Coefficients of Ergodicity and the Scrambling Index

Mahmud Akelbek *, Steve Kirkland [1]

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2

Abstract

For a primitive stochastic matrix S, upper bounds on the second largest modulus of an eigenvalue of S are very important, because they determine the asymptotic rate of convergence of the sequence of powers of the corresponding matrix. In this paper, we introduce the definition of the scrambling index for a primitive digraph. The scrambling index of a primitive digraph D is the smallest positive integer k such that for every pair of vertices u and v, there is a vertex w such that we can get to w from u and v in D by directed walks of length k; it is denoted by $k(D)$. We investigate the scrambling index for primitive digraphs, and give an upper bound on the scrambling index of a primitive digraph in terms of the order and the girth of the digraph. By doing so we provide an attainable upper bound on the second largest modulus of eigenvalues of a primitive matrix that make use of the scrambling index.

AMS classification: 15A42; 15A51; 05C20; 05C50

Key words: Scrambling index; Primitive digraph; Coefficient of ergodicity

1 Introduction

Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$,

* corresponding author.
Email addresses: akelbek@math.uregina.ca (Mahmud Akelbek), kirkland@math.uregina.ca (Steve Kirkland).

[1] Research supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada under grant OGP0138251.

arc set $E = E(D)$ and order n. Loops are permitted but multiple arcs are not. A $u \to v$ *walk* in a digraph D is a sequence of vertices $u, u_1, \ldots, u_t, v \ \in V(D)$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_t, v) \ \in \ E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a $u \to v$ walk where $u = v$. A *cycle* is a closed $u \to v$ walk with distinct vertices except for $u = v$.

The *length of a walk* W is the number of arcs in W. The notation $u \xrightarrow{k} v$ is used to indicate that there is a $u \to v$ walk of length k. The *distance* from vertex u to vertex v in D, is the length of a shortest walk from u to v, and denoted by $d(u, v)$. A *p-cycle* is a cycle of length p, denoted C_p. If the digraph D has at least one cycle, the length of a shortest cycle in D is called the *girth* of D, denoted $s(D)$.

A digraph D is called *primitive* if for some positive integer t there is a walk of length exactly t from each vertex u to each vertex v. If D is primitive the smallest such t is called the *exponent* of D, denoted by $\exp(D)$. There are numerous upper bounds on the exponent of a primitive digraph. One of the well known result on the exponent is due to Dulmage and Mendelsohn [2].

Proposition 1.1 *Let D be a primitive digraph with n vertices, and s be the girth of D. Then*

$$\exp(D) \leq n + s(n - 2).$$

A digraph D is primitive if and only of its strongly connected and the greatest common divisor of all cycle lengths in D is equal to one.

For a digraph D with n vertices, we define the *adjacency matrix* of D to be the $n \times n$ matrix $A(D) = (a_{ij})$, where $a_{ij} = 1$ if there is an arc from vertex i to vertex j, and $a_{ij} = 0$ otherwise. For a positive integer r, the *r-th power* of a digraph D, denoted by D^r, is the digraph on the same vertex set and with an arc from vertex i to vertex j if and only if $i \xrightarrow{r} j$ in D. It is easy to see that $(D(A))^r = D(A^r)$.

For a positive integer $r \geq 1$, the (i, j)-th entry of the matrix A^r is positive if and only if $i \xrightarrow{r} j$ in the digraph D. Since most of the time we are only interested in the existence of such walks, not the number of different directed walks from i to j, we interpret A as a Boolean $(0, 1)$-matrix, unless stated otherwise. We denote by J, O, and I the *all* 1*'s matrix*, the *all* 0*'s matrix* and the *identity matrix*, respectively.

For vertices u, v and w of a digraph D, if $(u, w), (v, w) \in E(D)$, then vertex w is called a *common out-neighbour* of vertices u and v. The *scrambling index* of a primitive digraph is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D. In other words, it is the smallest positive integer k such that each pair of vertices has a common out-neighbour in D^k. The scrambling index of D will be denoted by $k(D)$. An analogous definition can be given for nonnegative matrices. The *scrambling index* of a primitive matrix A is the smallest positive integer k such that any two rows of A^k have at least one positive element in a coincident position, and will be denoted by $k(A)$. The scrambling index of a primitive matrix A can also defined as the smallest positive integer k such that $A^k(A^T)^k = J$.

In 2006, Cho and Kim [5] introduced the competition index of a digraph. They define the *row graph* $R(A)$ of a Boolean matrix A. It is a graph whose vertices are the rows of A, and two vertices in $R(A)$ are adjacent if and only if their corresponding rows have a nonzero entry in the same column of A. The *competition index*, denoted $cindex(D)$, is the smallest positive integer q such that $R(A^q) = R(A^{q+m})$ for some positive integer m. For a primitive digraph D, $cindex(D)$ is the smallest integer q such that $R(A^r)$ is a complete graph for any $r \geq q$.

Cho and Kim's [5] definition of the competition index is the same as our definition of the scrambling index in the case of primitive digraphs. In [5], the authors present the following result about the competition index.

Proposition 1.2 (H.H. Cho, H.K. Kim) *Let D be a primitive digraph of order n (≥ 3) with girth s.*

(1) *If n is odd, then $cindex(D) \leq n + \frac{(n-3)s}{2}$.*

(2) *If n is even, then $cindex(D) \leq n - 1 + \frac{(n-2)s}{2}$.*

In section 2, we present the motivation to consider the scrambling index, and give an attainable upper bound on the second largest modulus of the eigenvalues of a stochastic matrix S by using the scrambling index. In section 3, we give an upper bound on the scrambling index $k(D)$ of a primitive digraph D in terms of the order n and the girth s of D.

2 Coefficients of ergodicity

The spectral radius of A is the largest modulus of the eigenvalues of A, denoted by $\rho(A)$. For a primitive matrix A, by the Perron-Frobenius Theorem we know that the spectral radius $\rho(A)$ is a simple eigenvalue of A and the modulus of every other eigenvalue is strictly less than $\rho(A)$. Any primitive matrix is diagonally similar to a scalar multiple of a stochastic matrix. Thus we will only consider primitive stochastic matrices. For a primitive stochastic matrix S, the powers of S converge to a rank one positive matrix, and the rate of convergence is governed by the second largest modulus of the eigenvalues of S. There are numerous results giving the upper bounds on the second largest modulus of eigenvalues of primitive matrices (see [4,7,9,11,13]).

In 1979, Seneta [9] introduced the general concept of coefficients of ergodicity for an $n \times n$ stochastic matrix and he showed that coefficients of ergodicity provide an upper bound on the moduli of non-unit eigenvalues of a stochastic matrix.

An explicit expression for a coefficient of ergodicity in terms of the entries of the given matrix is the well known Dobrushin or delta coefficient, denoted by $\tau_1(\cdot)$, which according to Seneta [10], was first introduced by Dobrushin [3] and Paz [6].

For an $n \times n$ stochastic matrix S, the coefficient of ergodicity is defined

$$\tau_1(S) = \frac{1}{2}\max_{i,j} \sum_{l=1}^{n} |s_{il} - s_{jl}|. \tag{1}$$

It is also shown (see [10,4,8]) that

$$|\lambda| \leq \tau_1(S), \tag{2}$$

where λ is a non-unit eigenvalue of S. Seneta [10] had the following result.

Proposition 2.1 *Let S be a stochastic matrix. Then $\tau_1(S) < 1$ if and only if no two rows are orthogonal, or equivalently, if any two rows have at least one positive element in a coincident position.*

By the Perron-Frobenius theorem we know that $|\lambda| < 1$ for any non-unit eigenvalues of S. In that case the coefficient of ergodicity in (1) does not provide any new information. Therefore we are interested in the case that $\tau_1(S) < 1$.

Seneta called a matrix that satisfies the conditions of Proposition 2.1 as a *scrambling matrix*. An irreducible scrambling matrix is also a primitive matrix. Motivated by Seneta's work, we introduce the scrambling index of a primitive digraph.

Definition 2.2 *The scrambling index of a primitive matrix A is the smallest positive integer k such that A^k is a scrambling matrix.*

Let S be a primitive stochastic matrix with scrambling index $k(S) = k$ and λ be an eigenvalue of S; then S^k is also a primitive stochastic matrix and λ^k is an eigenvalue of S^k. By applying Proposition 2.1 and (2) to the matrix S^k, we have the following result.

Theorem 2.3 *Let $S = (s_{ij})$ be an $n \times n$ primitive stochastic matrix with scrambling index $k(S) = k$ and suppose that λ is a non-unit eigenvalue of S. Then*

$$|\lambda| \leq (\tau_1(S^k))^{1/k}, \tag{3}$$

and

$$\tau_1(S^k) < 1,$$

where $\tau_1(S^k) = \frac{1}{2}\max\limits_{i,j} \sum_{l=1}^{n} |s_{il}^{(k)} - s_{jl}^{(k)}|$.

Formula 3 in Theorem 2.3 gives an attainable upper bound on the modulus of non-unit eigenvalues of primitive stochastic matrix. For a stochastic matrix S, we denote the maximum modulus of the non-unit eigenvalues of S simply by $\xi(S)$. Consider the following example.

