Spin systems dynamics and faults detection in threshold networks

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We consider an agent on a fixed but arbitrary node of a known threshold network, with the task of detecting an unknown missing link/node. We obtain analytic formulas for the probability of success, when the agent's tool is the free evolution of a single excitation on an XX spin system paired with the network. We completely characterize the parameters allowing for an advantageous solution. From the results emerges an optimal (deterministic) algorithm for quantum search, therefore gaining a quadratic speed-up with respect to the optimal classical analogue, and in line with well-known results in quantum computation. When attempting to detect a faulty node, the chosen setting appears to be very fragile and the probability of success too small to be of any direct use.

I. INTRODUCTION

Searching and traversing graphs with the use of a quantum dynamics (discrete or continuous) is a topic of wide study. The main findings on the algorithmic side exhibit a quadratic gain (producing then a class of Grover-like results), when used to search marked vertices in hypercubes, lattices, and more general objects [11], and an exponential one when transferring information to a distant site by free evolution [9]. With essentially the same formalism, the dynamics of spin systems forms the basis of concrete proposals for implementing communication buses in a variety of nanotechnology devices [2]. The central point is always a unitary operator reflecting the topology of a graph whose vertices are encoded in pure states or a (time-independent) Hamiltonian describing the interactions of particles in the actual physical network, and thus representing a noiseless quantum channel.

Considering these two contexts together, we take a parsimonious agent located on a specific node of a known threshold network, attempting to detect an unknown missing link/node. We obtain analytic formulas for the probability of success, when the missing link/node is searched by letting evolve a single excitation from the given vertex. For specific parameters, we have an optimal algorithm for quantum search in a deterministic fashion. Our working ground is an XX (or, equivalently, isotropic XY) system with homogeneous couplings and site dependent magnetic fields [3] (a proposed setting for distributed implementation consists of cavities connected by optical fibers [8]).

Threshold graphs are applied to the synchronization of parallel processes, set packing, scheduling, and to define the Guttman scale in the area of statistical surveys [10]. Formally, a *threshold graph* can be constructed from the one-vertex graph by repeatedly adding a single vertex of two possible types: an *isolated vertex*, *i.e.* a vertex without edges; a *dominating vertex*, *i.e.* a vertex connected to all other vertices. The *creation sequence* of a threshold graph G on n vertices is a word $x(G) \in \{0,1\}^n$ recursively defined as follows: $x(G) = x(G-i)\,y$, where $y = 0$ if i is isolated and $y = 1$ if i is dominating. A threshold graph is then characterized by its creation sequence, a property with important consequences. In fact, the information required to store a threshold graph on n vertices is at most n bits, two threshold graphs are isomorphic if and only if they have the same degree sequence, and the recognition can be done in linear time. Even if the above definition is somehow restrictive, it is valuable that one can construct arbitrarily large threshold networks with approximately any prescribed degree distribution, including the scale-free one [6].

The dynamics of the first excitation sector in the XX model is governed by the (combinatorial) Laplacian [3]. The *Laplacian* of a graph $G = (V, E)$ is the matrix L with entries $[L]_{i,j} = d(i)$ if $i = j$, $[L]_{i,j} = -1$ if $\{i,j\} \in E$ and $[L]_{i,j} = 0$ if $\{i,j\} \notin E$; here $d(i) = |\{j : \{i,j\} \in E\}|$ is the *degree* of $i \in V$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ be the eigenvalues of L arranged in the nonincreasing order. A graph is a threshold graph if and only if $\lambda_j = |\{i : d(i) \geq j\}|$, with $j = 1, ..., n$ [5]. In other words, the degree sequence and the nonzero eigenvalues of a threshold graph G are (Ferrer's) conjugate partitions of $2|E|$. For example, let us consider the word 0011011. The graph G has degree sequence $\{2,4,4,5,5,6,6\}$ and eigenvalues $\{7,7,6,6,4,2,0\}$.

