On Hadamard Diagonalizable Graphs

S. Barik, S. Fallat*and S. Kirkland†
Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan, Canada S4S 0A2

Abstract

Of interest here is a characterization of the undirected graphs G such that the Laplacian matrix associated with G can be diagonalized by some Hadamard matrix. Many interesting and fundamental properties are presented for such graphs along with a partial characterization of the cographs that have this property

Key words. Graph; Laplacian matrix; Hadamard matrix; Adjacency matrix; Hadamard diagonalizable graph; Cograph.

AMS subject classifications. 05C50, 15A18.

1 Introduction

Throughout this article we consider only simple graphs. Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \cdots, n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is defined as the $n \times n$ matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, \text{if } i \text{ and } j \text{ are adjacent in } G, \\ 0, \text{ otherwise.} \end{cases}$$

*Research supported in part by an NSERC research grant. sfallat@math.uregina.ca
†NSERC Grant No. OGP0138251. kirkland@math.uregina.ca

The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix of vertex degrees of G. It is well known that $L(G)$ is a singular positive semidefinite matrix. Throughout, the *spectrum of* G is defined as

$$S(G) = (\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)),$$

where $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ are the eigenvalues of $L(G)$ arranged in nondecreasing order. For any graph G, $\lambda_1(G) = 0$, with the all ones eigenvector $\mathbb{1}$ as a corresponding eigenvector. There is an extensive literature on Laplacian matrices in general, and on their spectra in particular. We refer the interested reader to the survey articles [10], [11] and [13], and the references therein, for further information (see also [4, 8]). Henceforth we say λ is an *eigenvalue of* G ($\lambda \in S(G)$) to mean that λ is an eigenvalue of $L(G)$. A graph G is said to be *Laplacian integral*, if each element of $S(G)$ is an integer.

One of the motivations for considering Laplacian eigenvalues arises from considering a vertex cut in the graph G. Specifically, if the vertex set V is partitioned into two nonempty sets A and B, of sizes k and $n-k$ respectively, then letting $\mathcal{E}$ denote the collection of edges that have one end vertex in A and the other in B, a standard pair of inequalities (see [6, 13]) assert that

$$\lambda_2(G) \leq \frac{n|\mathcal{E}|}{k(n-k)} \leq \lambda_n(G).$$

These inequalities are established by considering the n-vector $x_{A,B}$ which has an entry $n-k$ in each position corresponding to a vertex in A, and an entry $-k$ in each position corresponding to a vertex in B. Observing that $x_{A,B} \perp \mathbb{1}$, one then notes that $\frac{x_{A,B}^T L(G) x_{A,B}}{x_{A,B}^T x_{A,B}} = \frac{n|\mathcal{E}|}{k(n-k)}$, and appeals to a standard result on symmetric matrices. Note that equality holds in either the lower bound or the upper bound only if the vector x is an eigenvector of L.

A good deal of the literature on Laplacian spectra for graphs focuses on how the combinatorial structure of a graph is reflected in one or more of its Laplacian eigenvalues. In this paper, we turn our attention to the eigen*vectors* of the Laplacian matrix for a graph; while there is some existing literature in this area (see for instance [2] and [12]), the volume of such results is rather less extensive than that on Laplacian eigenvalues.

In this paper we deal extensively with the case that a connected graph G on n vertices has the property that $L(G)$ admits eigenvectors whose entries consist entirely of 1s and -1s. Observe that since any such eigenvector x is orthogonal to $\mathbb{1}$, the 1s and -1s in x generate a vertex cut of G into two subsets of cardinality $n/2$.

Recall that an $n \times n$ matrix $H = [h_{ij}]$ is a *Hadamard matrix* of order n if the entries of H are either $+1$ or -1 and such that $HH^T = nI$. That is, a $(+1, -1)$-matrix is a Hadamard matrix if the inner product of two distinct rows is 0 and the inner product of a row with itself is n. It is known that for a Hadamard matrix H of order n, $|det(H)| = n^{\frac{n}{2}}$, and evidently $H^{-1} = \frac{1}{n}H^T$ for such an H.

It is also easy to check that if the rows and columns of a Hadamard matrix are permuted, the matrix remains a Hadamard matrix. Further, if any row or column is multiplied by -1, the property of being a Hadamard matrix is retained. Thus, it is always possible to arrange to have the first row and first column of a Hadamard matrix contain only $+1$ entries. A Hadamard matrix in this form is said to be *normalized.*

It is known that a necessary condition for the existence of an $n \times n$ Hadamard matrix is that $n = 1, 2, 4k$ for some positive integer k. The following much studied conjecture addresses the sufficiency of this condition.

Conjecture 1 (Hadamard) *An $n \times n$ Hadamard matrix exists for $n = 1, n = 2$, and $n = 4k$ for any $k \in \mathbb{N}$.*

We say that a graph G is *Hadamard diagonalizable* if it has the property that $L(G)$ is diagonalized by some Hadamard matrix. In this paper, we investigate the following question: Which graphs are Hadamard diagonalizable? As noted above, any $(1, -1)$ eigenvector x of $L(G)$ can be thought of as corresponding to a vertex cut of V into two subsets A and B, each of cardinality $\frac{n}{2}$, where A is the set of vertices of G for which the corresponding entry in the eigenvector is 1, and B is the set of vertices of G for which the corresponding entry in the eigenvector is -1. If such an eigenvector x corresponds to eigenvalue λ, we find from the eigen-equation that each vertex in A is adjacent to $\frac{\lambda}{2}$ vertices in B, and vice versa. Thus, a $(1, -1)$ eigenvector for L corresponds to an 'evenly balanced' vertex cut in G. Suppose now that we have two $(1, -1)$ eigenvectors x and

y, with corresponding vertex cuts A, B and A', B', respectively. Then the inner product $x^T y$ can be written as

$$x^T y = |A \cap A'| + |B \cap B'| - |A \cap B'| - |B \cap A'|.$$

In particular, if x and y are orthogonal, then

$$|A \cap A'| + |B \cap B'| = |A \cap B'| + |B \cap A'|. \quad (1)$$

Thus, by asking for a graph G to be Hadamard diagonalizable, we are asking for G to possess a system of n evenly balanced cuts with the additional property that for any pair of distinct cuts A, B and A', B', their pairwise intersections satisfy (1).

In the sequel, we develop a number of basic properties of Hadamard diagonalizable graphs, and determine all such graphs on at most 12 vertices. We also give a detailed discussion of Hadamard diagonalizable cographs. It will transpire that all graphs that will be of interest to us are regular, so that any conclusions drawn regarding the eigenspaces of $L(G)$ apply equally to $A(G)$.

We note in passing that requiring certain eigenvectors to have entries only from a restricted set of values (for example, $\{-1, 0, 1\}$) is not novel to this present work. In fact, structured eigenspaces have been studied, particularly for the adjacency matrix of G (see, for example, [1, 15]), and as such demonstrating the existence of a structured eigenbases has become an interesting and important topic in spectral graph theory. In particular, it is noted in [15] that for a more straightforward eigenvector analysis it is desirable to achieve an eigenspace that is structurally simple. In [15] it is proved that every cograph admits a simply structured eigenspace basis (eigenvectors entries come from the set $\{-1, 0, 1\}$) for the eigenvalues 0 and -1 with respect to the adjacency matrix, and with such an eigenbasis, interesting constructions may be obtained for producing certain cographs. In addition, in [1] it is shown that the null space of the adjacency matrix of a forest has a basis consisting of vectors with entries from the set $\{-1, 0, 1\}$. Furthermore, in [1] it is suggested that the existence of a special bases (like those described above) are generally easier to handle from a computation point of view.

2 Preliminaries

In this section, we give a few definitions and background results that will be needed for our subsequent discussion.

If all the vertices of a graph G have the same degree, then we say that the graph is *regular*. A graph G is *bipartite* if its vertex set can be partitioned into two sets in such a way that no edge is incident with two vertices in the same set. A graph G is called a *cograph*, also known as a *decomposable graph* if and only if no induced subgraph of G is isomorphic to P_4, the path on 4 vertices. In [12], it is proved that any cograph is Laplacian integral.

Suppose that G is a graph on n vertices, with Laplacian spectrum $0 \equiv \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let G^c denote the complement of a graph G. Then the Laplacian spectrum of G^c is $0, n-\lambda_n, n-\lambda_{n-1}, \ldots, n-\lambda_2$ (see [12]). If $G=(V_1,E_1)$ and $H=(V_2,E_2)$ are two graphs on disjoint sets of m and n vertices, respectively, their *union* is the graph $G+H=(V_1\cup V_2, E_1\cup E_2)$, and their *join* is $G\vee H=(G^c+H^c)^c$, the graph on $m+n$ vertices obtained from $G+H$ by adding new edges from each vertex of G to every vertex of H.

