AN LMI APPROACH TO AUTOMATIC LOOP-SHAPING OF QFT CONTROLLERS

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Abstract: Quantitative Feedback Theory (QFT) is one of the most effective methods of robust controller design. In QFT design, we can consider the phase information of the perturbed plant so it is less conservative than H_∞ and μ-synthesis methods. In this paper, we want to overcome the major drawback of QFT method, i.e., lack of an automated technique for loop-shaping. Clearly such an automatic process must involve some sort of optimization, and while recent results on convex optimization have found fruitful applications in other areas of control theory we have tried to use LMI theory for automating the loop-shaping step of QFT design.

Keywords: Loop-shaping, Linear Matrix Inequalities, Quantitative Feedback Theory, Optimization.

1. INTRODUCTION

Quantitative Feedback Theory (QFT) is one of the most effective methods of robust controller design. Particularly it allows us to obtain less conservative controllers than other robust controller design methods like H_∞ and μ-synthesis. One feature that distinguishes QFT from other frequency-domain methods, such as H_∞ and LQG/LTR, is its ability to deal directly with uncertainty models and robust performance criteria. This is achieved by translating robust performance specifications and uncertainty models into so-called QFT bounds. These bounds, typically displayed on a Nichols chart plot, serve as a guide for shaping the nominal loop transfer function which involves the manipulation of gain, poles and zeros. This design process is executed efficiently using computer aided design software and is effective for "simple" problems, but QFT designers can benefit from an algorithm that automatically provides a first-cut solution to the loop-shaping problem. In addition, an automatic loop-shaping facility would enhance the capabilities of the expert QFT designer. Automatic loop-shaping algorithms have been proposed over the past twenty years and this paper reports on a new version.

In recent years, convex programming and LMI theory has been used widely in solving some important control problems, so we tried to use this method for solving QFT loop-shaping problem for the very first time.

This paper is not based upon any previous work, although methods mentioned in (Gahinet, et al., 1994) and (Krishnan, et al., 1977) inspired us to find our method.

2. THE QFT DESIGN TECHNIQUE

The general QFT problem is how to design controller C(s) and pre-filter F(s) such that for a given set of uncertain plants $P \in \{P\}$ with perturbed parameters $\alpha \in \Omega$ the following specifications are satisfied:

(i) Robust Stability:

$$T_R = FT = \frac{FCP}{1+CP} = \frac{FL}{1+L}$$

must be exponentially stable $\forall \alpha \in \Omega$.

(ii) Robust tracking performance: Two time functions *a(t)* and *b(t)*, are given and a command input *r(t)* (for example a step function) that specify the output tolerance of *y(t)* in the form:

$$a(t) \le y(t) \le b(t) \qquad \forall P \in \{P\} \tag{1}$$

These tracking specifications in the time domain can be translated into the frequency domain upper and lower bounds for $T_R(j\omega)$, that satisfies:

$$A(\omega) \le \left| T_R(j\omega) \right| \le B(\omega) \quad \textit{in dB units} \tag{2}$$

(iii) Output disturbance rejection specification: A function *D(ω)* is given that specifies the output disturbance rejection specifications in this form:

$$\left| T_d(j\omega) \right| \le D(\omega) \qquad \forall P \in \{P\} \tag{3}$$

where $T_d(s) = \frac{y_d(s)}{d(s)} = \frac{1}{1+C(s)P(s)} \equiv S(s)$

and *d(s)* is the output disturbance function.

(iv) *Input disturbance rejection specification:* A function *D'(ω)* is given that specifies the input disturbance rejection specification in this form:

$$\left|T'_d(j\omega)\right| \le D'(\omega) \qquad \forall P \in \{P\} \tag{4}$$

where $T'_d(s) = \frac{y_d(s)}{d'(s)} = \frac{P(s)}{1+G(s)P(s)} = S(s).P(s)$

and *d'(s)* is the input disturbance function.

