# Modelling, Analysis & Design by Velocity-Based Linearisation Families

D.J.Leith, W.E.Leithead[1]

## 1. Introduction

Whilst nonlinear dynamic systems are widespread, the analysis and design of such systems remains relatively difficult. In contrast, techniques for the analysis and design of linear time-invariant systems are rather better developed even though systems with genuinely linear time-invariant dynamics do not, in reality, exist. It is, therefore, attractive to adopt a divide and conquer philosophy whereby the analysis and design of a nonlinear system is decomposed into the analysis and design of a family of linear time-invariant systems. Perhaps the most widespread approach is to approximate a nonlinear system, locally to an equilibrium operating point, by its series expansion linearisation. Whilst the linearisation is only valid locally to a specific equilibrium operating point, this approach has the considerable advantage that it maintains continuity with established linear analysis techniques for which a substantial body of experience has been accumulated.

In contrast to the conventional series expansion linearisation approach, the velocity-based analysis and design framework associates a linear system with *every* operating point of a nonlinear system, not just the equilibrium operating points. The approach thereby relaxes the restriction to near equilibrium operation of conventional linearisation approaches whilst maintaining the continuity with linear methods. In this note, the velocity-based framework is briefly summarised and a number of its principal features are highlighted.

## 2. Velocity-based linearisation families

Consider the nonlinear dynamic system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \qquad (1)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$. Suppose that the nonlinear system, (1), is evolving along a trajectory, $(\mathbf{x}(t), \mathbf{r}(t))$, and at time, $t_1$, the trajectory has reached the point, $(\mathbf{x}_1, \mathbf{r}_1)$. In the vicinity of the operating point, $(\mathbf{x}_1, \mathbf{r}_1)$, the solution, $\mathbf{x}(t)$, to the nonlinear system is approximated (Leith & Leithead 1998a) by the solution, $\hat{\mathbf{x}}(t)$, to the linear system

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \qquad (2)$$

$$\dot{\hat{\mathbf{w}}} = \nabla_x\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\mathbf{w}} + \nabla_r\mathbf{F}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \qquad (3)$$

$$\dot{\hat{\mathbf{y}}} = \nabla_x\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\,\hat{\mathbf{w}} + \nabla_r\mathbf{G}(\mathbf{x}_1, \mathbf{r}_1)\,\dot{\mathbf{r}} \qquad (4)$$

There exists a velocity-based linearisation, (2)-(4), for every operating point. It follows that a velocity-based linearisation family, with members defined by (2)-(4), can be associated with the nonlinear system, (1). Whilst the solution to an individual velocity-based linearisation is only accurate in the vicinity of the corresponding operating point, the solutions to the members of the velocity-based linearisation family can be pieced together to globally approximate, to an arbitrary degree of accuracy, the solution to the nonlinear system (Leith & Leithead 1998a). Hence, the velocity-based linearisation family embodies the entire dynamics of the nonlinear system, (1), with no loss of information and provides an alternative representation of the nonlinear system.

The relationship between the nonlinear system, (1), and its associated velocity-based linearisation family is direct. Differentiating (1), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w} \qquad (5)$$

$$\dot{\mathbf{w}} = \nabla_x\mathbf{F}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{F}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \qquad (6)$$

$$\dot{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x}, \mathbf{r})\mathbf{w} + \nabla_r\mathbf{G}(\mathbf{x}, \mathbf{r})\,\dot{\mathbf{r}} \qquad (7)$$

Clearly, the velocity-based linearisation, (2)-(4), is simply the frozen form of (5)-(7) at the operating point, $(\mathbf{x}_1, \mathbf{r}_1)$. It should be noted that the differentiation step here is purely formal in nature and does not require differentiation of noisy measurements.