Suppose that the orders of G and H are m and n, respectively. Observe that the Laplacian matrix of $G\vee H$ can be written as

$$L(G\vee H)=\begin{bmatrix} nI+L(G) & -J \\ -J^T & mI+L(H) \end{bmatrix}.$$

If $x \perp \mathbb{1}$ is any eigenvector of $L(G)$ corresponding to an eigenvalue λ_i, $i>1$, then we have that

$$L(G\vee H)\begin{bmatrix} x \\ \mathbf{0} \end{bmatrix} = (n+\lambda_i)\begin{bmatrix} x \\ \mathbf{0} \end{bmatrix}.$$

Thus we see that $n+\lambda_i$ is an eigenvalue of $L(G\vee H)$. In a similar manner it follows that $m+\mu_i$ is also an eigenvalue of $L(G\vee H)$, for each eigenvalue μ_i, $i>1$, of $L(H)$. As 0 is an eigenvalue of $L(G\vee H)$ and the trace is the sum of the eigenvalues, we conclude that $m+n$ is also an eigenvalue of $L(G\vee H)$. Thus we have the following result from Merris [12, Theorem 2.1].

Theorem 2 Merris [12] *Let G and H be two graphs on m and n vertices, respectively. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ be the eigenvalues of $L(G)$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the eigenvalue of $L(H)$. Then the eigenvalues of $L(G \vee H)$ are*

$$0,\ m+n,\ \lambda_2+n, \lambda_3+n, \ldots, \lambda_m+n,\ \mu_2+m, \mu_3+m, \ldots, \mu_n+m.$$

The eigenvalue $m+n$ of $L(G \vee H)$ corresponds to an eigenvector Y with

$$Y(v) = \begin{cases} -n \text{ if } v \in G, \\ m \text{ if } v \in H. \end{cases}$$

Given two matrices $R = [r_{ij}]$ and S, the *tensor product* of R and S is defined to be the partitioned matrix $[r_{ij}S]$ and is denoted by $R \otimes S$. Given two graphs G and H, the *Cartesian product* of G and H is defined as the graph $G \square H$ with vertex set $V(G) \times V(H)$. Vertices (u_i, v_j) and (u_r, v_s) are adjacent in $G \square H$ if either $u_i = u_r$ and $\{v_j, v_s\} \in E(H)$ or $\{u_i, u_r\} \in E(G)$ and $v_j = v_s$. Fiedler [7] observed that

$$L(F \square H) = L(F) \otimes I + I \otimes L(H).$$

Thus we have the following result which completely describes the spectrum of the Cartesian product of two graphs (see also [9]).

Theorem 3 (Fiedler [7]) *Let G and H be graphs with*

$$S(G) = (\lambda_1, \ldots, \lambda_m) \quad \text{and} \quad S(H) = (\mu_1, \ldots, \mu_n).$$

Then the eigenvalues of $L(F \square H)$ are

$$\lambda_i + \mu_j, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Moreover, if X_i is an eigenvector of $L(G)$ affording λ_i and Y_j is an eigenvector of $L(H)$ affording μ_j, then $X_i \otimes Y_j$ is an eigenvector of $L(G \square H)$ affording $\lambda_i + \mu_j$.

We have the following is useful observations about Hadamard matrices. If H is a normalized Hadamard matrix of order $4k$, then every row (column) except the first has $2k$ minus ones and $2k$ plus ones, further k minus ones in any row (column) overlap with k minus ones in each other row (column). Also, note that, given Hadamard matrices H_1 of order n and H_2 of order m the tensor product of these two matrices, $H_1 \otimes H_2$, is also a Hadamard matrix, of order nm.

3 Basic properties of Hadamard diagonalizable graphs

We begin with the following basic, but useful, result.

Lemma 4 *A graph G is Hadamard diagonalizable if and only if there is a normalized Hadamard matrix that diagonalizes $L(G)$.*

Proof. Clearly if there is a normalized Hadamard matrix that diagonalizes $L(G)$, then G is Hadamard diagonalizable.

Suppose now that there is a Hadamard matrix H that diagonalizes $L(G)$, and note that each column of H is an eigenvector for $L(G)$. If G is connected, then the null space of $L(G)$ is spanned by $\mathbb{1}$; thus, some column of H is either $\mathbb{1}$ or $-\mathbb{1}$. It now follows that there is a signature matrix S such that HS is a normalized Hadamard matrix that diagonalizes $L(G)$.

If G has $k \geq 2$ connected components, say $G_1, \ldots, G_k$, it follows that $L(G)$ can be written as $\begin{bmatrix} L(G_1) & 0 & \ldots & 0 \\ 0 & L(G_2) & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & L(G_k) \end{bmatrix}$. Letting the orders of $G_1, \ldots, G_k$ be $n_1, \ldots, n_k$, respectively, we find that any $(1, -1)$ null vector for $L(G)$ is of the form $\begin{bmatrix} (-1)^{a_1}\mathbb{1}_{n_1} \\ (-1)^{a_2}\mathbb{1}_{n_2} \\ \vdots \\ (-1)^{a_k}\mathbb{1}_{n_k} \end{bmatrix}$ for some collection of integers $a_1, \ldots, a_k$. In particular, there is a column of H that is of that form. Let S denote the signature matrix $S = \begin{bmatrix} (-1)^{a_1}I_{n_1} & 0 & \ldots & 0 \\ 0 & (-1)^{a_2}I_{n_2} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & (-1)^{a_k}I_{n_k} \end{bmatrix}$.

Observe that SH is a Hadamard matrix with a columns of all ones, and that SH diagonalizes $SL(G)S$. But from the block diagonal structure of $L(G)$, it follows that $SL(G)S = L(G)$. Hence SH diagonalizes $L(G)$, and it follows that there is a normalized Hadamard matrix that diagonalizes $L(G)$. $\square$

For each $n \geq 1$, we use K_n to denote the complete graph on n vertices, that is, the graph with all possible edges, while K_n^c will be referred to as an *empty graph.* For $n = 2$, there exits only one Hadamard matrix and K_2 is the only graph diagonalizable by that, excluding the empty graph. Also, it is easy to check that for $n = 4$, excluding the empty graph, $K_2 + K_2$, $K_{2,2}$ and K_4 are the only graphs which are diagonalizable by the Hadamard matrix of order 4.

The following observations show that given any Hadamard matrix H of order $n = 4k, k \geq 1$, both K_{4k} and $K_{2k,2k}$ are diagonalizable by H.

Observation 1 *Let H be a normalized Hadamard matrix of order $n = 4k, k \geq 1$. Then K_n is diagonalizable by H.*

Proof. Let H be a normalized Hadamard matrix of order $n = 4k, k \geq 1$. We write H as $H = \left[\begin{array}{c|c} \mathbb{1} & \tilde{H} \end{array}\right]$ and let $\mathbf{D}$ denote the diagonal matrix $\mathbf{D} = \left[\begin{array}{c|c} 0 & \mathbf{0}^T \\ \hline \mathbf{0} & nI \end{array}\right]$.

Observe that

$$H\mathbf{D}H^T = n\tilde{H}\tilde{H}^T \text{ and } nI = HH^T = J + \tilde{H}\tilde{H}^T.$$

Thus we have, $H\mathbf{D}H^{-1} = \frac{1}{n}H\mathbf{D}H^T = nI - J = L(K_n)$. ∎

Observation 2 *Let H be a normalized Hadamard matrix of order $n = 4k, k \geq 1$. Then there is a permutation matrix P such that $K_{2k,2k}$ is diagonalizable by the Hadamard matrix PH.*

Proof. Let H be a normalized Hadamard matrix of order $n = 4k, k \geq 1$. By permuting the rows of H if necessary, we can write H in the form

$$H = \left[\begin{array}{cc|c} \mathbb{1} & \mathbb{1} & \\ & & \tilde{H} \\ \mathbb{1} & -\mathbb{1} & \end{array}\right].$$

Let $\mathbf{D}$ be the diagonal matrix defined as $\mathbf{D} = \left[\begin{array}{c|c|c} 0 & 0 & \mathbf{0}^T \\ \hline 0 & n & \mathbf{0}^T \\ \hline \mathbf{0} & \mathbf{0} & 2kI \end{array}\right]$.