In the classical QFT design, the above specifications will be transformed into a set of boundaries for some pre-specified frequencies (called trial frequencies) in Nichols chart and we have to specify the gain, poles and zeros of controller *C(s)* such that the open-loop transfer function $L_0(s) = P_0(\alpha_0, s).C(s)$ lies above the boundaries in each trial frequency (note that $P_0(\alpha_0, s)$ is the nominal plant). In this paper, we will introduce a method for converting this problem to an LMI problem that will be discussed in section IV.

3. LMI THEORY

Consider the problem of minimizing a linear function of a variable $x \in \Re^n$ subject to a matrix inequality:

Minimize $c^T x$
Subject to $F(x) \ge 0$ (5)

where $F(x) \hat{=} F_0 + \sum_{i=1}^{m} x_i F_i$.

The problem data are the vector $c \in \Re^n$ and m+1 symmetric matrices $F_0, \ldots, F_m \in \Re^{nxn}$ The inequality sign in $F(x) \ge 0$ means that F(x) is positive semi-definite, i.e., $z^T F(x) z \ge 0$ for all $z \in \Re^n$. We call the inequality $F(x) \ge 0$ a linear matrix inequality and the above problem a semi-definite program or an LMI problem. It is also called a convex optimization problem since its objective and constraints are convex, i.e.:
If $F(x) \ge 0$ and $F(y) \ge 0$ then for all λ, $0 \le \lambda \le 1$ then $F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y)$.

For example, linear programming problem:

Minimize $c^T x$
Subject to $Ax + b \ge 0$

is an LMI problem.
The most attractive feature of LMI theory is that LMI problems just have global minimums and there are no local minimums. Another important feature is that we can convert various LMI problems into a single LMI problem. Suppose that we have k LMIs $F_i(x) \ge 0$ (i=1,..,k). These k LMIs are equivalent to the LMI $F(x) \ge 0$ in which:.

$$F(x) = \begin{bmatrix} F_1(x) & \underline{O} & \ldots & \underline{O} \\ \underline{O} & F_2(x) & \ldots & \underline{O} \\ \ldots & \ldots & .. & \ldots \\ \underline{O} & \underline{O} & .. & F_k(x) \end{bmatrix}$$

In general, LMI problems do not have analytical solutions but there are some efficient numerical methods (like interior point method) for solving these problems, so all we have to do is to transform our optimization problem into one of standard LMI problems that has been mentioned in (Boyd, et al., 2002).

4. AUTOMATIC LOOP-SHAPING

In this section we will show how to convert the QFT loop-shaping problem into an LMI problem. We first start with the robust tracking specification. As mentioned above, we want to find a controller *C(s)* such that:

$$a(t) \le y(t) \le b(t) \;\; \forall \alpha \in \Omega, \forall t \ge 0 \tag{6}$$

or equivalently, so that:

$$\left(y(t) - \frac{a(t)+b(t)}{2}\right)^2 \le \left(\frac{b(t)-a(t)}{2}\right)^2 \;\; \forall \alpha \in \Omega, \forall t \ge 0 \tag{7}$$

An exact frequency-domain equivalent to the above time-domain inequality is unknown. A slightly weaker condition to

$$\int_0^\infty \left(y(t) - \frac{a(t)+b(t)}{2}\right)^2 \le \int_0^\infty \left(\frac{b(t)-a(t)}{2}\right)^2 \;\; \forall \alpha \in \Omega, \forall t \ge 0 \tag{8}$$

It is noted that the integration operation in (8) converts the original L_∞ (amplitude) into constraints to an L_2 (energy) constraint. Therefore satisfaction of (8) does not necessarily imply satisfaction of (7). Here *a(t)* and *b(t)* are upper and lower step-response specifications respectively. Also y(t) is the closed-loop response output as a function of the plant parametric uncertainty vector $\alpha \in \Omega$. A rigorous frequency-domain translation of bounds such as in (8) can be made negligible (Krishnan, et al., 1977). It will be shown that the relaxed problem is roughly equivalent to the solution of a frequency-domain sensitivity reduction problem. Let:

$$\frac{a(t)+b(t)}{2}=y_0(t)\ ,\ \frac{b(t)-a(t)}{2}=\upsilon(t)\quad \forall\alpha\in\Omega,\forall\omega \tag{9}$$