Example 2.4

Let A be the following $n \times n$ $(n \geq 3)$ stochastic matrix

$$A = \begin{bmatrix} a & 1-a & O^T_{1\times(n-2)} \\ 0 & 1-a & \frac{a}{n-2}e^T_{n-2} \\ e_{n-2} & O_{(n-2)\times 1} & O_{(n-2)\times(n-2)} \end{bmatrix},$$

where $a \in \mathbb{R}$, $0 < a < 1$, e_{n-2} is an $(n-2)$-dimensional column vector with all entries 1 and $O_{n\times n}$ is an $n \times n$ zero matrix. Then

$$A^2 = \begin{bmatrix} a^2 & 1-a & \frac{a(1-a)}{n-2}e^T_{n-2} \\ a & (1-a)^2 & \frac{a(1-a)}{n-2}e^T_{n-2} \\ ae_{n-2} & (1-a)e_{n-2} & O_{(n-2)\times(n-2)} \end{bmatrix}.$$

It is easy to see that A is a primitive stochastic matrix and that the scrambling index of A is $k(A) = 2$. The eigenvalues of A are 1, $\pm\sqrt{a-a^2}i$ and 0 with multiplicity $n-3$. Then $\xi(A) = \sqrt{a-a^2}$. By Theorem 2.3 we get $\tau_1(A^2) = (a-a^2)$. Comparing $\xi(A)$ with $\tau_1(A^2)$, we have $\xi(A) = (\tau_1(A^2))^{1/2}$.

3 Scrambling Index of a Primitive Digraph

In this section, we introduce the scrambling index of a primitive digraph and give an upper bound on the scrambling index in terms of the order and the girth of the digraph.

3.1 Introduction

Definition 3.1 *The scrambling index of a primitive digraph D, denoted by $k(D)$, is the smallest positive integer k such that for every pair of vertices u and v, we can get to a vertex w from both u and v in the digraph D by directed walks of length k.*

For a vertex $u \in V(D)$, we define the *local scrambling index* of vertex u as

$$k_u(D) = \min\{k : \ u \text{ has common out-neighbour with every other vertex in } D^k\}.$$

For $u, v \in V(D)$ $(u \neq v)$, define

$$k_{u,v}(D) = \min\{k : \ u \xrightarrow{k} w \text{ and } v \ \xrightarrow{k} w, \text{ for some } w \in V(D)\}.$$

Then

$$k(D) = \max_{u \in V(D)} \{k_u(D)\},$$

and

$$k(D) = \max_{u,v \in V(D)} \{k_{u,v}(D)\}.$$

From the definitions of $k(D)$, $k_u(D)$ and $k_{u,v}(D)$, we have

$$k_{u,v}(D) \leq k_u(D) \leq k(D).$$

The scrambling index gives another characterization of primitivity. For a primitive digraph D, by the definition of the scrambling index and the exponent it is easy to see that $k(D) \leq \exp(D)$.

3.2 The local scrambling index

We begin with a useful lemma.

Lemma 3.2 *Let p and s be positive integers such that $\gcd(p, s) = 1$ and $p > s \geq 2$. Then for each t, $1 \leq t \leq \max\{s-1, \lfloor p/2 \rfloor\}$, the equation $xp+ys = t$ has a unique integral solution (x, y) with $|x| \leq \lfloor s/2 \rfloor$ and $|y| \leq \lfloor p/2 \rfloor$.*

Proof. First consider the case that s is even. Then p is odd. We proceed by contradiction. Suppose that (x_1, y_1) is an integral solution of equation $xp +$

$ys = t$ with minimum absolute value of x_1. We claim that $|x_1| \le \lfloor \frac{s}{2} \rfloor$ and $|y_1| \le \lfloor \frac{p}{2} \rfloor$.

Suppose to the contrary that $|x_1| > \lfloor \frac{s}{2} \rfloor$, so that $|x_1| \ge \frac{s+2}{2}$. Consider the case that $x_1 \ge \frac{s+2}{2}$, so that $y_1 < 0$. Since $x_1 p + y_1 s = t$, we have

$$y_1 s = t - x_1 p. \tag{4}$$

If $s-1 = \max\{s-1, \lfloor p/2 \rfloor\}$, then $t \le s-1$. By (4) we have

$$y_1 s \le s - 1 - x_1 p \le s - 1 - \frac{sp}{2} - p.$$

Solving for y_1 we obtain

$$y_1 \le -\frac{p}{2} + (1 - \frac{1}{s} - \frac{p}{s}).$$

Since $1 - \frac{1}{s} - \frac{p}{s} < 1$, then $y_1 \le -\lfloor \frac{p}{2} \rfloor$.

If $\lfloor \frac{p}{2} \rfloor = \max\{s-1, \lfloor \frac{p}{2} \rfloor\}$, then $t \le \frac{p-1}{2}$. By (4) we have

$$y_1 s \le \frac{p-1}{2} - (\frac{s+2}{2})p,$$

solving for y_1, gives us

$$y_1 \le -\frac{p}{2} - \frac{p+1}{2s}.$$

Since $\frac{p+1}{2s} > 0$, then $y_1 \le -\lfloor \frac{p}{2} \rfloor$. Let $x' = x_1 - s$ and $y' = y_1 + p$, then $x'p + y's = t$, $|x'| < |x_1|$ and $|y'| \le |y_1|$, contradicting with the minimality of $|x_1|$. The arguments for the case that $x_1 \le -\frac{s+1}{2}$ and for the case that s is odd are similar and omitted.

Next we consider the uniqueness. Suppose for $i = 1, 2$ that (x_i, y_i) is an integral solution of the equation $xp + ys = t$ for some t, where $|x_i| \le \lfloor s/2 \rfloor$ and $|y_i| \le \lfloor p/2 \rfloor$. Then

$$x_i p + y_i s = t,$$

where $i = 1, 2$, and hence

$$(x_1 - x_2)p + (y_1 - y_2)s = 0. \tag{5}$$

Since $\gcd(s, p) = 1$, one of s and p is odd. Without loss of generality, suppose s is odd. Since $|x_i| \le \lfloor s/2 \rfloor$, $i = 1, 2$, then $|x_1 - x_2| \le |x_1| + |x_2| < s$. Suppose $\gcd((x_1 - x_2), s) = l$, then $1 \le l \le |x_1 - x_2|$. Let $s = ls'$ and $x_1 - x_2 = x'l$. It is easy to see that $s' \ge 2$, otherwise $s' = 1$, then $s = l$, but we have $l \le |x_1 - x_2| < s$. Substitute $s = ls'$ and $x_1 - x_2 = x'l$ in (5), we have $x'lp + (y_2 - y_1)ls' = 0$. Cancelling by l we get $x'p + (y_2 - y_1)s' = 0$. Since $\gcd(x', s') = 1$, then $x' | (y_2 - y_1)$, $p = (\frac{y_2 - y_1}{x'})s'$ and $\gcd(s, p) = s'$. This is a contradiction to $\gcd(s, p) = 1$. $\square$

Henceforth, we say (x, y) *is a solution of equation* $xp + ys = t$ *with minimum absolute value* to mean that $|x| \leq \lfloor s/2 \rfloor$, $|y| \leq \lfloor p/2 \rfloor$ and $xp + ys = t$.

Let D be a primitive digraph, and let s and p be two different cycle lengths in D. Suppose that $\gcd(s, p) = 1$, and that $2 \leq s < p \leq n$. For $u, v \in V(D)$, we can find a vertex $w \in V(D)$ such that there are directed walks from u to w and v to w such that both walks meet cycles of lengths s and p. Denote the lengths of these directed walks by $l(u, w)$ and $l(v, w)$. We say that w is a *double-cycle vertex* of u and v, and we let

$$l_{u,v} = \max\{l(u, w), l(v, w)\}.$$

Note that any vertex is a double-cycle vertex, and that there are many possible $l_{u,v}$'s. However, when using $l_{u,v}$, for specific digraphs we will make a good choice of the double-cycle vertex w and good choices of $l(u, w)$ and $l(v, w)$. In particular, we do not necessarily choose w so as to minimize $\max\{l(u, w), l(v, w)\}$. Without loss of generality, suppose that $l(u, w) \geq l(v, w)$ and $l(u, w) - l(v, w) \equiv r (\mathrm{mod}\, s)$, where $r \in \{0, 1, 2, \cdots, s - 1\}$. Then

$$l(u, w) - l(v, w) = ts + r, \quad (t \in \mathbb{Z},\ t \geq 0),$$

$$l(u, w) - (l(v, w) + ts) = r. \tag{6}$$

When $r \in \{1, 2, \cdots, s - 1\}$, since $(s, p) = 1$, by Lemma 3.2 there exist $x, y \in \mathbb{Z}$ with $x \leq \lfloor \frac{s}{2} \rfloor$ and $y \leq \lfloor \frac{p}{2} \rfloor$ such that either

$$xp - ys = r \quad \text{or} \quad ys - xp = r.$$

If

$$xp - ys = r, \tag{7}$$

then from (6) and (7) we have

$$xp + l(v, w) + ts = ys + l(u, w).$$

That is

$$u \overset{l(u,w)+ys}{\longrightarrow} w \quad \text{and} \quad v \overset{l(v,w)+xp+ts}{\longrightarrow} w. \tag{8}$$

When $r = 0$, from (6) we have $l(u, w) = l(v, w) + ts$, where $t \in \mathbb{Z}$ and $t \geq 0$. Hence

$$u \overset{l(u,w)}{\longrightarrow} w \quad \text{and} \quad v \overset{l(v,w)+ts}{\longrightarrow} w. \tag{9}$$

Therefore by (8) and (9) we obtain $k_{u,v}(D) \leq ys + l(u, w)$.

Similarly, if $ys - xp = r$, then we have

$$k_{u,v}(D) \leq xp + l(u, w). \tag{10}$$

From the above we have the following lemma.

Lemma 3.3 *Let D be a primitive digraph, and let s and p be two different cycles lengths in D. Suppose that $2 \le s < p \le n$ and $\gcd(s,p) = 1$. Then*

$$k_{u,v}(D) \le \min\{|y|s, |x|p\} + l_{u,v}, \tag{11}$$

where (x, y) is the integer solution of the equation $xp + ys = r$ with minimum absolute value and where $|l(u,w) - l(v,w)| \equiv r(\bmod s)$.

