{I}$ is dense in $\mathcal{LPV}$. If we fix the dimension of $\operatorname{span}\Theta_1 \subset \operatorname{span}\Theta - \theta_0$ and define

$$\mathcal{LPV}(r) := \{(\Theta,\Theta_1,A) \in \mathcal{LPV} \mid \dim\operatorname{span}\Theta - \theta_0 = r\}\,,$$

then $\mathcal{I}\cap\mathcal{LPV}(r)$ is open and dense in $\mathcal{LPV}(r)$ with respect to the relative topology.

Assuming irreducibility we define for the triple (Θ,Θ_1,A) the norms v_θ as in Section 4. Then define the constants

$$c^+(\theta,A,\mathcal{U}) := \max\,\{v_\theta(x) \mid \|x\| = 1\}\,,\quad c^-(\theta,A,\mathcal{U}) := \min\,\{v_\theta(x) \mid \|x\| = 1\}\,. \tag{5.14}$$

Denote by $v_{\theta,\eta}$ the operator norms from $(\mathbb{K}^n, v_\theta)$ to $(\mathbb{K}^n, v_\eta)$. Note that we have for arbitrary $B \in \mathbb{K}^{n\times n}$ that

$$\frac{c^-(\theta,A,\mathcal{U})}{c^+(\eta,A,\mathcal{U})} v_{\theta,\eta}(B) \le \|B\| \le \frac{c^+(\theta,A,\mathcal{U})}{c^-(\eta,A,\mathcal{U})} v_{\theta,\eta}(B)\,.$$

Theorem 5.1. *Let $Q \subset \mathcal{I}$ be compact. For each triple $(A,\Theta,\Theta_1) \in Q$ consider the system (2.1). Then there exist constants $C_-, C_+ > 0$ such that*

$$C_- \le \frac{c^+(\theta,A,\mathcal{U})}{c^-(\eta,A,\mathcal{U})} \le C_+\,,\quad \textit{for all } (\Theta,\Theta_1,A) \in Q,\ \theta,\eta \in \Theta\,.$$

Proof: We begin by showing the existence of C_+. Assume that there exist sequences $\{(\Theta_k,\Theta_{1k},A_k)\}_{k\in\mathbb{N}} \subset Q$, $\{\theta_k \in \Theta_k\}_{k\in\mathbb{N}}$, $\{\eta_k \in \Theta_k\}_{k\in\mathbb{N}}$ such that

$$\frac{c^+(\theta_k,A_k,\mathcal{U}_k)}{c^-(\eta_k,A_k,\mathcal{U}_k)} \to \infty\,.$$

Without loss of generality we may assume that $(\Theta_k,\Theta_{1k},A_k) \to (\Theta,\Theta_1,A) \in Q$, $\theta_k \to \theta \in \Theta$ and $\eta_k \to \eta \in \Theta$. For all $k \in \mathbb{N}$ choose $S_k \in \mathcal{S}_\infty(\theta_k,A_k,\mathcal{U}_k)$ such that $\|S_k\| = c^+(\theta_k,A_k,\mathcal{U}_k)$ and define $\tilde{S}_k := S_k/\|S_k\|$. Without loss of generality we may assume that $\tilde{S}_k \to \tilde{S}$, $\left\|\tilde{S}\right\| = 1$. Let $\varepsilon > 0$ and $T > 0$ be the constants for $\mathcal{R}(\eta,\theta,A,\mathcal{U})$ obtained from an application of [12, Lemma 3.1] as in the proof of Lemma 4.3 (i). Fix an arbitrary $x_0 \in \mathbb{K}^n$, $\|x_0\| = 1$. Then by convergence for all k large enough there exists an $R_k \in \mathcal{R}_{t_k}(\eta_k, \theta_k, A_k, \mathcal{U}_k)$ with $t_k \leq T + \varepsilon$ such that

$$\left\| \tilde{S} R_k x_0 \right\| \geq \frac{\varepsilon}{2}\,.$$

Define $T_k := e^{-\hat{\rho}(A_k, \mathcal{U}_k) t_k} S_k R_k \in \mathcal{S}_\infty(\eta_k, A_k, \mathcal{U}_k)$. Then we obtain

$$v_{\eta_k}(x_0) \geq \|T_k x_0\| = \frac{\|S_k\|}{e^{\hat{\rho}(A_k, \mathcal{U}_k) t_k}} \left\| \tilde{S}_k R_k x_0 \right\| \geq \frac{\|S_k\|}{e^{\hat{\rho}(A_k, \mathcal{U}_k) t_k}} \left(\left\| \tilde{S} R_k x_0 \right\| - \left\| \tilde{S} - \tilde{S}_k \right\| \|R_k x_0\| \right)$$

and hence for all k large enough we have

$$\begin{aligned} \frac{c^+(\theta_k, A_k, \mathcal{U}_k)}{v_{\eta_k}(x_0)} &\leq e^{\hat{\rho}(A_k, \mathcal{U}_k) t_k} \left(\left\| \tilde{S} R_k x_0 \right\| - \left\| \tilde{S} - \tilde{S}_k \right\| \|R_k x_0\| \right)^{-1} \\ &\leq e^{\hat{\rho}(A_k, \mathcal{U}_k) t_k} \frac{4}{\varepsilon}. \end{aligned}$$