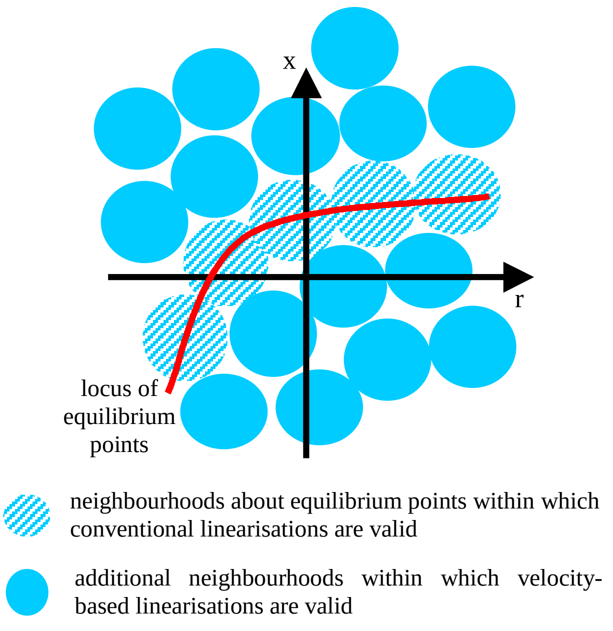


*Figure 1 - Validity of conventional and velocity-based linearisations*

[1] Department of Electronic & Electrical Engineering, University of Strathclyde, 50 George St., Glasgow G1 1QE. Tel. 0141 548 2407, Fax. 0141 548 4203, Email. doug@icu.strath.ac.uk

*Example – Skid-to-turn missile*

Consider a missile with pitch dynamics (Leith & Leithead 1998d)

$$\dot{q} = \frac{M}{I_{yy}}, \quad \dot{\alpha} = \frac{Z}{mV} + q$$

with

$$Z = \bar{q}\, SC_L, \quad M = \bar{q}\, ScC_M$$

where, $q$ is body axis pitch rate, $\alpha$ the angle of incidence, $V$ airspeed, $\bar{q}$ is dynamic pressure, $\overline{M}$ is the mach number, $\delta$ is the effective elevator deflection, $Z$ is the normal force, $M$ is pitching moment, $m$ , $I_{yy}$, $S$, $c$ are constants. The aerodynamic force and moment coefficients, $Z$ and $M$, depend on $\alpha$, $\delta$ and $\overline{M}$ .

Assuming that the short period approximation is accurate and $\partial Z/\partial\delta$ is sufficiently small that its contribution to the pitch dynamics can be neglected, the pitch dynamics in velocity-based form are

$$\dot{\mathbf{x}} = \mathbf{w}$$
$$\dot{\mathbf{w}} = \begin{bmatrix} 0 & M_\alpha(\rho) \\ 1 & Z_\alpha(\rho) \end{bmatrix}\mathbf{w} + \begin{bmatrix} M_\delta(\rho) \\ 0 \end{bmatrix}\dot{\delta} \quad (8)$$

with

$$M_\alpha(\rho) = \frac{1}{I_{yy}}\frac{\partial M}{\partial\alpha}, \quad M_\delta(\rho) = \frac{1}{I_{yy}}\frac{\partial M}{\partial\delta}, \; Z_\alpha(\rho) = \frac{1}{mV}\frac{\partial Z}{\partial\alpha} \mathrm{w}$$

here $\mathbf{x}=[q \;\; \alpha]^{\mathrm{T}}$, $\mathbf{w} = [\dot{q} \quad \dot{\alpha}]^{\mathrm{T}}$, $\rho = [V \;\; \bar{q} \;\; \overline{M} \;\; \alpha \;\; \delta]^T$. The members of the velocity-based linearisation family associated with the nonlinear dynamics are obtained by simply "freezing" the scheduling variable, $\rho$, in (8). The transfer function, of the velocity-based linearisation corresponding to the value $\rho_1$ of the scheduling variable, is

$$\dot{\alpha}(s) = \frac{M_\delta(\rho_1)}{s^2 - Z_\alpha(\rho_1)s - M_\alpha(\rho_1)}\dot{\delta}(s)$$

It should be noted that, whilst the incidence and elevator angles are related at equilibrium operating points, this is not the case at non-equilibrium operating points. Since the incidence and elevator angles, $\alpha$ and $\delta$, are elements of the scheduling variable, $\rho$, the velocity-based linearisation family contains members which do not correspond to the series expansion linearisation about *any* equilibrium point. It follows that, for example, a gain-scheduled controller designed on the basis of the equilibrium dynamics may not be valid during aggressive manoeuvring which takes the missile far from equilibrium.