Thus, we have $H\mathbf{D}H^T = n \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1}^T & -\mathbb{1}^T \end{bmatrix} + 2k\tilde{H}\tilde{H}^T$.

Now $HH^T = nI$ gives

$$J + \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1}^T & -\mathbb{1}^T \end{bmatrix} + \tilde{H}\tilde{H}^T = nI.$$

This implies that $\begin{bmatrix} 2J & \mathbf{0} \\ \mathbf{0} & 2J \end{bmatrix} + \tilde{H}\tilde{H}^T = nI$.

Thus $H\mathbf{D}H^T = n \begin{bmatrix} J & -J \\ -J & J \end{bmatrix} + 2k\Big(nI - \begin{bmatrix} 2J & \mathbf{0} \\ \mathbf{0} & 2J \end{bmatrix}\Big) = n \begin{bmatrix} J & -J \\ -J & J \end{bmatrix} + \frac{n^2}{2}I - n \begin{bmatrix} J & \mathbf{0} \\ \mathbf{0} & J \end{bmatrix}$.

Hence $H\mathbf{D}H^{-1} = \frac{1}{n}H\mathbf{D}H^T = 2kI - \begin{bmatrix} \mathbf{0} & J \\ J & \mathbf{0} \end{bmatrix} = L(K_{2k,2k})$.

The conclusion now follows. □

Our next result shows that all Hadamard diagonalizable graphs are regular and Laplacian integral. Further, the Laplacian eigenvalues of such graphs are even integers.

Theorem 5 *Let G be a graph of order n which is Hadamard diagonalizable. Then G is regular and all its Laplacian eigenvalues are even integers.*

Proof. Let $H^T L(G) H = \mathbf{D^*}$, for some diagonal matrix $\mathbf{D^*}$. That is, $L(G)H = H\mathbf{D}$, where $\mathbf{D} = \frac{1}{n}\mathbf{D^*}$.

Fix an index i with $i = 1, 2, \ldots, n$. There exists a corresponding signature matrix $\mathcal{S}_i$ such that $e_i^T H\mathcal{S}_i = \mathbb{1}^T$; observe that $H\mathcal{S}_i$ is another Hadamard matrix.

Now, $e_i^T L(G)H\mathcal{S}_i = e_i^T H\mathbf{D}\mathcal{S}_i = e_i^T H\mathcal{S}_i\mathbf{D} = \mathbb{1}^T\mathbf{D}$. Thus

$$e_i^T L(G)H\mathcal{S}_i\mathbb{1} = \mathbb{1}^T\mathbf{D}\mathbb{1} = \sum_{j=1}^{n} \lambda_j(G). \tag{2}$$

Since HS_i is a Hadamard matrix, its rows are pairwise orthogonal. Since the i-th row of HS_i is $\mathbb{1}^T$, it follows that $H\mathcal{S}_i\mathbb{1} = \mathbb{1}^T\mathbb{1}e_i = ne_i$. Thus

$$e_i^T L(G)H\mathcal{S}_i\mathbb{1} = e_i^T L(G)[H\mathcal{S}_i\mathbb{1}] = e_i^T L(G)[\mathbb{1}^T\mathbb{1}e_i] = ne_i^T L(G)e_i. \tag{3}$$

From Equation 2 and Equation 3, we have

$$d_i = e_i^T L(G) e_i = \frac{1}{n}\sum_{j=1}^{n} \lambda_j(G),$$

where d_i denotes the degree of the i-th vertex in G. It now follows that G is a regular graph with regularity $r = \frac{1}{n}\sum_{j=1}^{n} \lambda_j(G)$.

Next we shall prove that all the nonzero eigenvalues of $L(G)$ are even integers. Let λ be a nonzero eigenvalue of $L(G)$.

Since $L(G)H = H\mathbf{D}$, we have $L(G)h_i = \lambda h_i$, for some column h_i of H and $\lambda = \mathbf{d}_i$, the i-th diagonal entry of $\mathbf{D}$. Note that, h_i consists of $\frac{n}{2}$ entries of$+1$ and $\frac{n}{2}$ entries of -1 and the corresponding vertices form a cut in G. Thus using some permutation operations, we can write

$$\begin{bmatrix} L(G_1) + D_1 & -A \\ -A^T & D_2 + L(G_2) \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix} = \lambda \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}, \tag{4}$$

where G_1 and G_2 are induced subgraphs of G corresponding to the positive and negative vertices of G valuated by the eigenvector h_i.

From Equation 4, we have $[L(G_1)+D_1]\mathbb{1} + A\mathbb{1} = \lambda\mathbb{1}$ and $[L(G_2)+D_2]\mathbb{1} + A^T\mathbb{1} = \lambda\mathbb{1}$. Thus, $2D_1\mathbb{1} = 2D_2\mathbb{1} = \lambda\mathbb{1}$ and hence λ is an even integer. □

A consequence of Theorem 5 is that for any Hadamard diagonalizable graph G, $L(G)$ is just a scalar translate of $A(G)$. Hence any conclusions drawn on the eigenspaces associated with $L(G)$ apply equally to the eigenspaces of $A(G)$ as well. This then frames our eigenspace analysis within the context of existing such work like that in [15].

Lemma 6 *Let G_1 and G_2 be two graphs. If $G_1 + G_2$ is Hadamard diagonalizable, then G_1 and G_2 satisfy the following properties.*

(i) G_1 and G_2 both are regular graphs of same order and same regularity.

(ii) G_1 and G_2 both have even eigenvalues.

(iii) G_1 and G_2 share the same eigenvalues.

Proof. Let G_1 be of order m and G_2 be of order n. We have

$$L(G_1 + G_2) = \begin{bmatrix} L(G_1) & 0 \\ 0 & L(G_2) \end{bmatrix}.$$

There is a normalized Hadamard matrix H whose columns are eigenvectors of $L(G_1+G_2)$. Thus two of the columns of H are $\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix}$ and $\begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$, which serve as null vectors of $L(G_1+G_2)$. Since these two columns of H are orthogonal, we have $m = n$. Further, by Theorem 5, G_1+G_2 is regular and has even eigenvalues, thus both G_1 and G_2 are regular and of same regularity having even eigenvalues. Thus, we have (i) and (ii).

Let λ be an eigenvalue of $L(G_1+G_2)$, and let x be a column of H that serves as an eigenvector of $L(G_1+G_2)$ corresponding to λ. We can write x as $x = \begin{bmatrix} u \\ v \end{bmatrix}$, where each of u and v is a $(1, -1)$ vector. Since $L(G_1+G_2)x = \lambda x$ we have $L(G_1)u = \lambda u$ and $L(G_2)v = \lambda v$. Hence both $L(G_1)$ and $L(G_2)$ have λ as an eigenvalue. Since the spectrum of $L(G_1+G_2)$ consists of the union of the spectra of $L(G_1)$ and $L(G_2)$, we have (iii). $\square$

Lemma 7 *Let G be a Hadamard diagonalizable graph. Then G^c, $G+G$, and $G \vee G$ are also Hadamard diagonalizable.*

Proof. Suppose that $L(G)$ is diagonalizable by a Hadamard matrix, say H. It is easy to see that $L(G^c)$ is diagonalizable by the same Hadamard matrix H. Then we can see that the matrices $L(G+G)$ and $L(G \vee G)$ are diagonalizable by the Hadamard matrix $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$. $\square$

Note that the converse of the above lemma is not always true. For example, consider $G = K_6$. Both $K_6 + K_6$ and $K_6 \vee K_6$ are diagonalizable by a Hadamard matrix of order 12, but since there does not exists any Hadamard matrix of order 6, K_6 is not Hadamard diagonalizable.

