Log Convexity of Rate Region in 802.11e WLANs

Douglas J. Leith, Vijay G. Subramanian
Hamilton Institute, National University of Ireland Maynooth

Abstract—In this paper we establish the log convexity of the rate region in 802.11 WLANs. This generalises previous results for Aloha networks and has immediate implications for optimisation based approaches to the analysis and design of 802.11 wireless networks.

I. INTRODUCTION

In this paper we consider the log convexity of the rate region in 802.11 WLANs. The rate region is defined as the set of achievable throughputs and we begin by noting that the 802.11 rate region is well known to be non-convex. This is illustrated, for example, in Figure 1 for a simple two station WLAN. The shaded region indicates the set of achievable rate pairs \((s_1, s_2)\) where \(s_i\) is the throughput of station \(i\), \(i \in \{1, 2\}\). It can be seen from this figure that the maximum network throughput achievable when only a single station transmits (the extreme point along the x- or y-axes) is greater than that when both stations are active (e.g. the extreme point along the \(y = x\) line). This non-convex behaviour occurs because in 802.11 there is some probability of colliding transmissions when multiple stations are active, and so of lost transmission opportunities. However, also shown in Figure 2 is the same data but now replotted as the log rate region i.e. the set of achievable pairs \((\log s_1, \log s_2)\). Evidently, the log rate region is convex. Our main result in this paper is to establish that this behaviour is true in general, not just in this particular example. That is, although the 802.11 rate region is non-convex, it is nevertheless log convex. This has immediate implications for optimisation based approaches to the analysis and design of 802.11 wireless networks.

In a WLAN context, rate region properties have mainly been studied for Aloha networks. An early result by Post [5] establishes the log convexity of the rate region in slotted Aloha WLANs. More recently, the log convexity of the Aloha rate region in general mesh network settings has been established by several authors [7], [2], [3], [1], [8] in the context of utility optimisation. However all of these results make the standard Aloha assumption of equal idle and busy slot durations whereas in 802.11 WLANs highly unequal slot durations are the norm e.g. it is not uncommon to have busy slot durations that are 1000 times larger than the PHY idle slot duration.

In this paper we show how existence and uniqueness of fair solutions follows from log-convexity.

II. NETWORK MODEL

The 802.11e standard extends and subsumes the standard 802.11 DCF (Distributed Coordinated Function) contention mechanism by allowing the adjustment of MAC parameters that were previously fixed. With 802.11, on detecting the wireless medium to be idle for a period \(DIFS\), each station initializes a counter to a random number selected uniformly from the interval \([0, CW-1]\) where \(CW\) is the contention window. Time is slotted and this counter is decremented each slot that the medium is idle. An important feature is that the countdown halts when the medium becomes busy and only resumes after the medium is idle again for a period \(DIFS\). On the counter reaching zero, the station transmits a packet. If a collision occurs (two or more stations transmit simultaneously), \(CW\) is set to \(\min(2 \times CW, CW_{max})\) and the process repeated. On a successful transmission, \(CW\) is reset

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to the value $CW_{\text{min}}$ and a new countdown starts for the next packet. The 802.11e MAC enables the values of DIFS (called AIFS in 802.11e), $CW_{\text{min}}$ and $CW_{\text{max}}$ to be set on a per class basis for each station. Note that throughout this paper we restrict attention to situations where AIFS has the legacy value DIFS. In addition, 802.11e adds a TXOP mechanism that specifies the duration during which a station can keep transmitting without releasing the channel once it wins a transmission opportunity. In order not to release the channel, a SIFS interval is inserted between each packet-ACK pair. A successful transmission round consists of multiple packets and ACKs. By adjusting this time, the number of packets that may be transmitted by a station at each transmission opportunity can be controlled. A salient feature of the TXOP operation is that, if a large TXOP is assigned and there are not enough packets to be transmitted, the TXOP period is ended immediately to avoid wasting bandwidth.

