On Padé Approximations, Quadratic Stability and Discretization of Switched Linear Systems

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Abstract

In this note we consider the stability preserving properties of diagonal Padé approximations to the matrix exponential. We show that while diagonal Padé approximations preserve quadratic stability when going from continuous-time to discrete-time, the converse is not true. We discuss the implications of this result for discretizing switched linear systems. We also show that for continuous-time switched systems which are exponentially stable, but not quadratically stable, a Padé approximation may not preserve stability.

Keywords: Padé Approximations, Quadratic Stability, Switched Linear Systems and Discretization.

1. Introduction

The Diagonal Padé approximations to the exponential function are known to map the open left half of the complex plane to the open interior of the unit disk [3]. This gives rise to a correspondence between continuous-time stable LTI (linear time invariant) systems and their discrete-time stable counterparts (a fact that is often exploited in the systems and control community [6]). Perhaps the best known map of this kind is the first order diagonal Padé approximant (also known as the bilinear or Tustin map [3]). The bilinear map is known not only to preserve stability, but also preserve quadratic Lyapunov functions. That is, a positive definite matrix P satisfying $A_c^*P + PA_c < 0$ will also satisfy $A_d^*PA_d - P < 0$ where A_d is the mapping

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of A_c under the bilinear transform [6] with some sampling time h [1]. This makes it extremely useful when transforming a continuous-time switching system:

$$\dot{x} = A_c(t)x, \qquad A_c(t) \in \mathscr{A}_c \tag{1}$$

into an approximate discrete-time counterpart[1],

$$x(k+1) = A_d(k)x(k), \quad A_d(k) \in \mathscr{A}_d \tag{2}$$

because, the existence of a common positive definite matrix P satisfying $A_c^*P + PA_c < 0$ for all all $A_c \in \mathscr{A}_c$ implies that the same P satisfies $A_d^*PA_d - P < 0$ for all $A_d \in \mathscr{A}_d$. Thus quadratic stability of the continuous-time switching system implies quadratic stability of the discrete-time counterpart. This property is useful in obtaining results in discrete-time from their continuous-time counterparts [6], and in providing a robust method to obtain a stable discrete-time switching system from a continuous-time one.

Our objective in this present note is to determine whether this property is preserved by higher order (more accurate) Padé approximants. From the point of view of discretization, low order approximants are not always satisfactory, and one often chooses higher order Padé approximations in real applications. Later we present an example of a exponentially stable continuous-time switching system for which a discretisation based on a first order Pade approximation is unstable, but, discretizations based on second order approximations are stable for any sampling time. Also, it is well known that the first order Padé approximation (the bilinear approximation) can map a negative real eigenvalue to a negative eigenvalue if the sampling time is large. In such situations, while stability is preserved, qualitative behavior is not preserved even for LTI systems; a non-oscillatory continuous mode is transformed into an oscillatory discrete-time mode. In this context we establish the following facts concerning general diagonal Padé approximations.

(i) Consider an LTI system $\Sigma_c : \dot{x} = A_c x$ and let $\Sigma_d : x(k+1) = A_d x(k)$ be any discrete-time system obtained from Σ_c using any diagonal Padé approximation and any sampling time. If V is any quadratic Lyapunov function for Σ_c then, V is a quadratic Lyapunov function for Σ_d.

(ii) The converse of the statement in (i) is only true for first order Padé approximations.

[1] Discretization error is zero, only at sampling instants.

(iii) Consider a switched system $\Sigma_{sc} : \dot{x} = A_{sc}(t)x,\ A_{sc}(t) \in \{A_{c1},...,A_{cn}\}$ and let $\Sigma_d : x(k+1) = A_{sd}(k)x(k),\ A_{sd}(k) \in \{A_{d1},...,A_{dn}\}$ be a discrete-time switched system obtained from Σ_{sc} using any diagonal Padé approximations and any sampling times. If V is any quadratic Lyapunov function for Σ_{sc} then, V is a quadratic Lyapunov function for Σ_{sd}.

(iv) The converse of the statement in (iii) is only true for first order Padé approximations.

(v) Consider an exponentially stable switched system $\Sigma_{sc} : \dot{x} = A_{sc}(t)x,\ A_{sc}(t) \in \{A_{c1},...,A_{cn}\}$. Let $\Sigma_d : x(k+1) = A_{sd}(k)x(k),\ A_{sd}(k) \in \{A_{d1},...,A_{dn}\}$ be a discrete-time switched system obtained from Σ_{sc} using a p'th order diagonal Padé approximation. Then, Σ_{ds} may be unstable, even when $p = 1$.

These results are quite subtle, but we believe that they are important for a number or reasons. Discretization of switched systems is a relatively new research direction in the control systems community. To the best of our knowledge, few papers exist on this topic; for example see [7]. In the context of such studies, our results say that quadratic stability is robust with respect to diagonal Padé approximations. That is, quadratic stability is always preserved, even when the sampling time is poorly chosen. This is an important fact when building simulators of switched linear systems. Our results also indicate that Padé approximations do not, in general, preserve the stability properties of exponentially (but not quadratically) stable systems. In such cases, building a (stability preserving) discrete-time simulation model of such systems that preserve stability is non-trivial and remains an open question.