Note that the number $k_{u,v}(D)$ not only depends on $l_{u,v}$, it also depends on r. Since $k(D) = \max_{u,v}\{k_{u,v}(D)\}$, we have

$$k(D) \le \max_{u,v}\{\min\{|y|s, |x|p\} + l_{u,v}\}, \tag{12}$$

where x and y satisfy the conditions of Lemma 3.3. By Lemma 3.2 we have $|y| \le \lfloor \frac{p}{2} \rfloor$ and $|x| \le \lfloor \frac{s}{2} \rfloor$.

Theorem 3.4 *Let D be a primitive digraph, let p and s be different cycles lengths of D. Suppose that $\gcd(p,s) = 1$ and $2 \le s < p \le n$. Then*

$$k(D) \le \min\{\lfloor \frac{p}{2} \rfloor s,\ \lfloor \frac{s}{2} \rfloor p\} + \max_{\substack{u,v \\ u \ne v}}\{l_{u,v}\}, \tag{13}$$

where $l_{u,v} = \max\{l(u,w), l(v,w)\}$, $l(u,w)$ and $l(v,w)$ are the lengths of directed walks from u to w and v to w that meet with cycles of lengths p and s.

3.3 The scrambling index of a primitive digraph with a Hamilton cycle

In this section we consider the scrambling index of primitive digraphs with a Hamilton cycle.

Theorem 3.5 *Let D be a primitive digraph of order n with a Hamilton cycle, and let the girth of D be s, where $1 \le s \le n-1$. If $\gcd(n,s) = 1$, then*

$$k(D) \le n - s + \begin{cases} (\frac{s-1}{2})n, & \text{when } s \text{ is odd,} \\ (\frac{n-1}{2})s, & \text{when } s \text{ is even.} \end{cases}$$

Proof. When $s = 1$, it is easy to see that $k(D) \le n-1$. Next we consider the case that $s \ge 2$. For $u, v \in V(D)$, there exist $v_i, v_j \in C_s$ such that $u \xrightarrow{l} v_i$ and $v \xrightarrow{l} v_j$, where $0 \le l \le n-s$. Then $k_{u,v}(D) \le k_{v_iv_j}(D) + l$. Therefore it suffices to show that $k_{v_i,v_j}(D) \le (\frac{s-1}{2})n$ when s is odd and $k_{v_i,v_j}(D) \le (\frac{n-1}{2})s$ when s is even for all $v_i, v_j \in C_s$. If $v_i = v_j$, then $k_{v_i,v_j} = 0$. Next we consider the case where $v_i \ne v_j$.

Case 1. s is odd. Since $(s,n)=1$, by Lemma 3.2 for each $t\in\{1,2,\cdots,s-1\}$, there exist positive integers x and y such that

$$xn-ys=t \quad \text{or} \quad ys-xn=t,$$

where $x\leq\lfloor\frac{s}{2}\rfloor$ and $y\leq\lfloor\frac{n}{2}\rfloor$.

Suppose $d(v_i,v_j)=t$, where $1\leq t\leq s-1$. If $xn-ys=t$, then

$$v_i\xrightarrow{t}v_j\xrightarrow{ys}v_j \text{ and}$$

$$v_j\xrightarrow{xn}v_j.$$

Therefore we have

$$k_{v_i,v_j}(D)\leq xn\leq(\frac{s-1}{2})n.$$

If $ys-xn=t$, since $v_i\xrightarrow{t}v_j$ and $v_i,v_j\in C_s$, then $v_j\xrightarrow{s-t}v_i$. We also have $xn-(y-1)s=s-t$. Therefore we get

$$v_j\xrightarrow{s-t}v_i\xrightarrow{(y-1)s}v_i \text{ and}$$

$$v_i\xrightarrow{xn}v_i.$$

Thus $k_{v_i,v_j}(D)\leq xn\leq(\frac{s-1}{2})n$.

Case 2. s is even. Then n is odd. First consider all the pairs of vertices v_i and v_j such that $v_i\xrightarrow{\frac{s}{2}}v_j$. The integer solution of equation $xn+ys=\frac{s}{2}$ with minimum absolute value is $x=\frac{s}{2}$ and $y=-\frac{n-1}{2}$.

If $v_i\xrightarrow{\frac{s}{2}}v_j$, we have $v_j\xrightarrow{n-\frac{s}{2}}v_i$. Since $\frac{s}{2}n-(\frac{n-1}{2})s=\frac{s}{2}$, then

$$(\frac{n-1}{2})s-(\frac{s}{2}-1)n=n-\frac{s}{2}.$$

Therefore

$$v_j\xrightarrow{n-\frac{s}{2}}v_i\xrightarrow{(\frac{s}{2}-1)n}v_i \quad\text{and}$$

$$v_i\xrightarrow{(\frac{n-1}{2})s}v_i.$$

In that case $k_{v_i,v_j}(D)=(\frac{n-1}{2})s$.

Next suppose $d(v_i,v_j)=r<\frac{s}{2}$, where $s>2$. By Lemma 3.2 there exist positive integers x and y with $x\leq\frac{s}{2}$ and $y\leq\frac{n-1}{2}$ such that either $xn-ys=r$ or $ys-xn=r$. We claim that if $xn-ys=r$, then $y\leq\frac{n-1}{2}-1$. To see this note that if $y=\frac{n-1}{2}$, then $xn-ys=(x-\frac{s}{2})n+\frac{s}{2}$ and $|(x-\frac{s}{2})n+\frac{s}{2}|>r$. Analogously we find that if $ys-xn=r$, then $x\leq\frac{s}{2}-1$. Hence $k_{v_i,v_j}(D)\leq\{(\frac{n-1}{2}-1)s,(\frac{s}{2}-1)n\}+\frac{s}{2}-1<(\frac{n-1}{2})s$. □

Denote

$$k(n,s) = \begin{cases} (\frac{s-1}{2})n, & \text{when } s \text{ is odd,} \\ (\frac{n-1}{2})s, & \text{when } s \text{ is even,} \end{cases}$$

and

$$K(n,s) = k(n,s) + n - s.$$

Let $D_{s,n}$ denote a digraph with a Hamilton cycle and unique cycle of length s, where the Hamilton cycle is $1 \to n \to n-1 \to \cdots \to 2 \to 1$ and the cycle of length s is $1 \to s \to s-1 \to \cdots \to 2 \to 1$ as shown in Figure 1.

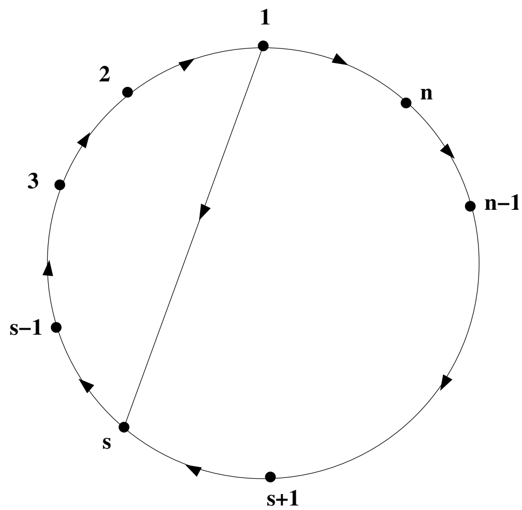

Fig. 1. $D_{s,n}$

Corollary 3.6 *Let D be a primitive digraph of order n with a Hamilton cycle, and let the girth of D be s, where $1 \leq s \leq n-1$ and $\gcd(s,n) = 1$. If $k(D) = K(n,s)$, then D contains a subgraph isomorphic to $D_{s,n}$.*

Proof. By Theorem 3.5, we know that $k_{v_i,v_j}(D) \leq k(n,s)$ for $v_i, v_j \in C_s$. We claim that the cycle of length s is formed from s consecutive vertices on the Hamilton cycle. Otherwise for any two vertices u and v of D, we can get to vertices s_1 and s_2 on the cycle C_s by directed walks of lengths less than $n-s$. Then $k_{u,v}(D) < k(n,s) + n - s$. This is contradiction to $k(D) = K(n,s)$.

Suppose $v_s, v_{s-1}, \cdots, v_1$ are the s consecutive vertices on the Hamilton cycle that form the cycle C_s, and let the Hamilton cycle be $v_1 \to v_n \stackrel{n-s}{\to} v_s \to v_{s-1} \cdots v_1$. Then there is an arc from vertex v_1 to vertex v_s. Otherwise any arc from v_1 to v_i, $2 \leq i \leq s-1$, will produce a cycle with length less than s. This is a contradiction, since the girth of D is s. $\Box$

Next we consider the digraph $D_{s,n}$, and throughout in this paper we label $D_{s,n}$ as in Figure 1. By the definition of the digraph $D_{s,n}$ it is obvious that $D_{s,n}$ is primitive if and only if $\gcd(s,n) = 1$. Cho and Kim [5] have obtained the formula for $k(D_{s,n})$ when n is even and $s > n/2$. We will find the exact value of $k(D_{s,n})$ for all cases and give the list of all pairs of vertices u and v of $D_{s,n}$ such that $k_{u,v}(D_{s,n}) = k(D_{s,n})$. For $D_{s,n}$, if $s = 1$, it is very easy to see that $k_{n,i}(D_{s,n}) = n - 1$ for each $i \neq n$, and for all $i, j \neq n$, $k_{i,j}(D_{s,n}) < n - 1$.

Lemma 3.7 *Suppose that* $\gcd(s, n) = 1$, *and* $s \geq 2$. *For vertices* u, $v \in V(D_{s,n})$, $u, v \neq n$, *then* $k_{u,v}(D_{s,n}) < K(n, s)$.