Lemma 8 *Let G_1 and G_2 be two Hadamard diagonalizable graphs on m and n vertices. Then $G_1 \square G_2$ is also Hadamard diagonalizable.*

Proof. Let $L(G_1)$ and $L(G_2)$ be diagonalizable by the Hadamard matrices H_1 and H_2 of order m and n, respectively, say with $H_i^{-1}L(G_i)H_i = \mathbf{D}_i, i = 1, 2$. Note that $H_1 \otimes H_2$ is a Hadamard matrix, and that $(H_1 \otimes H_2)^{-1} = H_1^{-1} \otimes H_2^{-1}$. We thus have

$$(H_1 \otimes H_2)^{-1} L(G_1 \square G_2) H_1 \otimes H_2 = (H_1 \otimes H_2)^{-1}(L(G_1) \otimes I + I \otimes L(G_2)) H_1 \otimes H_2 =$$

$$H_1^{-1}L(G_1)H_1 \otimes I + I \otimes H_2^{-1}L(G_2)H_2 = \mathbf{D}_1 \otimes I + I \otimes \mathbf{D}_2.$$

Hence $H_1 \otimes H_2$ diagonalizes $L(G_1 \Box G_2)$. □

Using Theorem 5 and Lemmas 6, 7 and 8, we are able to determine all Hadamard diagonalizable graphs on 8 vertices as follows:

a) the only 0-regular graph on 8 vertices is the empty graph, which is Hadamard diagonalizable;

b) the only 1-regular graph on 8 vertices is $K_2 + K_2 + K_2 + K_2$, which is Hadamard diagonalizable;

c) the only 2-regular graphs on 8 vertices are $(K_{2,2}) + (K_{2,2})$, and C_8; the former is Hadamard diagonalizable, while the latter fails to be Laplacian integral, and hence is not Hadamard diagonalizable;

d) there are five connected 3-regular graphs on 8 vertices [14]; of those only $(K_{2,2})\Box K_2$ has Laplacian spectrum consisting of even integers, and it is Hadamard diagonalizable;

e) the only disconnected 3-regular graph on 8 vertices is $K_4 + K_4$, which is Hadamard diagonalizable;

f) noting that the 4-regular (respectively, 5-regular, 6-regular, 7-regular) graphs on 8 vertices are the complements of the 3-regular (respectively, 2-regular, 1-regular, 0-regular) graphs on 8 vertices, we find that the remaining Hadamard diagonalizable graphs on 8 vertices are $((K_{2,2})\Box K_2)^c = K_4 \Box K_2, (K_4 + K_4)^c, ((K_{2,2}) + (K_{2,2}))^c, (K_2 + K_2 + K_2 + K_2)^c$ and K_8.

Hence we have identified all 10 graphs of order 8 that are Hadamard diagonalizable.

Before we determine all graphs of order 12 that are Hadamard diagonalizable, the following will be useful.

Observation 3 *Here we list all of the regular graphs on six vertices and their eigenvalues. Each is listed according to its degree of regularity, r.*

(i) $r = 0$: $G = K_6^c$; $\{0, 0, 0, 0, 0, 0, \}$,

(ii) $r = 1$: $K_2 + K_2 + K_2$; $\{0, 0, 0, 2, 2, 2\}$,

(iii) $r = 2$: $K_3 + K_3$; $\{0, 0, 3, 3, 3, 3\}$ *or* C_6; $\{0, 1, 1, 3, 3, 4\}$,

(iv) $r = 3$: $K_{3,3}$; $\{0, 3, 3, 3, 3, 6\}$ *or* C_6^c; $\{0, 2, 3, 3, 5, 5\}$,

(v) $r = 4$: $(K_2 + K_2 + K_2)^c$; $\{0, 4, 4, 4, 6, 6\}$,

(vi) $r = 5$: $G = K_6$; $\{0, 6, 6, 6, 6, 6\}$.

We begin by considering the disconnected case first.

Lemma 9 *The only disconnected graphs of order* 12 *that are Hadamard diagonalizable are* K_{12}^c *and* $K_6 + K_6$.

Proof. If G is a disconnected graph on 12 vertices that is Hadamard diagonalizable, then, by Lemma 6, we may write $G = G_1 + G_2$, where G_1 and G_2 are both regular graphs on 6 vertices with the same degree of regularity and with common even integer eigenvalues. Working through the above list it is not difficult to deduce that the only cases of interest are: G_1 and G_2 are either both empty or both complete; or G_1 and G_2 (or their complements) are both $K_2 + K_2 + K_2$ (that is, cases (i), (ii), (v), and (vi)). It is not difficult to conclude that in the latter cases, it is impossible for the null space of such a Laplacian matrix to be made up of (1,-1) orthogonal vectors. The former case coincides with our proposed conclusion. This completes the proof. □

We are now in a position to complete the case of graphs on 12 vertices that are Hadamard diagonalizable. It is worth noting that the only connected graphs on 12 vertices that are Hadamard diagonalizable are the complements of the graphs above.

Proposition 10 *The only connected graphs of order* 12 *that are Hadamard diagonalizable are* K_{12} *and* $K_{6,6}$.

Proof. Suppose G is a connected graph of order 12 that is Hadamard diagonalizable. Then G is a regular graph and has all even integer eigenvalues. Suppose 12 is an eigenvalue of G. Then G^c is disconnected, and so, by Lemma 9 we conclude that G must be one of K_{12} or $K_{6,6}$.

Now, assume G is such a graph and 12 is not an eigenvalue of G, and assume that H is a 12×12 Hadamard matrix the diagonalizes L. The remainder of the argument will depend on the smallest positive eigenvalue of G.

Case 1: Suppose the smallest positive eigenvalue of G is 2. Then we may assume, without loss of generality, that an eigenvector for 2 is of the form $\begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$ and that the Laplacian for G is of the form:

$$L = \begin{bmatrix} L_1 + I & -I \\ -I & L_2 + I \end{bmatrix},$$

where L_1 and L_2 are the Laplacians for two graphs G_1 and G_2 where both are of order 6 and both are regular of the same degree of regularity. Suppose that G_1 has an eigenvector v orthogonal to $\mathbb{1}$ corresponding to an eigenvalue $\lambda \leq 1$. Letting $u = \begin{bmatrix} v \\ 0 \end{bmatrix}$, we find that u is orthogonal to the all ones vector of order 12, and that $0 < u^T L u = (\lambda+1) u^T u \leq 2u^T u$. If $\lambda < 1$, then the smallest positive eigenvalue of L is less than 2, contrary to our hypothesis. If $\lambda = 1$, then it follows that u must be an eigenvector of L corresponding to the eigenvalue 2, which, by inspecting the structure of L and u, is impossible. A similar argument applies to G_2, and so we deduce that for both G_1 and G_2, zero is a simple eigenvalue, and all remaining eigenvalues exceed 1.

Subcase 1.1 Both G_1 and G_2 are K_6. Then $L(G) = \begin{bmatrix} 7I - J & -I \\ -I & 7I - J \end{bmatrix}$, where the diagonal blocks are both 6×6. We may write H as $\begin{bmatrix} \mathbb{1} & \mathbb{1} & H_1 \\ \mathbb{1} & -\mathbb{1} & H_2 \end{bmatrix}$, where necessarily the columns of the 6×10 matrices H_1 and H_2 are all orthogonal to $\mathbb{1}$. Then each column of H is an eigenvector of $\begin{bmatrix} 6I - J & 0 \\ 0 & 6I - J \end{bmatrix}$, so that H diagonalizes $L(K_6 + K_6)$. It now follows that H must also diagonalize $\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$, where again the diagonal blocks are 6×6. Observe that any eigenvector of $\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$ is either of the form $\begin{bmatrix} w \\ w \end{bmatrix}$ for some vector w, or of the form $\begin{bmatrix} w \\ -w \end{bmatrix}$ for some vector w. It now follows that we can permute the columns of H so that it has the form $\begin{bmatrix} \mathbb{1} & \mathbb{1} & W_1 & W_2 \\ \mathbb{1} & -\mathbb{1} & W_1 & -W_2 \end{bmatrix}$, where W_1, W_2 are both 6×5. But in this case, it follows that $\begin{bmatrix} \mathbb{1} & W_1 \end{bmatrix}$ is a 6×6 Hadamard matrix, a contradiction.

Subcase 1.2 Suppose that 2 is not a simple eigenvalue of G. Then it follows that there are (1,-1) vectors x, y such that $x^T \mathbb{1} = y^T \mathbb{1} = 0$ and $w = \begin{bmatrix} x \\ y \end{bmatrix}$ is a column of H. Then $24 = w^T L w = x^T L_1 x + y^T L_2 y + x^T x + y^T y - 2x^T y$. Since the smallest positive eigenvalue of both L_1 and L_2 must be strictly greater than 1, we see from Observation 3 that in fact the smallest positive eigenvalues of L_1 and L_2 are at least 2. We conclude that $x^T y \geq 6 = \sqrt{(x^T x)(y^T y)}$, and so applying the Cauchy-Schwarz inequality, $x^T y = 6 = \sqrt{(x^T x)(y^T y)}$; recalling the characterization of the equality case in the Cauchy-Schwarz inequality, we find that necessarily $x = y$ and further (again referring to Observation 3) we must have $G_1 = G_2 = C_6^c$.