The consequences of our observations go beyond numerical simulation. In many applications one converts a continuous-time switched system to a discrete-time equivalent before embarking on control design. Our results indicate that one must exhibit extreme caution in discretizing a continuous-time switched system model. In particular, care is needed in assuming that properties of the original continuous-time problem are inherited from properties of the-discrete time approximation [4]. In fact, stability of the discrete-time model does not necessarily imply stability of the continuous-time one: *even for discrete-time systems that are quadratically stable*. Our results also pose questions for model order reduction of switched linear systems. Again this is a relatively new area of study of considerable interest in the VLSI community. In such applications, where the ultimate objective is numerical simulation, stability may be preserved in the reduction of the continuous-time model to another lower order continuous-time model, only for it to be lost in the discretization step.

2. Mathematical Preliminaries

The following definitions and results are useful in developing the main result, Theorem 1, which is given in Section 3.

Notation : A square matrix A_c is said to be Hurwitz stable if all of its eigenvalues lie in the open left-half of the complex plane. A square matrix A_d is said to be Schur stable if all its eigenvalues lie in the open interior of the unit disc. The notation M^* is used to denote the complex conjugate transpose of a general square matrix M; M is hermitian if $M^* = M$. A hermitian matrix P is said to be positive (negative) definite if $x^*Px > 0$ ($x^*Px < 0$) for all non-zero x and we denote this by $P > 0$ ($P < 0$). In all of the following definitions, $P = P^* > 0$.

A matrix P is a Lyapunov matrix for a Hurwitz stable matrix A_c if $A_c^*P + PA_c < 0$. In this case, $V(x) = x^*Px$ is a quadratic Lyapunov function (QLF) for the continuous-time LTI system $\dot{x}(t) = A_c x(t)$. A matrix P is a Stein matrix for a Schur stable matrix A_d if $A_d^*PA_d - P < 0$. In this case, $V(x) = x^*Px$ is a quadratic Lyapunov function for the discrete-time LTI system $x(k+1) = A_d x(k)$.

Given a finite set of Hurwitz stable matrices $\mathscr{A}_c$ a matrix P is a common Lyapunov matrix (CLM) for $\mathscr{A}_c$ if $A_c^*P + PA_c < 0$ for all A_c in $\mathscr{A}_c$. In this case, we say that the continuous-time switching system (1) is quadratically stable (QS) with Lyapunov function $V(x) = x^*Px$ and V is a common quadratic Lyapunov function (CQLF) for $\mathscr{A}_c$.

Given a finite set of Schur stable matrices $\mathscr{A}_d$ a matrix P is a common Stein matrix (CSM) for $\mathscr{A}_d$ if $A_d^*PA_d - P < 0$ for all A_d in $\mathscr{A}_d$. In this case, we say that the discrete-time switching system (2) is quadratically stable (QS) with Lyapunov function $V(x) = x^*Px$ and V is a common quadratic Lyapunov function (CQLF) for $\mathscr{A}_d$.

Our primary interest in this note is to examine the invariance of quadratic Lyapunov functions under diagonal Padé approximations to the matrix exponential. Recall the definition of the diagonal Padé approximations to the exponential function.

Definition 1. *(Diagonal Padé Approximations) [3][12]: The p^{th} order diagonal Padé approximation to the exponential function e^s is the rational function C_p defined by*

$$C_p(s) = \frac{Q_p(s)}{Q_p(-s)} \tag{3}$$

where

$$Q_p(s) = \sum_{k=0}^{p} c_k s^k \quad \textit{and} \quad c_k = \frac{(2p-k)!p!}{(2p)!k!(p-k)!}. \tag{4}$$

Thus the p^{th} order diagonal Padé approximation to $e^{A_c h}$, the matrix exponential with sampling time h, is given by

$$C_p(A_c h) = Q_p(A_c h) Q_p^{-1}(-A_c h) \tag{5}$$

where $Q_p(A_c h) = \sum_{k=0}^{p} c_k (A_c h)^k$.

Much is known about diagonal Padé maps in the context of LTI systems. In particular, the fact that such approximations map the open left half of the complex plane to the interior of the unit disc is widely exploited in systems and control. This implies the well known fact that these maps preserve stability of LTI systems as stated formally in the following lemma.

Lemma 1. *[3] (Preservation of stability) Suppose that A_c is a Hurwitz stable matrix and, for any sampling time $h > 0$, let $A_d = C_p(A_c h)$ be a diagonal Padé approximation of $e^{A_c h}$ of any order p. Then A_d is Schur stable.*

A special diagonal Padé approximation is the first order approximation. This is also sometimes referred to as the bilinear (or Tustin) transform.

Definition 2. *(Bilinear transform) [3][12]: The first order diagonal Padé approximation to the matrix exponential with sampling time h is defined by:*

$$C_1(A_c h) = \left(I + A_c \frac{h}{2}\right)\left(I - A_c \frac{h}{2}\right)^{-1}. \tag{6}$$

This approximation is known to not only preserve stability, but also to preserve quadratic Lyapunov functions [1, 2, 6]; namely if P is a Lyapunov matrix for A_c then it is also a Stein matrix for $A_d = C_1(A_c h)$. The converse statement is also true. Actually, we have the following known result which is a special case of Lemma 3 below.