Proof. For a pair of vertices u and v in $D_{s,n}$, if $u, v \neq n$, then there are vertices s_1 and s_2 on cycle C_s such that $l(u, s_1) = l(v, s_2) < n - s$. By Theorem 3.5, we know that $k_{s_1,s_2}(D_{s,n}) \leq k(n, s)$. Therefore $k_{u,v}(D_{s,n}) \leq k_{s_1,s_2}(D_{s,n}) + l(u, s_1) \leq k(n, s) + l(u, s_1) < k(n, s) + n - s = K(n, s)$. □

By Lemma 3.7 we know that the upper bound on the scrambling index for the digraph $D_{s,n}$ is achieved for vertex n and some vertex (or vertices) u ($\neq n$) in $D_{s,n}$. Also notice that for vertices n and u ($\neq n$), the local scrambling index $k_{n,u}(D_{s,n})$ is attained with s as the double-cycle vertex. Therefore for vertices n and u ($\neq n$) in $D_{s,n}$, we always choose vertex s as the double-cycle vertex. Below we explain how to find $k_{n,u}(D_{s,n})$.

Remark 3.8

We consider the following two cases.

(a) Vertex u is not on the cycle C_s. From the digraph we know that there are unique directed paths from vertices n and u to vertex s, and $d(n, s) = n - s$, $d(u, s) = u - s$, $d(n, u) = n - u$.

Suppose $d(n, u) \equiv d'(\text{mod} s)$, so that $d(n, u) = d' + ts$ for some nonnegative integer t. If $d' = 0$, then $d(n, u) = t's$ for some positive integer t'. Hence $d(n, s) = d(u, s) + t's$. In that case we have

$$n \xrightarrow{d(n,s)} s \quad \text{and}$$

$$u \xrightarrow{d(u,s)} s \xrightarrow{t's} s.$$

Since the directed walks from vertices n and u to vertex s are unique, then we have $k_{n,u}(D_{s,n}) = d(n, s) = n - s$.

If $d' \geq 1$, then by Lemma 3.2, there exist unique positive integers x and y with minimum absolute value such that $xn - ys = d'$ or $ys - xn = d'$, where $|x| \leq \lfloor \frac{s}{2} \rfloor$ and $|y| \leq \lfloor \frac{n}{2} \rfloor$. Without loss of generality suppose that $xn - ys = d'$. Then

$$n \xrightarrow{d(n,s)} s \xrightarrow{ys} s \quad \text{and}$$
$$v \xrightarrow{d(u,s)} s \xrightarrow{xn+ts} s.$$

Hence $k_{n,u}(D) = ys + d(n, s)$.

(b) Vertex u is on the cycle C_s. Then $d(n,s) = n - s$, and there are exactly two different directed paths from vertex u to vertex s. They are $u \stackrel{u-1}{\rightarrow} 1 \stackrel{1}{\rightarrow} s$ and $u \stackrel{u-1}{\rightarrow} 1 \stackrel{1}{\rightarrow} n \stackrel{n-s}{\rightarrow} s$. Let $d_1 = u$ and $d_2 = n - s - u$.

Suppose $d_i \equiv d_i'(\mathrm{mod} s)$, so that $d_i = d_i' + ts$ for some nonnegative integer t, where $i = 1, 2$. For each d_i', $i = 1, 2$, similar to (a), we can find directed walks from vertices n and u to vertex s of the same lengths. Denote the lengths of these directed walks by $f_{n,u}^{(i)}$, $i = 1, 2$. In that case, $k_{n,u}(D) = \min\{f_{n,u}^{(i)}, i = 1, 2\}$.

Lemma 3.9 *Let $D = D_{s,n}$. Then for all vertices u and v in D, $l_{u,v}(D) \leq \max\{n - s, \lfloor \frac{n}{2} \rfloor\}$.*

Proof. Let $u, v \in V(D)$ and w be a double-cycle vertex of vertices u and v.

Case 1. If $u, v \in C_s$, then either $d(u,v) \leq \lfloor \frac{s}{2} \rfloor$ or $d(v,u) \leq \lfloor \frac{s}{2} \rfloor$. Without loss of generality, suppose we have $d(u,v) \leq \lfloor \frac{s}{2} \rfloor$. Then let $w = v$. We have $l_{u,v}(D) \leq \lfloor \frac{s}{2} \rfloor < \lfloor \frac{n}{2} \rfloor$.

Case 2. If $u, v \notin C_s$, take $w = s$. Then $l_{u,v}(D) \leq n - s$.

Case 3. If $u \in C_s,\ v \notin C_s$, consider the following two cases. If $s \leq \lceil \frac{n+1}{2} \rceil$, take $w = s$. Then $d(v,s) \leq n - s, d(u,s) \leq s - 1 \leq n - s$.

If $s > \lceil \frac{n+1}{2} \rceil$ and $d(v,u) \leq \lfloor \frac{n}{2} \rfloor$, then let $w = u$. Otherwise we have $s > \lceil \frac{n+1}{2} \rceil$ and $d(u,v) \leq \lfloor \frac{n}{2} \rfloor$. From the digraph we know that $d(v,s) \leq n - s$ and $d(u,s) \leq d(u,v) \leq \lfloor \frac{n}{2} \rfloor$. In that case let $w = s$, and we have $l_{u,v} \leq \max\{n - s, \lfloor \frac{n}{2} \rfloor\}$. □

Theorem 3.10 *Let $D = D_{s,n}$ and* $\gcd(s,n) = 1$*, where $2 \leq s \leq n - 1$. Then*

$$k(D) = K(n,s). \tag{14}$$

Proof. For a pair of vertices u and v in D, if $u, v \neq n$, then by Lemma 3.7 we have $k_{u,v}(D) < K(n,s)$.

Next we consider all the pairs of vertices n and u, and show that $k_{n,u}(D) = K(n,s)$ for some vertices u in D. We consider the following three cases.

Case 1. s is odd and n is even. We have

$$(\frac{n}{2})s - (\frac{s-1}{2})n = \frac{n}{2}.$$

Let $\frac{n}{2} \equiv r(\mathrm{mod} s)$, so that $\frac{n}{2} = r + t's$ for some nonnegative integer t'. Then

$$(\frac{n}{2} - t')s - (\frac{s-1}{2})n = r, \tag{15}$$

and

$$(\frac{s-1}{2})n-(\frac{n}{2}-t'-1)s=s-r. \tag{16}$$

Case 1.1. $d(n,u)\equiv r(\text{mod} s)$. Since $\frac{n}{2}=r+t's$, $n=2r+2t's$. Let $h=2t'$, then $n=2r+hs$. From the digraph we know that $d(n,u)\equiv r(\text{mod} s)$ when $u=n-r-ts$, where $t\in\{0,1,2,\cdots,h\}$. When $t\in\{0,1,2,\cdots,h-1\}$, vertex $n-r-ts\notin C_s$ and $d(n,s)-d(u,s)=r+ts$. Using Remark 3.8 (a) and (15) we have

$$k_{n,n-r-ts}(D)=(\frac{s-1}{2})n+n-s.$$

Suppose that $t=h$, so that $n-r-ts=r$. Since $r<s$, vertex $r\in C_s$. Then as in Remark 3.8 (b) there are two different directed walks from vertex r to vertex s; they are $r\xrightarrow{r}s$ and $r\xrightarrow{r+n-s}s$.

For the directed path whose length is r, if $n-s>r$, then $n-s-r=r+(2t'-1)s$ or $n-s-(r+(2t'-1)s)=r$. Then by (15) we have

$$n\xrightarrow{n-s}s\xrightarrow{(\frac{s-1}{2})n}s\quad\text{and}$$
$$r\xrightarrow{r}s\xrightarrow{(\frac{n}{2}-t')s+(2t'-1)s}s.$$

If $n-s<r$, then $n=2r$, $r-(n-s)=s-r$ and

$$(\frac{s-1}{2})n-(\frac{n}{2}-1)s=s-r. \tag{17}$$

Therefore by (17) we have

$$n\xrightarrow{n-s}s\xrightarrow{(\frac{s-1}{2})n}s\quad\text{and}$$
$$r\xrightarrow{r}s\xrightarrow{(\frac{n}{2}-1)s}s.$$

In this case let $f^{(1)}_{n,r}=k(n,s)+n-s$.

For the directed path whose length is $n-s+r$, by (15) we have

$$n\xrightarrow{n-s}s\xrightarrow{(\frac{n}{2}-t')s}s\quad\text{and}$$
$$r\xrightarrow{n-s+r}s\xrightarrow{(\frac{s-1}{2})n}s.$$

Let $f^{(2)}_{n,r}=(\frac{s-1}{2})n+n-s+r$. Therefore by Remark 3.8 (b), we have $k_{n,r}(D_{s,n})=\min\{f^{(1)}_{n,r},f^{(2)}_{n,r}\}=k(n,s)+n-s$.

Case 1.2. $d(n,u)\not\equiv r(\text{mod} s)$. Then $d(n,u)\neq\frac{n}{2}$, and by Lemma 3.9 we have $l_{nu}\le\{\frac{n}{2}-1,n-s\}$. By Lemma 3.3 and (16) we have

$$k_{n,u}(D)\le(\frac{n}{2}-t'-1)s+l_{nu}<k(n,s)+n-s$$

as desired.

Case 2. s is odd and n is odd. We have

$$(\frac{n-1}{2})s-(\frac{s-1}{2})n=\frac{n-s}{2}.$$

Let $\frac{n-s}{2}\equiv r(\mathrm{mod} s)$, so that $\frac{n-s}{2}=r+t's$ for some nonnegative integer t'. Then

$$(\frac{n-1}{2}-t')s-(\frac{s-1}{2})n=r, \tag{18}$$

and

$$(\frac{s-1}{2})n-(\frac{n-1}{2}-t'-1)s=s-r. \tag{19}$$

Case 2.1. $d(n,u)\equiv r(\mathrm{mod} s)$. Since $\frac{n-s}{2}=r+t's$, we have $n=2r+(2t'+1)s$. Let $h=2t'+1$, then $n=2r+hs$. From the digraph we know that $d(n,u)\equiv r(\mathrm{mod} s)$ when $u=n-r-ts$, where $t\in\{0,1,2,\cdots,h\}$. When $t\in\{0,1,2,\cdots,h-1\}$, vertex $n-r-ts\notin C_s$, and $d(n,s)-d(u,s)=r+ts$. Using Remark 3.8 (a) and (18), we have

$$k_{n,n-r-ts}(D)=(\frac{s-1}{2})n+n-s.$$

When $t=h$, then $n-r-ts=r$. Since $r<s$, vertex $r\in C_s$. Then as in Remark 3.8 (b), there are two different directed walks from vertex r to vertex s, and they are $r\xrightarrow{r}s$ and $r\xrightarrow{r+n-s}s$.