## 3. Velocity-based modelling & analysis

The velocity-based linearisation family is an alternative representation of a nonlinear system which involves no loss of information. This representation

- Supports divide and conquer methodologies whereby the analysis/design of a nonlinear system is decomposed into the analysis/design of a number of linear systems. Continuity is thereby maintained with well-established linear methods.
- Is not confined to near equilibrium operation but rather accommodates both transitions between equilibrium operating points and sustained operation far from equilibrium (in marked contrast to conventional linearisation-based representations).
- Does not require an equilibrium operating point to be determined in order to analyse a system: indeed, no distinction is made between equilibrium and non-equilibrium operating points. Trimming, which is highly non-trivial for complex nonlinear systems, is therefore not required.
- Avoids the numerical differentiation associated with conventional numerical linearisation of a nonlinear model about an equilibrium point. Numerical differentiation is an undesirable ill-conditioned operation. The velocity-based linearisation associated with an operating point is obtained simply by "freezing" the velocity form of the system.
- Is particularly appropriate for complex nonlinear integrated systems since, in contrast to conventional linearisation-based representations it supports modular analysis and design. (The equilibrium operating point of a sub-system is dependent on the characteristics of the overall system in which it is embedded. Hence, conventional linearisation-based analysis does *not* support de-coupled modular analysis of complex systems. In contrast, the velocity-based representation of a sub-system is entirely independent of its equilibrium points. Moreover, the velocity-based representation of a combination of sub-systems is just the combination of the velocity-based representations of the individual sub-systems (Leith & Leithead 1998f) . Hence, analysis and design results for a specific sub-system can be integrated in a direct and transparent manner with those obtained for other sub-systems).

## 4. Velocity-based control design

The velocity-based linearisation of the feedback combination of a plant and controller is simply the feedback combination of the velocity-based linearisations of the plant and controller. This immediately suggests a control design procedure of the form (Leith & Leithead 1998b)

1. Determine the velocity-based linearisation family associated with the nonlinear plant dynamics.
2. Design a linear controller corresponding to each member of the plant velocity-based linearisation family.
3. Implement a nonlinear controller with velocity-based linearisation family corresponding to the linear controller family designed at step 2.

This design procedure retains a divide and conquer approach and maintains the continuity with linear design methods which is an important feature of the conventional gain-scheduling approach. However, in contrast to the conventional gain-scheduling approach, the resulting nonlinear controller

- Is valid throughout the operating envelope of the plant, not just in the vicinity of the equilibrium operating points.
- Does *not* inherently involve a slow variation requirement. Rather, a number of factors which are a function of the controller design, and which the designer is free to adjust, contribute towards any restriction on the rate of evolution of the trajectories. A primary factor which contributes towards any slow variation requirement is the degree of similarity between the dynamics of the velocity-based linearisations, of the closed-loop system, associated with different operating points. This determines the degree of nonlinearity of the closed-loop system and is largely dependent on the performance specification.
- May be designed such that the closed-loop velocity-based linearisations have the same input-output dynamics at every operating point. In this case the gain-scheduled controller may be interpreted as a dynamic inversion controller (Leith & Leithead 1998c). The dynamic inversion controller is valid globally and involves no slow variation constraint whatsoever. The velocity-based dynamic inversion approach is a direct generalisation of linear pole-zero inversion which decomposes the nonlinear inversion problem into a number of straightforward linear pole-zero inversion tasks. In contrast to Feedback Linearisation, the velocity-based dynamic inverse does not require full state feedback.

## 5. For further details, see


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