Lemma 2. *[2] (Preservation of Lyapunov functions) Suppose that A_c is a Hurwitz stable matrix and, for any sampling time $h > 0$, let $A_d = C_1(A_c h)$ be the first order diagonal Padé approximation (bilinear transform) of $e^{A_c h}$. Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for A_d.*

As we shall see, bilinear transforms play a key role in studying general diagonal Padé approximations. In particular, a *complex* version of this map that inherits some of the above properties will be very useful in what follows.

Lemma 3. *(The complex bilinear transform) Let A_c be a Hurwitz stable matrix and for any complex number λ with Re$(\lambda) > 0$, define the matrix*

$$A_d = (\lambda I + A_c)(\lambda^* I - A_c)^{-1} . \tag{7}$$

Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for A_d.

Proof : Consider any matrix $P = P^* > 0$. When A_d is given by (7), the Stein inequality $A_d^* P A_d - P < 0$ can be expressed as

$$(\lambda^* I - A_c)^{-*}(\lambda I + A_c)^* P(\lambda I + A_c)(\lambda^* I - A_c)^{-1} - P < 0 .$$

Post-multiplication by $\lambda^* I - A_c$ and pre-multiplication by $(\lambda^* I - A_c)^*$ results in the following equivalent inequality

$$(\lambda I + A_c)^* P(\lambda I + A_c) - (\lambda^* I - A_c)^* P(\lambda^* I - A_c) < 0 ,$$

which simplifies to

$$(\lambda + \lambda^*)(PA_c + A_c^* P) < 0 .$$

Since $\lambda + \lambda^* > 0$ this last inequality is equivalent to the Lyapunov inequality $PA_c + A_c^* P < 0$. Thus P is a Lyapunov matrix for A_c if and only if it is a Stein matrix for A_d. ■

The final basic result that we shall need concerns common Stein matrices for discrete-time systems. A proof of this (well known) lemma is given in the Appendix.

Lemma 4. *If P is a CSM for $A_1, \cdots, A_m$ then P is a Stein matrix for the matrix product $\prod_{i=1}^m A_i$.*

3. Main Result

We now present the main result of the paper: Theorem 1. A main consequence of this result is that common quadratic Lyapunov functions are preserved by all diagonal Padé discretizations for all sampling times. Thus, quadratic stability is preserved under all diagonal Padé discretizations of a quadratically stable continuous-time switched system. This result is stated formally in Corollary 1.

Theorem 1. *Suppose that A_c is a Hurwitz stable matrix and A_d is any p^{th} order Padé approximation to $e^{A_c h}$ for any $h > 0$. If P is a Lyapunov matrix for A_c then, P is a Stein matrix for A_d.*

Proof: Consider any matrix P which is a Lyapunov matrix for A_c. Recall that $A_d = Q_p(A_c h)Q_p^{-1}(-A_c h)$. Since the coefficients of the polynomial Q_p are real,

$$Q_p(sh) = kh^p \prod_{j=1}^{n} (\alpha_j + s) \prod_{i=1}^{m} (\lambda_i + s)(\lambda_i^* + s)$$

for some $k \neq 0$, where $2m + n = p$, the real numbers $-h\alpha_j, j = 1, \cdots, n$ are the real zeros of Q_p and the complex numbers $-h\lambda_i, -h\lambda_i^*,\ i = 1, \cdots, m$ are the non-real zeros of Q_p. Since all the zeros of Q_p have negative real parts ([3][12]) we must have $\alpha_j > 0$ for all j and $Re(\lambda_i) > 0$ for all i. It now follows that A_d can be expressed as

$$A_d = \left(\prod_{j=1}^{n} (\alpha_j I + A_c)\right)\left(\prod_{i=1}^{m} (\lambda_i I + A_c)(\lambda_i^* I + A_c)\right)\left(\prod_{i=1}^{m} (\lambda_i I - A_c)(\lambda_i^* I - A_c)\right)^{-1}\left(\prod_{j=1}^{n} (\alpha_j I - A_c)\right)^{-1}$$

which, due to commutativity of the factors, can be expressed as

$$A_d = \left(\prod_{j=1}^{n} (\alpha_j I + A_c)(\alpha_j^* I - A_c)^{-1}\right)\left(\prod_{i=1}^{m} (\lambda_i I + A_c)(\lambda_i^* I - A_c)^{-1}\right)\left(\prod_{i=1}^{m} (\lambda_i^* I + A_c)(\lambda_i I - A_c)^{-1}\right).$$

Hence A_d is a product of bilinear terms of the form $(\lambda I + A_c)(\lambda^* I - A_c)^{-1}$ where $Re(\lambda) > 0$. Since P is a Lyapunov matrix for A_c, it follows from Lemma 3 that P is a Stein matrix for each of the bilinear terms. Thus A_d is a product of a bunch of matrices each of which have P as a Stein matrix. It now follows from Lemma 4 that P is a Stein matrix for A_d. ■

The above theorem is illustrated in Figure 3. If we denote the convex cone of all positive definite matrices satisfying $A_c^T P + PA_c < 0$ by $\mathscr{L}_{A_c}$, and the convex cone of all positive definite matrices satisfying $A_d^T PA_c - P < 0$ by $\mathscr{S}_{A_d}$, this theorem