Observe that the largest eigenvalue of G is bounded above by the largest eigenvalue of C_6^c plus the largest eigenvalue of $\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$. Referring to Observation 3, it follows that the largest eigenvalue of G is at most 7, and since G must have even integer eigenvalues, we conclude that the largest eigenvalue of G is 6. Given that both $G_1 = G_2 = C_6^c$, we find that the trace of L is 48 and the trace of L^2 is 240. Letting the multiplicities of 2 and 4 as eigenvalues of L be k_1, k_2, respectively, we have the linear system $48 = 2k_1 + 4k_2 + 6(11 - k_1 - k_2), 240 = 4k_1 + 16k_2 + 36(11 - k_1 - k_2)$; solving that system yields $k_1 = k_2 = 3$, and we see that the only allowed list of eigenvalues for G is $\{0, 2, 2, 2, 4, 4, 4, 6, 6, 6, 6, 6\}$. From the development above we observe that the only way for a column of H of the form $w = \begin{bmatrix} x \\ y \end{bmatrix}$ with $x^T \mathbb{1} = y^T \mathbb{1} = 0$ to be an eigenvector for the eigenvalue 2 is if $x = y$ (from our Cauchy-Schwarz argument above) and in addition x is an eigenvector of C_6^c for the eigenvalue 2. Since 2 is a simple eigenvalue of C_6^c and occurs as an eigenvalue of L with multiplicity three, this is clearly impossible.

Subcase 1.3 Suppose that 2 is a simple eigenvalue of G. We claim that in this case, any eigenvector v of G_1 that is orthogonal to $\mathbb{1}$ must correspond to an eigenvalue λ of G_1 with $\lambda > 1$. To see the claim, observe that if $\lambda \leq 1$, then the vector $u = \begin{bmatrix} v \\ 0 \end{bmatrix}$ satisfies $u^T L(G) u \leq 2u^T u$ and $u^T \mathbb{1} = 0$, so that necessarily u is an eigenvector of $L(G)$ corresponding to the eigenvalue 2, contrary to the hypothesis that 2 is a simple eigenvalue of $L(G)$.

A similar argument holds for G_2, and so referring to Observation 3, we find that G_1 and G_2 must be among the list $\{K_{3,3}, C_6^c, (K_2+K_2+K_2)^c\}$. In addition, we can exclude the case that both $G_1 = G_2 = C_6^c$, since if this were the case, the argument in Subcase 1.2 proves that two would be a multiple eigenvalue of G. Suppose first that $G_1, G_2 \in \{K_{3,3}, C_6^c\}$ but not both are C_6^c. In this case the largest eigenvalue of G is 8. Denote the multiplicities of the eigenvalues 4 and 6 of L by k_1 and k_2, respectively. Using the fact that the traces of L and L^2 are 48 and 240, respectively, we arrive at the linear system $48 = 2 + 4k_1 + 6k_2 + 8(10 - k_1 - k_2); 240 = 4 + 16k_1 + 36k_2 + 64(10 - k_1 - k_2)$. Solving the system yields $k_1 = 9, k_2 = -1$, certainly a contradiction. On the other hand, assume that both G_1 and G_2 are $(K_2+K_2+K_2)^c$. Again, working with the traces of L and L^2 (which are 60 and 360, respectively) it follows that the only allowed spectrum of G is $\{0, 2, 4, 4, 4, 6, 6, 6, 6, 6, 8, 8\}$. Using the fact that there is no (1,-1) eigenvector of G_1 for the eigenvalue 6, it follows that there is not a (1,-1) eigenvector of G for the eigenvalue 8, a contradiction.

So we conclude that 2 is not a eigenvalue for any such G. Hence, by considering complements, we may also rule out the possibility of 10 being an eigenvalue for any such graph G.

Case 2: Suppose that the smallest eigenvalue of G is 4. Then we may assume, without loss of generality, that an eigenvector for 4 is of the form $\begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$ and that the Laplacian for G is of the form:

$$L = \begin{bmatrix} L_1 + 2I & -I - P^T \\ -I - P & L_2 + 2I \end{bmatrix},$$

where L_1 and L_2 are the Laplacians for two graphs G_1 and G_2, where both are of order 6 and both are regular of the same degree of regularity, and P is a permutation matrix with zero trace. Furthermore, it is not difficult to verify that the smallest positive eigenvalues for G_1 and G_2 must both be at least two. Hence $G_1, G_2 \in \{K_{3,3}, C_6^c, (K_2+K_2+K_2)^c, K_6\}$. Thus the largest eigenvalue of G is at least 8, and if it exceeds 8, then this eigenvalue must be at least 10, but we have ruled out all such graphs G in the above cases. So the largest eigenvalue of G must be exactly 8.

Let x be an eigenvector for $\lambda = 6$ of L_1. Then the largest eigenvalue of L will be 8 if $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is an eigenvector for L, that is, if $Px = -x$ for such an eigenvector x. If G_1 is K_6, then there are five linearly independent eigenvectors for $\lambda = 6$. Furthermore, it is not possible that $Px = -x$, for each of these five eigenvectors. This rules out the case of both G_1 and G_2 being K_6.

Suppose both G_1 and G_2 are $(K_2+K_2+K_2)^c$. Then we may assume that the eigenvectors for L_1 corresponding to 6 are of the form: $x_1 = \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \mathbb{1} \\ 0 \\ -\mathbb{1} \end{bmatrix}$. It is not difficult to verify that it is impossible for $Px_1 = -x_1$ and for $Px_2 = -x_2$ simultaneously.

For the remaining cases, G_1 and G_2 must be one of $\{K_{3,3}, C_6^c\}$. Then G is regular of degree 5, and using the fact that the largest eigenvalue of L is 8, it follows that the only allowed spectrum for G is $\{0, 4, 4, 4, 4, 4, 4, 6, 6, 8, 8, 8\}$. We finish the argument by considering three separate cases.

Suppose that both G_1 and G_2 are C_6^c. Then we may assume that the (unique) (1,-1) eigenvector of L_1 corresponding to the eigenvalue 2 is of the form $x = \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$. Then the only way 4 can be the smallest eigenvalue of L is if $Px = -x$, and hence P must be of the form:

$$P = \begin{bmatrix} 0 & Q \\ R & 0 \end{bmatrix},$$

for some permutation matrices Q, R. Now we may draw a similar conclusion for L_2 and the (unique) (1,-1) eigenvector corresponding to the eigenvalue 2. From this, we can completely determine the form of G, and observe that the eigenvectors of L corresponding to the eigenvalues $0, 4, 4, 4$ are given by:

$$\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ -\mathbb{1} \\ -\mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \\ \mathbb{1} \\ -\mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \\ -\mathbb{1} \\ \mathbb{1} \end{bmatrix}.$$

Consider a (1,-1) eigenvector of L corresponding to eigenvalue 6 (or 8), partitioned as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Setting $s_i = \mathbb{1}^T x_i$, we find from the orthogonality condition for eigenvectors associated with distinct eigenvalues, that $s_i = 0$ for all i, but this is a contradiction, as each x_i has order 3.

A similar argument applies if both G_1 and G_2 are $K_{3,3}$. So, finally suppose that G_1 is C_6^c and that G_2 is $K_{3,3}$. Following similar reasoning as above, we deduce the existence of the following four eigenvectors of L corresponding to the eigenvalues $0, 4, 4, 8$:

$$\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \\ \mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ \mathbb{1} \\ -\mathbb{1} \\ -\mathbb{1} \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \mathbb{1} \\ -\mathbb{1} \end{bmatrix}.$$

Now considering a (1,-1) eigenvector of L for $\lambda = 6$ (which is distinct from $0, 4$ and 8, and so orthogonal to eigenvectors associated with those eigenvalues), we arrive at a contradiction.

So we conclude that 4 is not a eigenvalue for any such G. Hence, by considering complements, we may also rule out the possibility of 8 being an eigenvalue for any such graph G.

Case 3: The only nonzero eigenvalue of G is 6. However, no such regular connected graph on 12 vertices has this property.

This completes the proof. □

4 Eigenspaces for regular cographs

The class of *complement reducible graphs* (or *cographs* for short) consists of those graphs that can be constructed from isolated vertices by a sequence of operations of unions and complements. Equivalently, a graph is cograph if and only if it has no induced P_4 subgraphs. If G is a connected cograph, it is known (and not difficult to show, by induction on the number of vertices) that G can be written as $G_1 \vee \ldots \vee G_k$, where $G_1, \ldots, G_k$ are disconnected cographs of lower order. The paper [3] surveys a number of results on cographs. It is known that any cograph is Laplacian integral (see [5, 12]); in light of our discussion in this paper, it is natural to wonder which cographs are Hadamard diagonalizable. In this section, we address that question.

