Algebraic Connectivity for Vertex-Deleted Subgraphs, and a Notion of Vertex Centrality

Steve Kirkland
Hamilton Institute
National University of Ireland Maynooth
Ireland
stephen.kirkland@nuim.ie
phone: +353-(0)1-708-6797; fax: +353-(0)1-708-6269

Abstract

Let G be a connected graph, suppose that v is a vertex of G, and denote the subgraph formed from G by deleting vertex v by $G \setminus v$. Denote the algebraic connectivities of G and $G \setminus v$ by $\alpha(G)$ and $\alpha(G \setminus v)$, respectively. In this paper, we consider the functions $\phi(v) = \alpha(G) - \alpha(G \setminus v)$ and $\kappa(v) = \frac{\alpha(G \setminus v)}{\alpha(G)}$, provide attainable upper and lower bounds on both functions, and characterise the equality cases in those bounds. The function κ yields a measure of vertex centrality, and we apply that measure to analyse certain graphs arising from food webs.

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1. Introduction and Preliminaries

Let G be a connected graph on n vertices, and label the vertices of G with the numbers $1, \ldots, n$. The *Laplacian matrix for* G is the $n \times n$ matrix $L(G) = D - A$, where A is the $(0, 1)$ adjacency matrix for G, and D is the diagonal matrix of vertex degrees. It is well-known that $L(G)$ is a symmetric positive semi-definite matrix, and that $\mathbf{1}$, the vector of all ones, is a null vector for $L(G)$. In the case that G is connected, then 0 is a simple eigenvalue of the matrix $L(G)$; more generally, it is not difficult to establish that the multiplicity of 0 as an eigenvalue of $L(G)$ coincides with the number of connected components in G. Order the eigenvalues of $L(G)$ in nondecreasing order. The *algebraic connectivity of* G, denoted henceforth by $\alpha(G)$, is the second smallest eigenvalue of the Laplacian matrix $L(G)$. Thus it follows that G is connected if and only if $\alpha(G) > 0$. As is shown in [6], we have $\alpha(G) = \min\{u^T L(G)u | u^T \mathbf{1} = 0, u^T u = 1\}$. We note also that an eigenvector of $L(G)$ associated with $\alpha(G)$ is known as a *Fiedler vector*.

As the term "algebraic connectivity" suggests, $\alpha(G)$ provides an algebraic measure of how connected the graph G is; there is a wealth of results to support that statement, beginning with the pioneering work of Fielder on the subject [6]. The surveys [1] and [9] provide overviews of work on algebraic connectivity and Fiedler vectors for graphs.

Given a connected graph G and a vertex v of G, we let $G \setminus v$ denote the subgraph formed from G by deleting v and all edges incident with it. In this paper, we consider the relationship between $\alpha(G)$ and $\alpha(G \setminus v)$. The following key result from [6] serves as a starting point for our investigation.

Theorem 1.1. *Let G be a graph, and suppose that v is a vertex of G. Then*

$$\alpha(G) \leq \alpha(G \setminus v) + 1. \tag{1.1}$$

Theorem 1.1 provides some information on the relationship between $\alpha(G)$ and $\alpha(G \setminus v)$, but as the following example shows, it is possible for $\alpha(G \setminus v)$ to be greater than, equal to, or less than $\alpha(G)$, depending on the choice of the vertex to be deleted.

Example 1.2. Consider the graph H formed from the path on 5 vertices by appending a pendant vertex at the centre of the path (see Figure 1 below). It is straightforward to determine that $\alpha(H) = \frac{3-\sqrt{5}}{2}$ (which is approximately 0.3820). If v is any of the non-pendant vertices $2, 3, 4$ of H, then $H \setminus v$ is disconnected, so that $\alpha(H \setminus v) = 0$. If v is vertex 6 in Figure 1, then $\alpha(H \setminus v) = \frac{3-\sqrt{5}}{2}$. Finally, if v is either of the pendant vertices $1, 5$, we find from a Matlab computation that $\alpha(H \setminus v)$ is approximately 0.5188.

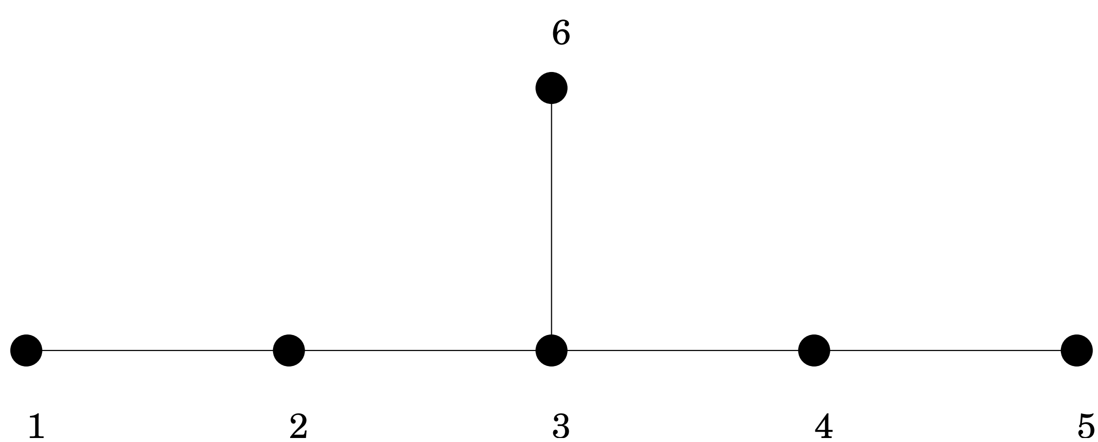

Figure 1: Graph for Example 1.2

In the light of the above considerations, we make the following two definitions. Given a connected graph G, for each vertex v of G, we define $\phi(v) = \alpha(G) - \alpha(G \setminus v)$, and we define $\kappa(v) = \frac{\alpha(G \setminus v)}{\alpha(G)}$; note that we suppress the explicit dependence on G for both functions. Evidently $\phi(v)$ is positive, 0, or negative according as $\kappa(v)$ is less than 1, equal to 1, or greater than 1, respectively. For succinctness, we focus on κ for the remainder of the discussion in this section. In the event that $\kappa(v) < 1$, we can take the interpretation that vertex v and its incident edges serve to increase the algebraic connectivity of G. Similarly, if $\kappa(v) > 1$, then v and its incident edges decrease the algebraic connectivity of G, while if $\kappa(v) = 1$, then v and its incident edges have no effect on the algebraic connectivity of G. Thus we find that κ can be used to provide a relative measure of the contribution to $\alpha(G)$ of a particular vertex (and its incident edges). Hence, κ can thought of as providing a measure of importance, or centrality (see [5]), for the vertices of the graph: those vertices with low values of κ may be viewed as being important to the connectivity properties of the graph (as measured by $\alpha(G)$), while those vertices with high κ values can be considered to diminish the connectivity of the graph.