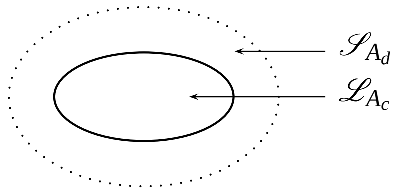

Figure 1: Illustration of Theorem 1.

establishes the fact that $\mathscr{L}_{A_c} \subseteq \mathscr{S}_{A_d}$. In other words, our main theorem states that if A_d is a diagonal Padé approximation of $e^{A_c h}$ for any $h > 0$ then a Lyapunov matrix for A_c is also a Stein matrix for A_d. Lemma 2 tells us that the converse of this statement is true for $p = 1$; namely for $p = 1$ we have that $\mathscr{L}_{A_c} = \mathscr{S}_{A_d}$. However, the converse of this statement is not necessarily true for $p \geq 2$; that is, for $p \geq 2$, a Stein matrix for A_d is not necessarily a Lyapunov matrix for A_c, and in general $\mathscr{L}_{A_c}$ is strictly contained in $\mathscr{S}_{A_d}$. This is demonstrated in the following example.

Example 1: Consider the Hurwitz stable matrix:

$$A_c = \begin{bmatrix} 1.56 & -100 \\ 0.1 & -4.44 \end{bmatrix}$$

Now consider the matrix A_d obtained under the 2^{nd} order diagonal Padé approximation of $e^{A_c h}$ with the discrete time step $h = 2$:

$$A_d = \begin{bmatrix} -0.039 & 0.4205 \\ -0.0004 & -0.0138 \end{bmatrix}$$

The matrix

$$P = \begin{bmatrix} 2.3294 & -0.0138 \\ -0.0138 & 2.7492 \end{bmatrix}$$

is a Stein matrix for A_d but is not a Lyapunov matrix for A_c.

The following corollary is easily deduced from the main theorem. This is probably the most useful result in the paper. It says that quadratic stability is preserved under all diagonal Padé discretizations of a quadratically stable continuous-time switched system.

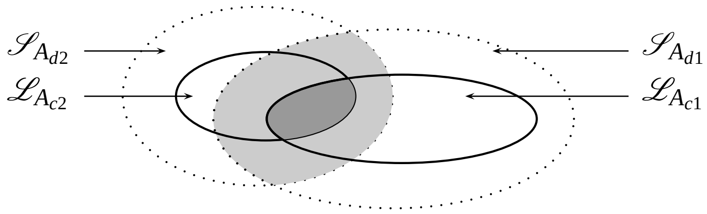

Figure 2: Two Padé approximations.

Corollary 1. *Suppose that $P = P^* > 0$ is a CLM for a finite set of matrices $\mathscr{A}_c$. Then P is CSM for any finite set of matrices $\mathscr{A}_d$, where each A_d in $\mathscr{A}_d$ is a diagonal Padé approximation of $e^{A_c h}$ of any order for some A_c in $\mathscr{A}_c$ and $h > 0$.*

Proof : If P is a CLM for $\mathscr{A}_c$ then, P is an Lyapunov matrix for every A_c in $\mathscr{A}_c$. It now follows from Theorem 1, that P is a Stein matrix for every A_d in $\mathscr{A}_d$. Hence P is a CSM for $\mathscr{A}_d$.

The last corollary shows that the diagonal Padé approximations preserve quadratic stability for switching systems. Thus, quadratic stability of a continuous-time switching system implies quadratic stability of a corresponding discrete-time switching system obtained via a diagonal Padé discretization. This is easily deduced by extending the situation in Figure 3 to multiple matrices, (see Figure 3).
However, it is very important to note that the corollary does not imply the converse. Namely, intersection of the discrete time sets $\mathscr{S}_{A_{d1}}$ and $\mathscr{S}_{A_{d2}}$ does not imply the intersection of the corresponding continuous time sets. In fact this converse is not true in general as the following example illustrates.

Example 2: Consider the Hurwitz stable matrices:

$$A_{c1} = \begin{bmatrix} 1.56 & -100 \\ 0.1 & -4.44 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

Since the matrix product $A_{c1}A_{c2}$ has negative real eigenvalues it follows that there is no CLM [9] for $\{A_{c1}, A_{c2}\}$. Now consider the matrices A_{d1}, A_{d2} obtained under the 2^{nd} order diagonal Padé approximation of $e^{A_{ci}h}$ with the discrete time step $h = 2$:

$$A_{d1} = \begin{bmatrix} -0.039 & 0.4205 \\ -0.0004 & -0.0138 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0.1429 & 0 \\ 0 & 0.8187 \end{bmatrix}.$$

These matrices have a CSM

$$P_d = \begin{bmatrix} 2.3294 & -0.0138 \\ -0.0138 & 2.7492 \end{bmatrix}.$$

Comment : Example 1, together with Corollary 1, illustrate the following facts. Let $\mathscr{A}_c$ be a finite set of Hurwitz stable matrices and $\mathscr{A}_d$ the corresponding finite set of Schur stable matrices obtained under diagonal Padé approximations for fixed p and h. If P is a CLM for $\mathscr{A}_c$ then P is a CSM for $\mathscr{A}_d$. However, as the example demonstrates, the existence of a CSM for $\mathscr{A}_d$ does not imply the existence of a CLM for $\mathscr{A}_c$.