For the directed path whose length is r, since $n-s-r=r+2t's$, then by (18) we have

$$n\xrightarrow{n-s}s\xrightarrow{(\frac{s-1}{2})n}s \text{ and}$$
$$r\xrightarrow{r}s\xrightarrow{(\frac{n-1}{2}-t')s+2t's}s.$$

Let $f^{(1)}_{n,r}=k(n,s)+n-s$.

For the directed path whose length is $n-s+r$, since $n-s+r-n-s=r$, then by (18) we have

$$n\xrightarrow{n-s}s\xrightarrow{(\frac{n-1}{2}-t')s}s \text{ and}$$
$$r\xrightarrow{n-s+r}s\xrightarrow{(\frac{s-1}{2})n}s.$$

Let $f^{(2)}_{n,r}=(\frac{s-1}{2})n+n-s+r$. Therefore $k_{n,r}(D)=\min\{f^{(1)}_{n,r},\ f^{(2)}_{n,r}\}=k(n,s)+n-s$.

Case 2.2. $d(n,u)\not\equiv r(\bmod s)$. We know from Lemma 3.9 that $l_{nu}\le\max\{\frac{n-1}{2},n-s\}$. Therefore by Lemma 3.3 and (19) we get

$$k_{nu}(D)\le\max\{\frac{n-1}{2},n-s\}+(\frac{n-1}{2}-t'-1)s<k(n,s)+n-s,$$

as desired.

Case 3. s is even and n is odd. We have

$$(\frac{s}{2})n-(\frac{n-1}{2})s=\frac{s}{2}. \tag{20}$$

Case 3.1. $d(n,u)\equiv\frac{s}{2}(\mathrm{mod}\, s)$. If $n>\frac{3s}{2}$, then $n-\frac{s}{2}>s$. Let $d(n,u)=\frac{s}{2}+t's$ for some nonnegative integer t'. Then

$$(\frac{s}{2})n-(\frac{n-1}{2}-t')s=\frac{s}{2}+t's. \tag{21}$$

If $u\notin C_s$, then by Remark 3.8(a) and (21) we have

$$k_{n,u}(D)=(\frac{n-1}{2}-t')s+n-s.$$

When $t'=0$, then $u=n-\frac{s}{2}$ and $k_{n,n-\frac{s}{2}}(D)=k(n,s)+n-s$.

If $u\in C_s$, then $t'\geq 1$. Otherwise if $t'=0$, then $u=n-\frac{s}{2}$. Since $n-\frac{s}{2}>s$, then $u\notin C_s$, this is a contradiction. In that case, $k_{n,u}(D)=(\frac{n-1}{2}-t')s+n-u=(\frac{n-1}{2})s+n-s-(t'-1)s-u<k(n,s)+n-s$.

If $n<\frac{3s}{2}$, then $n-\frac{s}{2}<s$, so that vertex $n-\frac{s}{2}\in C_s$. There is only one vertex such that $d(n,u)\equiv\frac{s}{2}(\mathrm{mod}\, s)$. It is vertex $n-\frac{s}{2}$. By Remark 3.8 (b), there are two directed walks from vertex $n-\frac{s}{2}$ to vertex s, and their lengths are $n-\frac{s}{2}$ and $n-\frac{s}{2}+n-s$. For the directed walk whose length is $n-\frac{s}{2}$ we have

$$n\overset{n-s}{\longrightarrow}s \ \text{ and } n-\frac{s}{2}\overset{n-\frac{s}{2}}{\longrightarrow}s.$$

Since $n-\frac{s}{2}-(n-s)=\frac{s}{2}$, then by (20) we have $f^{(2)}_{n,\frac{n}{2}-s}=(\frac{n-1}{2})s+n-\frac{s}{2}$.

For the directed path whose length $n-\frac{s}{2}+n-s$, we have $n-\frac{s}{2}\overset{n-\frac{s}{2}+n-s}{\longrightarrow}s$ and $n\overset{n-s}{\longrightarrow}s$. Note that $d(n-\frac{s}{2},n)=n-\frac{s}{2}$. By (20) we have

$$(\frac{n-1}{2})s-(\frac{s}{2}-1)n=n-\frac{s}{2}. \tag{22}$$

Then

$$n\overset{n-s}{\longrightarrow}s\overset{(\frac{n-1}{2})s}{\longrightarrow}s \ \text{ and}$$
$$n-\frac{s}{2}\overset{n-\frac{s}{2}+n-s}{\longrightarrow}s\overset{(\frac{s}{2}-1)n}{\longrightarrow}s.$$

Therefore $f^{(2)}_{n,\frac{n}{2}-s}=(\frac{s}{2}-1)n+n-s+n-\frac{s}{2}=k(n,s)+n-s$. Therefore $k_{n,n-\frac{s}{2}}(D)=\min\{f^{(1)}_{n,\frac{n}{2}-s},\ f^{(2)}_{n,\frac{n}{2}-s}\}=k(n,s)+n-s$.

Case 3.2. $d(n,u) \not\equiv \frac{s}{2}(\mathrm{mod} s)$. Then

$$k_{n,u}(D) \le (\frac{n-1}{2} - 1)s + \max\{n-s, \frac{n-1}{2}\} < k(n,s) + n - s.$$

□

Let r be the positive integer that is defined as follows

$$r \equiv \begin{cases} \frac{n}{2}(\mathrm{mod} s), & \text{if } s \text{ is odd and } n \text{ is even,} \\ \frac{n-s}{2}(\mathrm{mod} s), & \text{if both } s \text{ and } n \text{ are odd .} \end{cases} \quad (23)$$

From the proof of Theorem 3.10, we know all the pairs of vertices u and v in $D_{s,n}$ such that $k_{u,v}(D_{s,n}) = K(n,s)$.

Corollary 3.11 *Suppose that* $\gcd(s,n) = 1$*, and* $s \ge 2$*. Then for* $u, v \in V(D_{s,n})$*, without loss of generality take* $u > v$*,* $k_{u,v}(D_{s,n}) = K(n,s)$ *if and only if* $u = n$ *and*

(1) $v = n - r - ts$ *for some* $t \in \{0, 1, 2, \cdots, \frac{n-2r}{s}\}$*, when* s *is odd.*

(2) $v = n - \frac{s}{2}$*, when* s *is even.*

3.4 Upper bounds on the scrambling indices for arbitrary primitive digraphs

In this section we consider upper bounds on the scrambling indices for general primitive digraphs.

Lemma 3.12 *Let* D *be a primitive digraph with a Hamilton cycle and let the girth of* D *be* s*, where* $\gcd(n,s) = 1,\ 2 \le s < n$*. Then either the cycle* C_s *is formed from* s *consecutive vertices on the Hamilton cycle or there is another cycle of length* p *such that* $\gcd(s,p) = q$*, where* $q \le \frac{s}{2}$ *when* s *is even and* $q \le \frac{s}{3}$ *when* s *is odd.*

Proof. In the following we give the proof for the case that s is even. The case that s is odd is similar. Suppose the Hamilton cycle of D is

$$1 \to n \to n-1 \to \cdots \to 2 \to 1.$$

If D contains an arc from vertex i to vertex $i + s - 1(\mathrm{mod} n)$ for some i, then the cycle C_s is formed from s consecutive vertices on the Hamilton cycle. Otherwise the cycle C_s includes s vertices that are not all consecutive on the Hamilton cycle. Suppose (u,v) is an arc on cycle C_s, and that u and v are not consecutive vertices on the Hamilton cycle. Then there is a directed path from vertex v to vertex u through the Hamilton cycle. This directed path with the (u, v) arc forms a directed cycle. Denote this directed cycle by C_{uv}. Suppose p is the length of C_{uv}; then $s < p < n$.

If $\gcd(s, p) = q,\ q \leq \frac{s}{2}$, then we are done. Otherwise, suppose the arcs $(a_i, a_i'),\ i = 1, 2, \cdots, m$, are all the non-consecutive arcs on the cycle C_s in order, and p_i is the length of cycle $C_{a_i a_i'}$. Then

$$s = m + \sum_{i=1}^{m-1} (a_i' - a_{i+1}) + a_m^{'} - a_1,$$

and

$$p_i = \begin{cases} a_i' - a_i + 1, & \text{when } a_i' > a_i, \\ n - (a_i - a_i') + 1, & \text{when } a_i' < a_i \ . \end{cases}$$

Summing the p_i we obtain

$$\sum_{i=1}^{m} p_i = m + \sum_{i=1}^{m-1} (a_i' - a_{i+1}) + a_m^{'} - a_1 + tn \ = s + tn,$$

where t is the number of cycles $C_{a_i a_i'}$ with $a_i' < a_i$. Therefore $t \leq m \leq s$.

We claim that $t < s$. If $m < s$, then clearly $t < s$. If $m = s$, then $a_i^{'} = a_{i+1}$ for $i = 1, 2, \cdots, m-1$, $a_m^{'} = a_1$, and the cycle C_s is $a_1 \to a_2 \to \cdots \to a_m \to a_1$. Without loss of generality, suppose that $a_1 = \max\{a_i, i = 1, 2, \cdots, m\}$. We have $a_m < a_1$, so $t \leq m - 1 < s$. Hence $t < s$.