Theorem 11 *Let G be a cograph. Then there exists a basis B of eigenvectors of $L(G)$ such that each vector of B has at most two distinct nonzero entries.*

Proof. The proof follows directly by using Theorem 2, and applying induction. □

The converse of Theorem 11 fails in general, as there are graphs G such that $L(G)$ has a basis of eigenvectors such that each vector in the basis has at most two distinct nonzero entries, but G is not a cograph. The following result helps to establish that statement.

Lemma 12 *If G is a connected graph on $n \geq 3$ vertices, then $G \square K_2$ is not a cograph.*

By taking any connected Hadamard diagonalizable graph G on at least 4 vertices, we can produce a graph $G \square K_2$, which is also Hadamard diagonalizable and hence it has a basis of eigenvectors such that each vector in the basis has at most two distinct nonzero entries. However, $G \square K_2$ is not a cograph.

A natural question that arises here is to characterize all the Hadamard diagonalizable cographs. C_4 and K_4 are the only two Hadamard diagonalizable cographs of order 4.

As it has been discussed earlier a Hadamard diagonalizable graph is regular and all its Laplacian eigenvalues are even integers. So we have to consider the obvious necessary conditions in our search of regular Hadamard diagonalizable cographs.

The following lemma is useful in reaching our goal.

Lemma 13 *Let $G = G_1 \vee G_2$ be a regular cograph on n vertices that is Hadamard diagonalizable, where both G_1 and G_2 are disconnected. Then $|G_1| = |G_2| = \frac{n}{2}$ and both G_1 and G_2 are regular graphs with same degree of regularity.*

Proof. Consider G^c, which is also Hadamard diagonalizable. Since $G^c = G_1^C + G_2^c$, and since each of G_1^c and G_2^c is connected, we find from Lemma 6 that G_1^c and G_2^c have the same order and the same degree of regularity. The conclusion now follows. □

We say that a graph G has *property E* if, for each eigenvalue λ of $L(G)$, there is a corresponding eigenvector of $L(G)$ with every entry equal to either $+1$ or -1. Observe that any Hadamard diagonalizable graph has property E.

Proposition 14 *Let G be a regular connected cograph that has property E. Write G as $G = G_1 \vee G_2 \vee \ldots \vee G_k$, where, for each $i = 1, 2, \ldots, k$, G_i is a disconnected graph with n_i vertices. Then $n_1 = n_2 = \ldots = n_k$, each G_i is regular cograph, $G_1, \ldots, G_k$ all have the same degree of regularity. Further, the graphs $G_1, \ldots, G_k$ all share the same eigenvalues.*

Proof. Suppose without loss of generality that $n_1 \geq n_2 \geq \ldots \geq n_k$. Let $n = n_1 + \ldots + n_k$, and write $L(G)$ as

$$L(G) = \begin{bmatrix} L(G_1) + (n - n_1)I & -J & \ldots & -J \\ -J & L(G_2) + (n - n_2)I & \ldots & -J \\ \vdots & & \ddots & \vdots \\ -J & \ldots & -J & L(G_k) + (n - n_k)I \end{bmatrix}.$$

Since G_1 is disconnected, $L(G_1)$ has a null vector that is orthogonal to $\mathbb{1}$, and it follows that $n - n_1$ is an eigenvalue of $L(G)$. Let u be an eigenvector of $L(G)$ corresponding to $n - n_1$; appealing to property E, we may assume that u has entries either 1 and -1.

Partition u conformally with $L(G)$ as $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}$. From the eigen-equation and the fact that $\mathbb{1}^T u = 0$, we find that for each $i = 1, \ldots, k$, $(n - n_1)u_i = L(G_i)u_i + (n - n_i)u_i - \sum_{j \neq i} \mathbb{1}^T u_j = L(G_i)u_i + (n - n_i)u_i + \mathbb{1}^T u_i$. Consequently, we have $(n - n_1)\mathbb{1}^T u_i =$

$\mathbb{1}^T L(G_i)u_i + (n-n_i)\mathbb{1}^T u_i + \mathbb{1}^T u_i = (n-n_i+1)\mathbb{1}^T u_i$. Since $n_1 \geq n_i >> n_i - 1$ for each i, we conclude that $\mathbb{1}^T u_i = 0, i = 1, \ldots, k$. Again referring to the eigen-equation, we find that for each $i = 1, \ldots, k$, $L(G_i)u_i = -(n_1 - n_i)u_i$. As each $L(G_i)$ is positive semidefinite, it must be the case that $n_i = n_1, i = 1, \ldots, k$. Now, from the fact that G is regular, it follows that each G_i is regular, and that the graph $G_1, \ldots, G_k$ all have the same degree of regularity.

Next, we consider $L(G^c)$, with is a direct sum of the matrices $L(G_i^c) + n_1 I - J, i = 1, \ldots, k$. Note that since G has property E, so does G^c. Let λ be a nonzero eigenvalue of G^c, and let v be a $(1, -1)$ eigenvector of $L(G^c)$. Partition v conformally with $L(G^c)$ as $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}$. Then for each $i = 1, \ldots, k$, we have $L(G_i^c)v_i + n_1 v_i - Jv_i = \lambda v_i$. Consequently, $\lambda \mathbb{1}^T v_i = \mathbb{1}^T L(G_i^c)v_i + n_1 \mathbb{1}^T v_i - \mathbb{1}^T J v_i = 0$, so that $\mathbb{1}^T v_i = 0$. But then we have $L(G_i^c)v_i + n_1 v_i = \lambda v_i, i = 1, \ldots, k$. As the nonzero eigenvalues of $L(G^c)$ consist of the union of the nonzero eigenvalues of the $L(G_i^c) + n_1 I, i = 1, \ldots, k$, we deduce that $L(G_1^c), \ldots, L(G_k^c)$ all share the same eigenvalues. As each G_i is on n_1 vertices, it now follows that the graphs $G_1, \ldots, G_k$ all share the same eigenvalues. □

Lemma 15 *Let Γ_1, Γ_2 be two connected regular cographs with property E on $n \geq 2$ vertices. If $S(\Gamma_1) = S(\Gamma_2)$, then $\Gamma_1 = \Gamma_2$.*

Proof. We proceed by induction on n, and note that the result is readily established for $n = 2$.

Let

$$\Gamma_1 = G_1 \vee G_2 \vee \ldots \vee G_m \text{ and } \Gamma_2 = H_1 \vee H_2 \vee \ldots \vee H_k,$$

where both $G_i,\ i = 1, 2, \ldots, m$ and $H_j,\ j = 1, 2, \ldots, k$ are disconnected graphs. Since both Γ_1 and Γ_2 have property E, we find from Proposition 14 that

$$|G_1| = |G_2| = \ldots = |G_m| \text{ and } |H_1| = |H_2| = \ldots = |H_k|.$$

Observe that for $i = 1, 2, \ldots, m$, G_i is a regular cograph that satisfies property E, and that again appealing to Proposition 14, we have $S(G_1) = S(G_2) = \ldots = S(G_m)$. Thus, by using induction we have $G_1 = G_2 = \ldots = G_m$. Similarly, we find that $H_1 = H_2 = \ldots = H_k$.

Further, notice that the smallest nonzero eigenvalue of Γ_1 is $(m-1)|G_1|$ and the smallest nonzero eigenvalue of Γ_2 is $(k-1)|H_1|$. Since $S(\Gamma_1) = S(\Gamma_2)$, we have $(m-1)|G_1| = (k-1)|H_1|$. Hence $|G_1| = |H_1|$ as $m|G_1| = k|H_1|$. Thus we have $m = k$.

Consequently, we find that since Γ_1 and Γ_2 share the same eigenvalues, so do G_1 and H_1. As G_1^c and H_1^c are connected regular cographs of the same order, they also share the same eigenvalues, and again by the induction hypothesis, we find that $G_1^c = H_1^c$, so that $G_1 = H_1$. Since we have already shown that $m = k$, $G_1 = G_2 = \ldots = G_m$, and $H_1 = H_2 = \ldots = H_k$, we thus find that $\Gamma_1 = \Gamma_2$. ∎

We are now in a position to characterize the regular cographs with property E. We will show that a subset of these cographs will also be Hadamard diagonalizable.