Consider an 802.11e WLAN with $n$ stations. As usual, we divide time into MAC slots, where each MAC slot may consist either of a PHY idle slot, a successful transmission or a colliding transmission (where more than one station attempts to transmit simultaneously). Let $\tau_i$ denote the probability that station $i$ attempts a transmission. The mean throughput of station $i$ is given [4] by

$$s_i(T) = \frac{\tau_i \prod_{k \in N \setminus \{i\}} (1 - \tau_k) L_i}{\sigma P_{\text{idle}} + T_s P_{\text{suc}} + T_c (1 - P_{\text{idle}} - P_{\text{suc}})}$$

where $P_{\text{idle}} = \sum_{i \in N} \tau_i \prod_{k \in N \setminus \{i\}} (1 - \tau_k)$ and $P_{\text{suc}} = \sum_{i \in N} \tau_i \prod_{k \in N \setminus \{i\}} (1 - \tau_k)$. $T_s$ is the PHY idle slot duration, $T_c$ is the duration of a successful transmission and $T_c$ is the duration of a collision.

It will prove useful to work in terms of the quantity $x_i = \tau_i / (1 - \tau_i)$ rather than $\tau_i$. With this transformation we have that $P_{\text{idle}} = 1 / \prod_{k \in N} (1 + x_k)$ and $P_{\text{suc}} = \sum_{i \in N} x_i / \prod_{k \in N} (1 + x_k)$ and so

$$s_i(T) = \frac{x_i L_i / T_e}{\sigma / T_e - 1 + (T_s / T_e - 1) \sum_{i \in N} x_i + \prod_{k \in N} (1 + x_k)}$$

**Definition 1: Rate Region.** The rate region is the set $R(\bar{\tau})$ of achievable throughput vectors $S(T) = [s_1 \ldots s_n]^T$ as the vector $T$ of attempt probabilities ranges over $[0, \bar{\tau}]^n$, where $0 \leq \bar{\tau} \leq 1$.

In this paper we assume that the value of $\tau_i$ can be freely selected in the interval $[0, \bar{\tau}]$. This is an extremely mild assumption however. For example, suppose $CW_{\text{max}}$ is set equal to $CW_{\text{min}}$. Then $\tau = 2q / CW_{\text{min}}$, where $q$ is the probability that there is a packet available for transmission when the station wins a transmission opportunity and so is related to the packet arrival rate. When a station is saturated we have that $q = 1$. We note that the value of $q$ here is similar to the quantity in [4] also referred to as $q$. By adjusting $q$ (via the packet arrival process) and/or $CW_{\text{min}}$ it can be seen that the value of $\tau_i$ can be controlled as required.

**Definition 2: Log-convexity.** Recall that a set $C$ is convex if for any $s_1, s_2 \in C$ and $0 \leq \alpha \leq 1$, there exists an $s^* \in C$ such that $s^* = \alpha s_1 + (1 - \alpha) s_2$. Then a set $C$ is log convex if the set $\log C := \{\log s : s \in C\}$ is convex.

### III. LOG CONVEXITY

#### A. Log convexity

We begin in this section by assuming that $\bar{\tau} = 1$. This assumption is then later relaxed. For convenience we set $\alpha := \sigma / T_e$ with $\sigma \in [0, 1]$ and $K := T_s / T_e - 1$ with $K \geq 0$. Then the throughput expression can be written as

$$s_i(T) = \frac{x_i L_i / T_e}{X(T)}$$

where

$$X(T) := a + K \sum_{i \in N} x_i + \prod_{k = 2}^n \sum_{A \subseteq \{1, \ldots, n\}} \prod_{j \in A} x_j.$$

We know that the rate region $R(1)$ is non-convex, but ask whether it is log convex. Let $\log S(T) = [\log s_1 \ldots \log s_n]^T$. The rate region $R(1)$ is log convex if $\forall T^1, T^2 \in (0, 1)^n$, $\exists T^* \in (0, 1)^n$ such that

$$\alpha \log S(T^1) + (1 - \alpha) \log S(T^2) = \log S(T^*)$$

for every $\alpha \in [0, 1]$. Equivalently,

$$\frac{x_i^1}{X(T^1)} \alpha \frac{x_i^2}{X(T^2)} \text{ is log convex}$$

for all $i = 1 \ldots n$, or

$$\frac{x_i^1 \alpha (x_i^2)^{(1 - \alpha)}}{x_i^1} = \frac{X(T^1)^\alpha X(T^2)^{(1 - \alpha)}}{X(T^*)}$$

Note that here we restrict $T$ to $(0, 1)^n$ rather than $[0, 1]^n$. This involves no loss of generality since the $S(T)$ is a continuous function of $T$. Also note that the $L_i / T_e$ term cancels on both sides of the equation so the log convexity result is independent of this term.