Let $|\psi_t\rangle = U_t|\psi_0\rangle$, where $U_t \equiv \exp(-iLt)$, for $t \in \mathbb{R}^+$, and $|\psi_0\rangle$ is an element of an n dimensional Hilbert space, with standard basis $\{|i\rangle : i \in V\}$. Let $p_G(i,j,t) = |\langle j|U_t|i\rangle|^2$ be the probability that a single excitation travels from node i to node j after a free evolution of duration t (this is also called *fidelity*); we say that there is *perfect state transfer* when $p_G(i,j,t) = 1$, for $i \neq j$. Since the Laplacian spectrum of a threshold graph is integral, it follows that the dynamics governed by L is always periodic [2, 4], *i.e.*, there is t such that $p_G(i,i,t) = 1$, for every i.

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Firstly, we propose in Theorem 2 a complete characterization of the class of threshold graphs allowing for perfect state transfer. Secondly, with a direct application of this result, we describe how the evolution can be used to detect a missing link. The procedure gives a straightforward algorithm for (optimal) quantum search in certain threshold networks. Indeed, as a special case, we have a deterministic version of Grover's search [1]. Theorem 6 shows that the result can not be extended to a large subfamily of threshold graphs. As expected, the graphs for which the result holds can be associated to modulated chains of length 3, where the couplings are determined by the number of edges connecting vertices from certain subsets (namely, an equitable partition). This is a common situation when studying perfect state transfer [2]. Interference effects drive the evolution in a way that does not seem to be exploitable for searching (by using out technique), unless the missing links belong to the subsets.

II. FAULTY LINKS

For any $p \in \mathbb{N}$, let K_p denote the complete graph on p vertices, and let O_p denote the empty graph on p vertices. Let $\mathbf{0}_p$ and $\mathbf{1}_p$ denote the zero vector and all ones vector, respectively, both of order p. For $p, q \in \mathbb{N}$, we use $J_{p\times q}$ to denote the $p \times q$ all ones matrix; often the subscript will be suppressed when the order is evident from the context. We need two operations: the union and the joint. The *union*, denoted by $\cup$, consists of taking two graphs and looking at them as a single one whose connected components are exactly the graphs. The *join*, denoted by $\vee$, is the graph obtained by taking the union of two graphs, plus edges connecting all their respective vertices. The following lemma can be deduced from the basic properties of threshold graphs and it does not require a proof (see, *e.g.*, [5]). It is our main technical tool.

Lemma 1 *Let G be a connected graph on at least two vertices. Then G is a threshold graph if and only if one of the following two conditions is satisfied:*

(i) there are indices $m_1, \ldots, m_{2k} \in \mathbb{N}$ with $m_1 \geq 2$ such that $G = ((((O_{m_1} \vee K_{m_2}) \cup O_{m_3}) \vee K_{m_4}) \ldots) \vee K_{m_{2k}} \equiv \Gamma(m_1, \ldots, m_{2k})$;

(ii) there are indices $m_1, \ldots, m_{2k+1} \in \mathbb{N}$ with $m_1 \geq 2$ such that $G = ((((K_{m_1} \cup O_{m_2}) \vee K_{m_3}) \cup O_{m_4}) \ldots) \vee K_{m_{2k+1}} \equiv \Gamma(m_1, \ldots, m_{2k+1})$.

The vertices $1.,..,m_1$ correspond to the first subset, $m_1+1,...,m_1+m_2$ correspond to the second subset, etc.

The next theorem is the central result of this section. It gives a complete characterization of threshold graphs with perfect state transfer. As it is usually done when studying this topic, we write explicitly the eigensystem of the Laplacian. The parameterization is the one described in Lemma 1.

