Primitive Digraphs with the Largest Scrambling Index

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Abstract

The scrambling index of a primitive digraph D is the smallest positive integer k such that for every pair of vertices u and v, there is a vertex w such that we can get to w from u and v in D by directed walks of length k; it is denoted by $k(D)$. In [1] we gave the upper bound on $k(D)$ in terms of the order and the girth of a primitive digraph D. In this paper, we characterize all the primitive digraphs such that the scrambling index is equal to the upper bound.

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Key words: Scrambling index; Primitive digraph

1 Introduction

There are numerous results giving the upper bounds on the second largest modulus of eigenvalues of primitive stochastic matrices (see [3,5–8]). In [1], by using Seneta's [6] definition of coefficients of ergodicity, we have provided an attainable upper bound on the second largest modulus of eigenvalues of a primitive matrix that makes use of the so-called scrambling index (see below).

For vertices u, v and w of a digraph D, if $(u, w), (v, w) \in E(D)$, then vertex w is called a *common out-neighbour* of vertices u and v. The *scrambling index* of

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a primitive digraph is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. The scrambling index of D will be denoted by $k(D)$.

The main result in [1] is the following.

Theorem 1.1 **[1]** *D be a primitive digraph with n vertices and girth s. Then*

$$k(D) \leq K(n, s). \tag{1}$$

Equality holds if $D = D_{s,n}$ and $\gcd(n, s) = 1$. Where $D_{s,n}$ is a digraph as in Figure 1, $K(n, s) = k(n, s) + n - s$ and

$$k(n,s) = \begin{cases} (\frac{s-1}{2})n, & \text{when } s \text{ is odd,} \\ (\frac{n-1}{2})s, & \text{when } s \text{ is even.} \end{cases}$$

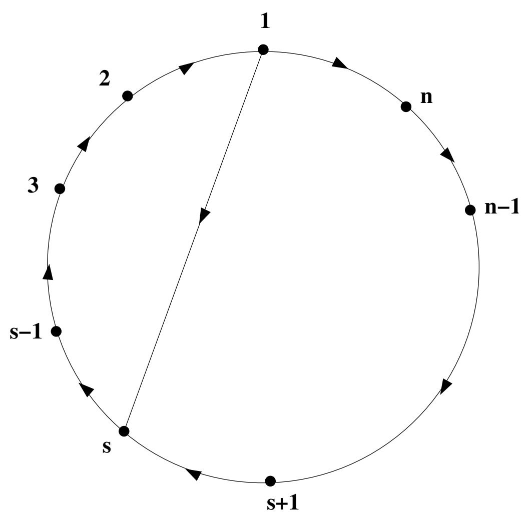

Fig. 1. $D_{s,n}$

In this paper, we characterize all the primitive digraphs D such that $k(D) = K(n, s)$.

2 Some results on scrambling index

For terminology and notation used here we follow [1] and [2].

Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$ and order n. Loops are permitted but multiple arcs are not. A $u \to v$ *walk* in a digraph D is a sequence of vertices $u, u_1, \ldots, u_t, v \in V(D)$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_t, v) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a $u \to v$ walk where $u = v$. A *cycle* is a closed $u \to v$ walk with distinct vertices except for $u = v$. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k. The *distance* from vertex u to vertex v in D, is the length of a shortest walk from u to v, and denoted by $d(u, v)$. A *p-cycle* is a cycle of length p, denoted C_p. If the digraph D has at least one cycle, the length of a shortest cycle in D is called the *girth* of D, denoted $s(D)$. The number of arcs entering (leaving) a vertex u is called the *in-degree (out-degree)* of u, denoted $deg^-(u)$ ($deg^+(u)$).

A digraph D is called *primitive* if for some positive integer t there is a walk of length exactly t from each vertex u to each vertex v. If D is primitive, the smallest such t is called the *exponent* of D, denoted by $\exp(D)$. A digraph D is primitive if and only if its strongly connected and the greatest common divisor of all cycle lengths in D is equal to one [2]. For a positive integer r, we define D^r to be the digraph with the same vertex set as D and arc (u, v) if and only if $u \xrightarrow{r} v$ in D. Consequently, the scrambling index is the smallest positive integer k such that each pair of vertices has a common out-neighbour in D^k.

For a vertices $u, v \in V(D)$ $(u \neq v)$, we define

$$k_{u,v}(D) = \min\{k : \; u \xrightarrow{k} w \text{ and } v \; \xrightarrow{k} w, \text{ for some } w \in V(D)\}.$$

Then

$$k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\}.$$

Lemma 2.1 **[1]** *Let p and s be positive integers such that $\gcd(p, s) = 1$ and $p > s \geq 2$. Then for each t, $1 \leq t \leq \max\{s-1, \lfloor p/2 \rfloor\}$, the equation $xp+ys = t$ has a unique integral solution (x, y) with $|x| \leq \lfloor s/2 \rfloor$ and $|y| \leq \lfloor p/2 \rfloor$.*