The by Parseval's theorem, a sufficient condition for satisfaction of (8) is:

$$\left|y(j\omega)-y_0(j\omega)\right|\le\left|v(j\omega)\right|\quad \forall\alpha\in\Omega,\forall\omega \tag{10}$$

We assume that *a(t)*,*b(t)* and hence *y(t)* are all Laplace transformable. For internal stability, every y(t) is required to have the same number N_z of non-minimum-phase zeros. Bode's sensitivity theorem (Bode, 1945), normalized with respect to y_0 shows that (temporarily dropping the argument $j\omega$:

$$\frac{y-y_0}{y_0}=\frac{FT-FT_0}{FT_0}=S\frac{P-P_0}{P_0} \tag{11}$$

where:

$P(s)=P(\alpha,s)$ is uncertain plant,

$P_0(s)=P(\alpha_0,s)$ is nominal plant,

$L_0(s)=P_0(\alpha,s).C(s)$ is nominal open-loop transfer function,

$L(s)=P(\alpha,s).C(s)$ is open-loop transfer function,

C(s) is the controller, and also:

$$S(s)=\frac{1}{1+L(s)},\ T(s)=\frac{L(s)}{1+L(s)}\ ,T_R(s)=F(s)\frac{L(s)}{1+L(s)}$$

Since the tracking specification imposes the constraint $\left|y-y_0\right|\le\left|\upsilon\right|$, we have from (10) and (11) that:

$$\left|S(j\omega)y_0(j\omega)\delta_{P_0}(j\omega)\right|\le\left|\upsilon(j\omega)\right| \tag{12}$$

is implied by:

$$\max_{p\in P}\left|S(j\omega)\right|\le\left|\frac{\upsilon(j\omega)}{y_0(j\omega)\delta_{P_0}(j\omega)}\right|\quad\forall\alpha\in\Omega,\forall\omega\in\Re \tag{13}$$

where: $\delta_{P_0}(\omega)\equiv\max_{p\in P}\left|\frac{P(j\omega)-P_0(j\omega)}{P_0(j\omega)}\right|$

Satisfaction of inequality (13) is a sufficient condition for the L_2 tracking specification to be met. Note that we can use equation 13 independent of the method we used for deriving *$y_0(j\omega)$* and *$\upsilon(j\omega)$.* In fact QFT designers has understood that finding the frequency domain equivalents of time domain specifications is not so difficult and for almost every practical time domain specifications we can find a 2nd order frequency domain equivalent. Therefore we can ignore equation (8) and obtain *$y_0(j\omega)$* and *$\upsilon(j\omega)$* using every method we prefer and then equation (13) can be used.

Now the robust tracking specification can be stated as follows:

$$\left|S(j\omega)\right|\le\frac{\left|\upsilon(j\omega)\right|}{\left|y_0(j\omega)\right|\delta_{P_0}(j\omega)}\equiv M_T(\omega)\quad\forall\alpha\in\Omega,\forall\omega \tag{14}$$

We also mentioned that robust output disturbance rejection specification is as follows:

$$\left|S(j\omega)\right|=\left|\frac{1}{1+L(j\omega)}\right|\le M_D(\omega)\quad\forall\alpha\in\Omega,\forall\omega \tag{15}$$

And we also want to have robust stability. It means that:

$$T_R(s)=F(s)\frac{C(s)P(s)}{1+C(s)P(s)} \tag{16}$$

must be exponentially stable $\forall\alpha\in\Omega$. It means that the maximum gain of T_R must be less than a priori defined value. The pre-filter *F(s)* is always a stable and is almost always a low pass filter, so its maximum gain is almost negligible and we can interpret this specification as finding controller *C(s)* such that the high frequency gain of *T(s)* will be less than a defined value. We can easily this specification into a limit over the maximum gain of sensitivity function *S(s)* Hence the robust stability specification can be stated as follows:

$$\left|S(j\omega)\right| \le M_S(\omega) \quad \forall \alpha \in \Omega, \forall \omega \qquad (17)$$

in which M_S is specified by designer.
Now we must find a controller C(s) such that equations (15),(16),(17) be satisfied simultaneously, so we define:

$$M(\omega) \equiv \min\{M_D(\omega), M_T(\omega), M_S(\omega)\} \in L_\infty \qquad (18)$$

