On Algebraic Connectivity Augmentation

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Abstract

Suppose that G is an undirected graph, and that H is a spanning subgraph of G^c whose edges induce a subgraph on p vertices. We consider the expression $\alpha(G \cup H) - \alpha(G)$, where α denotes the algebraic connectivity. Specifically, we provide upper and lower bounds on $\alpha(G \cup H) - \alpha(G)$ in terms of p, and characterise the corresponding equality cases. We also discuss the density of the expression $\alpha(G \cup H) - \alpha(G)$ in the interval $[0, p]$. A bound on $\alpha(G \cup H) - \alpha(G)$ is provided in a special case, and several examples are considered.

Key words: Graph, Laplacian matrix, Algebraic connectivity

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = \{1, \cdots, n\}$ and edge set E. Let $d(i)$ denote the degree of the vertex $i \in V$, and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is given by $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. It is easy

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to see that $L(G)$ is a positive semidefinite matrix with the smallest eigenvalue equal to 0 and corresponding null vector $\mathbf{1}$, the column vector of all ones. We denote the eigenvalues and the spectrum of $L(G)$ by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ and $Spec(L(G)) = \{\mu_1(G), \ldots, \mu_{n-1}(G), 0\}$, respectively.

Fielder ([3]) has shown that the second smallest eigenvalue of $L(G)$ is 0 if and only if G is disconnected. That eigenvalue is known as the *algebraic connectivity* of G and is denoted by $\alpha(G)$; an eigenvector of $L(G)$ associated with $\alpha(G)$ is called a *Fiedler vector*. The algebraic connectivity of a graph is a spectral invariant that has been extensively studied, in part because it reflects the connectivity of a graph in a different way than either the vertex connectivity, $\nu(G)$, or the edge connectivity. Also in [3], Fiedler proved that $\alpha(G) \leq \nu(G) \leq \delta(G)$, where $\delta(G)$ is the minimal degree of G. The surveys in [1], and [9] provide overviews of the literature on algebraic connectivity.

Before proceeding further, we introduce some terminology and notation. If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of vertices, their *graph sum* is $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \vee G_2$, of G_1 and G_2 is the graph obtained from $G_1 + G_2$ by adding new edges from each vertex in G_1 to every vertex of G_2. If G_1 and G_2 are graphs on k and m vertices respectively, with eigenvalues $\mu_1(G_1) \geq \mu_2(G_1) \geq \cdots \geq \mu_{k-1}(G_1) \geq \mu_k(G_1) = 0$ and $\mu_1(G_2) \geq \mu_2(G_2) \geq \cdots \geq \mu_{m-1}(G_2) \geq \mu_m(G_2) = 0$, respectively, then the eigenvalues of $L(G_1 \vee G_2)$ are given by $m + k, \mu_1(G_1) + m, \ldots, \mu_{k-1}(G_1) + m, \mu_1(G_2) + k, \ldots, \mu_{m-1}(G_2) + k, 0$. We note that for any graph G on n vertices, $\mu_1(G_1) \leq n$, with equality if and only if G is a join of two graphs. Further, if G_1 and G_2 are graphs with the same sets of vertices ($V_1 = V_2 = V$) and $E_1 \cap E_2 = \emptyset$, then their *union* $G_1 \cup G_2$ is the graph $(V, E_1 \cup E_2)$. Given a graph G with n vertices, its *complement*, denoted G^c, is the graph on the same vertex set as G whose edge set is the complement of that of G. The eigenvalues of G^c can be obtained as $\mu_{n-i}(G^c) = n - \mu_i(G)$, $\forall i, 1 \leq i \leq n$. The *complete graph* on n vertices, that is, the graph on n vertices with all possible edges, is denoted by K_n. We use O_m to denote the *empty graph* on m vertices – i.e. the graph on m vertices with no edges. The *complete bipartite graph* $K_{p,q}$ is the join of the empty graphs O_p and O_q. A zero matrix or vector will be denoted by 0, an all ones matrix will be denoted by J and an identity matrix will be denoted by I; usually the orders of these matrices will be determined by the context, but where that is not the case, the orders will be denoted by appropriate subscripts.

Suppose that we have a graph G, and that we construct a new graph $\hat{G}$ by adding an edge to G. Since adding an edge to G has the effect of adding a rank one positive semidefinite matrix to $L(G)$, it follows readily that $0 \leq \alpha(\hat{G}) - \alpha(G)$. It is shown in [4] that $\alpha(\hat{G}) - \alpha(G) \leq 2$, while in [8] it is shown that $\alpha(\hat{G}) - \alpha(G) = 2$ if and only if $\hat{G}$ is a complete graph (that fact appears without proof in [4] as an exercise). In a related vein, the so-called maximum algebraic connectivity augmentation problem has been introduced in [2]. That problem can be phrased as follows: given a graph G and $k \in I\!N$, add k edges not belonging to G so as to maximize the algebraic connectivity of the resulting augmented graph.

In this work, we study a variation of the algebraic connectivity augmentation problem. Given a graph G on n vertices with $G \neq K_n$, let $\widetilde{H}$ be a subgraph of G^c with no isolated vertices. Let p be the number of vertices induced by the edge set of $\widetilde{H}$, and set $H = \widetilde{H} + O_{n-p}$. It is straightforward to determine that $L(G \cup H) = L(G) + L(H)$, and from Theorem 4.3.1 in [5], we have that $\mu_j(G) + \mu_n(H) \leq \mu_j(G \cup H) \leq \mu_j(G) + \mu_1(H), \forall j = 1, \cdots, n$. As $\mu_1(H) \leq p$ and $\mu_n(H) = 0$, it now follows that

$$0 \leq \alpha(G \cup H) - \alpha(G) \leq p. \tag{1}$$

The inequalities (1) serve as a starting point for our work in the sequel.