Theorem 2 *Let G be a threshold graph. When $G \equiv \Gamma(m_1, \ldots, m_{2k})$ (resp. $G \equiv \Gamma(m_1, \ldots, m_{2k+1})$), $p_G(i, j, t) = 1$ if and only if $(i, j) = (1, 2)$ and in addition: $t = \frac{\pi}{2}$; $m_1 = 2$; $m_2 \equiv 2 \bmod 4$; and $m_j \equiv 0 \bmod 4, j = 3, \ldots, 2k$ (resp. $j = 3, \ldots, 2k+1$).*

Proof. Suppose that we have $m_1, \ldots, m_{2k} \in \mathbb{N}$ with $m_1 \geq 2$, and consider the Laplacian matrix L of $\Gamma(m_1, \ldots, m_{2k})$. For each $1 \leq l \leq 2k$, let $\sigma_l = \sum_{p=1}^{l} m_p$. Fix an odd index j between 1 and $2k$, and note that if $m_j \geq 2$, then for any vector $u \in \mathbb{R}^{m_j}$ such that $u \perp \mathbf{1}_{m_j}$, the vector $\left[\mathbf{0}_{\sigma_{j-1}} | u | \mathbf{0}_{\sigma_{2k}-\sigma_j}\right]^T$ is an eigenvector for L corresponding to the eigenvalue $\lambda_0(j) \equiv m_{j+1} + m_{j+3} + \ldots + m_{2k}$. Letting $u_1, \ldots, u_{m_j-1}$ be an orthonormal basis of $(\mathbf{1}_{m_j})^\perp$, we find that $\sum_{l=1}^{m_j-1} u_l u_l^T = I - \frac{1}{m_j} J_{m_j\times m_j}$. Note also that if j is odd and $2 \leq j \leq 2k$, then the vector

$$\overrightarrow{\lambda}_0(j) := \left[\begin{array}{c} \left(\frac{m_j}{\sigma_{j-1}\sigma_j}\right)^{1/2} \mathbf{1}_{\sigma_{j-1}} \\ \hline -\left(\frac{\sigma_{j-1}}{m_j\sigma_j}\right)^{1/2} \mathbf{1}_{m_j} \\ \hline \mathbf{0}_{\sigma_{2k}-\sigma_j} \end{array}\right] \qquad (1)$$

is also an eigenvector for L corresponding to $\lambda_0(j)$ that has length one and is orthogonal to the eigenvectors constructed above. Similarly, if j is an even index between 2 and $2k$, and $u_1, \ldots, u_{m_j-1}$ is an orthonormal basis of $(\mathbf{1}_{m_j})^\perp$, then the vectors $\left[\mathbf{0}_{\sigma_{j-1}} | u_l | \mathbf{0}_{\sigma_{2k}-\sigma_j}\right]^T$, with $l = 1, \ldots m_j - 1$, are eigenvectors for L corresponding to the eigenvalue $\lambda_1(j) \equiv \sigma_j + m_{j+2} + m_{j+4} + \ldots + m_{2k}$. Note further that the vector in Eq. (1) is an eigenvector for L corresponding to $\lambda_1(j)$.

Finally, we observe that $\mathbf{1}_{\sigma_{2k}}/\sqrt{\sigma_{2k}}$ is a null vector for L. It now follows that for each index j between 2 and $2k$, the matrix

$$\begin{aligned} P[\lambda_x(j)] &= \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & I - \frac{1}{m_j} J & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \\ &+ \overrightarrow{\lambda}_0(j) \left[\sqrt{\tfrac{m_j}{\sigma_{j-1}\sigma_j}} \mathbf{1}^T_{\sigma_{j-1}} \middle| -\sqrt{\tfrac{\sigma_{j-1}}{m_j\sigma_j}} \mathbf{1}^T_{m_j} \middle| \mathbf{0}^T_{\sigma_{2k}-\sigma_j} \right] \\ &= \left[\begin{array}{c|c|c} \frac{m_j}{\sigma_{j-1}\sigma_j} J_{\sigma_{j-1}\times\sigma_{j-1}} & -\frac{1}{\sigma_j} J & \mathbf{0} \\ \hline -\frac{1}{\sigma_j} J & I - \frac{1}{\sigma_j} J_{m_j\times m_j} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \end{aligned}$$