In this paper, we develop the theory around the functions $\phi(v)$ and $\kappa(v)$. Specifically, we give tight upper and lower bounds on both quantities, and characterise the equality cases in each. We also estimate the number of vertices for which $\phi(v) \geq 0$ (equivalently, $\kappa(v) \leq 1$). Finally, we consider several examples of graphs arising from food web data, compute the values of κ for those examples, and contextualise the results.

Throughout, we will used standard notation, terminology and results from graph theory and matrix theory. The reader is referred to [10] and [7] for the relevant material.

2. Bounds on $\phi(v)$

In this section, we consider the function $\phi(v)$, and provide some bounds on that quantity. It follows from (1.1) that for any connected graph G, and any vertex v of that graph, we have $\phi(v) \leq 1$. Our first result characterises the case of equality in (1.1). Here we denote the *neighbourhood of a vertex* v - i.e. the set of vertices in G adjacent to v - by $N(v)$, and we denote the degree of vertex v by $d_G(v)$. Given a vector w, we refer to the set of indices corresponding to its nonzero entries as the *support* of w.

Theorem 2.1. *Let G be a connected graph on $n \geq 3$ vertices, and let v be a vertex of G. We have $\phi(v) = 1$ if and only if there is a Fiedler vector for G whose support is a subset of $N(v)$.*

Proof: Suppose first that $\phi(v) = 1$ and let $d_G(v) = m$. The Laplacian matrix for G is given by $L(G) = \left[\begin{array}{c|c|c} L_{11} & L_{12} & 0 \\ \hline L_{21} & L_{22}+I & -\mathbf{1} \\ \hline 0^T & -\mathbf{1}^T & m \end{array}\right]$, where the last row and column corresponds to vertex v, the m rows and column preceding that correspond to $N(v)$, and where $L(G \setminus v) = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right]$. Let w be a Fiedler vector for $L(G \setminus v)$, partitioned as $\left[\begin{array}{c} w_1 \\ \hline w_2 \end{array}\right]$. Letting z be the vector $\left[\begin{array}{c} w_1 \\ \hline w_2 \\ \hline 0 \end{array}\right]$, we find that $\alpha(G\setminus v)z^Tz + w_1^Tw_1 + w_2^Tw_2 = (\alpha(G\setminus v)+1)z^Tz = \alpha(G)z^Tz \leq z^TL(G)z = w^TL(G\setminus v)w + w_2^Tw_2 = \alpha(G\setminus v)w^Tw + w_2^Tw_2 = \alpha(G\setminus v)z^Tz + w_2^Tw_2$ (here the inequality follows from the fact that for any vector u orthogonal to $\mathbf{1}, \alpha(G)u^Tu \leq u^TL(G)u$). We now conclude that necessarily $w_1 = 0$ and that

z must be a Fiedler vector for G. Hence z is a Fielder vector for G whose support is a subset of $N(v)$.

Conversely, suppose that G has a Fiedler vector y whose support is a subset of $N(v)$. Write $L(G)$ as $\left[\begin{array}{c|c} \tilde{L}+D & -x \\ \hline -x^T & m \end{array}\right]$, where $\tilde{L}$ is the Laplacian matrix for $G \setminus v$ and where D is a diagonal matrix with ones in the diagonal positions corresponding to vertices in $N(v)$, and zeros elsewhere. Write y as $y = \left[\begin{array}{c} \tilde{y} \\ \hline 0 \end{array}\right]$. Since $y^T L(G) y = \alpha(G) y^T y$, we have $\tilde{y}^T \tilde{L} \tilde{y} + \tilde{y}^T D \tilde{y} = \alpha(G) \tilde{y}^T \tilde{y}$. Hence, $\tilde{y}^T \tilde{L} \tilde{y} = \alpha(G) \tilde{y}^T \tilde{y} - \tilde{y}^T D \tilde{y} = (\alpha(G) - 1) \tilde{y}^T \tilde{y}$, the last inequality following from the fact that y (and hence $\tilde{y}$) has support in $N(v)$. Thus we find that $\alpha(G \setminus v) \le \alpha(G) - 1$, and since we always have $\alpha(G \setminus v) \ge \alpha(G) - 1$, we conclude that $\phi(v) = 1$.

$\square$

Suppose that we have graphs G_1 and G_2. The *join* of G_1 and G_2, which we denote $G_1 \vee G_2$, is the graph formed from the disjoint union of G_1 and G_2 by adding all possible edges between vertices of G_1 and vertices of G_2. From Theorem 2.1, it is straightforward to determine that if we take a graph H and an isolated vertex v and form $G = H \vee \{v\}$, then $\alpha(G) = \alpha(H) + 1$, so that equality holds in (1.1). In particular, we find that for a graph on n vertices having a vertex v of degree $n-1$, we have $\phi(v) = 1$. What can we say about $\phi(v)$ if the degree of vertex v is $n-2$? The following example suggests that the answer to that question can be quite subtle.

Here we use the notation $i \sim j$ to denote the fact that vertex i is adjacent to vertex j; by a mild abuse of that notation, we use $G \setminus \{\, i \sim j\}$ to denote the graph formed from G by deleting the edge between vertex i and vertex j.

Example 2.2. Suppose that $n \ge 4$ is even. Let P_{n-1} denote the path on $n-1$ vertices, labeled so that $i \sim i+1, i = 1, \ldots, n-2$. Consider the graph on n vertices given by $(P_{n-1} \vee \{n\}) \setminus \{\frac{n}{2} \sim n\}$; evidently vertex n has degree $n-2$ in that graph. In this example, we will show that if $n = 4, 6, 8$ then $\phi(n) = 1$, while if $n \ge 10$, then $\phi(n) < 1$.

Let $G = P_{n-1} \vee \{n\}$; certainly $\alpha(G) = \alpha(P_{n-1}) + 1$ by Theorem 2.1. We have the following eigenvalues for $L(G)$: $0; n$; and $\lambda_j = 2(1 - \cos(\frac{\pi j}{n-1})) + 1, j = 1, \ldots, n-2$. Further, we have the following orthonormal basis of eigenvectors for $L(G)$: for $\lambda_j, w_j = \sqrt{\frac{2}{n-1}} \begin{bmatrix} \cos(\frac{\theta_j}{2}) \\ \cos(\frac{3\theta_j}{2}) \\ \vdots \\ \cos(\frac{(2n-3)\theta_j}{2}) \\ 0 \end{bmatrix}$, where $\theta_j = \frac{\pi j}{n-1}, j = 1, \ldots, n-2$; for $0, w_{n-1} = \frac{1}{\sqrt{n-1}} \mathbf{1}$; and for $n, w_n = \left[\begin{array}{c} \frac{-1}{\sqrt{n(n-1)}} \mathbf{1} \\ \hline \sqrt{\frac{n-1}{n}} \end{array}\right]$. Let Q be the orthogonal matrix given by $Q = \left[\begin{array}{cccc} w_1 & w_2 & \ldots & w_n \end{array}\right]$.