This paper is organized as follows. In Section 2, we characterize the equality cases for the upper and lower bounds in (1). Section 3 provides an upper bound on $\alpha(G \cup H) - \alpha(G)$ in the case that $G \cup H$ is not a complete graph, and provides examples of classes of graphs for which the upper bound of (1) is approached. Section 4 discusses the density of $\alpha(G \cup H) - \alpha(G)$ in the interval $[0, p]$.

2 Extreme values of $\alpha(G \cup H) - \alpha(G)$

We begin by characterizing the equality case in the upper bound afforded by (1).

Theorem 2.1 *Let G be a graph on n vertices, and let H be a subgraph of G^c of the form $H = \tilde{H} + O_{n-p}$, where $\tilde{H}$ is a connected graph on p vertices. We have $\alpha(G \cup H) - \alpha(G) = p$ if and only if $G \cup H = K_n$ and $\tilde{H}$ is a join of two graphs.*

PROOF. First suppose that $\alpha(G \cup H) = \alpha(G) + p$, and let v be a Fiedler vector of G. Since $L(G \cup H) = L(G) + L(H)$, we have

$$(\alpha(G) + p)v^T v \leq v^T(L(G \cup H))v = v^T(L(G) + L(H))v =$$

$$v^T L(G)v + v^T L(H)v = \alpha(G)v^T v + v^T L(H)v \leq \alpha(G)v^T v + pv^T v.$$

Consequently

$$v^T(L(G \cup H))v = v^T L(G)v + v^T L(H)v = \alpha(G)v^T v + pv^T v. \tag{2}$$

From the fact that $H = \tilde{H} + O_{n-p}$, it follows that the largest eigenvalue of $L(H)$ is at most p; from (2) we deduce that in fact p is an eigenvalue of $L(H)$, and that v is an associated eigenvector for p. It now follows that p is an eigenvalue of $L(\tilde{H})$. Recall that for any graph on p vertices, p is a Laplacian eigenvalue if and only if the graph is a join of two graphs of smaller order. It now follows that there are graphs H_q and H_{p-q} on q and $p-q$ vertices, respectively, such that $\tilde{H}$ has the form $\tilde{H} = H_{p-q} \vee H_q$; writing $L(H)$ as

$$L(H) = \left[\begin{array}{c|c|c} L(H_q) & -J & 0 \\ \hline -J & L(H_{p-q}) & 0 \\ \hline 0 & 0 & 0 \end{array}\right],$$

we find that v can be taken to be a scalar multiple of the vector w given by

$$w = \begin{bmatrix} (p-q)\mathbf{1}_q \\ -q\mathbf{1}_{(p-q)} \\ 0_{(n-p)} \end{bmatrix}.$$

Next, we write $L(G)$ as

$$L(G) = \left[\begin{array}{c|c|c} L_{1,1} & 0 & L_{1,3} \\ \hline 0 & L_{2,2} & L_{2,3} \\ \hline L_{3,1} & L_{3,2} & L_{3,3} \end{array}\right].$$

Since w is an eigenvector of $L(G)$ corresponding to $\alpha(G)$, we have

$$L(G)w = \left[\begin{array}{c|c|c} L_{1,1} & 0 & L_{1,3} \\ \hline 0 & L_{2,2} & L_{2,3} \\ \hline L_{3,1} & L_{3,2} & L_{3,3} \end{array}\right] \begin{bmatrix} (p-q)\mathbf{1}_q \\ -q\mathbf{1}_{(p-q)} \\ 0_{n-p} \end{bmatrix} = \alpha(G) \begin{bmatrix} (p-q)\mathbf{1}_q \\ -q\mathbf{1}_{(p-q)} \\ 0_{n-p} \end{bmatrix}. \tag{3}$$

Evidently (3) holds if and only if

$$(p-q)L_{1,1}\mathbf{1}_q = (p-q)\alpha(G)\mathbf{1}_q,$$

$$-qL_{2,2}\mathbf{1}_{(p-q)} = -q\alpha(G)\mathbf{1}_{(p-q)}, \text{ and}$$

$$(p-q)L_{3,1}\mathbf{1}_q - qL_{3,2}\mathbf{1}_{(p-q)} = 0.$$

Since $p \neq q$, we have in particular that

$$L_{1,1}\mathbf{1}_q = \alpha(G)\mathbf{1}_q.$$

Let $L_{1,1} = L_0 + D$, where $L_0\mathbf{1}_q = 0$ and D is the diagonal matrix such that $D\mathbf{1}_q + L_{1,3}\mathbf{1}_{n-p} = 0$. So then

$$\alpha(G)\mathbf{1}_q = L_{1,1}\mathbf{1}_q = (L_0 + D)\mathbf{1}_q = L_0\mathbf{1}_q + D\mathbf{1}_q = D\mathbf{1}_q,$$

and consequently,

$$\alpha(G)\mathbf{1}_q = D\mathbf{1}_q = d\mathbf{1}_q$$

for some suitable integer d. Hence $\alpha(G) = d$, and in the graph G, each vertex in the set $\{1, \ldots, q\}$ is adjacent to exactly d vertices in the set $\{p+1, \ldots, n\}$. Thus we have that $\alpha(G \cup H) = d + p$, while the minimum degree of $G \cup H$ is at most $p - 1 + d$. Recalling a result of Fiedler [3], which states that the only graphs for which the algebraic connectivity exceeds the vertex connectivity are the complete graphs, we conclude that the graph $G \cup H$ must be the complete graph K_n.