with $x = 0, 1$, is an orthogonal idempotent (*i.e.*, a projector) for L corresponding to the eigenvalue $\lambda_0(j)$ or $\lambda_1(j)$ according as j is odd or even, respectively. Here the $(1, 1)$ block is of order σ_{j-1}, the $(2, 2)$ block is of order m_j, and $\mathbf{0}$ denotes a zero block whose order is determined from the context. It now follows that for any real $t \geq 0$, we can use the orthogonal idempotents to write

$$U_t = e^{-it\lambda_0(1)} \left[\begin{array}{c|c} I - \frac{1}{m_1} J_{m_1 \times m_1} & 0 \\ \hline 0 & 0 \end{array}\right] + \sum_{j\geq 3, j,\ \mathrm{odd}} e^{-it\lambda_0(j)} P[\lambda_0(j)] + \sum_{j\geq 2, j,\ \mathrm{even}} e^{-it\lambda_1(j)} P[\lambda_i(j)] + \frac{1}{\sigma_{2k}} J. \quad (2)$$

(We note that an equivalent expression for U_t appears in Theorem 3.1 of [7].)

Now we consider an off-diagonal entry z of U_t; for concreteness, we take z to be in the upper triangle of the matrix, and we suppose that z is in a column that lies in the j_0-th subset of the partitioning that is induced by the indices $m_1, \ldots, m_{2k}$. (We observe in passing that all such offdiagonal entries are equal.) From Eq. (2), we find that

$$z = e^{-it\lambda_{p_{j_0}}(j_0)} \left(\frac{-1}{\sigma_{j_0}}\right) + \sum_{j=j_0+1}^{2k} e^{-it\lambda_{p_j}(j)} \frac{m_j}{\sigma_{j-1}\sigma_j} + \frac{1}{\sigma_{2k}},$$

where for each index j, p_j is 0 or 1 according as j is odd or even, respectively. Consequently,

$$|z| \leq \frac{1}{\sigma_{j_0}} + \sum_{j=j_0+1}^{2k} \frac{m_j}{\sigma_{j-1}\sigma_j} + \frac{1}{\sigma_{2k}}.$$

Note that $m_{2k}/(\sigma_{2k-1}\sigma_{2k}) + 1/\sigma_{2k} = 1/\sigma_{2k-1}$. Thus, the summation giving $|z|$ telescopes, so that

$$\sum_{j=j_0+1}^{2k} \frac{m_j}{\sigma_{j-1}\sigma_j} + \frac{1}{\sigma_{2k}} = \frac{1}{\sigma_{j_0}}.$$

Hence, $|z| \leq 2/\sigma_{j_0}$. In particular, since $m_1 \geq 2$, we see that $|z| < 1$ if $j_0 \geq 2$.

Suppose now that $|z| = 1$. Then necessarily $j_0 = 1$, and, since $1 \leq 2/m_1$, we find that $m_1 = 2$. Indeed, it must also be the case that z is the $(1,2)$ entry of U_t. From the consideration of the equality case in the triangle inequality, we find that $|z| = 1$ if and only if, in addition to $m_1 = 2$ we have:

(i) $e^{-it\lambda_0(1)} = -1$, while

(ii) $e^{-it\lambda_0(j)} = 1$, for all odd $j \geq 3$ and

(iii) $e^{-it\lambda_1(j)} = 1$, for all even j.

From *(i)* we have $e^{-2ti} = -1$, while from *(ii)* it follows that $e^{-itm_{2l}} = 1$ for each $l = 2, \ldots, k$. Applying these conditions, in conjunction with *(iii)*, it then follows that $e^{-itm_2} = -1$, while $e^{-itm_{2l+1}} = 1$, for $l = 1, \ldots, k-1$. Hence there are integers $p_1, \ldots, p_{2k}$ such that

$$\frac{t}{\pi} = \frac{2p_1+1}{2} = \frac{2p_2+1}{m_2} = \frac{2p_j}{m_j},$$

for $j = 3, \ldots, 2k$. We thus find that $m_2 = (4p_2+2)/(2p_1+1)$ while $m_j = 4p_j/(2p_1+1)$, $j = 3, \ldots, 2k$. We now deduce that $m_2 \equiv 2 \bmod 4$, and $m_j \equiv 0 \bmod 4$ for $j = 3, \ldots, 2k$.