Let D be a primitive digraph, and let s and p be two different cycle lengths in D and $\gcd(s, p) = 1$, where $2 \leq s < p \leq n$. For $u, v \in V(D)$, we can find a vertex $w \in V(D)$ such that there are directed walks from u to w and v to w such that both walks meet cycles of lengths s and p. Denote the lengths of these directed walks by $l(u, w)$ and $l(v, w)$. We say that w is a *double-cycle vertex* of u and v, and we let

$$l_{u,v} = \max\{l(u, w), l(v, w)\}.$$

Lemma 2.2 **[1]** *Let D be a primitive digraph, and let s and p be two different cycles lengths in D. Suppose that $2 \leq s < p \leq n$ and $\gcd(s, p) = 1$. Then*

$$k_{u,v}(D) \leq \min\{|y|s, |x|p\} + l_{u,v}, \tag{2}$$

where (x, y) is the integer solution of the equation $xp + ys = r$ with minimum absolute value and where $|l(u, w) - l(v, w)| \equiv r (\bmod s)$.

Corollary 2.3 **[1]** *Let D be a primitive digraph of order n with a Hamilton cycle, and let the girth of D be s, where $1 \leq s \leq n-1$ and $\gcd(s,n) = 1$. If $k(D) = K(n,s)$, then D contains a subgraph isomorphic to $D_{s,n}$.*

Lemma 2.4 **[1]** *Let $D = D_{s,n}$. Then for all vertices u and v in D, $l_{u,v}(D) \leq \max\{n-s, \lfloor \frac{n}{2} \rfloor\}$.*

Let r be the positive integer that is defined as follows

$$r \equiv \begin{cases} \frac{n}{2}(\bmod s), & \text{if } s \text{ is odd and } n \text{ is even,} \\ \frac{n-s}{2}(\bmod s), & \text{if both } s \text{ and } n \text{ are odd .} \end{cases} \quad (3)$$

Corollary 2.5 **[1]** *Suppose that $\gcd(s,n) = 1$, and $s \geq 2$. Then for $u, v \in V(D_{s,n})$, without loss of generality take $u > v$, $k_{u,v}(D_{s,n}) = K(n,s)$ if and only if $u = n$ and*

(1) $v = n - r - ts$ for some $t \in \{0, 1, 2, \cdots, \frac{n-2r}{s}\}$, when s is odd.

(2) $v = n - \frac{s}{2}$, when s is even.

Lemma 2.6 **[1]** *Let D be a primitive digraph with a Hamilton cycle and let the girth of D be s, where $\gcd(n,s) = 1$, $2 \leq s < n$. Then either the cycle C_s is formed from s consecutive vertices on the Hamilton cycle or there is another cycle of length p such that $\gcd(s,p) = q$, where $q \leq \frac{s}{2}$ when s is even and $q \leq \frac{s}{3}$ when s is odd.*

Lemma 2.7 **[1]** *Let D be a primitive digraph with n vertices, and suppose that s is the girth of D with $s \geq 2$. If there is another cycle of length p, $s < p \leq n$, such that $\gcd(s,p) = 1$, then*

$$k(D) \leq K(n,s). \quad (4)$$

Furthermore, if $p < n$, then $k(D) < K(n,s)$.

Let D be a primitive digraph and $L(D) = \{s, a_1, \cdots, a_r\}$ be the set of distinct cycle lengths of D, where $s < a_1 < \cdots < a_r$.

Lemma 2.8 **[1]** *Let D be a primitive digraph with n vertices, and s be the girth of D with $s \geq 2$. Let $L(D) = \{s, a_1, \cdots, a_r\}$. If $\gcd(s, a_i) \neq 1$ for each $i = 1, 2, \cdots, r$, Then*

$$k(D) < K(n,s).$$

Corollary 2.9 **[1]** *Let D be a primitive digraph of order n, and s be the girth of D with $s \geq 2$. If there is a cycle of length p, $s < p \leq n$, such that*

$\gcd(s,p) < s/3$ *or* $\gcd(s,p) \leq s/3$ *and* $C_s \cap C_p \neq \emptyset$*, then*

$$k(D) < K(n,s).$$

3 Characterization of primitive digraphs with $k(D) = K(n,s)$

3.1 Properties of a primitive digraph D with $k(D) = K(n,s)$

Let D be a primitive digraph with n vertices, s be the girth of D, and $k(D) = K(n,s)$. Then by Lemma 2.7 and Lemma 2.8 there is a cycle of length p, $s < p \leq n$, such that $\gcd(s,p) = 1$ and $p = n$. Since D contains a Hamilton cycle, then by Corollary 2.3 D contains $D_{s,n}$ as a subgraph. From the above, we conclude the following.

Theorem 3.1 *Let D be a primitive digraph with n vertices, let the girth of D be $s \geq 2$, and suppose that $k(D) = K(n,s)$. Then*

(1) *There is no cycle of length p, $s < p < n$, such that* $\gcd(s,p) = 1$.

(2) *D contains $D_{s,n}$ as a subgraph and* $\gcd(s,n) = 1$.

