On Q-spectral integral variation

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Abstract

Let G be a graph with two non adjacent vertices and G' the graph constructed from G by adding an edge between them. It is known that the trace of Q' is 2 plus the trace of Q, where Q and Q' are the signless Laplacian matrices of G and G' respectively. So, the sum of the Q'-eigenvalues of G' is the sum of the the Q-eigenvalues of G plus two. It is said that Q-spectral integral variation occurs when either only one Q-eigenvalue is increased by two or two Q-eigenvalues are increased by 1 each one. In this article we present some conditions for the occurrence of Q-spectral integral variation under the addition of an edge to a graph G.

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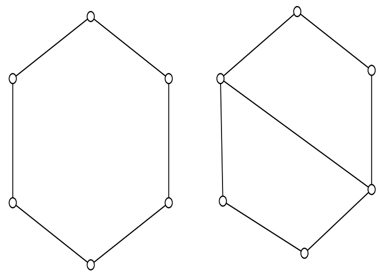
1 Preliminaries

Let G be a simple graph on n vertices. The *signless Laplacian matrix* of G is $Q(G) = A(G) + D(G)$, or simply Q, where $A(G)$ is its adjacency matrix and $D(G) = diag(d_1, \ldots, d_n)$ is the diagonal matrix of the vertex degrees in G [1]. The *spectrum of* $Q(G)$, $Sp_Q(G) = (q_1, \ldots, q_{n-1}, q_n)$, is the sequence of the eigenvalues of $Q(G)$ displayed in non-increasing order $q_1 \geq \ldots \geq q_{n-1} \geq q_n$. A graph G is Q-*integral* if the spectrum of $Q(G)$ consists only of integers [2].

In a graph G, when an edge e is inserted between 2 non adjacent vertices v_1 and v_2, their degrees d_1 and d_2 are incremented 1 each one in the resulting graph $G' = G + e$. Then the trace of the matrix $Q' = Q(G + e)$ is equal to the trace of the matrix $Q(G)$ increased by 2, that is $tr(Q(G + e)) = tr(Q(G)) + 2$. Consequently, if $q'_1 \geq q'_2 \geq \cdots q'_n \geq 0$ are the Q-eigenvalues of G' then $\sum q'_i = 2 + \sum q_i$. Note that $Q(G') = Q(G) + [e_1 + e_2][e_1 + e_2]^T$, where e_i denotes the ith standard unit basis vector, $i = 1, 2$. Since $H = [e_1 + e_2][e_1 + e_2]^T$ is a positive semidefinite matrix and $tr(H) = 2$, none of the eigenvalues of $Q(G)$ can decrease if we add the edge $e = \{v_1, v_2\}$ to G. If one of the Q-eigenvalues is increased by 2 or if two Q-eigenvalues are increased by 1, we say there is either Q-*spectral integral variation in one place* or Q-*spectral integral variation in two places*. So, if G is a Q-integral graph and there is Q-spectral integral variation when we add an edge to G, then $G + e$ is also a Q-integral graph. There are pairs of Q-integral graphs where one of them is obtained from the other by the addition of an edge linking 2 non adjacent vertices (see [3]).

Example 1.1 The graphs G_1 and G_2 shown in the figure bellow reflect this situation. Their Q-spectra are, respectively, $Sp(Q(G_1)) = (4, 3^{(2)}, 1^{(2)}, 0)$ and $Sp(Q(G_2)) = (5, 3^{(2)}, 2, 1, 0)$



The spectral integral variation for the Laplacian matrix was completely characterized, (see Fan [4] and Kirkland [5]). However, in the case of the adjacency matrix $A(G)$, it is impossible to have any alteration in only one of their eigenvalues when an edge is inserted between 2 non adjacent vertices of

G. It happens because the trace of $A(G)$ is zero, so it is also the sum of their eigenvalues. Then, in this situation, there can be no spectral integral variation in one place. In 2005, Pan *et al.* [6] proved that there is no rational variation in two places for $A(G)$.

2 Q-spectral integral variation

For the Q matrix we have the following result.

Proposition 2.1 *Let G be a connected graph on $n \geq 3$ vertices and let v_1 and v_2 be non adjacent vertices in G. Then, there is no Q-spectral integral variation in only one place when we add the edge $\{v_1, v_2\}$ to G.*

From now on, the results for Q-spectral integral variation are similar to those given by Kirkland [5] for the spectral integral variation in two places for the Laplacian matrix. Note that if G is a connected graph with $n \geq 3$ vertices, v_1 and v_2 are non adjacent vertices in G and $G' = G + e$, where $e = \{v_1, v_2\}$, then $q_1(G') > q_1(G)$.

Lemma 2.2 *Let G be a connected graph on $n \geq 3$ vertices and let v_1 and v_2 be non adjacent vertices in G. Let $d_{1,2} = |N(1) \cap N(2)|$ where $N(i)$ is the set of neighbors of v_i in G. If occurs Q-spectral integral variation when we add the edge $e = \{v_1, v_2\}$ to G and the Q-eigenvalue $q_k, k > 1$, has been increased, then:*

$$q_1 + q_k = d_1 + d_2 + 1; \tag{1}$$

and

$$q_1 q_k = d_1 d_2 - d_{1,2}. \tag{2}$$

Corollary 2.3 *Let G be a connected graph on $n \geq 3$ vertices and let v_1 and v_2 be non adjacent vertices in G. If there is a Q-spectral integral variation when we add the edge $e = \{v_1, v_2\}$ to G then the Q-eigenvalues of G that have been changed are:*

$$q_1 = \frac{d_1 + d_2 + 1 + \sqrt{(d_1 + d_2 + 1)^2 - 4(d_1 d_2 - t)}}{2} \tag{3}$$

and

$$q_k = \frac{d_1 + d_2 + 1 - \sqrt{(d_1 + d_2 + 1)^2 - 4(d_1 d_2 - t)}}{2}, \tag{4}$$

where $t = |N(1) \cap N(2)|$.