Next, we consider the values of $t \in [0, 2\pi]$ for which the $(1,2)$ entry of U_t has modulus one, and note that since L has integer eigenvalues, an offdiagonal entry of U_t has modulus one if and only if the corresponding entry in $\exp(-i(t+2q\pi)L)$ has modulus one for any integer q. From the foregoing we find that the only possible values of $t \in [0, 2\pi]$ for which the $(1,2)$ entry of U_t has modulus one are $t = \pi/2$ and $t = 3\pi/2$. Since $m_2 \equiv 2 \bmod 4$ and $m_j \equiv 0 \bmod 4$, for $j = 3, \ldots, 2k$, we find readily that conditions i)-iii) hold when $t = \pi/2$ or $t = 3\pi/2$, so that the $(1,2)$ entries of $exp(-i\pi L/2)$ and $exp(-i3\pi L/2)$ have modulus one. The second part of the theorem follows along the same lines. ■

Corollary 3 *Let G be a threshold graph on $n = 4m$ vertices with $m_1 = 2$ and $m_2 = 4m-2$. The pair $\{i,j\}$, with i and j in the set of size m_1, can be found with certainty by the use of $O(n-1)$ evolutions induced by L, each one of time $\pi/2$.*

Proof. Consider K_n, the complete graph on n vertices. A special case of Theorem 2 is $K_n^- := K_n - \{i,j\}$, for two distinct arbitrary vertices i and j. When $n = 4m$ ($m \geq 1$), $p_{K_n^-}(i,j,\pi/2) = 1$ and $p_{K_n^-}(k,l,\pi/2) = 0$, for every pair $\{k,l\}$, with $k,l \neq i,j$. Since $|E(K_n)| \to n^2$ for $n \to \infty$, $n-1$ applications of $U_{\pi/2}$ implement a deterministic and optimal version of Grover's search [1]. ■

In practice, we are given a network modeled by a complete K_n that is missing a single unknown link $\{i,j\}$. By starting the dynamics on each possible vertex (or, equivalently, particle) of K_n^-, we can determine the missing edge by letting the entire system evolve for a time $\pi/2$ and then perform a projective measurement at the same vertex. The dynamics governs the behaviour of a walker on the network. If the walker has moved, then the new position is vertex j and the link $\{i,j\}$ is missing.

The same reasoning may be applied to find a missing matching. Recall that a *matching* is a set of vertex-disjoint edges. A matching is *perfect* if it includes all vertices.

Corollary 4 *Let K_n be a graph on $n = 4m$ vertices. A deleted matching from K_n of size $k \leq 2m$ can be found with certainty by the use of exactly $n/2-1$ evolutions induced by L, each one of time $\pi/2$.*

It is just matter of running the dynamics from an arbitrary vertex i and detecting the missing edge $\{i,j\}$ in the matching. Then, we pass to a vertex $k \neq i,j$, and so on and so forth. A deterministic search requires at most $\sum_{i=3:\ \mathrm{odd}}^{n-1} i = n^2/4 - 1$ steps. For threshold graphs, Theorem 2 specifies completely the detectable links. From the perspective of a direct application to network search based on the described method (free-evolution on a spin system), the theorem shows that the complete graph is the only threshold graph in which every deleted link can be actually found. For all other threshold graphs there are some undetectable links. Even if the same procedure can be certainly generalized to any graph, since the search is performed in cliques (*i.e.*, complete subgraphs), a complete knowledge of cliques is necessary and a mechanism to induce evolution only in desired cliques is needed. Still, it is useful to remark that implementing such a mechanism (which would turn on and off the interactions between different cliques) permits to transfer an excitation to any desired vertex.