In the following we only consider primitive digraphs that contain $D_{s,n}$ as a subgraph, and we label the digraph D as in Figure 1. For $D_{s,n}$, by Corollary 2.5 we know all the pairs of vertices $u, v \in V(D_{s,n})$ such that $k_{u,v}(D_{s,n}) = K(n,s)$.

Proposition 3.2 [4] *The t-th power of a cycle of length p is the disjoint union of* $\gcd(p,t)$ *cycles of length $p/\gcd(p,t)$.*

Definition 3.3 *If the digraph D contains at least two different cycles, then the distance between two different cycles in D is defined as follows*

$$d(C', C'') = \min\{d(u,v) | u \in C',\ v \in C''\},$$

where C' and C'' are different cycles in D.

Lemma 3.4 *Let $D = D_{s,n}$,* $\gcd(n,s) = 1$*, and let t be a positive integer such that $t|s$. Then*

(i) The digraph D^t contains a Hamilton cycle and t disjoint cycles of length s/t.

(ii) Every cycle of length s/t is formed from s/t consecutive vertices on the Hamilton cycle in D^t.

Denote the t cycles of length s/t in D^t by $H_1,\ H_2, \cdots, H_t$ in order as in Figure 2, and we say that H_i and $H_{(i+1)(\bmod t)}$, where $i = 1, 2, \cdots, t$, are neighbour cycles in D^t. We also have the following:

(iii) The distance between two neighbour cycles of length s/t in D^t is either $\lceil \frac{n-s}{t} \rceil$ or $\lceil \frac{n-s}{t} \rceil + 1$.

Proof: (i) Since $\gcd(s, n) = 1$, then $\gcd(t, n) = 1$. Therefore by Lemma 3.2, we know that D^t contains a Hamilton cycle and t disjoint cycles of length s/t.

(ii) For vertices $i,\ 1 \leq i \leq t$, we have $i + pt \in C_s$, $0 \leq p \leq \frac{s}{t} - 1$. Also we have

$$i \xrightarrow{t} i + t \xrightarrow{t} i + 2t \xrightarrow{t} \cdots \xrightarrow{t} i + (\frac{s}{t} - 1)t \xrightarrow{t} i.$$

Therefore every cycle of length s/t is formed from s/t consecutive vertices on the Hamilton cycle in D^t.

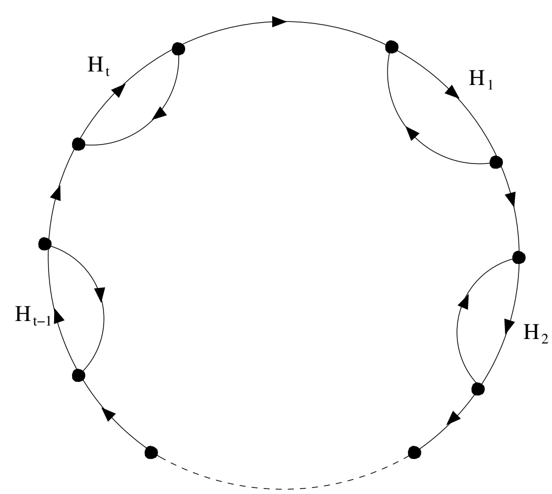

Fig. 2. D^t

(iii) There are two different types of directed paths of length t in $D_{s,n}$. One type contains the arc $1 \to s$, and the other type does not contain the arc $1 \to s$. Observing D^t, we know that every arc in the Hamilton cycle in D^t corresponds to a directed path of length t in $D_{s,n}$ that does not contain the arc $1 \to s$, and all the other arcs, we call them shortly s-arcs, correspond to directed paths of length t in $D_{s,n}$ that contain the arc $1 \to s$. Also notice that if $u_1 \to u_2$ is an s-arc, then $1 \leq u_1 \leq t$ and $s - (t-1) \leq u_2 \leq s$.

Let $d(H_i, H_{(i+1)(\bmod t)}) = q$ for some i, then there exist a vertex $u \in H_i$ and a vertex $v \in H_{(i+1)(\bmod t)}$ such that $d(u, v) = q$ in D^t. From the digraph D^t, we know that $\deg^+(u) = 2$ and $\deg^-(v) = 2$. Hence u is the starting vertex of an $s - arc$ and v is the ending vertex of an $s - arc$. Therefore $1 \leq u \leq t$ and $s - (t-1) \leq v \leq s$.

Since in D^t, we have $u \xrightarrow{q} v$, then in $D_{s,n}$ we have $u \xrightarrow{qt} v$ and this directed walk does not go through the arc $1 \to s$.