Next, let $\tilde{G}$ be the graph formed from G by deleting the edge $\frac{n}{2} \sim n$ (here $\frac{n}{2}$ corresponds to the middle vertex of P_{n-1}). Letting $\tilde{L}$ be the corresponding Laplacian matrix, we have $\tilde{L} = L - (e_{\frac{n}{2}} -$

$e_n)(e_{\frac{n}{2}} - e_n)^T$. Consequently, we find that $Q^T\tilde{L}Q = Q^T L Q - Q^T(e_{\frac{n}{2}} - e_n)(e_{\frac{n}{2}} - e_n)^T Q$. Let x^T be the vector $\left[\begin{array}{ccccccccc} 0 & -\sqrt{\frac{2}{n-1}} & 0 & \sqrt{\frac{2}{n-1}} & 0 & \dots & 0 & (-1)^{\frac{n-2}{2}}\sqrt{\frac{2}{n-1}} & 0 & -\sqrt{\frac{n}{n-1}} \end{array}\right]$. Then $Q^T(e_{\frac{n}{2}} - e_n) = x$, and we find that $Q^T\tilde{L}Q = \begin{bmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_{n-2} & & \\ & & & 0 & \\ & & & & n \end{bmatrix} - xx^T$. This last is permutationally similar to $\left[\begin{array}{ccccc|c} \lambda_1 & & & & & \\ & \lambda_3 & & & & \\ & & \ddots & & & 0 \\ & & & \lambda_{n-3} & & \\ & & & & 0 & \\ \hline & & 0 & & & M \end{array}\right]$, where $M = \begin{bmatrix} \lambda_2 & & & & \\ & \lambda_4 & & & \\ & & \ddots & & \\ & & & \lambda_{n-2} & \\ & & & & n \end{bmatrix} - yy^T$, and where $y^T = \left[\begin{array}{ccccc} -\sqrt{\frac{2}{n-1}} & \sqrt{\frac{2}{n-1}} & \dots & (-1)^{\frac{n-2}{2}}\sqrt{\frac{2}{n-1}} & -\sqrt{\frac{n}{n-1}} \end{array}\right]$.

In order to estimate $\alpha(\tilde{G})$, we need to estimate the smallest eigenvalue of M. By considering the leading 2×2 principal submatrix of M, and applying interlacing, it follows that the smallest eigenvalue of M is bounded above by $\frac{1}{2}\left(\lambda_2 + \lambda_4 - \frac{4}{n-1} - \sqrt{(\lambda_2 - \lambda_4)^2 + \frac{16}{(n-1)^2}}\right)$. This last is, in turn, less than $\frac{1}{2}(\lambda_2 + \lambda_4) - \frac{4}{n-1}$, so if $\frac{1}{2}(\lambda_2 + \lambda_4) - \frac{4}{n-1} < \lambda_1$ then we have $\alpha(\tilde{G}) < \alpha(P_{n-1}) + 1$.

Note that the inequality $\frac{1}{2}(\lambda_2 + \lambda_4) - \frac{4}{n-1} < \lambda_1$ is equivalent to

$$2\cos(\frac{\pi}{n-1})(1 - \cos(\frac{\pi}{n-1}))(2\cos(\frac{\pi}{n-1}) + 1)^2 < \frac{4}{n-1}. \tag{2.2}$$

Since $\cos(\frac{\pi}{n-1}) > 1 - \frac{\pi^2}{2(n-1)^2}$, we find that if $9\frac{\pi^2}{(n-1)^2} < \frac{4}{n-1}$, then certainly (2.2) holds. It now follows that if $n \geq 24$, then (2.2) holds, which yields $\alpha(\tilde{G}) < \alpha(P_{n-1}) + 1$. For $n = 14, \dots, 22$, computations on Matlab show that $\frac{1}{2}\left(\lambda_2 + \lambda_4 - \frac{4}{n-1} - \sqrt{(\lambda_2 - \lambda_4)^2 + \frac{16}{(n-1)^2}}\right) < \lambda_1$, so that $\alpha(\tilde{G}) < \alpha(P_{n-1}) + 1$ for these values of n.

Finally, computations with the corresponding Laplacian matrices reveal that for $n = 10, 12$, we have $\alpha(\tilde{G}) < \alpha(P_{n-1}) + 1$, while for $n = 4, 6, 8$, we have $\alpha(\tilde{G}) = \alpha(P_{n-1}) + 1$.

Consequently, it follows that for the graph $\tilde{G}$, we have $\phi(n) = 1$ provided that $n = 4, 6$, or 8, while $\phi(v) < 1$ for $n \geq 10$.

Our next result provides an attainable lower bound on ϕ which serves as a companion to (1.1). Here K_m denotes the complete graph on m vertices, while the disjoint union of graphs G and H is denoted by $G \cup H$.

Theorem 2.3. *Let G be a connected graph on $n \geq 3$ vertices. Then for any vertex v of $G, \phi(v) \geq -(n-2)$. Equality holds in the lower bound if and only if $G = (K_{n-2} \cup K_1) \vee K_1$, and v is the pendant vertex of that graph.*

Proof: Suppose first that $G \setminus v \neq K_{n-1}$. Then $\alpha(G \setminus v) \leq n-3$ (see [6], for example), and so we find that $\phi(v) = \alpha(G) - \alpha(G \setminus v) > -(n-3) > -(n-2)$. Next, suppose that $G \setminus v = K_{n-1}$, and that $d_G(v) = m$. It is straightforward to determine that $\alpha(G) = m$, so we find that $\phi(v) =$

$\alpha(G) - \alpha(G \setminus v) = m - (n-1) \geq -(n-2)$, as desired. Moreover, we also find that $\phi(v) = -(n-2)$ if and only if $G \setminus v = K_{n-1}$ and $m = 1$. $\square$

The following result will prove useful in some of our subsequent analysis.