III. FAULTY NODES

Here we consider the problem of the previous section but for vertices. Given a threshold graph G, let $\hat{G} = G - j$, for some vertex $j \in V$, and let $\hat{U}_t = \exp(-iL(\hat{G})t)$. We ask whether it is plausible that if G has perfect state transfer, we are then able to detect the missing vertex. The idea is based on taking advantage of the graph structure, apart from the edge between the two vertices in the set of size m_1. After a technical lemma, we will prove that the offdiagonal entries of $\hat{U}_t$, while bounded away from 1 in modulus, can have relatively large magnitude, something which does not help to give specific information about the deleted vertex. As a consequence, we do not obtain sufficient information about j. The negative result highlights an important role for special symmetries. We shall give a proof for $G = \Gamma(m_1, m_2, \ldots, m_{2k})$. The theorem for the case $\Gamma(m_1, m_2, \ldots, m_{2k+1})$ has a parallel statement.

Lemma 5 *Let $a \in \mathbb{N}$ be odd. Then*

(i) $\max\{\min\{-\cos 2t, -\cos at, \cos(a-2)t\} | t \in [0, 2\pi]\} = \cos(\pi/a)$*;*

(ii) $\max\{\min\{-\cos 2t, \cos at, -\cos(a-2)t\} | t \in [0, 2\pi]\} = \cos(\pi/a)$*.*

Proof. Set $a = 2m+1$. If

$$t \in S = \left[\frac{m}{2m+1}\pi, \frac{m+1}{2m+1}\pi\right] \cup \left[\frac{3m+1}{2m+1}\pi, \frac{3m+2}{2m+1}\pi\right],$$

then $-\cos 2t < \cos(\pi/a)$. On the other hand, if $t \notin S$ then $|\cos(a-2)t| \leq \cos(\pi/a)$. Thus, we find that

$$\begin{aligned} &\min\{-\cos 2t, -\cos at, \cos(a-2)t\}, \\ &\min\{-\cos 2t, \cos at, -\cos(a-2)t\} \leq \cos\frac{\pi}{a}, \end{aligned}$$

for all $t \in [0, 2\pi]$. Let $b = (m+1)/(2m+1)$ and $c = m/(2m+1)$. Next we show that for each of the functions above, the value $\cos(\pi/a)$ is attainable for some t. If m is odd/even then

$$\begin{aligned} \cos(\pi/a) = &\{-\cos 2(c\pi), -\cos a(c\pi), \cos(a-2)(c\pi)\}, \\ &\{-\cos 2(b\pi), -\cos a(b\pi), \cos(a-2)(b\pi)\} \end{aligned}$$

respectively; in a similar way, if m is even/odd

$$\begin{aligned} \cos(\pi/a) = &\{-\cos 2(c\pi), \cos a(c\pi), -\cos(a-2)(c\pi)\}, \\ &\{-\cos 2(b\pi), \cos a(b\pi), -\cos(a-2)(b\pi)\}. \end{aligned}$$

■

Theorem 6 *Let $m_1, \ldots, m_{2k} \in \mathbb{N}$ such that $m_1 = 2$, $m_2 \equiv 2 \bmod 4$, and $m_l \equiv 0 \bmod 4$, for $l = 3, \ldots, 2k$. Let $i \neq j$ and $g = \left(\frac{1}{2} + \frac{m_2}{\sigma_1\sigma_2} + \frac{1}{\sigma_{2k}}\right)^2$.*

(i) If $\hat{G} = \Gamma(m_1 - 1, m_2, \ldots, m_{2k})$ then $|[\hat{U}_t]_{i,j}| \leq 2/(m_2+1)$;

(ii) If l is even and $\hat{G} = \Gamma(m_1, \ldots, m_{l-1}, m_l - 1, m_{l+1}, \ldots, m_{2k})$ then

$$|[\hat{U}_t]_{i,j}| \leq \left[g - \left(1 - \cos\frac{\pi}{1 + \sum_{i=2:\ even}^{2k} m_i}\right)\frac{m_2}{\sigma_1\sigma_2\sigma_{2k}}\right]^{1/2} + \frac{1}{2} - \frac{m_2}{\sigma_1\sigma_2} - \frac{1}{\sigma_{2k}}; \tag{3}$$