In $D_{s,n}$, the directed path from vertex u to vertex v without going through the arc $1 \to s$ is of the form $u \xrightarrow{l_1} 1 \xrightarrow{1} n \xrightarrow{n-s} s \xrightarrow{l_2} v$, where $l_1, l_2 \le t-1$. Thus

$$\begin{aligned} n-s+1 \le & \, qt \le n-s+1+(t-1)+(t-1), \quad \text{and} \\ n-s+1 \le & \, qt \le n-s+(t-1)+t. \end{aligned}$$

Hence

$$\left\lceil \frac{n-s}{t} \right\rceil \le q \le \left\lceil \frac{n-s}{t} \right\rceil + 1.$$

Therefore the distance between any two neighbour cycles of length s/t is $\lceil \frac{n-s}{t} \rceil$ or $\lceil \frac{n-s}{t} \rceil + 1$. □

3.2 The case s is even

Lemma 3.5 *Let D be a primitive digraph that contains $D_{s,n}$ as a subgraph, where s is the girth of D,* $\gcd(n,s) = 1$ *and s is even. If D contains another cycle of length p, where $s \le p < n$. Then $k(D) < K(n,s)$.*

Proof. Let C_p be the cycle of length p in the primitive digraph D.

Case 1: Suppose $\gcd(s,p) = r$, with $r < \frac{s}{3}$. Then by Corollary 2.9 we have $k(D) < K(n,s)$.

Case 2: Suppose $\gcd(s,p) = \frac{s}{3}$. If $C_s \cap C_p \neq \emptyset$, we are also done by Corollary 2.9. If $C_s \cap C_p = \emptyset$, consider $D^{\frac{s}{3}}$. There are $\frac{s}{3}$ cycles of length 3 and $\frac{s}{3}$ cycles of length $\frac{3p}{s}$. Let $p' = \frac{3p}{s}$. For $u, v \in V(D^{\frac{s}{3}})$, $l_{uv} \le n-3$. Hence

$$\begin{aligned} k_{u,v}(D^{\frac{s}{3}}) \le & (\frac{3-1}{2})p' + n - 3 \\ = & \, p' + n - 3. \end{aligned}$$

Since $p \le n-s,\ p' \le \frac{3n}{s} - 3$, we have

$$k_{u,v}(D) \le \frac{s}{3}(n+p'-3) \le \frac{ns}{3} + n - 2s < k(n,s) + n - s.$$

Case 3. $\gcd(s,p) = \frac{s}{2}$. Since s is even, then n is odd. We know there is only one pair of vertices $u, v \in V(D_{s,n})$ such that $k_{u,v}(D_{s,n}) = k(n,s) + n - s$, and they are vertex n and $n-\frac{s}{2}$. Consider the digraph $D^{\frac{s}{2}}$. It is easy to see that vertices n and $n-\frac{s}{2}$ are consecutive vertices on the Hamilton cycle in the digraph $D^{\frac{s}{2}}$, and there are $\frac{s}{2}$ cycles of length 2 and $\frac{s}{2}$ cycles of length p' respectively, where $p'=\frac{2p}{s}$ and p' is odd (since $p=\frac{s}{2}p'$). Let $p'=2t+1$ for some nonnegative integer t. For vertex $n-\frac{s}{2}$, we can find a vertex w such that the directed walk from vertex $n-\frac{s}{2}$ to vertex w is a path through both cycles of length 2 and p', and $l(n-\frac{s}{2},w)\leq n-p'$. Since in $D^{\frac{s}{2}}$, we have $n\xrightarrow{1}n-\frac{s}{2}$. Then $l(n,w)-l(n-\frac{s}{2},w)=1$ and $l(n,w)\leq n-p'+1$. Therefore in the digraph $D^{\frac{s}{2}}$, we have

$$n\xrightarrow{l(n,w)+2t}w \quad \text{and}$$
$$n-\frac{s}{2}\xrightarrow{l(n-\frac{s}{2},w)+p'}w.$$

Thus $k_{n,n-\frac{s}{2}}(D^{\frac{s}{2}})\leq n$; and hence

$$k_{n,n-\frac{s}{2}}(D)\leq(\frac{s}{2})n<k(n,s)+n-s.$$

Case 4. $\gcd(s,p)=s$. Suppose $p=ts$, where $1\leq t<\frac{n}{s}$.

If $t=1$, then $p=s$. If the cycle C_p is formed from s vertices that are not consecutive on the Hamilton cycle, then by Lemma 2.6, there exists another cycle of length q such that $\gcd(s,q)\leq\frac{s}{2}$. For this case, from the previous results we know that $k_{n,n-\frac{s}{2}}(D)<k(n,s)+n-s$.

If the cycle C_p is formed by joining vertex i to vertex $(i+s-1)(\bmod n)$, where $i\neq 1$, then consider the subgraph $D_{p,n}$. Note that since $i\neq 1$, although $p=s$, but $C_p\neq C_s$. Therefore $D_{p,n}\neq D_{s,n}$. In $D_{p,n}$, the upper bound is attained for only one pair of vertices, and they are vertex $i-1$ and vertex $(i+s-2)(\bmod n)$. Since $i-1\neq n$, we have $k_{n,n-\frac{s}{2}}(D_{p,n})<K(n,s)$. Therefore in the digraph D, we also have