4. A Converse Result

We have seen that if P is a Lyapunov matrix for A_c then, for any positive integer p, P is a Stein matrix for the p^{th} order Padé approximation of $e^{A_c h}$ for **all** $h > 0$ that is,

$$A_d(h)^* P A_d(h) - P < 0 \qquad \text{for all } h > 0\,,$$

where $A_d(h)$ is a diagonal Padé approximation (of any fixed order) to $e^{A_c h}$. The next lemma tells us that to achieve a converse result we need the following additional condition to hold,

$$\lim_{h \to 0} \frac{A_d(h)^* P A_d(h) - P}{h} < 0\,. \tag{8}$$

Lemma 5. *Suppose that, for all $h > 0$, the matrix $A_d(h)$ is a Padé approximation (of any fixed order) to $e^{A_c h}$. Then P is a Lyapunov matrix for A_c if and only if P is a Stein matrix for $A_d(h)$ for all $h > 0$ and (8) holds.*

Proof: In view of our previous results, we can prove this result if we show that

$$\lim_{h \to 0} \frac{A_d(h)^* P A_d(h) - P}{h} = P A_c + A_c^* P\,. \tag{9}$$

To demonstrate this limit, first recall that $A_d(h) = Q_p(A_c h) Q_p(-A_c h)^{-1}$ and

$$Q_p(A_c h) = I + \frac{1}{2}(A_c h) + h^2 D_p(A_c h)$$

where D_p is a polynomial. Hence

$$\lim_{h\to 0} Q_p(-A_c h) = I$$

and

$$\lim_{h\to 0} \frac{Q_p(A_c h)^* P Q_p(A_c h) - Q_p(-A_c h)^* P Q_p(-A_c h)}{h} = PA_c + A_c^* P$$

Since

$$A_d(h)^* P A_d(h) - P = Q_p(-A_c h)^{-*} \left[Q_p(hA_c)^* P Q_p(hA_c) - Q_p(-A_c h)^* P Q_p(-A_c h) \right] Q_p(-A_c h)^{-1}$$

we obtain the desired result (9). ■

5. Implications of Main Result

The starting point for our work was the recently published paper [6]. One of the main results of that paper was the fact that the bilinear transform preserves quadratic stability when applied to continuous-time switched systems. We have shown that this property also holds for general diagonal Padé approximations (although the converse statement is not true). This is an important observation due to the fact that while the bilinear transform is stability preserving, it is not always a good approximation to the matrix exponential. Our result says that “more accurate" approximations are also stability preserving when going from continuous-time to discrete-time.

Two potential applications of this result are immediate. First, stable discrete-time LTI systems can be obtained from their continuous-time counterparts in a manner akin to that described in [6]. Secondly, our results provide a method to discretize quadratically stable linear switched system in a manner that preserves stability; see [7] for a recent paper on this topic. That is, given a quadratically stable switched linear system, a discrete-time counterpart obtained using diagonal Padé approximations to the matrix exponential, will also be quadratically stable. Since this property is true for all orders of approximation, and for all sampling times, then our main result says that *quadratic stability is robustly preserved under Padé discretizations or any order.*

In the context of the previous comment, it is important to realize that the robust stability preserving property of Padé approximations is a unique feature of quadratically stable systems. It was recently shown that non-quadratic Lyapunov functions may not be preserved under the bilinear transform with sampling time $h = 2$. This fact was first demonstrated in [6], where it was proven that unlike quadratic Lyapunov functions, ∞-norm and 1-norm type Lyapunov functions are not necessarily preserved under the bilinear mapping with $h = 2$. In fact the situation may be worse as the following example illustrates.

Example 3: Consider a continuous-time switching system described by (1) with $\mathscr{A}_c = \{A_{c1}, A_{c2}, A_{c3}\}$ where

$$A_{c1} = \begin{bmatrix} -19.00 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -0.10 \end{bmatrix}, \quad A_{c2} = \begin{bmatrix} -19 & 0 & 0 \\ -10 & -9 & 0 \\ -18.75 & 0 & -0.10 \end{bmatrix}, \quad A_{c3} = \begin{bmatrix} -19.00 & 0 & 18.75 \\ 0 & -9 & 8.75 \\ 0 & 0 & -0.10 \end{bmatrix}.$$

Using the ideas in [8] (also see Theorem 2 in the next section) it can be shown that this continuous-time switching system is globally exponentially stable. It follows from the results of Dayawansa and Martin [5] that this switching system has a Lyapunov function (though this is not necessarily quadratic). Now consider a discrete-time approximation to the above system. We assume that switching is restricted to only occur at multiples of the sampling time $h = 0.25$. Using the first order Pad'e approximation, we obtain a discrete-time switching system described described by (2) with $\mathscr{A}_d = \{A_{d1}, A_{d2}, A_{d3}\}$ where

$$A_{di} = (I - \frac{1}{8}A_{ci})^{-1}(I + \frac{1}{8}A_{ci}), \qquad i = 1, 2, 3.$$

that is,

$$A_{d1} \approx \begin{bmatrix} -0.40 & 0 & 0 \\ 0 & -0.06 & 0 \\ 0 & 0 & 0.98 \end{bmatrix}, A_{d2} \approx \begin{bmatrix} -0.40 & 0 & 0 \\ -0.35 & -0.06 & 0 \\ -1.37 & 0 & 0.98 \end{bmatrix}, A_{d3} \approx \begin{bmatrix} -0.40 & 0 & 1.37 \\ 0 & -0.06 & 1.01 \\ 0 & 0 & 0.98 \end{bmatrix}.$$

We now claim that the discrete-time switching system is unstable. To see this we simply consider the incremental switching sequence $A_{d3} \to A_{d2} \to A_{d1}$; then the dynamics of the system evolve according to the product

$$A_d = A_{d1}A_{d2}A_{d3}.$$

Since the eigenvalues of A_d are approximately $\{-0.002, -0.060, -1.035\}$, then with one eigenvalue outside the unit disc, this switching sequence, repeated periodically results in an unstable system.