Lemma 2.4. *Let $G \neq K_n$ be a connected graph on $n \geq 3$ vertices, and let $d_G(n) = m$. Write the Laplacian matrix $L(G)$ as*

$$L(G) = \left[\begin{array}{c|c} \tilde{L} + D & -x \\ \hline -x^T & m \end{array}\right],$$

where $\tilde{L}$ is the Laplacian matrix for $G \backslash n$. Let w be a Fiedler vector for $L(G)$, written as $w = \left[\begin{array}{c} \tilde{w} \\ \hline w_n \end{array}\right]$. Then

$$\alpha(G \setminus n) \leq \alpha(G) + \frac{(m - \frac{n-2}{n-1}\alpha(G))w_n^2 - \tilde{w}^T D \tilde{w}}{\tilde{w}^T \tilde{w} - \frac{w_n^2}{n-1}} = \alpha(G) + \frac{(m - \frac{n-2}{n-1}\alpha(G))w_n^2 - \sum_{i \sim v} w_i^2}{\tilde{w}^T \tilde{w} - \frac{w_n^2}{n-1}}. \quad (2.3)$$

In particular, if $mw_n^2 \leq \tilde{w}^T D \tilde{w} + \alpha(G) w_n^2 \frac{n-2}{n-1}$, then $\phi(n) \geq 0$; further, if strict inequality holds in the former, then $\phi(n) > 0$.

Proof: Note that $\tilde{w}^T \mathbf{1} = -w_n$, and consider the vector $u = \tilde{w} + \frac{w_n}{n-1}\mathbf{1}$. Note that in fact u is not the zero vector, otherwise it follows that $G = K_n$, contrary to our hypothesis. Evidently $u^T \mathbf{1} = 0, u^T u = \tilde{w}^T \tilde{w} - \frac{w_n^2}{n-1}$, and $u^T \tilde{L} u = \tilde{w}^T \tilde{L} \tilde{w}$. Since w is a Fielder vector for $L(G)$, we find that $\tilde{w}^T \tilde{L} \tilde{w} + \tilde{w}^T D \tilde{w} - 2w_n x^T \tilde{w} = \alpha(G)(\tilde{w}^T \tilde{w} + w_n^2)$, and also that $-x^T \tilde{w} + mw_n = \alpha(G) w_n$. Combining these, we find that $\frac{u^T \tilde{L} u}{u^T u} = \alpha(G) + \frac{(m - \frac{n-2}{n-1}\alpha(G))w_n^2 - \tilde{w}^T D \tilde{w}}{\tilde{w}^T \tilde{w} - \frac{w_n^2}{n-1}}$. The conclusions now follow upon noting that $\alpha(G \setminus n) \leq \frac{u^T \tilde{L} u}{u^T u}$. $\square$

Our next two remarks show how certain entries in a Fielder vector can yield information on ϕ.

Remark 2.5. Let G be a connected graph, and suppose that v is a vertex for which there is a Fiedler vector whose entry corresponding to v has minimum absolute value. Then $\phi(v) \geq 0$. This is clear if G is a complete graph; in the case that G is not a complete graph, we adopt the notation of Lemma 2.4. Referring to (2.3), it follows that $\phi(v) \geq \frac{\sum_{i \sim v} w_i^2 - mw_v^2 + \frac{n-2}{n-1}\alpha(G) w_v^2}{\tilde{w}^T \tilde{w} - \frac{w_v^2}{n-1}}$. From the hypothesis that w_v has minimum absolute value in the Fiedler vector w, we find that $\sum_{i \sim v} w_i^2 \geq mw_v^2$. Hence, $\phi(v) \geq 0$.

For a connected graph G with Fiedler vector w, we say that a vertex v is a *characteristic vertex* if $w_v = 0$ and there is a vertex i such that $i \sim v$ and $w_i \neq 0$.

Remark 2.6. Let G be a connected graph on three or more vertices. If v is a characteristic vertex of G, then $\phi(v) > 0$. This is clear if G is complete, so suppose that G is not a complete graph. As v is a characteristic vertex, we have $w_v = 0$ and $\sum_{i \sim v} w_i^2 > 0$. From (2.3), we find that $\phi(v) \geq \frac{\sum_{i \sim v} w_i^2}{\tilde{w}^T \tilde{w}} > 0$.

Remark 2.7. Let G be a connected graph on at least three vertices. If $\alpha(G)$ has multiplicity two or more as an eigenvalue of $L(G)$, then $\phi(v) \geq 0$ for every vertex v of G. To see this, note that since $\alpha(G)$ has multiplicity at least two, for each vertex v of G, there is a Fielder vector whose entry corresponding to v is 0. The conclusion now follows from Remark 2.5.

In light of the remarks made in Section 1, we may take the view that a vertex v of a graph does not diminish the algebraic connectivity provided that $\phi(v) \geq 0$ (equivalently, $\kappa(v) \leq 1$). Our next result discusses the number of such vertices.

Theorem 2.8. *Let G be a connected graph on $n \geq 3$ vertices, and suppose that for some $k \in I\!N$, we have $\alpha(G) > (k-1)\frac{n-1}{n-2}$. Then there are at least k vertices of G, say $u_1, \ldots u_k$, such that $\phi(u_i) \geq 0, i = 1, \ldots, k$.*

Proof: The conclusion is readily verified if $G = K_n$, so we suppose henceforth that G is not complete. Let w be a Fiedler vector for G, and partition the vertex set of G into subsets $S_1, \ldots, S_r$ so that: i) if $v_1, v_2 \in S_i$, then $|w_{v_1}| = |w_{v_2}|$; and ii) if $v_1 \in S_i, v_2 \in S_j$, and $i < j$, then $|w_{v_1}| < |w_{v_2}|$.

We claim that for each $i = 1, \ldots, r-1$, if $|\cup_{j=1}^{i} S_j| < k$, then for each $u \in \cup_{j=1}^{i+1} S_j, \phi(u) \geq 0$. We prove the claim by induction on i, and begin with the case $i = 1$.

Suppose that $|S_1| < k$. Since each vertex in S_1 corresponds to an entry in w of minimum absolute, from Remark 2.5, we see that if $u \in S_1$, then $\phi(u) \geq 0$. Next consider a vertex $u \in S_2$, and let $d_G(u) = m$. Suppose that u is adjacent to p vertices in S_1, and note that $0 \leq p \leq k-1$, the latter inequality following from the hypothesis that $|S_1| < k$. Then from the hypothesis on $\alpha(G)$, we find that $\sum_{i \sim u} w_i^2 + \alpha(G) w_u^2 \frac{n-2}{n-1} > (m-p)w_u^2 + (k-1)w_u^2 = (m+k-p-1)w_u^2 \geq m w_u^2$. It now follows from (2.3) that $\phi(u) > 0$. Hence, for each $u \in S_1 \cup S_2, \phi(u) \geq 0$.