$$k_{n,n-\frac{s}{2}}(D)<k(n,s)+n-s.$$

Now suppose that $t>1$, then $s<\frac{n}{2}$. If $C_s\cap C_p\neq\emptyset$, there is at least one vertex w belonging to the cycle C_p such that $s+1\leq w\leq n-\frac{s}{2}-1$. Otherwise the cycle C_p only has to contain vertices between vertex s to vertex 1 and n to $n-\frac{s}{2}+1$. But there are only $s+\frac{s}{2}$ such vertices and $s+\frac{s}{2}<p$. Hence for vertices $n-\frac{s}{2}$, we have $l(n-\frac{s}{2},w)<n-\frac{3s}{2}$. Then $l(n,w)<n-s$ and $l(n,w)-l(n-\frac{s}{2})=\frac{s}{2}$. In $D_{s,n}$, when $n>\frac{3s}{2}$, we get

$$n\xrightarrow{n-s}s\xrightarrow{(\frac{n-1}{2})s}s \quad \text{and}$$

$$n-\frac{s}{2}\xrightarrow{n-\frac{3s}{2}}s\xrightarrow{\frac{s}{2}n}s.$$

When $n < \frac{3s}{2}$, we have

$$n \xrightarrow{n-s} s \xrightarrow{(\frac{n-1}{2})s} s \quad \text{and}$$
$$n - \frac{s}{2} \xrightarrow{n-\frac{s}{2}+n-s} s \xrightarrow{(\frac{s}{2}-1)n} s.$$

Note that $\frac{n-1}{2} \geq \frac{n-1}{s} \geq t$ and let $\frac{n-1}{2} = t + t'$. Then $(\frac{n-1}{2})s = p + t's$, where $p = st$. Hence

$$n \xrightarrow{l(n,w)} w \xrightarrow{p+t's} w \quad \text{and}$$
$$n - \frac{s}{2} \xrightarrow{l(n-\frac{s}{2},w)} w \xrightarrow{\frac{s}{2}n} w.$$

Therefore $k_{n,n-\frac{s}{2}}(D) \leq l(n,w) + p + t's < k(n,s) + n - s$.

If $C_s \cap C_p = \emptyset$, for vertex $n - \frac{s}{2}$ we can find a vertex $w \in C_p$ such that $l(n-\frac{s}{2}, w) \leq n-s-p$. Then $l(n,w) \leq n-s-p+\frac{s}{2}$ and $l(n,w)-l(n-\frac{s}{2},w) = \frac{s}{2}$. Since $\frac{n-1}{2} \geq \frac{n-1}{s} \geq t$, let $\frac{n-1}{2} \equiv t'(\bmod t)$. For a nonnegative integer h we have $\frac{n-1}{2} = th + t'$. If $t' = 0$, then $(\frac{n-1}{2})s = hts = hp$, and so

$$n \xrightarrow{l(n,w)} w \xrightarrow{hp} w \quad \text{and}$$

$$n - \frac{s}{2} \xrightarrow{l(n-\frac{s}{2},w)} w \xrightarrow{\frac{s}{2}n} w.$$

Therefore $k_{n,n-\frac{s}{2}}(D) \leq hp + l(n,w) < k(n,s) + n - s$.

If $t' \neq 0$, $t > t' > 0$, we know that

$$\frac{s}{2}n - (\frac{n-1}{2})s = \frac{s}{2},$$

or equivalently

$$(th + t')s - \frac{s}{2}n = -\frac{s}{2}.$$

Adding $(t - t')s$ on both sides, we get

$$hts + t's + (t-t')s - \frac{s}{2}n = -\frac{s}{2} + (t-t')s,$$

or

$$(h+1)ts - (\frac{s}{2}n + (t-t'-1)s) = \frac{s}{2}.$$

Therefore we have

$$n \xrightarrow{l(n,w)} w \xrightarrow{\frac{s}{2}n+(t-t'-1)s} w \quad \text{and}$$
$$n - \frac{s}{2} \xrightarrow{l(n-\frac{s}{2},w)} w \xrightarrow{(h+1)p} w.$$

Then $k_{n,n-\frac{s}{2}}(D) \leq \frac{s}{2}n + (t-t'-1)s + l(n,w) \leq \frac{s}{2}n + (t-t'-1)s + n - s - p = (\frac{n-1}{2})s + n - s - t's < k(n,s) + n - s$, as desired. □

Theorem 3.6 *Let D be a primitive digraph of order n and girth s, where s is even. Then $k(D) = K(n,s)$ if and only if $D = D_{s,n}$ and* $\gcd(n,s) = 1$.

3.3 The case s is odd

Lemma 3.7 *Let D be a primitive digraph that contains $D_{s,n}$ as a subgraph, where* $\gcd(n,s) = 1$*, s is odd and $s \geq 3$. If D contains a cycle of length p with* $\gcd(s,p) \leq \frac{s}{3}$*, then $k(D) < K(n,s)$.*

Proof. Case 1. $\gcd(s,p) = l,\ l < \frac{s}{3}$. Then by Corollary 2.9 $k(D) < k(n,s) + n - s$.