(iii) If $l \geq 3$ is odd and $\hat{G} = \Gamma(m_1, \ldots, m_{l-1}, m_l - 1, m_{l+1}, \ldots, m_{2k})$ then

$$|[\hat{U}_t]_{i,j}| \leq \left[g - \left(1 - \cos\frac{\pi}{\sum_{i=1:\ odd}^{l} m_i - 1}\right)\frac{m_2 m_{l+1}}{\sigma_1\sigma_2\sigma_l\sigma_{l+1}}\right]^{1/2} + \frac{1}{2} - \frac{m_2}{\sigma_1\sigma_2} - \frac{m_{l+1}}{\sigma_l\sigma_{l+1}}. \tag{4}$$

Proof. Let z be an offdiagonal entry of $\hat{U}_t$. As in the proof of Theorem 2, we find that if $z \neq [\hat{U}_t]_{1,2}, [\hat{U}_t]_{2,1}$ then $|z| \leq 2/\sigma_2 \leq 2/3$. Thus, for the remainder of the proof, we assume *wlog* that z is in the $(1,2)$ position. Recall that $\sigma_l = \sum_{p=1}^{l} m_p$.

(i) Note that $\Gamma(m_1 - 1, m_2, \ldots, m_{2k}) = \Gamma(m_2 + 1, m_3, \ldots, m_{2k})$. From the proof of Theorem 2, it follows that $|z| \leq 2/(m_2+1)$.

(ii) Suppose that l is even and that $\hat{G} = \Gamma(m_1, \ldots, m_{l-1}, m_l - 1, m_{l+1}, \ldots, m_{2k})$. Set $a = 1 + m_2 + m_4 + \ldots + m_{2k}$. Suppose that we have positive parameters $\alpha\beta, \gamma$ such that $\alpha \geq \beta, \gamma$, and note that $\left|-\alpha e^{-it(a-2)} + \beta e^{-ita} + \gamma\right|^2 = \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\gamma\cos(a-2)t + 2\beta\gamma\cos at - 2\alpha\beta\cos 2t$. From Lemma 5, for each t, one of $-\cos 2t$, $\cos at$ and $-\cos(a-2)t$ is bounded above by $\cos(\pi/a)$. It now follows that $\left|-\alpha e^{-it(a-2)} + \beta e^{-ita} + \gamma\right| \leq (\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\gamma + 2\beta\gamma\cos\frac{\pi}{a} + 2\alpha\beta)^{1/2} = ((\alpha+\beta+\gamma)^2 - (1 - \cos\frac{\pi}{a})\beta\gamma)^{1/2}$.

Taking $\alpha = 1/\sigma_1 = 1/2$, $\beta = m_2/(\sigma_1\sigma_2)$, and $\gamma = 1/\sigma_{2k}$, applying the triangle inequality as in the proof of Theorem 2, we have

$$\begin{aligned}|z| &\leq \left|\frac{-1}{2}e^{-it(a-2)} + \frac{m_2}{\sigma_1\sigma_2}e^{-ita} + \frac{1}{\sigma_{2k}}\right| + \left|\sum_{j=3}^{2k}\frac{m_j}{\sigma_{j-1}\sigma_j}\right| \\ &\leq \left|\frac{1}{2}e^{-it(a-2)} + \frac{m_2}{\sigma_1\sigma_2}e^{-ita} + \frac{1}{\sigma_{2k}}\right| + 1 \\ &\quad -\frac{1}{2} - \frac{m_2}{\sigma_1\sigma_2} - \frac{1}{\sigma_{2k}},\end{aligned}$$

which implies Eq. (3).