Conversely, suppose that $H = (H_{p-q} \vee H_q) + O_{n-p}$ and that $G \cup H = K_n$. Then $\alpha(G \cup H) = n$, while $\alpha(G) = n - \mu_1(G^c)$. Since $G^c = H$, and since the structure of H yields that $\mu_1(H) = p$, we find that $\alpha(G) = n - p$. Thus $\alpha(G \cup H) - \alpha(G) = p$, as desired. ■

We now characterize the equality case in the lower bound of (1).

Theorem 2.2 *Suppose that we have graphs G and $H = \tilde{H} + O_{n-p}$, where G is connected, and $\tilde{H}$ is a connected subgraph of G^c on p vertices. Then $\alpha(G \cup H) = \alpha(G)$ if and only if there is a Fiedler vector of G whose entries are constant on the vertices of $\tilde{H}$.*

PROOF. Suppose first that there is a Fiedler vector of G, say v, whose entries are constant on the vertices of $\tilde{H}$. Then $L(H)v = 0$, so that $L(G \cup H)v = L(G)v + L(H)v = \alpha(G)v$. Hence $\alpha(G \cup H) \leq \alpha(G)$; since $G \cup H$ is formed from G by adding edges, we also have $\alpha(G \cup H) \geq \alpha(G)$, whence $\alpha(G \cup H) = \alpha(G)$.

Conversely, suppose that $\alpha(G \cup H) = \alpha(G)$, and let w be a Fiedler vector for $G \cup H$, say with $||w|| = 1$. Then $\alpha(G) = \alpha(G \cup H) = w^T L(G)w + w^T L(H)w$. Since w is orthogonal to $\mathbf{1}$ and G is connected, we find that $w^T L(G)w \geq \alpha(G)$. Hence we have $\alpha(G) = w^T L(G)w + w^T L(H)w \geq \alpha(G) + w^T L(H)w$, from which we conclude that w is a null vector for $L(H)$ and w is also a Fielder vector for G. Since $\tilde{H}$ is connected, it now follows that w is constant on the vertices of $\tilde{H}$. ■

In contrast to Theorem 2.1, which provides a purely graph-theoretic characterization of the equality case in the upper bound of (1), the corresponding result in Theorem 2.2 for the lower bound of (1) is dependent upon the structure of the Fiedler vectors of the graph in question. Our next example provides some conditions that are more combinatorial in nature and are sufficient to yield equality in the lower bound of (1).

Example 2.3 Suppose that we have graphs H_1, H_2, H_3 on m_1, m_2, and m_3 vertices, respectively, and suppose further that $\alpha(H_3) \geq m_3 - m_1 - m_2$. Construct the graph $G = (H_1 + H_2) \vee H_3$. It is shown in [6] that the algebraic and vertex connectivities of G coincide (with a common value of m_3), and that any non-complete graph for which the algebraic and vertex connectivities are equal is construct in that manner.

We may write the Laplacian matrix for G as

$$L(G) = \left[\begin{array}{c|c|c} L(H_1) + m_3 I & 0 & -J \\ \hline 0 & L(H_2) + m_3 I & -J \\ \hline -J & -J & L(H_3) + (m_1 + m_2)I \end{array}\right].$$

It follows that the vector $w = \left[\begin{array}{c} m_2 \mathbf{1}_{m_1} \\ \hline -m_1 \mathbf{1}_{m_2} \\ \hline 0 \end{array}\right]$ is a Fiedler vector for G. Note that w is constant on the first m_1 vertices of G, and also on the next m_2 vertices of G. Let $\tilde{H}$ be any spanning subgraph of H_1^c, and let $H = \tilde{H} + O_{m_2+m_3}$. Referring to Theorem 2.2, we see that necessarily $\alpha(G \cup H) = \alpha(G)$.

Recall that two vertices in a graph are *duplicates* if they have precisely the same neighbourhoods, while a set of vertices in a graph is an *independent set* if it induces an empty graph. Our final result of this section addresses the lower bound of (1) in the setting of duplicate vertices.

Proposition 2.4 *Let G be a connected graph on vertices $1, \ldots, n$, and suppose that vertices $1, \ldots, p$ form an independent set of $p \geq 2$ duplicate vertices. Let $\tilde{H}$ be a connected graph on vertices $1, \ldots, p$, and let $H = \tilde{H} + O_{n-p}$. Then $\alpha(G \cup H) = \alpha(G)$ if and only if $G \neq O_p \vee G_0$, where G_0 is a graph on $n - p$ vertices such that $\alpha(G_0) > n - 2p$.*

PROOF. First we suppose that $G = O_p \vee G_0$, where G_0 is on $n - p$ vertices, and $\alpha(G_0) > n - 2p$. Then $L(G)$ can be written as $L(G) = \left[\begin{array}{c|c} (n-p)I & -J \\ \hline -J & L(G_0) + pI \end{array}\right]$.
It follows that $\alpha(G) = \min\{n - p, \alpha(G_0) + p\} = n - p$. Further, the eigenspace of $L(G)$ for the eigenvalue $n - p$ is spanned by the vectors $e_1 - e_j, j = 2, \ldots, p$. We deduce then that no Fiedler vector of G is constant on vertices $1, \ldots, p$. By Theorem 2.2 we now find that necessarily $\alpha(G \cup H) > \alpha(G)$.