Since $s | p_i,\ i = 1, 2, \cdots, m$, then $s | tn$ and $\gcd(s, n) = 1$, so $s | t$. But $t < s$, a contradiction. □

Lemma 3.13 *Let D be a primitive digraph with n vertices, and suppose that s is the girth of D with $s \geq 2$. If there is another cycle of length p, $s < p \leq n$, such that* $\gcd(s, p) = 1$, *then*

$$k(D) \leq K(n, s).$$

Proof. We consider the following three cases.

Case 1. $p = n$. By Theorem 3.5, we have the result.

Case 2. $p = n - 1$. Then the cycle C_s and the cycle C_p have at least $s - 1$ and at most s common vertices.

If the cycle C_s and the cycle C_p have s common vertices, then we consider the subgraph of D that contains C_s and C_p. As in the proof of Theorem 3.5 we have $k_{ij}(D) \le k(n-1, s)$ for $i, j \in C_s$. Hence

$$k(D) \le k(n-1, s) + n - s < k(n, s) + n - s.$$

If the cycles C_s and C_p have $s-1$ common vertices, then only one vertex of the cycle C_s does not belong to the cycle C_p, and we have $k_{ij}(D) \le k(n-1, s)+1$ for $i, j \in C_s$.

When $s \ge 4$, we have $k(D) \le k(n-1, s) + 1 + n - s < k(n, s) + n - s$.

When $s = 2$, the digraph D has a spanning subgraph D_1 as in Figure 2. For D_1, $k(D_1) = 2n - 4$. Hence $k(D) \le 2n - 4 < 2n - 3$

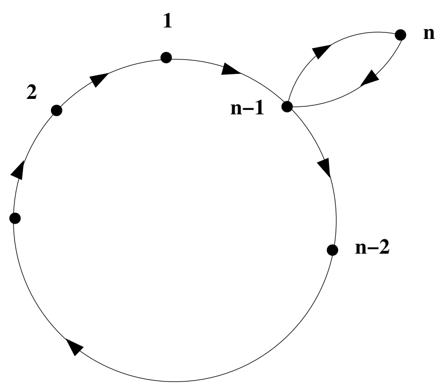

Fig. 2. D_1

When $s = 3$, the digraph D has a subgraph D_2 as in Figure 3. For $u, v \in V(D_2)$, we have $l_{u,v} \le n-3$. By Theorem 3.4 we get $k(D_2) \le \lfloor \frac{s}{2} \rfloor (n-1) + l_{u,v} \le 2n - 4$. Hence $k(D) \le 2n - 4 < 2n - 3$.

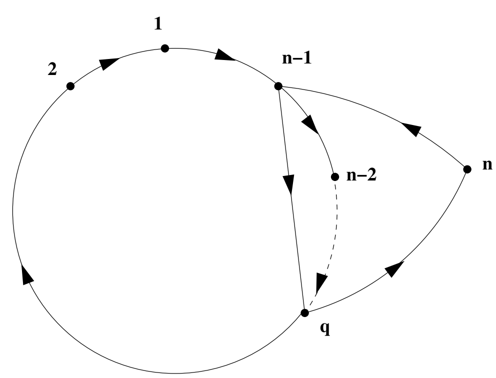

Fig. 3. D_2

Case 3. $p \leq n-2$. For $u, v \in V(D)$, we can find vertices $s_1, s_2 \in C_s$ such that

$$u \xrightarrow{n-s} s_1 \quad \text{and} \quad v \xrightarrow{n-s} s_2.$$

If there exists a vertex $w \in C_s \cap C_p$, then $l_{s_1,s_2} \leq s-1$. Otherwise $l_{s_1,s_2} \leq n-p$. Then $l_{u,v} \leq n-s+l_{s_1,s_2}$, and by Lemma 3.3 we have

$$k_{u,v}(D) \leq \min\{\lfloor \frac{p}{2} \rfloor s, \lfloor \frac{s}{2} \rfloor p\} + l_{u,v}.$$

Case 3.1. s is even. Then

$$k_{u,v}(D) \leq (\frac{p-1}{2})s + l_{u,v}. \tag{24}$$

Case 3.1.1. If $C_s \cap C_p \neq \emptyset$, we have $l_{u,v} \leq n-s+l_{s_1,s_2} \leq n-1$. Then

$$\begin{aligned} k_{u,v}(D) &\leq (\frac{p-1}{2})s + n - 1 \\ &\leq (\frac{n-3}{2})s + n - 1 \\ &= (\frac{n-1}{2})s - s + n - 1 < (\frac{n-1}{2})s + n - s. \end{aligned}$$

Case 3.1.2. If $C_s \cap C_p = \emptyset$, we have $l_{u,v} \leq n-s+l_{s_1,s_2} \leq n-s+n-p$. We consider the following two cases.

(a) $p < n-2$ and $s > 2$. Then

$$\begin{aligned} k_{u,v}(D) &\leq (\frac{p-1}{2})s + n - p + n - s \\ &= p(\frac{s}{2} - 1) + n - \frac{s}{2} + n - s \\ &< (n-2)(\frac{s}{2} - 1) + n - \frac{s}{2} + n - s \\ &= (\frac{n-1}{2})s - s + 2 + (n-s) \leq (\frac{n-1}{2})s + n - s. \end{aligned}$$

(b) $p \leq n-2$ and $s = 2$. In that case we need to show that $k_{u,v}(D) < 2n-3$ for $u, v \in V(D)$. Suppose vertices s_1 and s_2 are in C_s. Then there is a vertex w on the cycle C_p such that $\max\{l(s_1, w), l(s_2, w)\} \leq n-p$. Without loss of generality, suppose that $l(s_1, w) = \max\{l(s_1, w), l(s_2, w)\}$.

If $l(s_1, w) < n-p$, and $l(s_1, w)$ and $l(s_2, w)$ have the same parity, then $k_{s_1,s_2}(D) < n-p < n-2$. Otherwise $l(s_1, w) = 2t+1+l(s_2, w)$ for some nonnegative integer t. Then

$$s_1 \xrightarrow{p-1} s_1 \xrightarrow{l(s_1,w)} w \quad \text{and}$$

$$s_2 \xrightarrow{2t} s_2 \xrightarrow{l(s_2,w)} w \xrightarrow{p} w.$$

Hence $k_{s_1,s_2}(D) \leq l(s_1, w)+p-1 \leq n-2$. In that case, for vertices $u, v \in V(D)$, there exist vertices s' and s'' on the cycle C_s such that $\max\{l(u, s'), l(v, s'')\} < n-2$. Then $k_{u,v}(D) \leq n-2+n-2 = 2n-4 < 2n-3$.

If $l(s_1, w) = n-p$, then the digraph D has a spanning subgraph D_3 as in Figure 4. We have

$$s_1 \xrightarrow{p-1} s_1 \xrightarrow{n-p} w \quad \text{and}$$
$$s_2 \xrightarrow{n-p-1} w \xrightarrow{p} w.$$

Hence $k_{s_1,s_2}(D) = n-1$.

If for vertices u, v, there exist vertices s' and s'' on the cycle C_s such that

$\max\{l(u, s'), l(v, s'')\} < n-2$, then $k_{u,v}(D) < n-2+n-1 = 2n-3$.

Otherwise $\max\{l(u, s'), l(v, s'')\} = n-2$. Without loss of generality, suppose that $l(u, s') = n-2$. Then vertex v is on the directed walk from vertex u to vertex s'. If $l(v, s'')$ has the same parity as $n-2$, then $k_{u,v}(D) = n-2$. Otherwise $n-2 = 2t+1+l(v, s')$ for some nonnegative integer t. We have

$$u \xrightarrow{n-2+p} s' \quad \text{and}$$

$$v \xrightarrow{l(v,s')} s' \xrightarrow{2t+p+1} s'.$$

Hence $k_{u,v}(D) \leq n-2+p \leq n-2+n-2 < 2n-3$.

Case 3.2. s is odd ($s \geq 3$). Then

$$k_{u,v}(D) \leq (\frac{s-1}{2})p + l_{u,v}. \tag{25}$$

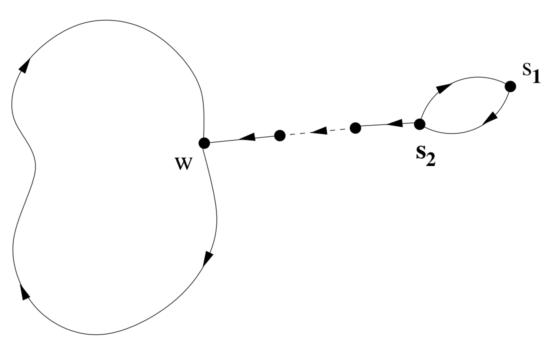

Fig. 4. D_3

Case 3.2.1. If $C_s \cap C_p \neq \emptyset$, we have $l_{u,v} \leq n-1$. Then we consider the following two cases.

(a) $p < n-2$. Then

$$\begin{aligned} k_{u,v}(D) &\leq (\frac{s-1}{2})p + n - 1 \\ &< (\frac{s-1}{2})(n-2) + n - 1 \\ &= (\frac{s-1}{2})n + n - s. \end{aligned}$$

(b) $p = n-2$. In that case we have $l_{u,v} \leq n - s + \frac{s-1}{2}$. Hence

$$\begin{aligned} k_{u,v}(D) &\leq (\frac{s-1}{2})(n-2) + n - s + \frac{s-1}{2} \\ &= (\frac{s-1}{2})n + n - s - \frac{s-1}{2} \\ &< (\frac{s-1}{2})n + n - s. \end{aligned}$$

Case 3.2.2. If $C_s \cap C_p = \emptyset$, we have $l_{u,v} \leq n - s + l_{s_1,s_2} \leq n - s + n - p$. Since $s \geq 3$, then $p < n-2$. We consider the following two cases.