Case 2. $\gcd(s,p) = \frac{s}{3}$. If $C_s \cap C_p \neq \emptyset$, we are done by Corollary 2.9. If $C_s \cap C_p = \emptyset$, consider $D^{\frac{s}{3}}$. There are $\frac{s}{3}$ cycles of length 3 and $\frac{s}{3}$ cycles of length $\frac{3p}{s}$, let $p' = \frac{3p}{s}$. For $u, v \in V(D^{\frac{s}{3}})$, we have $l_{uv} \leq n - 3$. Hence

$$k_{u,v}(D^{\frac{s}{3}}) \leq (\frac{3-1}{2})p' + n - 3$$

$$= p' + n - 3.$$

Since $p \leq n - s$ and $p' \leq \frac{3n}{s} - 3$, we get

$$k_{u,v}(D) \leq \frac{s}{3}(n + p' - 3) \leq \frac{ns}{3} + n - 2s < k(n,s) + n - s. \qquad \Box$$

Next we consider a primitive digraph D that contains $D_{s,n}$ as a subgraph, where $\gcd(s,n) = 1$ and s is odd, and where the digraph D also contains another cycle of length p with $\gcd(s,p) = s$.

Lemma 3.8 *Let D be a primitive digraph that contains $D_{s,n}$ as a subgraph, where* $\gcd(s,n) = 1$*, s is odd and $s \geq 3$. Suppose that the digraph D also contains another cycle of length p with* $\gcd(s,p) = s$*. If $C_s \cap C_p \neq \emptyset$, then $k(D) < K(n,s)$.*

Proof. Suppose that $p = ts$ and that u is a vertex of $D_{s,n}$ such that $k_{nu}(D) = (\frac{s-1}{2})n + n - s$.

If $u \notin C_s$, then in the digraph $D_{s,n}$ we have

$$n \xrightarrow{n-s} s \xrightarrow{(\frac{s-1}{2})n} s \quad \text{and}$$

$$u \xrightarrow{u-s} s \xrightarrow{ms} s,$$

where m is a positive integer such that $ms - (\frac{s-1}{2})n = n - u$.

If there is a vertex w such that $s + 1 \leq w \leq u$ and it belongs to the cycle C_p, then choose w as the double-cycle vertex of u and n. Then we have $l(u,w) < u - s,\ l(n,w) < n - s$ and $l(n,w) - l(u,w) = n - u$. Also since $ms > n > p$ and $p = ts$, then $ms = p + t's$ for some nonnegative integer t'. Then

$$n \xrightarrow{l(n,w)} w \xrightarrow{(\frac{s-1}{2})n} w \quad \text{and}$$
$$u \xrightarrow{l(u,w)} w \xrightarrow{p+t's} w.$$
Thus $k_{n,u}(D) \leq (\frac{s-1}{2})n + l(n,w) < k(n,s) + n - s$.

Otherwise there is an arc from vertex j , $u < j \leq n$, to vertex i, $1 \leq i \leq s$. Then we can get from vertex n to a vertex i on the cycle C_s in less than $n-s$ steps. Therefore $k_{n,u}(D) < k(n,s) + n - s$.

Next consider $u \in C_s$. If $p = s$, suppose that the cycle C_p is formed from s consecutive vertices as in Figure 3.

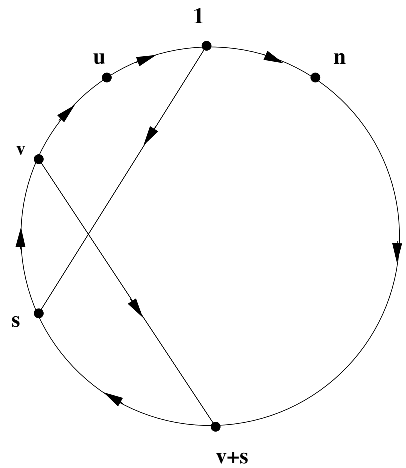

Fig. 3. $D_{s,n} \cup \{v \to v+s\}$

If $v = u+1$, then $l(n,w) < n-s$ and $l(u,w) = s \neq n-s$. Therefore $k_{n,u}(D) < k(n,s) + n - s$. If $v \neq u+1$, then consider the subgraph $D_{p,n}$. In $D_{p,n}$, for some vertex v' we have $k_{v-1,v'}(D_{p,n}) = K(n,s)$. Since $v-1 \neq u, n$, then $k_{n,u}(D_{p,n}) < k(n,s) + n - s$. Therefore $k_{n,u}(D) < k(n,s) + n - s$.

If the cycle C_p is not formed from s consecutive vertices, then by Lemma 2.6, there exists a cycle of length q such that $\gcd(s,q) \leq \frac{s}{3}$. In that case, by Lemma 3.7, we have $k(D) < k(n,s) + n - s$.

If $p > s$, then take the first vertex w on cycle C_p from vertex n as the double-cycle vertex of u and n. Since $p \geq 2s$, $l(n,w) \leq n - 2s$. Since $l(u,n) < s$, then $l(u,w) < n - s$.