(iii) Suppose that $l \geq 3$ is odd, and that $\hat{G} = \Gamma(m_1, \ldots, m_{l-1}, m_l - 1, m_{l+1}, \ldots, m_{2k})$. Set $a = m_1 + m_3 + \ldots + m_l - 1$. As in the proof of *(ii)*, we consider positive parameters $\alpha \geq \beta, \gamma$, and note that $\left|-\alpha e^{-it\lambda_0(1)} + \beta e^{-it\lambda_1(2)} + \gamma e^{-it\lambda_1(l+1)}\right| = \left|-\alpha + \beta e^{-2it} + \gamma e^{-iat}\right|$. As in *(ii)*, we deduce from Lemma 5 that $\left|-\alpha + \beta e^{-2it} + \gamma e^{-iat}\right| \leq ((\alpha + \beta + \gamma)^2 - (1 - \cos\frac{\pi}{a})\beta\gamma)^{1/2}$. Taking $\alpha = 1/\sigma_1 = 1/2$, $\beta = m_2/(\sigma_1\sigma_2)$, and $\gamma = m_{l+1}/(\sigma_l\sigma_{l+1})$, Eq. (4) follows. ■

Notice that in the context of Theorems 6, the $(1,2)$ entry of $\hat{U}_t$ can have large modulus. For example, it is straightforward to show that if $m_1 = 2$, $m_2 \equiv 2 \bmod 4$, and $m_l \equiv 0 \bmod 4$, for $l = 3, \ldots, 2k$ and $\hat{G} = \Gamma(m_1, m_2, \ldots, m_{2k-1}, m_{2k} - 1)$, then $|[\hat{U}_{\frac{\pi}{2}}]_{1,2}| = \left(1 - 2\left((\sigma_{2k} - 1)/\sigma_{2k}^2\right)\right)^{1/2}$. Similarly, if $m_1 = 2$, $m_2 \equiv 2 \bmod 4$, and $m_l \equiv 0 \bmod 4$, for $l = 3, \ldots, 2k+1$ and $\hat{G} = \Gamma(m_1, m_2, \ldots, m_{2k}, m_{2k+1} - 1)$, then $|[\hat{U}_{\frac{\pi}{2}}]_{1,2}| = \left(1 - 2\left((\sigma_{2k+1} - 1)/\sigma_{2k+1}^2\right)\right)^{1/2}$.

IV. CONCLUSIONS

We have shown that the dynamics on a network governed by the Laplacian, seen as the Hamiltonian restricted to the single excitation sector of an XX spin system, can detect and find a faulty link in the complete graph with a quadratic gain with respect to a deterministic method, and we have extended the observation to matchings. The result gives a way to perform optimal quantum search (on the complete graph), and a new insight into the related algorithms. Essentially we have a reinterpretation of a fact discovered in [3]. Our contribution is to have put the statement in a more general mathematical context, by giving a complete characterized of threshold graphs with perfect state transfer with respect to the XX model. Dealing with deleted nodes, we have shown that our method does not give any clear advantage. The basic idea of the paper is to use a free quantum evolution to search a missing item on a network. The method is different from the ones studies in [11] substantially because the algorithm does not require any control. After the set-up of the network, the system evolves without intermediate operations. The process is distributed because the nodes of the network are identified with spin particles. The measurements are local in the sense that they are concerned with the single sites, independently. That is why we can locate autonomous agents on the sites.

We have considered threshold graphs only, since these have integer Laplacian spectrum, and so possess periodic dynamics, *i.e.*, a necessary condition for perfect state transfer (see [2]). The vertices involved in the phenomenon are special. The symmetry with respect to these vertices can be exploited to create a "reference point" inside the graph. Looking ahead, one direction for further exploration is to determine what information about the topology of a spin system (paired with a network) can be obtained by a free evolution and final local measurements. Here, more than designing search algorithms, it is a matter to determine what kind of graphs have some sort of searching capability embedded in their structure. The method of the paper can be generalized to searching a missing link (or, equivalently, a marked link) in any network, with steps involving one clique at a time. The method works for vertices if we employ the notion of the line graph. However, we have shown that the direct detection of a missing vertex is not a natural task for the studied dynamics, at least on threshold graphs. It is an open question to determine whether the dynamics can help to find marked nodes, when taking a different initial probability distribution, and if we can obtain the quadratic speed-up in this case.

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