Now we suppose that the claim holds for some i_0 with $1 \leq i_0 \leq r-2$, and that $|\cup_{j=1}^{i_0+1} S_j| < k$. Then certainly $|\cup_{j=1}^{i_0} S_j| < k$, so by the induction hypothesis, for each $u \in \cup_{j=1}^{i_0} S_j, \phi(u) \geq 0$. Suppose now that $u \in S_{i_0+1}$, that $d_G(u) = m$, and that u is adjacent to p vertices in $\cup_{j=1}^{i_0} S_j$. As before, we find that $\sum_{i \sim u} w_i^2 + \alpha(G) w_u^2 \frac{n-2}{n-1} > (m-p)w_u^2 + (k-1)w_u^2 = (m+k-p-1)w_u^2 \geq m w_u^2$. Hence from (2.3), $\phi(u) > 0$, completing the proof of the induction step, and hence of the claim.

Next, we prove that there are at least k vertices whose corresponding value of ϕ is nonnegative. First note that by Remark 2.5, each vertex of S_1 yields a nonnegative value for ϕ, so in particular if $|S_1| \geq k$, then the desired conclusion follows. On the other hand, if $|S_1| < k$, then there is some index i_1 with $1 \leq i_1 \leq r-1$ such that $|\cup_{j=1}^{i_1} S_j| < k \leq |\cup_{j=1}^{i_1+1} S_j|$. By the claim, we then find that each vertex in $\cup_{j=1}^{i_1+1} S_j$ yields a nonnegative value for ϕ, and again the conclusion holds. □

Remark 2.9. Let u be one of the vertices that is shown in Theorem 2.8 to yield a nonnegative value for ϕ. A careful analysis of the proof of Theorem 2.8 shows that the only situation in which $\phi(u)$ can be 0 is if the Fiedler vector w has some zero entries, and in addition u is a vertex in S_1 such that $N(u) \subset S_1$.

The following consequence of Theorem 2.8 provides a lower bound on the number of vertices yielding a nonnegative value for ϕ.

Corollary 2.10. *Let G be a connected graph on $n \geq 4$ vertices. Then there are least $\lfloor \frac{\alpha(G)(n-2)}{n-1} \rfloor + 1$ vertices u for which $\phi(u) \geq 0$.*

Proof: Let $k = \lfloor \frac{\alpha(G)(n-2)}{n-1} \rfloor + 1$, so that $k - 1 \leq \frac{\alpha(G)(n-2)}{n-1}$. We claim that in fact $k - 1 < \frac{\alpha(G)(n-2)}{n-1}$, and note that once the claim is established, the conclusion will follow immediately from Theorem 2.8.

To see the claim, suppose to the contrary that $k - 1 = \frac{\alpha(G)(n-2)}{n-1}$. Then $\alpha(G) = \frac{(k-1)(n-1)}{n-2}$, and as $\alpha(G)$ is an algebraic integer, in fact $\alpha(G)$ must be an integer in this case. Since $n - 2 \geq 2$, and since $n - 1$ and $n - 2$ have no common factors, it must be the case that $n - 2$ divides $k - 1$. We conclude that $k - 1 = n - 2$, which in turn implies that $\alpha(G) = n - 1$. This last is impossible, since for any graph H on n vertices, either $\alpha(H) = n$ or $\alpha(H) \leq n - 2$ (see [6]). We thus conclude that the case $k - 1 = \frac{\alpha(G)(n-2)}{n-1}$ cannot arise, completing the proof of the claim. $\square$

Remark 2.11. Observe that for the graph $G = K_n \setminus \{1 \sim 2\}$, we have $\lfloor \frac{\alpha(G)(n-2)}{n-1} \rfloor + 1 = n - 2$. Note that for this graph G there are precisely $n - 2$ vertices u for which $\phi(u) \geq 0$.

3. Bounds on $\kappa(v)$

In this section, we consider connected graphs, and provide upper and lower bounds on κ; as above, we will suppress the explicit dependence of κ on the underlying graph. We begin with a couple of simple observations.

Remark 3.1. Let G be a graph such that $\alpha(G) \geq 1$. Then for any vertex v of G we have $\kappa(v) \geq 1 - \frac{1}{\alpha(G)}$. To see the inequality, note that from (1.1), we have $\kappa(v) = \frac{\alpha(G \setminus v)}{\alpha(G)} \geq \frac{\alpha(G)-1}{\alpha(G)} = 1 - \frac{1}{\alpha(G)}$, as desired.

Remark 3.2. Let G be a connected graph, and recall that a vertex u of G is a *cutpoint* if $G \setminus u$ is disconnected. It is straightforward to see that for any connected graph G, and any vertex v of G, $\kappa(v) \geq 0$, with equality if and only if v is a cutpoint.

Our first result of this section provides an attainable lower bound on κ when the vertex in question is not a cutpoint.

Theorem 3.3. *Let G be a connected graph G on $n \geq 3$ vertices, and suppose that v is a vertex of G that is not a cutpoint. Then $\kappa(v) \geq \frac{2-2\cos(\frac{\pi}{n-1})}{3-2\cos(\frac{\pi}{n-1})}$. Equality holds in the lower bound if and only if either*
a) $G = P_{n-1} \vee \{v\}$, or
b) $G = (P_{n-1} \vee \{n\}) \setminus \{\frac{n}{2} \sim n\}$, $n = 4, 6,$ or 8, and vertex v coincides with vertex n.

Proof: Since $\alpha(G) \le \alpha(G\backslash v)+1$, we have $\kappa(v) = \frac{\alpha(G\backslash v)}{\alpha(G)} \ge \frac{\alpha(G\backslash v)}{\alpha(G\backslash v)+1} \ge \frac{2-2\cos(\frac{\pi}{n-1})}{3-2\cos(\frac{\pi}{n-1})}$, the last inequality following from the fact that for any connected graph H on m vertices, $\alpha(H) \ge 2-2\cos(\frac{\pi}{m})$; we note also that $\alpha(H) = 2-2\cos(\frac{\pi}{m})$ if and only if $H = P_m$. Thus the desired inequality for $\kappa(v)$ holds.

Suppose now that $\kappa(v) = \frac{2-2\cos(\frac{\pi}{n-1})}{3-2\cos(\frac{\pi}{n-1})}$. Then necessarily we must have $G \setminus v = P_{n-1}$, and $\alpha(G) = \alpha(P_{n-1}) + 1$. Without loss of generality we take vertex v to be n. If $d_G(n) = n-1$, then we are in case a).

Suppose next that $d_G(n) \le n-2$. Let x be a Fiedler vector for P_{n-1}, normalized so that $x^T x = 1$, and let y be the vector formed from x by appending a 0 in the n-th position. Suppose that vertex i of G is not adjacent to vertex n. By considering the quadratic form $y^T L(G) y$, we find that $\alpha(G) \le y^T L(G) y = \alpha(P_{n-1}) + 1 - x_i^2$. In particular, we see that if $\alpha(G) = \alpha(P_{n-1}) + 1$, then necessarily $x_i = 0$. We conclude that n is even, $i = \frac{n}{2}$, and that vertex n is adjacent to every other vertex except $\frac{n}{2}$ in G. Hence $G = (P_{n-1} \vee \{n\}) \setminus \{\frac{n}{2} \sim n\}$, and referring to Example 2.2, we find that necessarily n is either $4, 6$ or 8, so that condition b) holds.