Now, suppose that $G \neq O_p \vee G_0$, where G_0 is a graph on $n-p$ vertices such that $\alpha(G_0) > n-2p$. Since vertices $1, \ldots, p$ are duplicates, and form an independent set, we find that $L(G)$ can be written as $L(G) = \left[\begin{array}{c|c|c} dI & -J & 0 \\ \hline -J & L_{22} & L_{23} \\ \hline 0 & L_{32} & L_{33} \end{array}\right]$, where d is the common degree of vertices $1, \ldots, p$. Observe that d is an eigenvalue of $L(G)$ (and so in particular, $\alpha(G) \leq d$), and that the vectors $e_1 - e_j, j = 2, \ldots, p$ form a linearly independent set of eigenvectors of $L(G)$ corresponding to the eigenvalue d. If $\alpha(G) < d$, then from the fact that eigenvectors of $L(G)$ corresponding to different eigenvalues are orthogonal, it follows that any Fiedler vector of G is orthogonal to each $e_1 - e_j, j = 2, \ldots, p$. Hence, any Fiedler vector for G is constant on vertices $1, \ldots, p$, so that by Theorem 2.2, $\alpha(G \cup H) = \alpha(G)$.

Suppose now that $\alpha(G) = d$. We first consider the case that $n-p-d \geq 1$. Observe that since $\alpha(G) = d$, we are in the situation that the vertex connectivity of G coincides with its algebraic connectivity. Appealing to the result of [6], it follows that $L(G)$ can be written as $L(G) = \left[\begin{array}{c|c|c} dI & 0 & -J \\ \hline 0 & L(G_1) + dI & -J \\ \hline -J & -J & L(G_2) + (n-d)I \end{array}\right]$.

Observe that the vector $\left[\begin{array}{c} (n-p-d)\mathbf{1}_p \\ \hline -p\mathbf{1}_{n-p-d} \\ \hline 0 \end{array}\right]$ is a Fielder vector for G that is constant on vertices $1, \ldots, p$. Again by Theorem 2.2, we have $\alpha(G \cup H) = \alpha(G)$.

Finally, if $\alpha(G) = d$ and $n-p-d = 0$, we find from the result of [6] that $L(G)$ can be written as $L(G) = \left[\begin{array}{c|c} (n-p)I & -J \\ \hline -J & L(G_0) + pI \end{array}\right]$. From our hypothesis on G, it must be the case that $\alpha(G_0) \leq n-2p$; since $d = \alpha(G) = \min\{n-p, \alpha(G_0)+p\}$, we find that $\alpha(G_0) = d-p$. Letting w be a Fielder vector for G_0, it now follows that the vector $\left[\begin{array}{c} 0 \\ \hline w \end{array}\right]$ is a Fielder vector for G that is constant on vertices $1, \ldots, p$. Hence, by Theorem 2.2, we have $\alpha(G \cup H) = \alpha(G)$. ∎

3 Behaviour of $\alpha(G \cup H) - \alpha(G)$ when the edges of H induce a star

From Theorem 2.1, it follows the equality in the upper bound of (1) can hold only if $G \cup H$ is a complete graph. Our next result provides an upper bound on $\alpha(G \cup H) - \alpha(G)$ in the case that $G \cup H$ is not complete, and the edges of H induce a star.

Theorem 3.1 *Let G be a graph on vertices $1, \ldots, n$, and suppose that vertex 1 of G has degree d. Select $p - 1 \geq 1$ vertices of G, say $u_1, \ldots, u_{p-1}$ none of which is adjacent to vertex 1 in G. Let H be the graph on vertices $1, \ldots, n$ whose only edges are those between vertex 1 and each of vertices $u_1, \ldots, u_{p-1}$. If $G \cup H \neq K_n$, then $\alpha(G \cup H) - \alpha(G) \leq p - \epsilon_0$, where ϵ_0 is the smallest positive root of the polynomial $d\epsilon(p - \epsilon) - (1 - \epsilon)^2(p - 1 - \epsilon)^2$.*

PROOF. Without loss of generality, we write $L(G)$ as

$$L(G) = \left[\begin{array}{c|c|c|c} d & {0^T}_{p-1} & -\mathbf{1}_d^T & 0^T \\ \hline 0_{p-1} & L_{22} & L_{23} & L_{24} \\ \hline -\mathbf{1}_d & L_{32} & L_{33} & L_{34} \\ \hline 0 & L_{42} & L_{43} & L_{44} \end{array}\right],$$

and $L(H)$ as

$$L(H) = \left[\begin{array}{c|c|c|c} p-1 & -\mathbf{1}_{p-1}^T & 0^T & 0^T \\ \hline -\mathbf{1}_{p-1} & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

Let the vector w, partitioned conformally with $L(H)$ be given by $w = \left[\begin{array}{c} \sqrt{\frac{p-1}{p}} \\ \hline \frac{-1}{\sqrt{p(p-1)}}\mathbf{1}_{p-1} \\ \hline 0 \\ \hline 0 \end{array}\right]$.

(We note in passing that w is a unit eigenvector of the matrix $L(H)$ corresponding to its spectral radius, p.) Let v be a Fiedler vector for G, normalised so that $||v|| = 1$ and $v^T w \geq 0$. Set $\theta = 1 - (v^T w)^2$, and let z be projection of v in the direction orthogonal to w, so that $v = \sqrt{1 - \theta} w + z$ and $||z||^2 = \theta$. Partitioning z conformally with $L(G)$ as $\begin{bmatrix} z_1 \\ \hline z_2 \\ \hline z_3 \\ \hline z_4 \end{bmatrix}$, we find that $\mathbf{1}_{p-1}^T z_2 = (p-1)z_1$, so that $z_2^T z_2 \geq (p-1)z_1^2$.