Clearly, by selecting a smaller sampling time one obtains a better approximation to the continuous-time system. However, selecting an appropriate sampling time is difficult for switched systems since sampling time is usually related to solution growth rates. While this is simple to calculate for an LTI system, bounds on the solution growth rates are usually very difficult to calculate for a switched system. On the other-hand, were the original system quadratically stable, then our main result implies that stability can never be lost by a bad or unlucky choice of sampling time.

5.1. A further comment on the counter example

Example 3 in the previous section indicates that our main result and its corollary do not, in general, extend to switched systems which are exponentially stable, but which do not have a quadratic Lyapunov function. An interesting question therefore to ask is how one discretizes a general, exponentially stable, switching system. In this section we give a preliminary result in this direction. Specifically, we take a closer look at Example 3, and ask the question as to how one might discretize the system in the example so that exponential stability is preserved irrespective of choice of sampling time. Our results can be summarised as follows:

(i) Even ordered Padé discretizations preserve exponential stability for the system class illustrated by Example 3. This is true for any even ordered approximation, and for any sampling time.

(ii) Odd ordered Padé discretizations preserve exponential stability provided the sampling time is smaller that a computable bound.

The above items say that even ordered Padé discretizations preserve stability in a robust manner; odd ordered ones do not. Example 3 is an example of a switching system of the form (1) where every matrix A_c in $\mathscr{A}_c$ has real negative eigenvalues and every pair of matrices in $\mathscr{A}_c$ have $n-1$ common eigenvectors (namely all such matrix pairs are pairwise triangularizable). It is shown in [8] that such systems are exponentially stable. This result follows from the following theorem in [8] which we give here to aid our discussion.

Theorem 2. *[8] Suppose $\mathscr{V} = \{v_1, \dots, v_{n+1}\}$ is a set of vectors in $\mathbb{R}^n$ with the property that any subset of n vectors is linearly independent. Let*

$$\mathscr{M} = \{M_i : i = 0, 1 \cdots, n\}$$

where $M_0 = [v_1 \ \cdots \ v_n]$ *and*

$$M_i = [v_1 \ ... \ v_{n+1} \ v_{i+1} \ ... \ v_n] \qquad \textit{for} \qquad i = 1, 2, \cdots, n, \tag{10}$$

that is, M_i *is obtained by replacing the i-th column in* M_0 *with the vector* v_{n+1}*. Let* $\mathscr{A}_c$ *be any finite subset of the following set of matrices:*

$$\{MDM^{-1} : M \in \mathscr{M} \textit{ and } D \textit{ is diagonal negative definite } \} \tag{11}$$

Then the continuous-time switching system (1) is globally exponentially stable.

Recently a discrete-time version of this result was obtained [10]. Namely a discrete-time switching system is exponentially stable if every pair of matrices in $\mathscr{A}_d$ share $n-1$ common eigenvectors, and if all eigenvalues are real, inside the unit circle, and positive [10] (i.e. there is no oscillatory behavior).
In both the discrete-time case and the continuous-time case, the same type of Lyapunov function is used to prove stability. Since Padé approximations are eigenvector preserving, it immediately follows that any approximations that map real negative eigenvalues to positive ones, will, by invoking the above result, preserve exponential stability.

Using the above observations we obtain our next result. To describe this result, consider any positive integer p and let

$$\bar{\alpha}_p = \begin{cases} \text{largest real zero of } Q_p \\ -\infty \text{ if } Q_p \text{ has no real zeros} \end{cases}$$

Since all real zeros of Q_p must be negative, we must have $\bar{\alpha}_p < 0$. When p is odd, Q_p must have at least one real zero; hence $\bar{\alpha}_p$ is finite. When p is even, we show later than Q_p does not have any real zeros; hence $\bar{\alpha}_p = -\infty$ for even p. To illustrate,

$$Q_1(s) = 1 + \frac{1}{2}s, \qquad Q_2(s) = 1 + \frac{1}{2}s + \frac{1}{12}s^2;$$

hence

$$\bar{\alpha}_1 = -2, \qquad \bar{\alpha}_2 = -\infty.$$

Theorem 3. *Suppose that* $\mathscr{A}_c$ *is set of matrices satisfying the hypotheses of Theorem 2 and let*

$$\underline{\alpha} = \min\{\alpha : \alpha \textit{ is an eigenvalue of } A_c \textit{ and } A_c \in \mathscr{A}_c\}.$$

Consider any positive integer p and define

$$\bar{h}_p = \begin{cases} \bar{\alpha}_p/\underline{\alpha} & \text{if } Q_p \text{ has a real zero} \\ \infty & \text{if } Q_p \text{ has no real zeros} \end{cases} \tag{12}$$

Let $\mathscr{A}_d$ be any finite subset of

$$\{C_p(hA_c) : A_c \in \mathscr{A}_c \text{ and } 0 < h < \bar{h}_p\}$$

Then the discrete-time switching system (2) is globally exponentially stable.