Finally, from Example 2.2, we find readily that if either a) or b) holds, then $\kappa(v) = \frac{2-2\cos(\frac{\pi}{n-1})}{3-2\cos(\frac{\pi}{n-1})}$. □

The following result will be useful in establishing an upper bound on κ.

Lemma 3.4. *Let G be a connected graph on $n \ge 3$ vertices, and suppose that v is a pendant vertex of G. If $\alpha(G) < 1$, then $\kappa(v) < n-1$.*

Proof: Without loss of generality, we take v to be vertex n, adjacent to vertex $n-1$ in G. Let w be a Fiedler vector for G, and note that from the equation $L(G)w = \alpha(G)w$ we find that $w_n = \frac{1}{1-\alpha(G)} w_{n-1}$. Referring to (2.3), it follows that $\alpha(G \setminus v) \le \alpha(G) + \frac{(1-\frac{n-2}{n-1}\alpha(G))\frac{1}{(1-\alpha(G))^2} w_{n-1}^2 - w_{n-1}^2}{\overline{w}^T\overline{w} + w_{n-1}^2 - \frac{1}{(n-1)(1-\alpha(G))^2} w_{n-1}^2}$, where $\overline{w}$ is the vector formed from w by deleting the entries in positions $n-1$ and n. From the Cauchy-Schwartz inequality, and the fact that $w^T\mathbf{1} = 0$, we find that $\overline{w}^T\overline{w} \ge \frac{(w_{n-1}+w_n)^2}{n-2} = \left(\frac{2-\alpha(G)}{1-\alpha(G)}\right)^2 \frac{w_{n-1}^2}{n-2}$. It now follows that $\alpha(G \setminus v) \le \alpha(G) + \frac{(1-\frac{n-2}{n-1}\alpha(G))\frac{1}{(1-\alpha(G))^2} - 1}{\left(\frac{2-\alpha(G)}{1-\alpha(G)}\right)^2 \frac{1}{n-2} + 1 - \frac{1}{(1-\alpha(G))^2(n-1)}}$. Simplifying this last inequality yields $\alpha(G \setminus v) \le \alpha(G) + \alpha(G)(n-2)\left(\frac{2-\alpha(G)-\frac{n-2}{n-1}}{(2-\alpha(G))^2 - \frac{n-2}{n-1} + (n-2)(1-\alpha(G))^2}\right) < \alpha(G)(n-1)$, the last inequality since $1 < 2-\alpha(G)$. The conclusion now follows. □

We now use Lemma 3.4 to prove an attainable upper bound on κ.

Theorem 3.5. *Let G be a connected graph on $n \ge 3$ vertices. Then for any vertex v of $G, \kappa(v) \le n-1$. Equality holds in the upper bound if and only if $G = (K_{n-2} \cup K_1) \vee K_1$, and v is the pendant vertex of that graph.*

Proof: Let $\tilde{G}$ be formed from G by (if necessary) deleting all but one edge incident with v, so that v is pendant in $\tilde{G}$. Since $\alpha(G) \ge \alpha(\tilde{G})$, we certainly have $\kappa(v) \le \frac{\alpha(G\backslash v)}{\alpha(\tilde{G})} = \frac{\alpha(\tilde{G}\backslash v)}{\alpha(\tilde{G})}$.

Let w be the vertex of $\tilde{G}$ that is adjacent to v. If $d_{\tilde{G}}(w) \le n-2$, then $\alpha(\tilde{G}) < 1$ (see [8] for example), and in that case, it follows from Lemma 3.4 that $\kappa(v) < n-1$. On the other hand, if

$d_{\tilde{G}}(w) = n-1$ (and hence $d_G(w) = n-1$), we have $\alpha(\tilde{G}) = 1$, so that $\frac{\alpha(\tilde{G}\setminus v)}{\alpha(\tilde{G})} = \alpha(\tilde{G} \setminus v) \le n-1$. Thus we have $\kappa(v) \le n-1$, as desired.

Suppose now that $\kappa(v) = n-1$. From the argument above, we find that necessarily $\alpha(G \setminus v) = n-1$, so that $G \setminus v = K_{n-1}$. It must also be the case that $\alpha(G) = 1$, from which we deduce that v must be pendant in G. Hence $G = (K_{n-2} \cup K_1) \vee K_1$, and v is the pendant vertex of G. The converse implication for the equality case is straightforward. □

Theorem 3.6. *Let G be a connected graph on $n \ge 3$ vertices. There is at least one vertex v of G such that $\kappa(v) \le \frac{n-1}{n}$.*

Proof: The conclusion follows immediately if $G = K_n$, so henceforth we assume that $G \ne K_n$.

Suppose to the contrary that for each vertex v of $G, \kappa(v) > \frac{n-1}{n}$. Let w be a Fiedler vector for G. Then referring to (2.3), it must be the case that for each vertex v of G,

$$\frac{(d_G(v) - \frac{n-2}{n-1}\alpha(G))w_v^2 - \sum_{i\sim v} w_i^2}{\sum_{i\ne v} w_i^2 - \frac{w_v^2}{n-1}} > \frac{-\alpha(G)}{n}. \tag{3.4}$$

Rearranging (3.4) yields

$$n(d_G(v) - \frac{n-2}{n-1}\alpha(G))w_v^2 - n\sum_{i\sim v} w_i^2 > \alpha(G)(\frac{n}{n-1}w_v^2 - \sum_{i=1}^{n} w_i^2). \tag{3.5}$$

Summing (3.5) over all vertices v in G yields

$$-\left(\frac{n^2-2n}{n-1}\right)\alpha(G)\sum_{i=1}^{n} w_i^2 > -\left(\frac{n^2-2n}{n-1}\right)\alpha(G)\sum_{i=1}^{n} w_i^2,$$

a contradiction.

We conclude that for some vertex v, it must be the case that $\kappa(v) \le \frac{n-1}{n}$. □

Corollary 3.7. *Let G be a connected graph on $n \ge 3$ vertices. Then $\min\{\kappa(v)|v \in G\} \le \frac{n-1}{n}$. Equality holds in the inequality if and only if $G = K_n$.*

Proof: The inequality follows immediately from Theorem 3.6, so we need only characterize the equality case. Observe that if $G = K_n$, then for each vertex v we have $G \setminus v = K_{n-1}$, so that $\kappa(v) = \frac{n-1}{n}$, and hence $\min\{\kappa(v)|v \in G\} = \frac{n-1}{n}$.