We begin by noting that if $v_1 = 0$, then it follows readily that $\alpha(G \cup H) - \alpha(G) \leq 1$. An application of the intermediate value theorem shows that the function $d\epsilon(p-\epsilon) - (1-\epsilon)^2(p-1-\epsilon)^2$ has a root in $(0,1]$, so that $\epsilon_0 \leq 1$. Thus if $v_1 = 0$, we have $\alpha(G \cup H) - \alpha(G) \leq 1 \leq p - \epsilon_0$, as desired.

Henceforth, we assume that $v_1 \neq 0$. It then follows that $\sqrt{1-\theta}\sqrt{\frac{p-1}{p}} + z_1 \neq 0$, and from the eigenequation $L(G)v = \alpha(G)v$, we find that $d - \alpha(G) = \frac{\mathbf{1}_d^T z_3}{\sqrt{1-\theta}\sqrt{\frac{p-1}{p}}+z_1}$. Also, from the fact that $\alpha(G \cup H) - \alpha(G) \leq v^T L(H) v$, we deduce that

$$\alpha(G \cup H) - \alpha(G) \leq (1-\theta)p + z_2^T z_2 - (p-1)z_1^2.$$

Defining ϵ via $p - \epsilon = \alpha(G \cup H) - \alpha(G)$, we thus find that

$$\epsilon \geq \theta p + (p-1)z_1^2 - z_2^T z_2.$$

Since $G \cup H \neq K_n$, we also have $\alpha(G \cup H) \leq d + p - 1$; using the fact that $d - \alpha(G) = \frac{\mathbf{1}_d^T z_3}{\sqrt{1-\theta}\sqrt{\frac{p-1}{p}}+z_1}$, it now follows that $p - \epsilon \leq p - 1 + \frac{\mathbf{1}_d^T z_3}{\sqrt{1-\theta}\sqrt{\frac{p-1}{p}}+z_1}$, so that $1 - \epsilon \leq \frac{\mathbf{1}_d^T z_3}{\sqrt{1-\theta}\sqrt{\frac{p-1}{p}}+z_1}$. Finally, observe that from the Cauchy-Schwarz inequality, we have $\mathbf{1}_d^T z_3 \leq \sqrt{d(\theta - z_1^2 - z_2^T z_2)}$.

Next, we want to estimate the maximum value M of the function $\frac{\sqrt{\theta - z_1^2 - z_2^T z_2}}{\sqrt{1-\theta}\sqrt{\frac{p-1}{p}}+z_1}$ subject to the constraints
i) $z_1^2 + z_2^T z_2 \leq \theta$;
ii) $z_2^T z_2 \geq (p-1)z_1^2$; and
iii) $\epsilon \geq \theta p + (p-1)z_1^2 - z_2^T z_2$.
From i) and iii) we find that $\theta \leq \frac{\epsilon - (p-1)z_1^2 + z_2^T z_2}{p} \leq \frac{\epsilon + \theta - p z_1^2}{p}$, which in turn yields $\theta \leq \frac{\epsilon - p z_1^2}{p-1}$. Since the function we seek to maximize is increasing in θ, and since $-z_2^T z_2 \leq -(p-1)z_1^2$, we find that

$$M \leq \max\left\{ \frac{\sqrt{\frac{\epsilon - p z_1^2}{p-1} - p z_1^2}}{\sqrt{\frac{p-1-\epsilon}{p} + z_1^2} + z_1} \middle| \frac{-\sqrt{\epsilon}}{p} \leq z_1 \leq \frac{\sqrt{\epsilon}}{p} \right\}$$

$$= \frac{1}{\sqrt{p-1}} \max\left\{ \frac{\sqrt{\epsilon - p^2 z_1^2}}{\sqrt{\frac{p-1-\epsilon}{p} + z_1^2} + z_1} \middle| \frac{-\sqrt{\epsilon}}{p} \leq z_1 \leq \frac{\sqrt{\epsilon}}{p} \right\}.$$

Let $f(z) = \frac{\sqrt{\epsilon - p^2 z^2}}{\sqrt{\frac{p-1-\epsilon}{p} + z^2} + z}$. Evidently $f(z)$ is maximized on $[\frac{-\sqrt{\epsilon}}{p}, \frac{\sqrt{\epsilon}}{p}]$ at a critical point, and a basic computation reveals that $f'(z) = 0$ only if $z^2 = \frac{\epsilon^2}{p^2((p-1)(p-\epsilon)+\epsilon)}$. Noting that the critical point corresponding to the negative value of z will necessarily yield the maximum value of f, it now follows that $f(z)$ is maximized at $\hat{z} = \frac{-\epsilon}{p\sqrt{(p-1)(p-\epsilon)+\epsilon}}$.