Proof : We first show that all the eigenvalues of the matrices in $\mathscr{A}_d$ must be positive, real and less than one. So, consider any matrix A_d in $\mathscr{A}_d$. This matrix can be expressed as $A_d = C_p(A_c h)$ where A_c is in $\mathscr{A}_c$ and $h < \bar{\alpha}_p/\underline{\alpha}$. From the description of $\mathscr{A}_c$ we have $A_c = MDM^{-1}$ where D is diagonal with negative diagonal elements, $\alpha_1, \cdots, \alpha_n$. Consider any $i = 1, \cdots, n$. Since α_i is an eigenvalue of A_c, it follows from the definition of $\underline{\alpha}$ that $\alpha_i \geq \underline{\alpha}$; hence $h\alpha_i \geq h\underline{\alpha}$. Recalling the requirement that $h < \bar{\alpha}_p/\underline{\alpha}$ and noting that $\underline{\alpha} < 0$ we must have $h\underline{\alpha} > \bar{\alpha}_p$; hence

$$h\alpha_i > \bar{\alpha}_p$$

Since $Q_p(s) \neq 0$ for $s > \bar{\alpha}_p$ where $\bar{\alpha})p < 0$ and $Q_p(0) = 1 > 0$, it follows from the continuity of Q_p that $Q_p(s) > 0$ for $s > \bar{\alpha}_p$; hence $Q_p(h\alpha_i) > 0$. Since $-h\alpha_i > 0$, we also have $Q_p(-h\alpha_i) > 0$. Hence $C_p(h\alpha_i) = Q_p(h\alpha_i)/Q_p(-h\alpha_i) > 0$. Since $h\alpha_i < 0$ and C_p maps the open left half plane into the open unit disk, we must also have $C_p(h\alpha_i) < 1$. Since $A_d = C_p(A_c h)$ and $A_c = MDM^{-1}$, we have

$$A_d = M\Lambda M^{-1}$$

where Λ is diagonal with diagonal elements

$$\Lambda_{ii} = C_p(h\alpha_i), \quad i = 1, \cdots, p$$

Hence $C_p(h\alpha_1), \ldots, C_p(h\alpha_p)$, are the eigenvalues of A_d and these eigenvalues are positive, real and less that one.

We will now show that

$$\mathscr{A}_d = \{e^{\tilde{A}_c} : \tilde{A}_c \in \tilde{\mathscr{A}_c}\} \tag{13}$$

where $\tilde{\mathscr{A}_c}$ is a set of matrices which satisfy the hypotheses of Theorem 2. This will imply that the continuous-time switching system

$$\dot{x} = \tilde{A}_c(t)x(t) \qquad \tilde{A}_c(t) \in \tilde{\mathscr{A}_c} \tag{14}$$

is globally exponentially stable. Relationship (13) tells us that the state of the discrete-time system (2) corresponds to the state at $t = 0, 1, 2 \cdots$ of the continuous-time system (14) switching at these times; this will imply that the discrete-time switching system is globally exponentially stable. To achieve the above goal, consider any $i = 1, \cdots, p$ and we let $\tilde{\alpha}_i = \ln[C_p(h\alpha_i)]$. Then $\tilde{\alpha}_i$ is negative real and

$$C_p(h\alpha_i) = e^{\tilde{\alpha}_i}. \tag{15}$$

Now consider $\tilde{A}_c = M\tilde{D}M^{-1}$ where $\tilde{D}$ is the diagonal matrix with negative diagonal elements $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_p$. Since $\tilde{A}_c = M\tilde{D}M^{-1}$ we also have $e^{\tilde{A}_c} = M\tilde{\Lambda}M^{-1}$ where $\tilde{\Lambda}$ is diagonal with diagonal elements

$$\tilde{\Lambda}_{ii} = e^{\tilde{\alpha}_i}, \quad i = 1, \cdots, p.$$

It follows from (15) that $\tilde{\Lambda} = \Lambda$; hence

$$A_d = e^{\tilde{A}_c}.$$

Since $\mathscr{A}_c$ is a finite set of matrices satisfying the hypotheses of Theorem 2, it now follows that $\mathscr{A}_d$ can be expressed as (13) where $\tilde{\mathscr{A}}_c$ is a finite set of matrices satisfying the hypotheses of Theorem 2. As explained above this now implies that the discrete-time switching system is globally exponentially stable. ■

Note that $\underline{\alpha}$ is the most negative eigenvalue of the matrices in $\mathscr{A}_{\underline{c}}$. In the example of the previous section, $\underline{\alpha} = -19$ whereas $\overline{\alpha}_p = \overline{\alpha}_1 = -2$; hence $\overline{h}_p = -2/-19 = 0.1053$. In this example, $h = 0.25 > \overline{h}_p$ and so the hypotheses of the above theorem are not satisfied. It is easily verified that had we, in Example 2, discretized with $h < 0.1053$, the corresponding discrete-time switching would be have been exponentially stable.