Then the above constraints are simultaneously satisfied if:

$$\left|S(j\omega)\right| \le M(\omega) \qquad \forall \alpha \in \Omega, \forall \omega \qquad (19)$$

Therefore, our aim is to find a controller *C(s)* such that inequality (19) will be satisfied. To do this, we define a real-rational function *W(jω)* as upper bound for *1/M(ω),* so (19) converts to:

$$\left|S(j\omega)\right| \le \frac{1}{W(j\omega)} \qquad \forall \alpha \in \Omega, \forall \omega$$

or:

$$\left|S(j\omega)W(j\omega)\right| \le 1 \qquad \forall \alpha \in \Omega, \forall \omega$$

or equivalently:

$$\left\|S(j\omega)W(j\omega)\right\|_\infty \le 1 \qquad \forall \alpha \in \Omega \qquad (20)$$

Note that this is not an ordinary sensitivity reduction problem because here *S(jω)* is perturbed sensitivity function, not nominal sensitivity function. Using similar procedure, we can transform the input disturbance rejection specification into the below inequality:

$$\left\|T(j\omega)W_T(j\omega)\right\|_\infty \le 1 \qquad \forall \alpha \in \Omega \qquad (21)$$

in which *T(jω)* is perturbed complementary sensitivity function. Because both inequalities must be satisfied simultaneously, we must find a controller C(s) such that:

$$\left\|\begin{matrix} SW \\ TW_T \end{matrix}\right\|_\infty \le 1 \qquad \forall \alpha \in \Omega \qquad (22)$$

The above problem is equivalent to infinite nominal sensitivity reduction problem and instead of solving it, we consider just finite numbers of uncertain parameters and convert the problem to a finite number of sensitivity reduction problems. For example if parameter α_1 varies between 1 and 2 we can consider just 1, 1.5, 2 (or more values depending on the problem). In (Gahinet, et al., 1994) a method for converting the sensitivity reduction into LMI problem has been introduced. We can use this method and change all the finite sensitivity reduction problems into LMI form and as mentioned in section 3 all of the simultaneous LMI problems can be transformed into one LMI problem and can be solved by the available packages like MATLAB LMI toolbox.
Note that using this method; the loop-shaping problem has been completely automated so there is no need for calculating and plotting the QFT boundaries in Nichols chart.

5. RESULTS

We show the effectiveness of this method using a benchmark problem of QFT theory. The problem is to design a controller for a DC motor whose uncertain transfer function is:

$$P(s) = \frac{K}{s(1+\tau_m s)(1+\tau_e s)}$$

in which $150 \le K \le 300$ and $0.012 \le \tau_m \le 0.020$ and $\tau_e = 0.001s$.
The closed-loop objectives are:

- Robust stability specifications:

$$\forall P(j\omega) \in P, \forall \omega > 0; \left|\frac{C(j\omega)P(j\omega)}{1+P(j\omega)C(j\omega)}\right| \le 1.1$$

- Tracking Specifications:

$$\forall P(j\omega) \in P, \forall \omega > 0; T_d(\omega) \le \left|F\frac{CP}{1+PC}\right| \le T_u(\omega)$$

with:

$$T_d(\omega) = \left|\frac{1}{1+\frac{3(j\omega)}{50}+\frac{3(j\omega)^2}{50^2}+\frac{3(j\omega)^3}{50^3}}\right|$$

and

$$T_u(\omega) = \left| \frac{1}{1 + \frac{(j\omega)}{120} + \frac{(j\omega)^2}{120^2}} \right|$$

- Input disturbance rejection specifications:

$$\forall P(j\omega) \in P, \forall \omega > 0; \left| \frac{P(j\omega)}{1 + P(j\omega)C(j\omega)} \right| \le \left| \frac{\frac{(j\omega)}{10}}{1 + \frac{(j\omega)}{10}} \right|$$

- Output disturbance rejection specifications:

$$\forall P(j\omega) \in P, \forall \omega > 0; \left| \frac{1}{1 + P(j\omega)C(j\omega)} \right| \le \left| \frac{(j\omega)}{10} \right|$$

Design frequencies are chosen as [0.01, 0.05, 0.1, 0.2, 1.0, 5.0, 10.0].