A straightforward computation shows that $\epsilon - p^2 \hat{z}^2 = \frac{(p-1)(p-\epsilon)\epsilon}{(p-1)(p-\epsilon)+\epsilon}$. A longer, but no less straightforward, computation shows that $\frac{p-1-\epsilon}{p} + \hat{z}^2 = \frac{(p-1)^2(p-\epsilon)^2}{p^2((p-1)(p-\epsilon)+\epsilon)}$. Assembling these identities now shows that

$$\frac{1}{\sqrt{p-1}} \max\left\{ \frac{\sqrt{\epsilon - p^2 z_1^2}}{\sqrt{\frac{p-1-\epsilon}{p} + z_1^2} + z_1} \middle| \frac{-\sqrt{\epsilon}}{p} \leq z_1 \leq \frac{\sqrt{\epsilon}}{p} \right\} = \frac{\sqrt{\epsilon(p-\epsilon)}}{p-1-\epsilon}.$$

Thus we see that $1-\epsilon \leq \frac{\sqrt{d\epsilon(p-\epsilon)}}{p-1-\epsilon}$. So, either $\epsilon > 1 \geq \epsilon_0$, or we have $d\epsilon(p-\epsilon) \geq (1-\epsilon)^2(p-1-\epsilon)^2$. We thus conclude that ϵ is bounded below by the smallest positive root ϵ_0 of the function $d\epsilon(p-\epsilon) - (1-\epsilon)^2(p-1-\epsilon)^2$. The desired conclusion now follows. ■

Remark 3.2 Suppose that $d \in I\!N$ is fixed, and that ϵ_0 is the smallest positive root of $d\epsilon(p-\epsilon) - (1-\epsilon)^2(p-1-\epsilon)^2$. Then necessarily we have $(1-\epsilon_0)^2 = \frac{d\epsilon_0(p-\epsilon_0)}{(p-1-\epsilon_0)^2} \leq \frac{dp}{(p-2)^2}$, since $\epsilon_0 \leq 1$. It follows that as p increases without bound, the corresponding value for ϵ_0 converges to 1.

Example 3.3 Fix $d, p \in I\!N$, and suppose that $m \geq p-1$. Let G be the graph whose Laplacian matrix has the form

$$\left[\begin{array}{c|c|c|c} d & 0^T & 0^T & -\mathbf{1}^T \\ \hline 0 & (p-1+d)I_m & -J & -J \\ \hline 0 & -J & (m+d)I_{p-1} & -J \\ \hline -\mathbf{1} & -J & -J & (m+p+d)I_d - J \end{array}\right].$$

We then find that $\alpha(G) = d$. Next, let H be the spanning subgraph of G^c

whose edges induce a $K_{1,p-1}$, with centre vertex 1 adjacent to each of vertices $m+2, \ldots, m+p$. Then

$$L(G \cup H) = \left[\begin{array}{c|c|c|c} p-1+d & 0^T & -\mathbf{1}^T & -\mathbf{1}^T \\ \hline 0 & (p-1+d)I_m & -J & -J \\ \hline -\mathbf{1} & -J & (m+d+1)I_{p-1} & -J \\ \hline -\mathbf{1} & -J & -J & (m+p+d)I_d - J \end{array}\right].$$

It now follows that $\alpha(G \cup H) = p-1+d$. In particular we find that $\alpha(G \cup H) - \alpha(G) = p-1$. Observe that by Remark 3.2, as $p \to \infty$, the upper bound of Theorem 3.1 is asymptotic to $p-1$. Hence the bound of Theorem 3.1 performs well for graphs of this type.

From Theorem 3.1 and Remark 3.2 find that if vertex 1 of G has degree d, and the edges of H induce a star with centre vertex 1, then $p - (\alpha(G \cup H) - \alpha(G))$ is bounded away from zero, even if p is large. Our next result provides a family of graphs for which $\alpha(G \cup H) - \alpha(G)$ can be made arbitrarily close to p.

Proposition 3.4 *Fix $p \in I\!N$ with $p \geq 2$, and suppose that $m \in I\!N$ with $m \geq p-1$. Let $G_m = K_{m,m} - \{e_1, \ldots, e_{p-1}\}$, where the edges $e_1, \ldots, e_{p-1}$ are all incident with vertex 1 of $K_{m,m}$. Let $H_m = K_{1,p-1} + O_{2m-p}$ be such that $G_m \cup H_m = K_{m,m}$. Then $\alpha(G_m \cup H_m) - \alpha(G_m) \to p$, as $m \to \infty$.*

PROOF. Throughout this proof, we fix m, and suppress the explicit dependence of G_m and H_m on m. First, we compute the spectrum of $L(G)$ by considering the spectrum of $L(G^c)$. Without loss of generality, we may write $L(G^c)$ as

$$L(G^c) = \left[\begin{array}{c|c|c|c} m+p-2 & -\mathbf{1} & -\mathbf{1} & 0 \\ \hline -\mathbf{1^T} & (m+1)I_{p-1} - J & 0 & -J \\ \hline -\mathbf{1^T} & 0 & mI_{m-1} - J & 0 \\ \hline 0^T & -J & 0 & mI_{m-p+1} - J \end{array}\right].$$

Let v_1 and v_2 be vectors of dimensions $m-1$ and $m-p+1$, respectively, such that v_1 is orthogonal to $\mathbf{1}_{m-1}$ and v_2 is orthogonal to $\mathbf{1}_{m-p+1}$. We define a vector w as $w = \left[\begin{array}{c} 0 \\ \hline 0_{p-1} \\ \hline v_1 \\ \hline v_2 \end{array}\right]$, and note that $L(G^c)w = mw$. Consequently, we see that $L(G^c)$ has m as an eigenvalue of multiplicity at least $(2m-p-2)$. Similarly, if v_3 is a vector of dimension $p-1$ such that v_3 is orthogonal to $\mathbf{1}_{p-1}$, then the vector $z = \left[\begin{array}{c} 0 \\ \hline v_3 \\ \hline 0_{m-1} \\ \hline 0_{m-p+1} \end{array}\right]$, has the property that $L(G^c)z = (m+1)z$. Hence $L(G^c)$ has $m+1$ as an eigenvalue of multiplicity at least $(p-2)$.