Before proceeding to the next result, we briefly digress to show that for p even, the polynomial Q_p has no real zeros (hence $\overline{h}_p = \infty$ whenever p is even). This conclusion is evident from the following theorem. Throughout the paper, the order of a diagonal Padé approximation 'p' has been defined the order of the polynomial Q_p. But for a more general case, $R(z)$ is a rational approximation to e^z of order 'q', if $e^z - R(z) = Cz^{q+1} + \mathscr{O}(z^{q+2})$ with $C \neq 0$. Theorem 4 provides the maximum attainable order of such rational approximations under some conditions.

Theorem 4. *[11] Suppose that a rational approximation to the exponential function is given by $R(z) = P_k(z)/Q_j(z)$, where the subscripts k and j denote the orders* *of the polynomials P_k and Q_j respectively. Let Q_j have only m different complex zeros. If in addition Q_j has a real zero then, the order q of R satisfies*

$$q \leq k+m+1.$$

If Q_j has no real zeros then,

$$q \leq k+m.$$

A Padé approximation P_k/Q_j is a special case of the rational approximations considered in the above theorem and its order is $q = j+k$ [11], where k and j denote the orders of the polynomials P_k and Q_j. Hence, if Q_j has only m different complex zeros and at least one real zero, it must satisfy $j+k \leq k+m+1$, that is,

$$j \leq m+1\,.$$

If Q_j had a real zero when j is even, it must have two real zeros and, since Q_j has at least m complex zeros, this yields the contradiction that $j \geq m+2$. Hence, for a Padè approximation P_k/Q_j with j even, Q_j has no real zeros.

Comment: The above results tell us that for even order Padé approximations we have $\overline{h}_p = \infty$. This yields the next result.

Theorem 5. *Suppose that $\mathscr{A}_c$ is a finite set of matrices satisfying the hypotheses of Theorem 2 and p is any even positive integer. Then, for any sampling time, the discrete-time switching system (2) obtained under the p^{th} order diagonal Padé approximation is globally exponentially stable.*

The key point in the proof of the last theorem is that even ordered Padé polynomials do not have real zeros. It immediately follows that stability is preserved for any choice of sampling interval. Odd ordered Padé polynomials, on the other hand, have some real zeros, and these zeros can cause difficulties in ensuring that negative real eigenvalues map to positive ones. To preserve stability in this case one must select a sampling time that is small enough. To illustrate this point let us consider again the Example 3. We assume that switching is restricted to only occur at multiples of the sampling time $h = 1$ (which is chosen to illustrate the assertions in Theorem 3). As can be seen from the Table 1, the first two odd order approximations lead to an unstable discrete time switching system.

Order	$\lambda_{max}(A_{d1}A_{d2}A_{d3})$	Comment
1	2.5819	Unstable
2	0.5957	Stable
3	1.0710	Unstable
4	0.6539	Stable

Table 1: Stability of some even and odd approximations for Example 3

Comment : The results of this section indicate that the selection of stable Padé discretizations is guided strongly by the knowledge of the Lyapunov function for the original switched system. This suggests the following interesting open question. Namely, to determine if in choosing a discretization method for exponentially stable continuous-time switched systems, knowledge of a Lyapunov function for the original continuous-time system is required.

6. Conclusions

In this paper we have shown that diagonal Padé approximations to the matrix exponential preserves quadratic Lyapunov functions between continuous-time and discrete-time switched systems. We have also shown that the converse is not true. Namely, it does not follow that the original continuous-time system is quadratically stable even if the discrete-time system has a quadratic Lyapunov function. Furthermore, it is easily seen that such approximations do not (in general) preserve stability when used to discretize switched systems that are stable (but not quadratically stable). Our results suggest a number of interesting research directions. An immediate question concerns discretization methods that preserve other types of stability, see for example [13, 14]. Since general Padé approximations can be thought of as products of complex bilinear transforms, an immediate question in this direction concerns the equivalent map for other types of Lyapunov functions. Namely, given a continuous-time system with some Lyapunov functions, what are the mappings from continuous-time to discrete-time that preserve the Lyapunov functions. A natural extension of this question concerns whether discretization methods can be developed for exponentially stable switched and nonlinear systems but which do not have a quadratic Lyapunov function.

Appendix 1: Proof of Lemma 4

Suppose that P is a common Stein matrix for two matrices A_1 and A_2, that is,

$$A_1^* P A_1 < P \qquad \text{and} \qquad A_2^* P A_2 < P$$

Pre-multiply the first inequality by A_2^* and post-multiply it by A_2 and use the second inequality to obtain

$$A_2^* A_1^* P A_1 A_2 \leq A_2^* P A_2 < P\,,$$

that is,

$$(A_1 A_2)^* P (A_1 A_2) < P\,,$$

which implies that P is a Stein matrix for the product $A_1 A_2$. This shows that the statement of the lemma is true for $m = 2$. Now assume that it is true for $m = k$ and then let $M_k = \prod_{i=1}^{k} A_i$. Since $M_{k+1} = M_k A_{k+1}$, it follows from the result for two matrices that P is a Stein matrix for M_{k+1}. Hence by induction the proposed lemma is true for all m. So it can concluded that if all the constituent matrices of a product have a CSM P then P is a Stein matrix for the product. ■

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