Suppose now that $\min\{\kappa(v)|v \in G\} = \frac{n-1}{n}$ and that $G \ne K_n$. We find from the proof of Theorem 3.6 that it must be that case that $\kappa(v) = \frac{n-1}{n}$ for every vertex v. Further, we also find that equality must hold in (2.3) for each vertex of G.

Let w be a Fiedler vector for G, and suppose without loss of generality that $w_n \ne 0$. Write $L(G)$ as $L(G) = \left[\begin{array}{c|c} \tilde{L}+D & -x \\ \hline -x^T & m \end{array}\right]$, where $\tilde{L}$ is the Laplacian matrix for $G\setminus n$, and write w as $w = \left[\begin{array}{c} \tilde{w} \\ \hline w_n \end{array}\right]$.

Since equality holds (2.3) for vertex n, it must be the case that $u = \tilde{w} + \frac{w_n}{n-1}\mathbf{1}$ is a Fiedler vector for $G \setminus n$.

Consequently, we have

$$\tilde{L}\tilde{w} = \frac{n-1}{n}\alpha(G)\tilde{w} + \frac{\alpha(G)}{n}w_n\mathbf{1}. \tag{3.6}$$

From the eigenequation for $L(G)$, we also find that

$$\tilde{L}\tilde{w} + D\tilde{w} - w_n x = \alpha(G)\tilde{w}, \tag{3.7}$$

and that

$$-x^T\tilde{w} + mw_n = \alpha(G)w_n. \tag{3.8}$$

We find from (3.6) and (3.7) that $D\tilde{w} - w_n x = \frac{\alpha(G)}{n}(\tilde{w} - w_n\mathbf{1})$. Multiplying this equation by x^T, using the facts that $x^T D = x^T$ and $x^T x = m$, we find that $x^T\tilde{w} - w_n m = \frac{\alpha(G)}{n}(x^T\tilde{w} - w_n m)$; from (3.8) we thus have $-\alpha(G)w_n = \frac{\alpha(G)}{n}(-\alpha(G)w_n)$. Since $w_n \neq 0$, it must be the case that $\alpha(G) = n$. Hence $G = K_n$, contrary to our hypothesis. $\square$

4. Applications of $\boldsymbol{\kappa}$ to food webs

In this section, we consider several graphs arising from food webs, and discuss the various values of κ for vertices in those graphs. A *food web* is a graph that is used to represent certain relationships within an ecosystem. The vertices of the graph correspond to the various species (or sometimes collections of similar species) present within an ecosystem; for each pair of vertices in a predator/prey or consumer/resource relationship, the graph contains an edge between the predator vertex and the prey vertex.

There is an obvious lack of symmetry in the relationship between predator and prey, and one might reflect that asymmetry by representing the ecosystem as a directed graph (with directed arcs oriented from a predator vertex to a prey vertex). However, since effects due to changes to predator and prey species can propagate throughout a food web, it has been argued (see [2] and [13]) that there is merit in considering a food web as an undirected graph, as described above. We adopt that convention here, and consider the food webs below as undirected graphs.

In this section, we consider four different food webs, and discuss the κ values for the vertices in each graph . (All computations were done in Matlab, and the results reported here are rounded to four decimal places.) We note in passing that in the food web context, $\kappa(v)$ measures the relative change in the algebraic connectivity due to the removal of the species corresponding to vertex v. Our approach here is inspired in part by [5], where various notions of vertex centrality are considered and compared for a number of food webs.

A common feature to all four examples is that each graph contains just a handful of vertices whose corresponding κ value is small, while almost all of the remaining vertices yield values of κ

close to 1. This observation suggests that many of the vertices in these graphs make at most a marginal contribution to the algebraic connectivity, while just a few vertices in these graphs can be considered central.

Example 4.1. Our first food web is based on data for the Ythan estuary in Scotland. The resulting graph G has 134 vertices, primarily representing birds, fishes, invertebrates, and metazoan parasites [3]. Computations yield the value $\alpha(G) = 0.6687$.

There are eight vertices of G for which the value of κ is equal to 0, and they correspond to the following categories: redshank, fatherlasher, flounder, ragworm, small crustacean, periwinkle, genus Enteropmorpha, and particulate organic matter. Evidently each of these eight vertices is a cutpoint of G, and an analysis of the graph shows that in fact each of these eight vertices includes at least one pendant vertex in its neighbourhood. For all remaining vertices of G, the corresponding computed values for κ lie between 0.9459 and 1.0307.

Example 4.2. Our next food web is associated with a Shoal grass ecosystem in the St. Marks National Wildlife Refuge, Florida. The vertices in the graph primarily represent macroinvertebrates, fishes, and birds [3]. The corresponding graph G has 48 vertices, and a computed value of $\alpha(G) = 1.5875$. For this graph, the four smallest values of κ are: 0.5396 (micro-epiphytes); 0.5446 (benthos-eating birds); 0.5981 (deposit-feeding peracaridan crustaceans); and 0.5992 (epiphyte-grazing amphipods). Note that these values are not especially close to the lower bound of Remark 3.1, as $1 - \frac{1}{\alpha(G)} = 0.3701$. All remaining vertices yield values of κ between 0.7282 and 1.1548. The graph in Figure 2 below shows the κ values in ascending order for the St. Marks food web.

Example 4.3. Our third food web example arises from data for a coral reef on the Puerto Rico - Virgin Islands shelf complex. The corresponding graph G has 50 vertices, primarily representing fishes, other vertebrates, invertebrates, and basal groups [4]. The computed value of the algebraic connectivity is $\alpha(G) = 1.9794$.

For this graph, the two smallest values of κ are: 0.4975 (large sharks/rays, carnivorous); and 0.5027 (benthic autotrophs). In contrast to Example 4.2, both of these κ values are close to the lower bound of Remark 3.1, namely $1 - \frac{1}{\alpha(G)} = 0.4948$. The largest value of κ is 2.9384 (kyphosidae, herbivorous); it is interesting to note that the vertex corresponding to this species is adjacent to just two others in G, namely the vertices yielding the smallest two κ values. All remaining values of κ lie between 0.9993 and 1.0007. Finally, we remark that the two smallest values for κ correspond, respectively, to the pair of entries in the Fiedler vector of smallest absolute value, while the largest value for κ corresponds to the entry in the Fiedler vector of maximum absolute value.