Further, since $L(G^c)$ has an orthogonal basis of eigenvectors, it follows that there are remaining eigenvectors of $L(G^c)$ of the form $\begin{bmatrix} \alpha \\ \beta\mathbf{1}_{p-1} \\ \gamma\mathbf{1}_{m-1} \\ \rho\mathbf{1}_{m-p+1} \end{bmatrix}$. Consequently, the remaining eigenvalues of $L(G^c)$ coincide with those of the 4×4 matrix

$$A = \begin{bmatrix} m+p-2 & -(p-1) & -(m-1) & 0 \\ -1 & m-p+2 & 0 & -m+p-1 \\ -1 & 0 & 1 & 0 \\ 0 & -(p-1) & 0 & p-1 \end{bmatrix}.$$

Some straightforward computations reveal that the eigenvalues of A are m, $\frac{m+p+\sqrt{(m+p)^2-8p+8}}{2}$, $\frac{m+p-\sqrt{(m+p)^2-8p+8}}{2}$ and 0. Therefore

$$Spec(L(G^c)) = \left\{(m+1)^{(p-2)}, m^{(2m-p-1)}, \frac{m+p\pm\sqrt{(m+p)^2-8p+8}}{2}, 0\right\}$$

(where the superscripts denote multiplicities) and consequently,

$$Spec(L(G)) = \left\{m^{(2m-p-1)}, (m-1)^{(p-2)}, \frac{3m-p\pm\sqrt{(m+p)^2-8p+8}}{2}, 0\right\}.$$

As, $\alpha(K_{m,m}) = \alpha(G \cup H) = m$ and $\alpha(G) = \frac{3m-p-\sqrt{(m+p)^2-8p+8}}{2}$, we conclude that $\alpha(G \cup H) - \alpha(G) = \frac{\sqrt{(m+p)^2-8p+8}-(m-p)}{2}$. The conclusion now follows, since $\frac{\sqrt{(m+p)^2-8p+8}-(m-p)}{2} \to p$ as $m \to \infty$. ■

Remark 3.5 Another family of graphs for which $\alpha(G \cup H) - \alpha(G)$ can be made arbitrarily close to p is given by $G_m = K_{m,\ldots,m} - \{e_1, \ldots, e_{p-1}\}$, where $K_{m,\cdots,m}$ is the complete multipartite graph (with vertex set partitioned into

p partite sets, each of cardinality m) and where the edges $e_1, \ldots, e_{p-1}$ are all incident with vertex 1 and each of them is incident to a vertex in a different partition of $K_{m,\cdots,m}$. Let be the graph $H_m = K_{1,p-1} + O_{p(m-1)}$ such that $G_m \cup H_m = K_{m,\cdots,m}$. For simplicity, fix m and suppress the index in G_m and H_m. The Laplacian matrix of G^c has the form

$$L(G^c) = \left[\begin{array}{c|c|c|c|c|c|c}
m+p-2 & -\mathbf{1}^T & -\mathbf{1}^T & 0^T & 0^T & \cdots & 0^T \\ \hline
 & & & -\mathbf{1}^T & 0^T & \cdots & 0^T \\
 & & & 0^T & -\mathbf{1}^T & & \vdots \\
-\mathbf{1} & mI_{p-1} & 0 & \vdots & \vdots & \ddots & 0^T \\
 & & & 0^T & 0^T & \cdots & -\mathbf{1}^T \\ \hline
-\mathbf{1} & 0 & mI - J_{m-1} & 0 & 0 & \cdots & 0 \\ \hline
0 & -\mathbf{1}0\cdots 0 & 0 & mI - J_{m-1} & 0 & \cdots & 0 \\ \hline
0 & 0-\mathbf{1}0\cdots 0 & 0 & 0 & mI - J_{m-1} & \ddots & 0 \\ \hline
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
0 & 0\cdots 0-\mathbf{1} & 0 & 0 & 0 & \cdots & mI - J_{m-1}
\end{array}\right].$$

Arguing analogously as in Proposition 3.4, we conclude that $L(G^c)$ has m as an eigenvalue of multiplicity at least $p(m-2)$. It is easy to see that the remaining eigenvalues of $L(G^c)$ coincide with those of the $2p \times 2p$ matrix

$$A = \left[\begin{array}{c|c} M & -(m-1)I_p \\ \hline -I_p & I_p \end{array}\right],$$

where

$$M = \begin{bmatrix} m+p-2 & -\mathbf{1} \\ -\mathbf{1}^T & mI_{p-1} \end{bmatrix}.$$

Suppose that λ is an eigenvalue of M associated with eigenvector x, and consider the vector $\left[\begin{array}{c} sx \\ \hline tx \end{array}\right]$. Then $A \left[\begin{array}{c} sx \\ \hline tx \end{array}\right] = \gamma \left[\begin{array}{c} sx \\ \hline tx \end{array}\right]$ if and only if

$$\begin{bmatrix} \lambda & -(m-1) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \gamma \begin{bmatrix} s \\ t \end{bmatrix}.$$

It now follows that $Spec(L(G^c)) =$

$$\left\{m^{(p(m-2)+1)}, \left(\frac{m+1+\sqrt{(m+1)^2-4}}{2}\right)^{(p-2)}, \left(\frac{m+1-\sqrt{(m+1)^2-4}}{2}\right)^{(p-2)},\right.$$

$$\left.\frac{m+p+\sqrt{(m+p)^2-4p}}{2}, \frac{m+p-\sqrt{(m+p)^2-4p}}{2}, 0\right\}$$

and consequently $\alpha(G) = \frac{2mp-m-p-\sqrt{(m+p)^2-4p}}{2}$. As $\alpha(K_{m,\cdots,m}) = \alpha(G \cup H) = (p-1)m$, we conclude that $\alpha(G \cup H) - \alpha(G) = \frac{\sqrt{(m+p)^2-4p}-(m-p)}{2}$. Then $\alpha(G \cup H) - \alpha(G) \to p$ as $m \to \infty$.