In the digraph $D_{s,n}$, there is a vertex u', $u < u' < n$, such that $d(u,n) = d(n,u') = n - u'$, $k_{n,u'}(D) = k(n,s) + n - s$ and

$$n \xrightarrow{n-s} s \xrightarrow{(\frac{s-1}{2})n} s \quad \text{and}$$

$$u' \xrightarrow{u'-s} s \xrightarrow{ms} s,$$

where $ms - (\frac{s-1}{2})n = n - u'$. Since $ms > n > p$, then $ms = p + ts$ for some nonnegative integer t. In the digraph D we have

$$n \xrightarrow{l(n,w)} w \xrightarrow{p+ts} w \quad \text{and}$$

$$u \xrightarrow{l(u,w)} w \xrightarrow{(\frac{s-1}{2})n} w,$$

where $l(u,w) - l(n,w) = n - u'$. Therefore $k_{n,u}(D) \leq (\frac{s-1}{2})n + l(u,w) < (\frac{s-1}{2})n + n - s$. □

Lemma 3.9 *Let D be a primitive digraph that contains $D_{s,n}$ as a subgraph, suppose that s is odd, $s \geq 3$, and that there is another cycle of length p such that $C_s \cap C_p = \emptyset$ and $\gcd(s,p) = s$. If the cycle of length p is not formed from p consecutive vertices on the Hamilton cycle, then $k(D) < K(n,s)$.*

Proof. Since the cycle of length p is not formed from p consecutive vertices on the Hamilton cycle, then there exists an arc from vertex i to vertex j, where $s+1 \leq i < j \leq n$ and $j > i+1$. Then for any two vertices $u, v \in V(D)$, we can get to vertices $s_1, s_2 \in C_s$ in less than $n - s - 1$ steps. Therefore $k(D) \leq k(n,s) + n - s - 1$. □

The only remaining case is that D is a digraph constructed from $D_{s,n}$ by adding an arc from vertex u to vertex $u + ms - 1$, where s is odd, $s \geq 3$, $s < u < n - ms + 1$ and m is a positive integer such that $1 \leq m \leq \frac{n-u+1}{s}$.

Recall that in (3) we define the positive integer r as follows

$$r \equiv \begin{cases} \frac{n}{2}(\text{mod}s), & \text{if } s \text{ is odd, } n \text{ is even,} \\ \frac{n-s}{2}(\text{mod}s), & \text{if both } s \text{ and } n \text{ are odd .} \end{cases}$$

In both cases $n - 2r$ can be divided by s. Let

$$h = \frac{n-2r}{s}. \tag{5}$$

Note that in $D_{s,n}$, $h+1$ is the number of pairs of vertices that attain the upper bound $K(n,s)$.

Lemma 3.10 *Let D be a digraph constructed from $D_{s,n}$, $s \geq 3$, by adding an arc from vertex u to vertex $u + ms - 1$, where $s < u < n - ms + 1$. Then $k_{n,n-r-ts}(D) = K(n,s)$ if and only if $u = n - r - ts + 1$ and $\frac{n+h}{2} - t - 1 \equiv 0(\text{mod}m)$.*

Proof. For the digraph $D = D_{s,n}$, the upper bound $K(n,s)$ on the scrambling index is attained by the pair of vertices n and $n-r-ts$, where $0 \le t \le \frac{n-2r}{s}$. We only consider those pairs of vertices.

Suppose that $u = n-r-ts+1$ for some t. From the digraph we know that

$$n \overset{r+ts-ms}{\longrightarrow} n-r-ts+ms \quad \text{and}$$

$$n-r-ts \overset{n-ms}{\longrightarrow} n-r-ts+ms,$$

and $n-ms-(r+ts-ms) = n-r-ts = r+(h-t)s$, since $n = 2r+hs$. When n is even,

$$(\frac{n+h}{2}-t)s - (\frac{s-1}{2})n = r+(h-t)s.$$

Suppose $m-1-q$ is the smallest nonnegative integer such that $(\frac{n+h}{2}-t+m-1-q)s$ can be divided by $p = ms$, where $0 \le q \le m-1$. Then

$$n \overset{r+ts-ms}{\longrightarrow} n-r-ts+ms \overset{(\frac{n+h}{2}-t+m-1-q)s}{\longrightarrow} n-r-ts+ms$$

and

$$n-r-ts \overset{n-ms}{\longrightarrow} n-r-ts+ms \overset{(\frac{s-1}{2})n+(m-1-q)s}{\longrightarrow} n-r-ts+ms.$$

Therefore $k_{n,n-r-ts}(D) = (\frac{s-1}{2})n + n - s - qs$.

Since $(\frac{n+h}{2}-t+m-1-q)s$ can be divided by $p = ms$, then

$$\frac{n+h}{2} - t - 1 \equiv q (\text{mod} m).$$

Therefore if $\frac{n+h}{2}-t-1 \equiv 0(\text{mod} m)$, we have

$$k_{n,n-r-ts}(D) = K(n,s).$$

If $\frac{n+h}{2}-t-1 \not\equiv 0(\text{mod} m)$, then $k_{n,n-r-ts} < K(n,s)$.