Where we have used that the sequence $\{R_k\}_{k \in \mathbb{N}}$ is bounded so that the last term on the right converges to zero by construction. This shows that $c^+(\theta_k, A_k, \mathcal{U}_k)/c^-(\eta_k, A_k, \mathcal{U}_k)$ is bounded because $t_k \leq T + \varepsilon$ and $\hat{\rho}(A_k, \mathcal{U}_k)$ is bounded by compactness of Q.

The proof for the existence of C_- follows the same lines. □

Proposition 5.1. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The map*

$$(\Theta, \Theta_1, A) \mapsto \hat{\rho}(\Theta, \Theta_1, A)$$

is locally Lipschitz continuous on the space $\mathcal{I}$.

Proof: Let $Q \subset I$ be compact and $(\Theta, \Theta_1, A), (\Theta', \Theta_1', A') \in Q$ with associated sets of parameter variations $\mathcal{U}, \mathcal{U}'$. It can be shown that there is a constant $\alpha > 0$ only depending on Q such that for all $u \in \mathcal{U}$ there is a $w \in \mathcal{U}'$ with

$$\left\| A\left(u\left(\cdot\right)\right) - A'\left(w\left(\cdot\right)\right) \right\|_\infty \leq \alpha d\left(\left(\Theta, \Theta_1, A\right), \left(\Theta', \Theta_1', A'\right)\right)\,.$$

Denote the evolution operator corresponding to $A\left(u\left(\cdot\right)\right)$ by $\Phi\left(t,s\right)$ and the one corresponding to $A'\left(w\left(\cdot\right)\right)$ by $\Psi\left(t,s\right)$. Furthermore denote by $v_{w(t),w(s)}$ the operator norms induced by the parameterized Lyapunov functions $v_{w(t)}, v_{w(s)}$ corresponding to the linear parameter varying system (Θ', Θ_1', A'). Note that we have for $0 \leq r \leq s \leq t$ that $v_{w(r),w(t)}(TR) \leq v_{w(s),w(t)}(T) v_{w(r),w(s)}(R)$. Then we have for $t = k \in \mathbb{N}$ that

$$\begin{aligned} v_{w(0),w(k)}(\Phi(k,0)) &\leq v_{w(0),w(k)}(\Psi(k,k-1)\Phi(k-1,0)) \\ &\quad + v_{w(0),w(k)}\left((\Phi(k,k-1) - \Psi(k,k-1))\Phi(k-1,0)\right) \\ &\leq \left[e^{\hat{\rho}(A',\mathcal{U}')} + C \left\| \Phi(k,k-1) - \Psi(k,k-1) \right\| \right] v_{w(0),w(k-1)}(\Phi(k-1,0)) \end{aligned}$$

where C is a constant independent of $(\Theta, \mathcal{F}, A) \in Q, \theta \in \Theta$ and where we have used the extremality property (2.3) to obtain the bound on the first term. The constant C exists by

Theorem 5.1. Furthermore, we obtain a bound for the difference $\|\Phi(k,k-1)-\Psi(k,k-1)\|$ linear in $d := d\left((\Theta,\Theta_1,A),(\Theta',\Theta_1',A')\right)$ so that for a suitable constant C_2 we obtain the inequality

$$v_{w(0),w(k)}(\Phi(k,0)) \quad \leq \quad \left[e^{\hat{\rho}(A',\mathcal{U}')} + CC_2 d\right] v_{w(0),w(k-1)}(\Phi(k-1,0))\,,$$

which implies by induction and another application of Theorem 5.1 that for all $k \in \mathbb{N}$ we have

$$\|\Phi(k,0)\| \leq C_3 v_{w(0),w(k)}(\Phi(k,0)) \leq C_3\left[e^{\hat{\rho}(A',\mathcal{U}')} + CC_2 d\right]^k .$$