(a) $p < n-2$ and $s > 3$. Then

$$\begin{aligned} k_{u,v}(D) &\leq (\frac{s-1}{2})p + n - p + n - s \\ &= p(\frac{s-1}{2} - 1) + 2n - s \\ &< (n-2)(\frac{s-1}{2} - 1) + 2n - s \\ &\leq (\frac{n-1}{2})s + n - s. \end{aligned}$$

(b) $p < n-2$ and $s = 3$. In that case we need to show that $k_{u,v}(D) < 2n-3$ for $u, v \in V(D)$. Since $\gcd(s,p) = 1$, then there exists a positive integer m such that $p = 3m+1$ or $p = 3m+2$. Let the cycle C_s be $s_1 \to s_2 \to s_3 \to s_1$. For a vertex s on the cycle C_s we can find a vertex w on the cycle C_p such that the directed walk from vertex s to vertex w does not contain the other two vertices of the cycle C_s and $l(s,w) \leq n-p-2$. Without loss of generality, suppose $s = s_3$; then $l(s_1,w) = l(s_3,w) + 2$ and $l(s_2,w) = l(s_3,w) + 1$. Hence $l(s_2,w) \leq n-p-1$ and $l(s_1,w) \leq n-p$.

$(b.1)$ $p = 3m+1$. Since $p - 3m = 1$, we have

$$s_1 \xrightarrow{3m} s_1 \xrightarrow{l(s_1,w)} w \quad \text{and}$$

$$s_2 \xrightarrow{l(s_2,w)} w \xrightarrow{p} w.$$

Therefore $k_{s_1,s_2}(D) \le l(s_2, w) + p \le n - p - 1 + p = n - 1$.

Also since $3(m+1) - p = 2$, we have

$$s_1 \xrightarrow{l(s_1,w)} w \xrightarrow{p} w \quad \text{and}$$

$$s_3 \xrightarrow{3(m+1)} s_3 \xrightarrow{l(s_3,w)} w.$$

Therefore $k_{s_1,s_3}(D) \le l(s_1, w) + p \le n - p + p = n$.

For $u, v \in V(D)$, we can find vertices s', s'' on the cycle C_3 such that $l(u, s') = l(v, s'') \le n - 3$. If $s' = s''$, then $k_{u,v}(D) \le n - 3$. If $s' = s_1$ and $s'' = s_2$, then $k_{u,v}(D) \le k_{s_1,s_2}(D) + n - 3 \le n - 3 + n - 1 = 2n - 4 < 2n - 3$. If $l(u, s_1) = l(v, s_3) < n-3$, then $k_{u,v}(D) < k_{s_1 s_3}(D) + n - 3 = n + n - 3 = 2n - 3$.

So the only remaining case is $l(u, s_1) = l(v, s_3) = n-3$. For the directed walks from vertices u and v to vertices s_1 and s_3, one of them does not go through the cycle C_3. Otherwise $l(u, s_1) = l(v, s_3) = n - 3 - ts < n - 3$ for some nonnegative integer t. Without loss of generality, suppose that the directed walk from vertex u to vertex s_1 does not go through the cycle C_3, and that $l(u, s_1) = n - 3$. Then the directed walk from v to s_3 also passes through the vertex s_1, and we have $l(v, s_1) = n - 5$. Since $l(u, s_1) = n - 3 > n - p$, the directed walk from vertex u to vertex s_1 also passes through the cycle C_p. Hence

$$u \xrightarrow{n-3+p} s_1 \quad \text{and}$$
$$v \xrightarrow{n-5} s_1 \xrightarrow{3(s+1)} s_1.$$

Therefore $k_{u,v}(D) \le n - 3 + p < 2n - 5$.

$(b.2)$ $p = 3m+2$. Since $p - 3m = 2$ we get

$$s_1 \xrightarrow{3m} s_1 \xrightarrow{l(s_1,w)} w \quad \text{and}$$

$$s_3 \xrightarrow{l(s_3,w)} w \xrightarrow{p} w.$$

Therefore $k_{s_1,s_3}(D) \le l(s_3, w) + p \le n - p - 2 + p = n - 2$.

Also since $3(m+1) - p = 1$, we have

$$s_1 \xrightarrow{l(s_1,w)} w \xrightarrow{p} w \quad \text{and}$$

$$s_2 \xrightarrow{3(m+1)} s_2 \xrightarrow{l(s_2,w)} w.$$

Therefore $k_{s_1,s_2}(D) \le l(s_1, w) + p \le n - p + p = n$.

For $u, v \in V(D)$, we can find vertices s', s'' on the cycle C_3 such that $l(u, s') = l(v, s'') \le n - 3$. If $s' = s''$, then $k_{u,v}(D) \le n - 3$. If $s' = s_1$ and $s'' = s_3$, then $k_{s_1,s_3}(D) \le n - 3 + n - 2 = 2n - 5 < 2n - 3$.

If $s' = s_1$ and $s'' = s_2$, and $l(u, s_1) = l(v, s_2) < n - 3$, then $k_{s_1,s_2}(D) < 2n - 3$.

So the only remaining case is $l(u, s_1) = l(v, s_2) = n-3$. For the directed walks from vertices u and v to vertices s_1 and s_2, one of them does not go through the cycle C_3. Otherwise $l(u, s_1) = l(v, s_2) = n-3-ts$ for some nonnegative integer t. Without loss of generality, suppose that the directed walk from vertex u to vertex s_1 does not go through the cycle C_3, and that $l(u, s_1) = n - 3$. Then the directed walk from v to s_2 also passes through the vertex s_1, and we have $l(v, s_1) = n - 4$. Since $l(u, s_1) = n - 3 > n - p$, the directed walk from vertex u to vertex s_1 also passes through the cycle C_p. Hence

$$u \xrightarrow{n-3+p} s_1 \quad \text{and}$$

$$v \xrightarrow{n-4} s_1 \xrightarrow{3(s+1)} s_1.$$

Therefore $k_{u,v}(D) \le n - 3 + p < 2n - 5$. □

From the proof of Lemma 3.13 we have the following result.

Corollary 3.14 *Let D be a primitive digraph with n vertices, and suppose that s is the girth of D with $s \ge 2$. If there is another cycle of length p, $s < p \le n - 1$, such that $\gcd(s, p) = 1$, then*

$$k(D) < K(n, s).$$

Let D be a primitive digraph with n vertices, and let $L(D) = \{s, a_1, \cdots, a_r\}$ be the set of distinct cycle lengths of D, where $2 \le s < a_1 < \cdots < a_r \le n$. Next we consider the case that $\gcd(s, a_i) \ne 1$ for each $i = 1, 2, \cdots, r$.

Lemma 3.15 *Let D be a primitive digraph with n vertices, and s be the girth of D with $s \ge 2$. Let $L(D) = \{s, a_1, a_2, \cdots, a_{r-1}, a_r\}$. If $\gcd(s, a_i) \ne 1$ for each $i = 1, 2, \cdots, r$, Then*

$$k(D) < K(n, s).$$

Proof. Since $\gcd(s, a_i) \ne 1$ for each $i = 1, 2, \cdots, r$, then s is not a prime number and $s \ge 6$. There exists a directed cycle of length p, $s < p \le a_r$, such that $\gcd(s, p) \le \frac{s}{3}$. Otherwise, if $\gcd(s, a_i)$ is equal to either s or $\frac{s}{2}$ for each i, then $\gcd(s, a_1, a_2, \cdots, a_r) \ge \frac{s}{2}$. This is a contradiction to the fact that $\gcd(s, a_1, a_2, \cdots, a_r) = 1$. Suppose $\gcd(s, p) = t$, where $2 \le t \le \frac{s}{3}$.

We know that if D is primitive, then D^t is also primitive. Further D^t contains t cycles of length $\frac{s}{t}$, and t cycles of length $\frac{p}{t}$. Let $s' = \frac{s}{t}$ and $p' = \frac{p}{t}$, then $\gcd(s', p') = 1$ and $s' < p' \leq \frac{n}{t}$. For $u, v \in V(D^t)$ we can find vertices $s_1, s_2 \in C_{s'}$ such that

$$u \xrightarrow{n-s'} s_1 \quad \text{and} \quad v \xrightarrow{n-s'} s_2.$$

Case 1. $C_{s'} \cap C_{p'} \neq \emptyset$ in D^t. There exists a vertex $w \in C_{s'} \cap C_{p'}$, then $l_{s_1,s_2} \leq s' - 1$.

When s' is even, then

$$\begin{aligned} k_{u,v}(D^t) &\leq (\frac{p'-1}{2})s' + n - 1 \\ &= (\frac{p-t}{2t})\frac{s}{t} + n - 1. \end{aligned}$$

Thus

$$k_{u,v}(D) \leq t k_{u,v}(D^t) \leq \frac{ps}{2t} - \frac{s}{2} + tn - t.$$

When s' is odd, Then

$$\begin{aligned} k_{u,v}(D^t) &\leq (\frac{s'-1}{2})p' + n - 1 \\ &= (\frac{s-t}{2t})\frac{p}{t} + n - 1. \end{aligned}$$

Thus

$$k_{u,v}(D) \leq t k_{u,v}(D^t) \leq \frac{ps}{2t} - \frac{p}{2} + tn - t.$$

Since $p > s$, $\frac{ps}{2t} - \frac{p}{2} + tn - t < \frac{ps}{2t} - \frac{s}{2} + tn - t$. Let $k(t) = \frac{ps}{2t} - \frac{s}{2} + tn - t,$ where $2 \leq t \leq \frac{s}{3}$. Note that $k(t)$ is concave up as a function of t on the interval $[2, \frac{s}{3}]$. Hence it attains its maximum at one of the end points. When $t = 2$, we have $k(2) = \frac{ps}{4} - \frac{s}{2} + 2n - 2 < (\frac{n-1}{2})s + n - s$.