Theorem 16 *Let $S_0 = \{K_m : m \geq 2\}$, m is even. For $i \in \mathbb{N}$, let $S_i = \{G^c \vee \ldots \vee G^c : G \in S_{i-1}$ and the number of joined copies of G^c is even$\}$. Then, Γ is a connected regular cograph with property E on $n \geq 2$ vertices if and only if $\Gamma \in S_i$ for some $i = 0, 1, 2, \ldots$.*

Proof. First we show by induction on i that if $\Gamma \in S_i$ for some $i \geq 0$, then Γ is a regular cograph with property E. Note first that if $\Gamma \in S_0$, then it satisfies property E, since then $\Gamma = K_m$ for some even m. Suppose now that $\Gamma \in S_i$ for some $i = 1, 2, \ldots$. Thus, $\Gamma = G^c \vee \ldots \vee G^c$ for some $G \in S_{i-1}$. Suppose that $|\Gamma| = n$ and $|G| = m$. Then, by the induction hypothesis, G is a connected regular cograph and satisfies property E. Note that the eigenvalues of Γ are $0, n$, and $n - \lambda$ for each nonzero eigenvalue of G. For each eigenvalue $\lambda \neq 0$ of G, there is a corresponding $(1, -1)$ eigenvector v; it follows that the vector $\begin{bmatrix} v \\ v \\ \vdots \\ v \end{bmatrix}$ serves as a $(1, -1)$ eigenvector of Γ for the eigenvalue $n - \lambda$. Further, since Γ is comprised of an even number of joined copies of G^c, it follows that $\begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \end{bmatrix}$ and

$\begin{bmatrix} \mathbb{1} \\ \mathbb{1} \end{bmatrix}$ serve as eigenvectors for Γ for eigenvalues n and 0, respectively. It now follows that Γ is a connected regular cograph with property E.

Conversely, suppose that Γ is a regular connected cograph on $n \geq 2$ vertices with property E. We show by induction on n that $\Gamma \in S_i$ for some $i \geq 0$. The case $n = 2$ is readily established. Suppose now that $n \geq 3$. Since Γ is a connected cograph, we have $\Gamma = G_1 \vee \ldots \vee G_k$, where G_j are disconnected graphs. By Proposition 14, we find that the graphs $G_1, \ldots, G_k$ are all of the same order, and share the same eigenvalues.

Thus, each G_i^c is a connected regular cograph and $S(G_i^c) = S(G_j^c)$ for all i and j. And hence using Lemma 15, $G_i^c = G_j^c$. This implies that $G_i = G_j$. Thus $\Gamma = G_1 \vee \ldots \vee G_1$. But G_1 is a regular cograph with property E, so by the induction hypothesis, $G_1^c \in S_i$ for some $i \geq 0$. Lastly, we consider the number of joined copies of G_1^c, say m, from which Γ is comprised. Observe that $n = |\Gamma|$ is an eigenvalue of Γ, and that the corresponding eigenspace is spanned by the vectors $\begin{bmatrix} \mathbb{1} \\ -\mathbb{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbb{1} \\ 0 \\ -\mathbb{1} \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbb{1} \\ 0 \\ \vdots \\ 0 \\ -\mathbb{1} \end{bmatrix}$. In order that this eigenspace contains a vector with entries 1 or -1, it must be the case that m is even. Hence $\Gamma \in S_{i+1}$. □

Theorem 17 *Let G be a connected regular cograph. Then there is a basis of* $(1, -1)$ *eigenvectors for G if and only if $G \in S_i$ for some $i = 0, 1, 2, \ldots$.*

Proof. Suppose first that there is a basis of $(1, -1)$ eigenvectors for G. Then in particular, G has property E, and so by Theorem 16, $G \in S_i$ for some $i \geq 0$.

To establish that each graph in each S_i has a $(1, -1)$ eigenbasis, we proceed by induction on i. If $G \in S_0$, then $G = K_m$ for some even m. We claim, by induction on m, that there is a $(1, -1)$ eigenbasis for K_m. This is obvious for $m = 2$, so suppose that $m \geq 4$ is even. It is enough to show that there is a basis for $\mathbb{1}_\perp$ (the orthogonal complement of $\{\mathbb{1}\}$) in $\mathbb{R}^m$ consisting of $(1, -1)$ vectors. From the induction hypothesis, there is a basis for $\mathbb{1}_\perp$ in $\mathbb{R}^{m-2}$ consisting of $(1,-1)$ vectors, say $u_1, \ldots, u_{m-3}$. It is then straightforward to determine that the vectors

$$\begin{bmatrix} u_i \\ 1 \\ -1 \end{bmatrix}, i = 1, \ldots, m-3, \begin{bmatrix} u_1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} u_2 \\ -1 \\ 1 \end{bmatrix}$$

form the desired basis for $\mathbb{1}_\perp$ in $\mathbb{R}^m$. It now follows that the graphs in S_0 have the necessary $(1,-1)$ eigenbases.

Suppose now that $i \geq 1$ and that $\Gamma \in S_i$. Then there is a graph $H \in S_i$ and an even m such that Γ^c is the m-fold union of H with itself. Suppose that $|\Gamma| = n, |H| = p$. Let $\lambda \neq 0$ be an eigenvalue of H with multiplicity k. From the induction hypothesis, there are $(1,-1)$ eigenvectors $w_1, \ldots, w_k$ that span the λ eigenspace for H. For each $j = 1, \ldots, k$, consider the collection C_j of m vectors

$$\begin{bmatrix} w_j \\ w_j \\ w_j \\ \vdots \\ w_j \end{bmatrix}, \begin{bmatrix} w_j \\ -w_j \\ w_j \\ \vdots \\ w_j \end{bmatrix}, \begin{bmatrix} w_j \\ w_j \\ -w_j \\ \vdots \\ w_j \end{bmatrix}, \ldots, \begin{bmatrix} w_j \\ w_j \\ w_j \\ \vdots \\ -w_j \end{bmatrix}.$$

It is straightforward to determine that the set of vectors in $C_1 \cup \ldots \cup C_k$ is linearly independent, and so forms a $(1,-1)$ eigenbasis for the λ-eigenspace of Γ^c. Hence these vectors also form a $(1,-1)$ eigenbasis for the $(n-\lambda)$-eigenspace of Γ. Finally, letting $u_1, \ldots, u_{m-1}$ be a $(1,-1)$ eigenbasis for the eigenspace of K_m corresponding to the eigenvalue m, we see that $u_i \otimes \mathbb{1}_p, i = 1, \ldots, m-1$, is a $(1,-1)$ eigenbasis for the n-eigenspace of Γ. The conclusion now follows. □

While Theorem 17 has obvious connections to our work on Hadamard diagonalizable graphs, it also has ramifications on the existing work on the eigenspace structures of cographs with respect to the adjacency matrix (see [15]).

Observe that if $\Gamma \in S_i$ for some $i \geq 0$, then there is a unique $(i+1)$-tuple of even integers that can be associated with Γ, in the following manner. If $\Gamma \in S_0$, then $G = K_{m_0}$ for some even m_0, and we write $\Gamma \equiv G(m_0)$. If $\Gamma \in S_i$ for some $i \geq 1$, then for some even integer m_i, Γ can be written as the m_i-fold join of the complement of a graph $G(m_0, m_1, \ldots, m_{i-1})$ with itself. In that case, we write $\Gamma \equiv G(m_0, m_1, \ldots, m_i)$.

Next, we claim that for each $i \geq 0$, and each collection of even integers $m_0, \ldots, m_i$, the graph $G(m_0, m_1, \ldots, m_i)$ has exactly $i+2$ distinct Laplacian eigenvalues. We establish the claim by induction on i, and note that for $i = 0, G(m_0) = K_{m_0}$, which has two distinct Laplacian eigenvalues, namely 0 and m_0. Suppose now that the statement holds for some $i_0 \geq 0$, and that we have a collection of even integers $m_0, \ldots, m_{i_0}, m_{i_0+1}$. Observe that $(G(m_0, m_1, \ldots, m_{i_0+1}))^c$ is a union of m_{i_0+1} copies of $G(m_0, m_1, \ldots, m_{i_0})$. From the induction hypothesis, we find that $(G(m_0, m_1, \ldots, m_{i_0+1}))^c$ has $i_0 + 2$ distinct eigenvalues; note also that the multiplicity of the eigenvalue 0 is m_{i_0+1}. Further, since the order of $G(m_0, m_1, \ldots, m_{i_0})$ is $m_0 m_1 \ldots m_{i_0}$, we find that the eigenvalues of $(G(m_0, m_1, \ldots, m_{i_0+1}))^c$ are all bounded above by $m_0 m_1 \ldots m_{i_0}$. Referring to the relationship between the spectrum of a graph and its complement described in Section 2, it now follows that $G(m_0, m_1, \ldots, m_{i_0+1})$ has $i_0 + 3$ distinct eigenvalues, completing the induction step, and the proof of the claim.