As the operators $\Phi(t,0), t \neq k$ are only small perturbations of some $\Phi(k,0)$ and as the constants C, C_2, C_3 were chosen independently of $(\Theta,\Theta_1,A),(\Theta',\Theta_1',A') \in Q$ and $u \in \mathcal{U}$ was arbitrary, this shows that

$$\hat{\rho}(A,\mathcal{U}) \leq \hat{\rho}(A',\mathcal{U}') + CC_2 d\,.$$

By symmetry we obtain

$$|\hat{\rho}(A,\mathcal{U}) - \hat{\rho}(A',\mathcal{U}')| \leq CC_2 d\left((\Theta,\Theta_1,A),(\Theta',\Theta_1',A')\right) .$$

This completes the proof. □

6 The Gelfand Formula

To complete the discussion of the exponential growth of linear parameter varying systems we discuss the analogon of the generalized spectral radius for our case. Again we assume given a fixed controllable linear parameter varying system (Θ,Θ_1,A) and we suppress the dependence on that particular system in our notation. In the case of linear inclusions the generalized spectral radius is defined via the long term behavior of the maximal spectral radius of evolution operators. This in some way reflects the sort of periodic motion that is possible in the inclusion. In our case periodicity of the underlying parameter variation is the natural assumption, which is analyzed in the sequel.

For $t \in \mathbb{R}_+$ we define the set of evolution operators corresponding to periodic $u \in \mathcal{U}$ by

$$\mathcal{P}_t := \bigcup_{\theta\in\Theta} \mathcal{R}_t(\theta,\theta) .$$

Then we may define the normalized supremum over the spectral radii by

$$\bar{\rho}_t := \sup\left\{\frac{1}{t}\log r(S) \mid S \in \mathcal{P}_t\right\}$$

and the supremum of the exponential growth rates obtainable by periodic parameter variations is defined by

$$\bar{\rho} := \limsup_{t\to\infty} \bar{\rho}_t \,.$$

As it is clear that $\bar{\rho}_t \leq \hat{\rho}_t$, we obtain immediately that $\bar{\rho} \leq \hat{\rho}$. We intend to show that these quantities are equal. To this end we need the following lemma.

Lemma 6.1. *Consider system (2.1) with (A1)–(A3). Then there exist $\theta \in \Theta$, $x \in \mathbb{K}^n, v_\theta(x) = 1$ and a sequence $\{t_k\}_{k\in\mathbb{N}}$, $t_k \geq 1, k \in \mathbb{N}$ such that there exist $S_k \in \mathcal{R}_{t_k}(\theta,\theta)$ with $e^{-\hat{\rho}t_k}S_k x \to x$ and*

$$\left| v_\theta(S_k x) - e^{\hat{\rho}t_k} \right| < \frac{1}{k} e^{\hat{\rho}t_k} \,. \tag{6.15}$$

Proof: We may assume that $\hat{\rho} = 0$. Pick an arbitrary $\theta_0 \in \Theta$ and $z \in \mathbb{K}^n$ such that $v_{\theta_0}(z) = 1$. By Proposition 4.2 (iv) there exists a $u_1 \in \mathcal{U}$ such that $u_0(0) = \theta_0$ and such that $v_{u_0(1)}(\Phi_{u_0}(1,0)z) = v_{\theta_0}(z) = 1$. Applying this argument again there exists $u_1 \in \mathcal{U}$ such that $u_1(0) = u_0(1)$ and so that $v_{u_1(1)}(\Phi_{u_1}(1,0)\Phi_{u_0}(1,0)z) = 1$. Repeating this argument inductively we obtain a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ and we may then define $u \in \mathcal{U}$ by $u(t) = u_k(t-k)$, if $t \in [k,k+1]$. By construction we have

$$v_{u(k)}(\Phi_u(k,0)z) = 1 \,, \quad k \in \mathbb{N}.$$

Also by the extremality condition (2.3) for the family $v_\theta, \theta \in \Theta$ we have for $t \geq 0$ arbitrary and $k \geq t$ that

$$1 = v_{u(k)}(\Phi_u(k,t)\Phi_u(t,0)z) \leq v_{u(t)}(\Phi_u(t,0)z) \leq v_{\theta_0}(z) = 1 \,,$$

so that in fact $v_{u(t)}(\Phi_u(t,0)z) \equiv 1$.

As Θ is compact there exists a strictly increasing sequence $s_k \to \infty$ such that $u(s_k) \to \theta$ for some $\theta \in \Theta$. Without loss of generality we may assume that $s_{k+1} - s_k \geq 1$ and $z_k := \Phi_u(s_k,0)z \to x$. Now we have

$$|v_\theta(x) - v_{u(s_k)}(z_k)| \leq |v_\theta(x) - v_\theta(z_k)| + |v_\theta(z_k) - v_{u(s_k)}(z_k)| \,.$$

The first term on the right hand side converges to 0 by convergence of z_k, the second by locally uniform convergence of $v_{u(s_k)}$ to v_θ, which is a consequence of Proposition 4.3. Thus we have $v_\theta(x) = 1$.