Suppose that $t = \frac{s}{3}$. If s is odd, then $s > 7$ and $k(\frac{s}{3}) = \frac{3p}{2} - \frac{s}{2} + \frac{sn}{3} - \frac{s}{3}$. Since $p \leq n$, we get

$$\frac{3p}{2} - \frac{s}{2} + \frac{sn}{3} - \frac{s}{3} \leq \frac{3n}{2} + \frac{sn}{3} - \frac{5s}{6} < (\frac{s-1}{2})n + n - s. \tag{26}$$

If s is even, then $s \geq 6$ and $k(\frac{s}{3}) = \frac{3p}{2} - \frac{s}{2} + \frac{sn}{3} - \frac{s}{3}$. Similarly we have

$$\frac{3p}{2} - \frac{s}{2} + \frac{sn}{3} - \frac{s}{3} \leq \frac{3n}{2} + \frac{sn}{3} - \frac{5s}{6} < (\frac{n-1}{2})s + n - s.$$

Case 2. $C_{s'} \cap C_{p'} = \emptyset$ in D^t. Then we can find a vertex w in $C_{p'}$ such that

$$s_1 \overset{n-tp'}{\longrightarrow} w \quad \text{and} \quad s_2 \overset{n-tp'+\frac{s'}{2}}{\longrightarrow} w.$$

When s' is even, we have

$$\begin{aligned} k_{u,v}(D^t) &\leq (\frac{p'-1}{2})s' + 2n - tp' - \frac{s'}{2} \\ &= (\frac{p-t}{2t})\frac{s}{t} + 2n - p - \frac{s}{2t}. \end{aligned}$$

Hence

$$k_{u,v}(D) \leq t k_{u,v}(D^t) \leq \frac{ps}{2t} + 2nt - pt - s. \tag{27}$$

When s' is odd, we have

$$\begin{aligned} k_{u,v}(D^t) &\leq (\frac{s'-1}{2})p' + n - tp' + \frac{s'}{2} + n - s' \\ &= \frac{sp}{2t^2} - \frac{p}{2t} + n - p + \frac{s}{2t} + n - \frac{s}{t}. \end{aligned}$$

Thus

$$k_{u,v}(D) \leq t k_{u,v}(D^t) \leq \frac{sp}{2t} + 2nt - pt - \frac{p}{2} - \frac{s}{2}. \tag{28}$$

Since $\frac{ps}{2t} + 2nt - pt - \frac{p}{2} - \frac{s}{2} \leq \frac{ps}{2t} + 2nt - pt - s$, we consider the expression $\frac{ps}{2t} + 2nt - pt - s$. Let $k(t) = \frac{ps}{2t} + 2nt - pt - s$. Then $k(t)$ is concave up on any compact subinterval of $\mathbb{R}^+$, so it attains its maximum at one of the end points.

If $s \geq 8$, then there exists a directed cycle of length p, such that $\gcd(s, p) \leq \frac{s}{4}$. Otherwise $\gcd(s, a_i)$ is equal to one of s, $\frac{s}{2}$ or $\frac{s}{3}$. Then $\gcd(s, a_1, a_2, \cdots, a_r) \geq \frac{s}{6}$. This is a contradiction to the fact that $\gcd(s, a_1, a_2, \cdots, a_r) = 1$. Thus we check at the two end points $t = 2$ and $t = \frac{s}{4}$.

If $t = 2$, $k_{u,v}(D) \leq \frac{ps}{4} + 4n - 2p - s = \frac{p(s-8)}{4} + 4n - s$. When $s = 8$, $k_{u,v}(D) \leq 4n - 8 < 5n - 12$ $(= (\frac{n-1}{2})s + n - s)$. When $s > 8$,

$$
\begin{aligned}
k_{u,v}(D) &\le \frac{p(s-8)}{4} + 4n - s \\
&\le \frac{(n-s)(s-8)}{4} + 4n - s \\
&< (\frac{s-1}{2})n + n - s.
\end{aligned}
$$

If $t = \frac{s}{4}$, we have $k_{u,v}(D) \le \frac{p(8-s)}{4} + \frac{ns}{2} - s$. When $s = 8$, $k_{u,v}(D) \le \frac{ns}{2} - s < (\frac{s-1}{2}) + n - s$. When $s > 8$,

$$
k_{u,v}(D) \le \frac{ns}{2} - \frac{p(s-8)}{4} - s \le \frac{ns}{2} - \frac{(s+2)(s-8)}{4} - s
$$

$$
= \frac{ns}{2} - \frac{s(s-2)}{4} + 4 < \ (\frac{s-1}{2})n + n - s.
$$

There is only one remaining case, namely $t = 2$ and $s = 6$. In that case, there exists a cycle of length p such that $\gcd(s,p) = 2$. Otherwise $\gcd(s, a_i) = 3$ for all $i = 1, 2, \cdots, r$, and $\gcd(s, a_1, \cdots, a_r) \ne 1$. This is a contradiction. Since $s = 6$, then $p \ge 8$. We have $t = 2$ and $s' = 3$, then by (28), we get $k_{u,v}(D) \le 4n - p - 3 \le 4n - 11 < 4n - 9$ $(= (\frac{n-1}{2})s + n - s)$. $\square$

From the proof of Lemma 3.15, we get the following corollary.

Corollary 3.16 *Let D be a primitive digraph of order n, and s be the girth of D with $s \ge 2$. If there is a cycle of length p, $s < p \le n$, such that* $\gcd(s,p) < s/3$ *or* $\gcd(s,p) \le s/3$ *and $C_s \cap C_p \ne \emptyset$, then*

$$
k(D) < K(n, s).
$$

From Lemma 3.13 and Lemma 3.15, we have the main result of this paper.

Theorem 3.17 *D be a primitive digraph with n vertices and girth s. Then*

$$
k(D) \le K(n, s). \tag{29}
$$

Since $k(D_{s,n}) = K(n,s)$, the upper bound in (29) is attainable. Comparing upper bounds on $k(D)$ in Theorem 3.17 with Cho and Kim's [5] result on $cindex(D)$, the upper bounds on $k(D)$ and $cindex(D)$ are the same when n is odd and s is even, and for all other cases, the upper bounds on $k(D)$ are less than the upper bounds on $cindex(D)$.

When $s = n - 1$, $K(n, n-1) = \left\lceil \frac{(n-1)^2+1}{2} \right\rceil$.

Theorem 3.18 *Let D be a primitive digraph of order n. Then*

$$k(D) \le \left\lceil \frac{(n-1)^2+1}{2} \right\rceil . \tag{30}$$

Equality holds if and only if $D = D_{n-1,n}$.

Proof. For a primitive digraph D, we have $s \le n-1$. Then by Theorem 3.10 and Theorem 3.17, we get the inequality in (30). When $s = n-1$, apart from labeling of the vertices, there are only two primitive digraphs; they are $D_{n-1,n}$ and $D_{n-1,n} \cup \{2 \to n\}$. By Theorem 3.10, we know that $k(D_{n-1,n}) = K(n, n-1)$. Let $D' = D_{n-1,n} \cup \{2 \to n\}$. By Corollary 3.11, we know that there is only one pair of vertices in $D_{n,n-1}$ that can attain the upper bound, and they are vertex n and some vertex u ($\neq 1$). Similarly, there is only one pair of vertices that can attain the upper bound in $D' - \{1 \to n\}$, and they are vertex 1 and some vertex v ($\neq n$). Therefore $k_{n,u}(D') < K(n,s)$ and $k_{1,v}(D') < K(n,s)$, and we can conclude that $k(D') < K(n,s)$. □

Remark: In a subsequent paper, we will give the characterization of primitive digraphs D with $k(D) = K(n,s)$.

References

[1] R.A. Brualdi, H.J. Ryser, *Combinatorial Matrix Theory, Encyclopedia of Mathematics and its Applications* 39, Cambridge University Press, Cambridge, 1991.

[2] A.L. Dulmage, N.S. Mendelsohn, The exponent of a primitive matrix, *Canad. Math. Bull.* 5 (1962), 241–244.

[3] R.L. Dobrushin, Central limit theorem for non-stationary Markov chains, I, II, *Theory Prob. Appl.*, 1, 65-80, 329-383.

[4] D.J. Hartfiel, U.G. Rothblum, Convergence of inhomogeneous products of matrices and coefficients of ergodicity, *Linear Algebra Appl.* 277 (1998), 1–9.

[5] H.H. Cho, H.H. Kim, Competition indices of digraphs, preprint, 2006.

[6] A. Paz, *Introduction to Probabilistic Automata*, Academic Press, New York, 1971.

[7] A. Rhodius, On explicit forms for ergodicity coefficients, *Linear Algebra Appl.* 194 (1993), 71–83.

[8] U.G. Rothblum, C.P. Tan, Upper bounds on the maximum modulus of subdominant eigenvalues of nonnegative matrices, *Linear Algebra Appl.* 66 (1985), 45–86.

[9] E. Seneta, Coefficients of ergodicity: structure and applications, *Adv. Appl. Prob.* 11 (1979), 576–590.

[10] E. Seneta, *Nonnegative Matrices and Markov Chains*, Springer-Verlag, New York, 1981.

[11] E. Seneta, Explicit forms for ergodicity coefficients of stochastic matrices, *Linear Algebra Appl.* 191 (1993), 245–252.

[12] C.P. Tan, A functional form for a particular coefficient of ergodicity, *J. Appl. Prob.* 19 (1982), 858–863.

[13] C.P. Tan, Coefficients of ergodicity with respect to vector norms, *J. Appl. Prob.* 20 (1983), 277–287.