We proceed by postulating that $x_i$ is of the form

$$x_i = \frac{(x_i^1)^\alpha (x_i^2)^{(1 - \alpha)}}{\delta}$$

since the right side of (3) does not depend on any particular $i$. Then the log convexity question is whether we can find $\delta > 0$ satisfying

$$\delta = \frac{X(T^1)^\alpha X(T^2)^{(1 - \alpha)}}{X(T^*)}$$

Substituting from (4) into (5) and defining $y_k = (x_k^1)^\alpha (x_k^2)^{(1 - \alpha)}$ we will need to solve

$$\delta = \frac{X(T^1)^\alpha X(T^2)^{(1 - \alpha)}}{a + K \sum_{i \in N} y_i \frac{y_i}{\delta} + \prod_{i \in N} (1 + \frac{y_i}{\delta}) - 1}$$

1Ignoring post backoff for simplicity.
Using that solution we find that there exists at least one solution \( \bar{\tau} \) of (6) that solves (6). Choosing \( \delta = 1 \) it can be seen that this lower bound lies within the range of the left-hand side of (6). Considering this left-hand side in more detail, it is a smooth function of \( \delta \) and it can be verified that its second derivative with respect to \( \delta \) is uniformly positive i.e. the left-hand side is a convex function of \( \delta \). It is unbounded and has range that includes\( [a + K \sum_{i \in N} y_i + \prod_{i \in N} (1 + y_i) - 1, \infty) \). This turning point partitions the real line and two solutions to (6) then exist, one of which due to the convexity of the function is unique. This turning point is log convex for every \( \alpha \geq 0 \) and \( \alpha \neq 1 \).

We have therefore established the following theorem.

**Theorem 1:** The rate region \( R(1) \) is log convex.

Using the results of [9] the existence of a max-min fair solution immediately follows.

### B. Constraints on \( \tau \)

We can extend the foregoing analysis to situations where the station attempt probability is constrained i.e. the vector \( \tau \) of attempt probabilities ranges over \( [0, \bar{\tau}]^n \), where \( 0 \leq \bar{\tau} \leq 1 \). Note that an upper bound on \( \tau_i \) of \( \bar{\tau} \) results in an upper bound \( \bar{\tau}/(1 - \bar{\tau}) \) on \( x_i \). Therefore if \( \tau^1, \tau^2 \in [0, \bar{\tau}]^n \), then \( x^1, x^2 \in [0, \bar{\tau}/(1 - \bar{\tau})]^n \) and for every \( \alpha \in [0, 1] \) we also have \( y \in [0, \bar{\tau}/(1 - \bar{\tau})]^n \). From the proof of Theorem 1 we know that there exists at least one \( \delta \geq 1 \) that solves (6). Using that solution we find that \( x^* = y/\delta \leq y \) so that \( x^* \in [0, \bar{\tau}/(1 - \bar{\tau})]^n \). Note that we can even have different values of \( \bar{\tau} \) for every \( \alpha \).

Therefore we have the following corollary.

**Corollary 1:** The rate region \( R(\bar{\tau}) \) is log convex for every \( 0 \leq \bar{\tau} \leq 1 \).

### IV. Discussion

These log convexity results allow us to immediately apply powerful optimisation results to the analysis and design of 802.11 WLANs. Specifically, we have that any optimisation of the form

\[
\max_S f(S) \\
\text{s.t. } S \in R(\bar{\tau}), h_i(S) \leq 0, i = 1, \ldots, m
\]

can be converted into an optimisation

\[
\max_S f'(\log S) \\
\text{s.t. } \log S \in \log R(\bar{\tau}), h'_i(\log S) \leq 0, i = 1, \ldots, m
\]

where \( f'(z) = f(\exp(z)) \) (so, in particular, \( f'(\log S) = f(S) \)).

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### V. Conclusions

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### References