Example 4.4. Our last food web example is derived from data for the Northeast US shelf ecosystem. Here the graph G has 81 vertices, again representing fishes, other vertebrates, invertebrates, and basal groups [4]. The computed value for the algebraic connectivity is $\alpha(G) = 7.5421$. For this graph, the eight smallest values of κ are: 0.8755 (cancer crabs); 0.8755 (other crabs); 0.8763 (clams, mussels); 0.8791 (scallops); 0.8793 (phytoplankton); 0.8812 (lobsters); 0.8851 (detritus); and 0.8852 (urchins). It is worth remarking that the vertices corresponding to cancer crabs, and to other crabs, have the same neighbourhoods in G. As in Example 4.3, these eight κ values are quite close to $1 - \frac{1}{\alpha(G)} = 0.8674$, the bound of Remark 3.1. The maximum value of κ is 1.2516 (snails), while all remaining vertices yield values of κ lie between 0.9928 and 1.0023.

We computed a Fiedler vector for this example (it is unique up to scalar multiple), and found the ordering of indices that places its entries in ascending order. We then arranged the values of κ

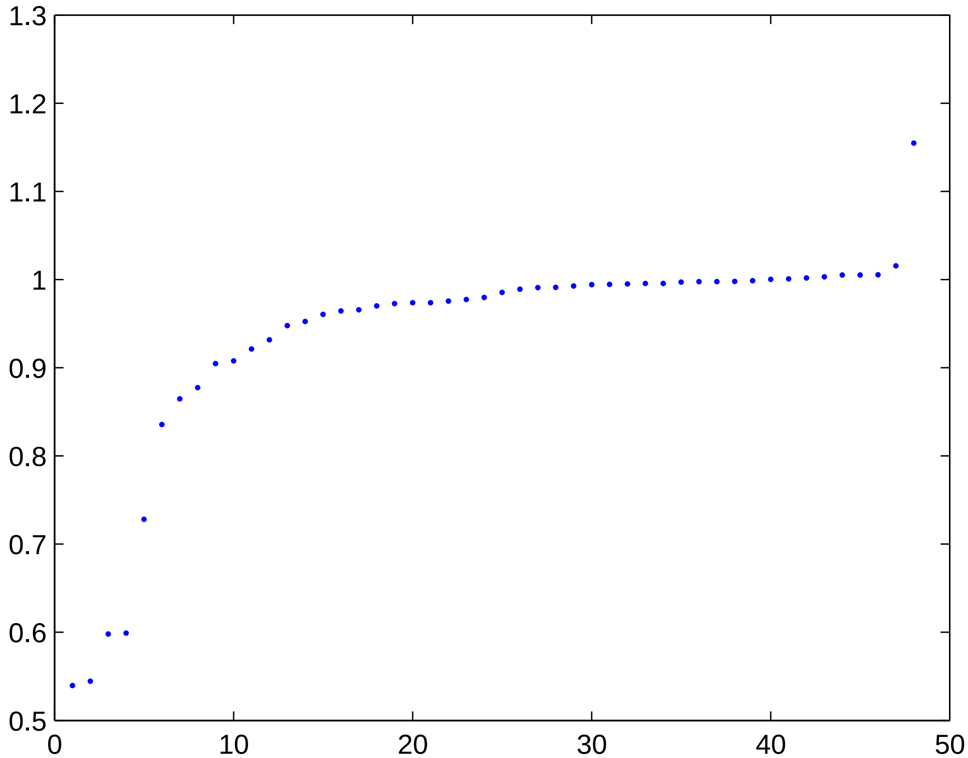

Figure 2: Increasingly sorted κ values for the St. Marks food web

according to the same ordering of indices; the results are displayed in Figure 3 below. From that Figure, we see that the minimum entry in the Fiedler vector corresponds to the maximum value for κ, while the next eight most negative entries in the Fiedler correspond to the eight smallest κ values noted above.

5. Conclusion

In this paper, we introduce the functions $\phi(v)$ and $\kappa(v)$, which measure the absolute and relative changes in the algebraic connectivity of a graph G upon deletion of a vertex v. We provide upper and lower bounds on both quantities, and characterise the equality cases in those bounds. We also give a lower bound on the number of vertices v such that $\phi(v) \geq 0$, and an upper bound on the minimum value of $\kappa(v)$ as v ranges over the vertices of the graph G. Finally, we explore a notion of vertex centrality arising from the function κ, and apply that notion to several graphs associated with food web data.

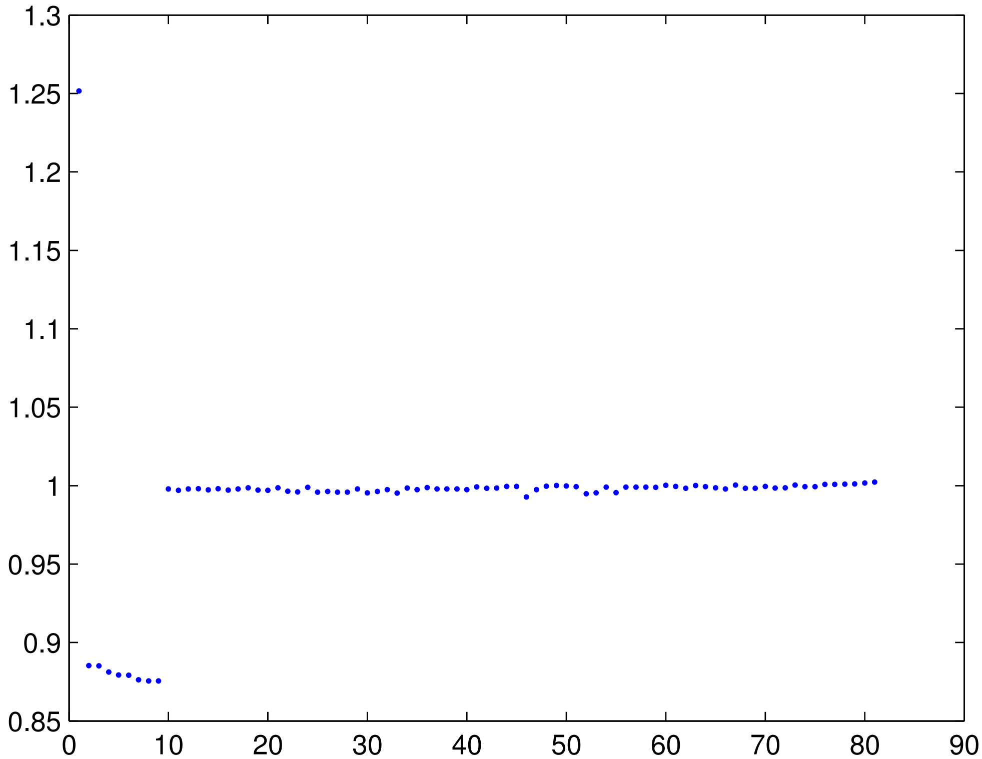

Figure 3: κ values sorted according to the Fiedler vector for the shelf ecosystem

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