Lemma 2.4 *Let G be a connected graph on $n \geq 3$ vertices and let v_1 and v_2 be non adjacent vertices in G. If occurs Q-spectral integral variation when we add the edge $e = \{v_1, v_2\}$ to G then there is an orthonormal basis of Q-eigenvectors such that $n-2$ among them are orthogonal to $e_1 + e_2$.*

Theorem 2.5 *Let G be a connected graph on $n \geq 3$ vertices and let v_1 and v_2 be non adjacent vertices in G. In order to occur Q-spectral integral variation when we add the edge $e = \{v_1, v_2\}$ to G and the eigenvalues q_1 and q_k be changed, it is necessary and sufficient that exists an orthonormal basis of Q-eigenvectors such that:*

(i) *$n-2$ eigenvectors are orthogonal to $\mathbf{e}_1 + \mathbf{e}_2$;*

(ii) *There is an eigenvector de Q associated to q_1 in the form* $\begin{bmatrix} a_1 \\ b_1 \\ \mathbf{u} \end{bmatrix}$,

with $(a_1 + b_1)^2 = 1 - \dfrac{1}{q_1 - q_k}$*;*

(iii) *There is an eigenvector de Q associated to q_k in the form* $\begin{bmatrix} a_2 \\ b_2 \\ \mathbf{v} \end{bmatrix}$,

with $(a_2 + b_2)^2 = 1 + \dfrac{1}{q_1 - q_k}$.

In the particular case when G is a regular graph we obtain a curious result.

Corollary 2.6 *Let G be a r-regular connected graph on $n \geq 3$ vertices and v_1 and v_2 non adjacent vertices in G. The only one case which Q-spectral integral variation occurs is when G is the cicle C_6 and the edge added on it is between two vertices without common neighbors.*

Theorem 2.7 *Let G be a connected graph on n vertices with signless Laplacian matrix given by*

$$Q = \begin{bmatrix} d_1 & 0 & \mathbf{1_r}^T & \mathbf{0_s}^T & \mathbf{1_t}^T & \mathbf{0_u}^T \\ 0 & d_2 & \mathbf{0_r}^T & \mathbf{1_s}^T & \mathbf{1_t}^T & \mathbf{0_u}^T \\ \mathbf{1_r} & \mathbf{0_r} & Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ \mathbf{0_s} & \mathbf{1_s} & Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ \mathbf{1_t} & -\mathbf{1_t} & Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ \mathbf{0_u} & \mathbf{0_u} & Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix},$$

where $t = d_{1,2}$, $r = d_1 - t$, $s = d_2 - t$ *and* $u = n - 2 - d_1 - d_2 + t$. *Form* G' *from* G *by adding the edge* $e = \{v_1, v_2\}$.

If occurs Q-spectral integral variation under the addition of that edge then for a_1 *and* b_1 *such that* $(a_1 + b_1)^2 = 1 - \dfrac{1}{q_1 - q_k}$ *and* $\begin{bmatrix} a_1 \\ b_1 \\ \mathbf{u} \end{bmatrix}$ *is a Q-eigenvector associated to* q_1*,we have*

$$Q_{11}(a_1 - (d_2 - q_1)b_1)\mathbf{1} + Q_{12}(b_1 - (d_1 - q_1)a_1)\mathbf{1} + Q_{13}((q_1 + 1 - d_1)a_1 + (q_1 + 1 - d_2)b_1)\mathbf{1}$$
$$= ((q_1 - detM)a_1 - q_1(d_2 - q_1)b_1)\mathbf{1};$$

$$Q_{21}(a_1 - (d_2 - q_1)b_1)\mathbf{1} + Q_{22}(b_1 - (d_1 - q_1)a_1)\mathbf{1} + Q_{23}((q_1 + 1 - d_1)a_1 + (q_1 + 1 - d_2)b_1)\mathbf{1}$$
$$= ((q_1 - detM)b_1 - q_1(d_1 - q_1)a_1)\mathbf{1};$$

$$Q_{31}(a_1 - (d_2 - q_1)b_1)\mathbf{1} + Q_{32}(b_1 - (d_1 - q_1)a_1)\mathbf{1} + Q_{33}((q_1 + 1 - d_1)a_1 + (q_1 + 1 - d_2)b_1)\mathbf{1}$$
$$= ((q_1(q_1 + 1 - d_1) - detM)a_1 + (q_1(q_1 + 1 - d_2) - detM)b_1)\mathbf{1};$$

$$Q_{41}((a_1 - (d_2 - q_1)b_1)\mathbf{1} + Q_{42}(b_1 - (d_1 - q_1)a_1)\mathbf{1} + Q_{43}((q_1 + 1 - d_1)a_1 + (q_1 + 1 - d_2)b_1)\mathbf{1}$$
$$= \mathbf{0}.$$

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