Next we consider all other pairs of vertices n and u such that $k_{n,u}(D_{s,n}) = K(n,s)$.

If $u \ne n-r-ts+1$, let $v = u+ms-1$. Consider the following three cases.

Case 1. $n-r-ts+1 < u$. We have

$$n \overset{n-v}{\to} v \quad \text{and}$$

$$n-r-ts \overset{n-r-ts+n-v}{\rightarrow} v.$$

In addition we have $n-r-ts+(n-v)-(n-v) = n-r-ts = r+(h-t)s$. Then we obtain

$$n \overset{n-v}{\longrightarrow} v \overset{(\frac{n+h}{2}-t+m-1-q)s}{\longrightarrow} v \quad \text{and}$$

$$n-r-ts \overset{n-r-ts+n-v}{\longrightarrow} v \overset{(\frac{s-1}{2})n+(m-1-q)s}{\longrightarrow} v.$$

Therefore $k_{n,n-r-ts}(D) = n-r-ts+(n-v)+(\frac{s-1}{2})n+(m-1-q)s < n-ms+(\frac{s-1}{2})n+(m-1-q)s = (\frac{s-1}{2})n+n-s-qs \leq k(n,s)+n-s$.

Case 2. $n-r-ts > v$. We have

$$n \overset{n-v}{\rightarrow} v \quad \text{and}$$

$$n-r-ts \overset{n-r-ts-v}{\rightarrow} v,$$

and $n-v-(n-r-ts-v) = r+ts$. Also

$$(\frac{n-h}{2}+t)s-(\frac{s-1}{2})n = r+ts.$$

Then

$$n \overset{n-v}{\longrightarrow} v \overset{(\frac{s-1}{2})n+(m-1-q)s}{\longrightarrow} v \quad \text{and}$$

$$n-r-ts \overset{n-r-ts-v}{\longrightarrow} v \overset{(\frac{n-h}{2}-t+m-1-q)s}{\longrightarrow} v.$$

Therefore $k_{n,n-r-ts}(D) = n-v+(\frac{s-1}{2})n+(m-1-q)s < n-ms+(\frac{s-1}{2})n+(m-1-q)s = (\frac{s-1}{2})n+n-s-qs \leq k(n,s)+n-s$.

Case 3. $u \leq n-r-ts \leq v$. Choose v as the double-cycle vertex of n and $n-r-ts$. Then

$$n \overset{n-v}{\longrightarrow} v \quad \text{and}$$

$$n-r-ts \overset{n-r-ts-u+1}{\longrightarrow} v.$$

If $n-v > n-r-ts-u+1$, since $n-v-(n-r-ts-u+1) = r+ts-(v-u+1) = r+(t-m)s$ and $v > ms$, then

$$\begin{aligned} k_{n,n-r-ts}(D) &\leq (\frac{s-1}{2})n+n-v+(m-1-q)s \\ &= (\frac{s-1}{2})n+n-s-v+ms-qs \\ &< k(n,s)+n-s. \end{aligned}$$

If $n-v < n-r-ts-u+1$, then $n-r-ts-u+1-(n-v) = -r-ts+v-u+1 = -r-ts+ms = s-r+(m-1-t)s$. Then

$$(\frac{s-1}{2})n-(\lfloor\frac{n}{2}\rfloor-t')s = s-r+(m-1-t)s$$

for some integer t'. Therefore

$$k_{n,n-r-ts}(D) \leq (\frac{s-1}{2})n + n - v + (m-1-q)s$$
$$= (\frac{s-1}{2})n + n - s - v + ms - qs < k(n,s) + n - s. \quad \square$$

Lemma 3.11 *Let D be a digraph constructed from $D_{s,n}$ $(s \geq 3)$ by adding arcs from vertex u_i to vertex $u_i + m_i s - 1$, where $u_i > s$, $m_i \geq 1$, $i = 1, 2$ and $u_1 \neq u_2$. Then $k(D) < K(n,s)$.*

Proof. Let D_i, $i = 1, 2$, be the subgraph of D that contains $D_{s,n}$ and the cycle of length $m_i s$, then by Lemma 3.10, we know that there is at most one pair of vertices, vertex n and vertex $u_i - 1$, such that $k_{n,u_i-1}(D_i) = K(n,s)$. Since $u_1 \neq u_2$, In the digraph D, we have $k_{n,u_i-1}(D) < K(n,s)$. $\square$

Concluding the above results, we have the following theorem.

Theorem 3.12 *Let D be a primitive digraph of order n and girth s, where s is odd and $s \geq 3$. Then $k(D) = K(n,s)$ if and only if $\gcd(n,s) = 1$ and $D = D_{s,n}$ or, $D = D_{s,n} \cup \{n - r - ts + 1 \rightarrow n - r - ts + ms\}$ for some $m \in \mathbb{N}$ and some $t \in \{1, 2, \cdots, \frac{n-2r}{s} - 1\}$ such that $\frac{n+h}{2} - t - 1 \equiv 0 (\bmod m)$, where r and h are as in* (3) *and* (5).

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