By convergence of $u(s_k)$ there exist $\sigma_k, \tau_k \to 0$ and $R_k \in \mathcal{R}_{\sigma_k}(\theta, u(s_k)), T_k \in \mathcal{R}_{\tau_k}(u(s_{k+1}),\theta)$. In particular, $R_k, T_k \to I$ as $k \to \infty$. Define $t_k := (s_{k+1} - s_k) + \tau_k + \sigma_k$. Then we obtain for $S_k := T_k \Phi_u(s_{k+1}, s_k) R_k \in \mathcal{R}_{t_k}(\theta,\theta)$ that

$$|v_\theta(S_k x) - 1| = |v_\theta(S_k x) - v_{u(s_{k+1})}(z_{k+1})|$$

$$\leq v_\theta(T_k \Phi_u(s_{k+1}, s_k) R_k x - \Phi_u(s_{k+1}, s_k) z_k) + |v_\theta(z_{k+1}) - v_{u(s_{k+1})}(z_{k+1})| \,.$$

As $k \to \infty$ the first term on the right goes to zero by the convergence of $T_k, R_k \to I$ and $z_k \to x$, while the second term on the right goes to zero by locally uniform convergence of $v_{u(s_k)}$ to v_θ.

In particular we see that $v_\theta(S_k x - z_{k+1}) \to 0$ which implies by convergence of z_k that $S_k x \to x$. The assertion follows by taking an appropriate subsequence of the sequence t_k. □

Theorem 6.1. *Consider a system of the form (2.1) given by the triple* (Θ, Θ_1, A). *Then*

$$\bar{\rho} = \hat{\rho} \,.$$

Proof: Without loss of generality we may assume that $\hat{\rho} = 0$.

We claim that we may assume (A1) to (A3). For reasons of space this is only shown for (A3). If $A(\Theta)$ is not irreducible then there exists a regular $T \in \mathbb{K}^{n\times n}$ such that all matrices $A_0 \in A(\Theta)$ can be transformed to

$$TA_0T^{-1} = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1d} \\ 0 & A_{22} & A_{23} & \dots & A_{2d} \\ 0 & 0 & A_{33} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \\ 0 & & \dots & 0 & A_{dd} \end{bmatrix},$$

where each of the sets $\{A_{ii}; A \in A(\Theta)\}, i = 1 \dots d$ is irreducible (or 0). Note that the linear parameter varying system defined by setting $A_i(\theta) = A_{ii}$ is irreducible. It is easy to see that

$$\hat{\rho}(A, \mathcal{U}) = \max_{i=1,\dots,d} \hat{\rho}(A_i, \mathcal{U}) \quad \text{and} \quad \bar{\rho}(A, \mathcal{U}) = \max_{i=1,\dots,d} \bar{\rho}(A_i, \mathcal{U}) \,. \tag{6.16}$$

So assume now that (A1) – (A3) hold. By Lemma 6.1 there exist $\theta \in \Theta, x \in \mathbb{K}^n, v_\theta(x) = 1$ and a sequence $S_k \in \mathcal{R}(\theta, \theta)$ such that $S_k x - x \to 0$. Then we have by [6, Lemma 2] for the eigenvalues $\lambda_i(k)$ of S_k that

$$0 \leq \min_{1\leq i\leq n} 1 - |\lambda_i\,(k)| \leq \min_{1\leq i\leq n} |1 - \lambda_i\,(k)| \leq C\|S_k x - x\|^{1/n} \,.$$

Denoting by $\tilde{\lambda}_k$ an eigenvalue of S_k for which the minimum on the left is attained we see that $|\tilde{\lambda}_k| \to 1$ as $k \to \infty$. As we have $|\tilde{\lambda}_k| \leq 1$ and $t_k \geq 1$ we obtain $\bar{\rho} \geq 1/t_k \log|\tilde{\lambda}_k| \geq \log|\tilde{\lambda}_k|$, and it follows that $\bar{\rho} \geq 0$. This completes the proof.

□

Corollary 6.1. *The map*

$$(\Theta, \Theta_1, A) \mapsto \hat{\rho}(\Theta, \Theta_1, A)$$

is continuous on $\mathcal{LPV}$.

Proof: The maps $\hat{\rho}_t, \bar{\rho}_t : \mathcal{LPV} \to \mathbb{R}$ are clearly continuous. Now $\hat{\rho} = \inf_{t>0} \hat{\rho}_t$, so that $\hat{\rho}$ is upper semicontinuous as the infimum of continuous functions. Conversely, $\bar{\rho} = \sup_{t>0} \bar{\rho}_t$ is lower semicontinuous. Now using Theorem 6.1 the function $\hat{\rho} = \bar{\rho}$ is continuous. □

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