Using the MATLAB LMI toolbox, the controller was determined to be:

$$C(s) = \frac{3.355\left(\frac{s}{0.630}+1\right)\left(\frac{s}{2.994}+1\right)\left(\frac{s}{2.994}+1\right)\left(\frac{s}{7.035}+1\right)\left(\frac{s}{7.035}+1\right)}{\left(\frac{s}{0.714}+1\right)\left(\frac{s}{0.822}+1\right)\left(\frac{s}{3.640}+1\right)\left(\frac{s}{20.254}+1\right)\left(\frac{s}{714.29}+1\right)}$$

In order to meet the closed-loop tracking requirement in the two degree of freedom structure, a suitable pre-filter must be obtained. This is designed according to the methodology described in (Houpis, et al., 1999). This pre-filter is given as:

$$F(s) = \frac{(1 + s/0.5)}{(1 + s/0.6)(1 + s/2.5)}$$

A collection of closed-loop Bode plots for the extreme plant parameter conditions are given in Figures 1 to 4 for expert's design and in figures 5 to 8 for our proposed method. As it can be seen in figures 5 to 8, the stability, performance and disturbance rejection requirements are satisfied.

It can also be seen that in comparison to the expert's design, the high frequency gain of $L_0(s)$ has been reduced. Also, the sensitivity and input disturbance responses are reduced specially in lower frequencies.
Although there is no direct implication for time domain performance, it can be seen that the corresponding time response is favorable, as shown in figure 6.

6. CONCLUSIONS

In this paper a method for automatic loop-shaping of QFT controllers has been introduced for the first time. The design process has been converted to an LMI problem which can be solved using efficient numerical methods. The results show the effectiveness of this new method.

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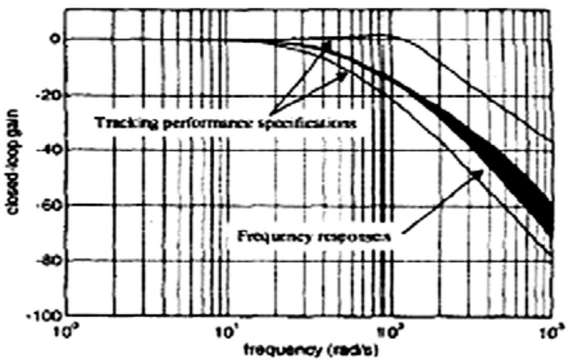