We conclude this section by observing that the graphs in Example 3.3, Proposition 3.4 and Remark 3.5 are particular cases of the class of graphs to which Theorem 3.1 applies.

4 Density of $\alpha(G \cup H) - \alpha(G)$

From (1), we see that if the edges of H induce a graph on p vertices, then necessarily $\alpha(G \cup H) - \alpha(G) \in [0, p]$. The result in this section shows that for any number r in $[0, p]$, there are graphs G and H as above such that $\alpha(G \cup H) - \alpha(G)$ can be made arbitrarily close to r. The theorem below is similar in spirit to results in [10] and [7], which deal with limit points for algebraic connectivity.

Theorem 4.1 *Fix $p \in I\!N$ with $p \geq 2$ and suppose that $r \in [0, p]$. Then there is a sequence of graphs G_n and H_n, each on, say k_n vertices, such that for each $n \in I\!N$:*
i) $H_n = K_{1,p-1} + O_{k_n - p}$;
ii) H_n is a subgraph of G_n^c;
iii) G_n is connected; and
iv) $\alpha(G_n \cup H_n) - \alpha(G_n) \to r$ as $n \to \infty$.

PROOF. We suppose first that $r > 0$. It is shown in [10] that there is a sequence of graphs Γ_n such that $\alpha(\Gamma_n)$ increases monotonically to the limit r as $n \to \infty$. For concreteness, we suppose that for each $n \in I\!N, \Gamma_n$ has $d_n > p$ vertices. For each $n \in I\!N$, let $m_n = d_n - p$, and consider the graph G_n given by

$$G_n = (K_{m_n,m_n} - \{e_1, e_2, \cdots, e_{p-1}\}) \vee \Gamma_n,$$

where the edges $e_1, e_2, \cdots, e_{p-1}$ are all incident with a common vertex. Letting $k_n = 2m_n + d_n$, and $H_n = K_{1,p-1} + O_{k_n-p}$, we see that $G_n \cup H_n = K_{m_n,m_n} \vee \Gamma_n$.

We thus find that $\alpha(G_n \cup H_n) = \min\{\alpha(K_{m_n,m_n}) + d_n, \alpha(\Gamma_n) + 2m_n\} = \min\{m_n + d_n, \alpha(\Gamma_n) + 2m_n\}$. Since $\alpha(\Gamma_n) \le p = m_n + d_n - 2m_n$, we see that $\alpha(G_n \cup H_n) = \alpha(\Gamma_n) + 2m_n$ for each $n \in I\!N$. From Proposition 3.4, for each $n \in I\!N$, we have $\alpha(K_{m_n,m_n} - \{e_1, e_2, \cdots, e_{p-1}\}) = \frac{3m_n - p - \sqrt{(m_n+p)^2 - 8p + 8}}{2}$. Hence $\alpha(G_n) = \min\{\frac{3m_n - p - \sqrt{(m_n+p)^2 - 8p+8}}{2} + d_n, \alpha(\Gamma_n) + 2m_n\}$. Observe that the inequality

$$\frac{3m_n - p - \sqrt{(m_n + p)^2 - 8p + 8}}{2} + d_n \le \alpha(\Gamma_n) + 2m_n$$

is equivalent to the condition

$$\frac{m_n + p - \sqrt{(m_n + p)^2 - 8p + 8}}{2} \le \alpha(\Gamma_n). \tag{4}$$

Note that as $n \to \infty$, the left side of (4) converges to 0 (since $m_n \to \infty$ as $n \to \infty$) while the right side converges to $r > 0$. We conclude that for all sufficiently large n, $\alpha(G_n) = \frac{3m_n - p - \sqrt{(m_n+p)^2 - 8p+8}}{2} + d_n$.

Consequently, for all sufficiently large values of n, we find that $\alpha(G_n \cup H_n) - \alpha(G_n) = \alpha(\Gamma_n) + 2m_n - \frac{3m_n - p - \sqrt{(m_n+p)^2 - 8p+8}}{2} - d_n$, or equivalently,

$$\alpha(G_n \cup H_n) - \alpha(G_n) = \alpha(\Gamma_n) + \frac{\sqrt{(m_n + p)^2 - 8p + 8} - (m_n + p)}{2}.$$

It now follows that $lim_{n \to \infty} \alpha(G_n \cup H_n) - \alpha(G_n) = r$.

Finally, we consider the case that $r = 0$. From the considerations above, we find that for each $i \in I\!N$, there are graphs G_{n_i}, H_{n_i} satisfying i)-iii) such that $|\alpha(G_{n_i} \cup H_{n_i}) - \alpha(G_{n_i}) - \frac{3}{2^{i+1}}| < \frac{1}{2^{i+1}}$ (this is because $\alpha(G \cup H) - \alpha(G)$ can be made arbitrarily close to $\frac{3}{2^{i+1}}$ via suitable choices of G and H). Hence we have $\alpha(G_{n_i} \cup H_{n_i}) - \alpha(G_{n_i}) \in (\frac{1}{2^i}, \frac{1}{2^{i-1}})$ for each $i \in I\!N$, from which it follows that $\alpha(G_{n_i} \cup H_{n_i}) - \alpha(G_{n_i})$ decreases monotonically to 0 as $i \to \infty$. ■

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