Our next result provides more detail on the nature of $(1, -1)$ eigenvectors for $G(m_0, m_1, \ldots, m_i)$.

Theorem 18 *Suppose that we have even integers $m_0, \ldots, m_i$. Label the distinct eigenvalues of $G(m_0, \ldots, m_i)$ as $0 = \mu_1 < \mu_2 < \ldots < \mu_{i+2}$. We have the following conclusions.*
a) The dimension of the eigenspace corresponding to $\mu_{\lfloor \frac{i}{2} \rfloor + 2}$ is $m_i m_{i-1} \ldots m_1 (m_0 - 1)$.
b) For each $l = 1, \ldots, \lfloor \frac{i}{2} \rfloor + 1$, the dimension of the eigenspace corresponding to μ_l is $m_i m_{i-1} \ldots m_{i+4-2l}(m_{i+3-2l} - 1)$ (here we interpret this quantity as 1 when $l = 1$). Further, every $(1, -1)$ eigenvector corresponding to μ_l has the form $w \otimes \mathbb{1}_{m_0 m_1 \ldots m_{i+2-2l}}$ for some $(1, -1)$ vector $w \in I\!R^{m_{i+3-2l} m_{i+4-2l} \ldots m_i}$.
c) For each $l = 1, \ldots i - \lfloor \frac{i}{2} \rfloor$, the dimension of the eigenspace corresponding to μ_{i+3-l} is $m_i m_{i-1} \ldots m_{i+3-2l}(m_{i+2-2l} - 1)$ (here we interpret this quantity as $m_i - 1$ when $l = 1$). Further, every $(1, -1)$ eigenvector corresponding to μ_{i+3-l} has the form $w \otimes \mathbb{1}_{m_0 m_1 \ldots m_{i+1-2l}}$ for some $(1, -1)$ vector $w \in I\!R^{m_{i+2-2l} m_{i+3-2l} \ldots m_i}$.

Proof. We prove all three assertions by induction on i, and note that the case $i = 0$ is straightforward. Suppose now that $i \geq 1$. Note that $G(m_0, \ldots, m_i)$ can be written as the m_i-fold join of the graph $G(m_0, \ldots, m_{i-1})^c$ with itself. Denoting the eigenvalues of $G(m_0, \ldots, m_{i-1})$ by $0 = \hat{\mu_1} < \hat{\mu_2} < \ldots < \hat{\mu_{i+1}}$; referring to the relationship between the spectrum of a graph and its complement described in Section 2, we find that $\mu_1 = 0$, and that $\mu_j = m_0 m_1 \ldots m_i - \hat{\mu_{i+3-j}}, j = 1, \ldots, i+1$. Further, for each $j = 1, \ldots, i$, the dimension of the eigenspace of $G(m_0, \ldots, m_i)$ corresponding to $\mu_j = m_0 m_1 \ldots m_i - \hat{\mu_{i+3-j}}$, is equal to the dimension of the eigenspace of $G(m_0, \ldots, m_{i-1})$ corresponding to $\hat{\mu_j}$, multiplied by m_i. Also, the dimension of the μ_{i+2} eigenspace of $G(m_0, \ldots, m_i)$ is $m_i - 1$, while the dimension of its null space is 1. Finally, we note that any $(1, -1)$ eigenvector of $G(m_0, \ldots, m_i)$ also serves as a $(1, -1)$ eigenvector of $G(m_0, \ldots, m_i)^c$, and hence of the m_i fold union of $G(m_0, \ldots, m_{i-1})$ with itself. The statements regarding the structure of $(1, -1)$ eigenvectors of $G(m_0, \ldots, m_i)$ now follow from the induction hypothesis, and corresponding statements regarding $(1, -1)$ eigenvectors of $G(m_0, \ldots, m_{i-1})$. □

Theorem 19 *Suppose that we have even integers $m_0, m_1, \ldots, m_i$ such that $G(m_0, m_1, \ldots, m_i)$ is a Hadamard diagonalizable graph. Then for each $k = 0, \ldots, i$, there exists a Hadamard matrix of order $\Pi_{j=k}^{i} m_j$.*

Proof. Here we keep the notation of Theorem 18. Suppose that H is a Hadamard matrix that diagonalizes $G(m_0, m_1, \ldots, m_i)$. Then in particular, the columns of H are a collection of orthogonal $(1, -1)$ eigenvectors for the Laplacian matrix of $G(m_0, m_1, \ldots, m_i)$.

Consider the set of m_i columns of H that correspond to the eigenvalues μ_1 and μ_{i+2}. From Theorem 18, it follows that these columns of H can be written as $w_j \otimes \mathbb{1}_{m_0 m_1 \ldots m_{i-1}}, j = 1, \ldots, m_i$, for some collection of vectors $w_j \in \mathbb{R}^{m_i}, j = 1, \ldots, m_i$. Since the columns of H are orthogonal $(1, -1)$ vectors, so are the vectors $w_1, \ldots, w_{m_i}$ - i.e. those vectors are the columns of a Hadamard matrix of order m_i. Hence there must exist a Hadamard matrix of that order.

Next, by considering the structure of the $(1, -1)$ vectors in the eigenspace corresponding to $\mu_l, l = 1, \ldots, \lfloor \frac{i}{2} \rfloor + 1$, and to $\mu_{i+3-l}, l = 1, \ldots i - \lfloor \frac{i}{2} \rfloor$, and applying a similar argument, we find that there also must also exist Hadamard matrices of orders $\Pi_{j=k}^{i} m_j$ for each $k = 0, \ldots, i$. □

In closing, we suspect that the converse to Theorem 19 is true. That is, under the hypothesis that for given even integers $m_0, m_1, \ldots, m_i$, if there exists a Hadamard matrix of order $\Pi_{j=k}^{i} m_j$, for each $k = 0, 1, 2, \ldots, i$, then $G(m_0, m_1, \ldots, m_i)$ is a Hadamard diagonalizable graph. If that were the case, then all regular cographs that are Hadamard diagonalizable would be completely described.

References

[1] S. Akbari, A. Alipour, E. Ghorbani, and G. Khosrovshahi, $\{-1, 0, 1\}$-Basis for the null space of a forest, *Linear Algebra and its Applications*, 414:506-511, (2006).

[2] T. Biyikoglu and J. Leydold, *Laplacian eigenvectors of graphs. Perron-Frobenius and Faber-Krahn type theorems,* Lecture Notes in Mathematics, 1915, Springer, Berlin, (2007).

[3] D. Corneil, H. Lerchs, and L. Burlingham, Complement reducible graphs, *Discrete Appl. Math.*, 3:163-174, (1981).

[4] D. M. Cvetkovic, M. Doob, and H. Sachs, *Spectra of Graphs,* Academic Press, (1979).

[5] S. Fallat, S. Kirkland, J. Molitierno, and M. Neumann, On graphs whose Laplacian matrices have distinct integer eigenvalues, *Journal of Graph Theory*, 50:162-174, (2005).

[6] S. Fallat, S. Kirkland, and S. Pati, On graphs with algebraic connectivity equal to minimum edge density, *Linear Algebra and its Applications*, 373:31-50, (2003).

[7] M. Fiedler, Algebraic connectivity of graphs, *Czechoslovak Math. J.*, 23(98):298-305, (1973).

[8] R. Grone, R. Merris and V.S. Sunder, The Laplacian spectrum of a graph, *SIAM J. on Matrix Analysis and Applications*, 11:218-238, (1990).

[9] W. Imrich and S. Klavžar, *Product graphs*, Wiley-Interscience, New York, (2000).

[10] R. Merris, Laplacian matrices of graphs: a survey, *Linear Algebra and its Applications*, 197/198: 143–176, (1994).

[11] R. Merris, A survey of graph Laplacians, *Linear and Multilinear Algebra*, 39:19-31, (1995).

[12] R. Merris, Laplacian graph eigenvectors, *Linear Algebra and its Applications*, 278:221-236, (1998).

[13] B. Mohar, Laplace eigenvalues of graphs–a survey, *Discrete Mathematics*, 109:171-183, (1992).

[14] M. Meringer, "Connected Regular Graphs." http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html#CRG.

[15] T. Sander, On certain eigenspaces of cographs, *The Electronic Journal of Combinatorics*, 15: #R140 (2008).