Figure 1 : Closed-loop response frequency

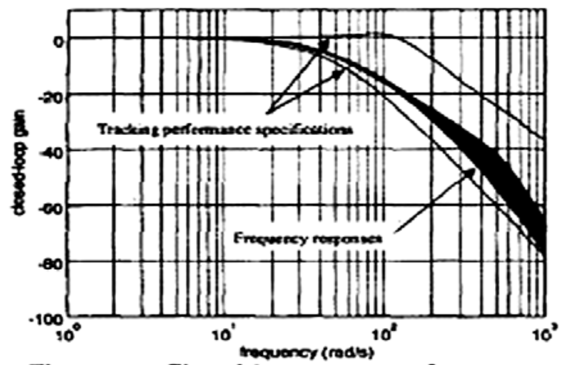

Figure 5 : Closed-loop response frequency

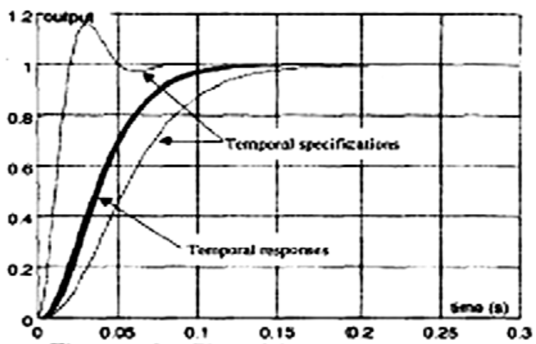

Figure 2 : Closed-loop step response

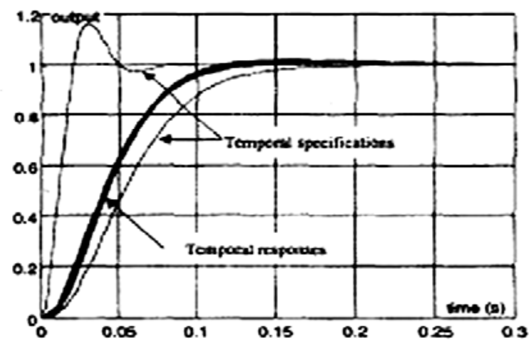

Figure 6 : Closed-loop step response

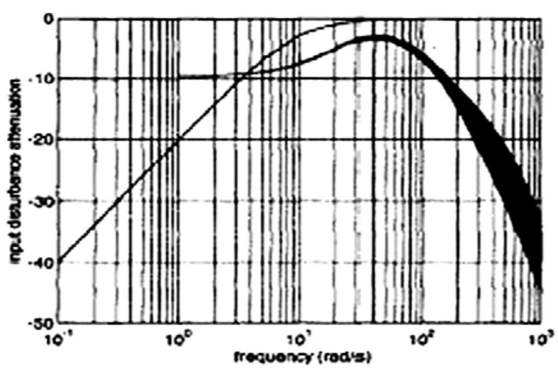

Figure 3 : Input disturbance response frequenc;

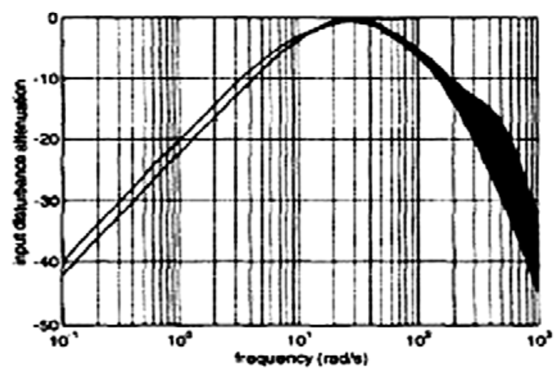

Figure 7 : Input disturbance response frequency

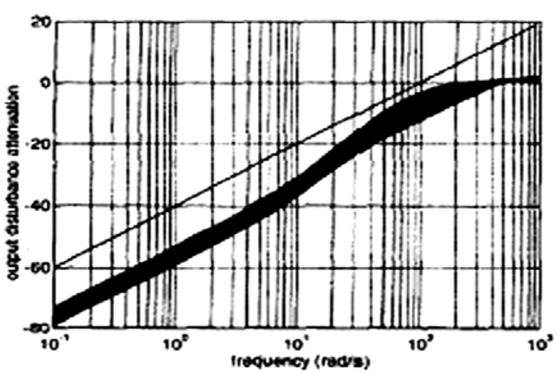

Figure 4 : Sensitivity response frequency

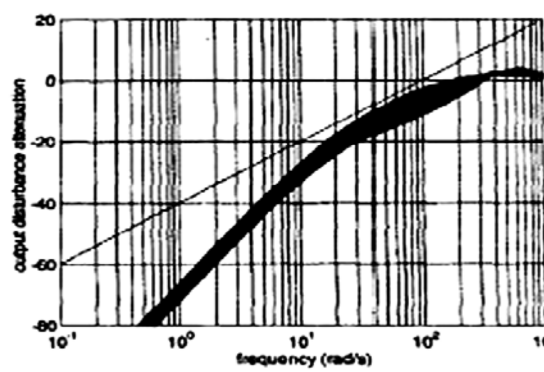

Figure 8 : Sensitivity response frequency