String Instability in Classes of Linear Time Invariant Formation Control with Limited Communication Range

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Abstract

This paper gives sufficient conditions for string instability in an array of linear time-invariant autonomous vehicles with communication constraints. The vehicles are controlled autonomously and are subject to a rigid or semi-rigid formation policy. The individual controllers are assumed to have a limited range of forward and backward communication with other vehicles. Sufficient conditions are given that imply a lower bound on the maximum peak of the frequency response magnitude of the transfer function mapping a disturbance to the leading vehicle to a vehicle in the chain. This lower bound quantifies the effect of spacing separation policy, intervehicle communication policy, and vehicle settling response performance. These results extend earlier works to the case of heterogeneous, non-nearest neighbor and semi-rigid formation situations.

Index Terms

Performance limitations, string stability, formation control, bullwhip effect, distributed systems.

I. Introduction

There has recently been extensive interest in a range of cooperative control problems, including those of controlling the formation of a large number of autonomous vehicles;

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see for example [22], [7], [21]. In general, these can include three, or even six degrees of freedom in some spacecraft attitude and position control problems. In such systems, it is often desirable to achieve tight (that is, rigid) formation control whilst moving at constant velocity. The simplest case of such formations is one dimensional systems, with one of the main examples being the control of intelligent vehicle highway systems [11].

As early as [17], a difficulty known as 'string instability' has been observed in tight formation control of long strings of vehicles based on local information. Here we use the term 'String instability' to describe the amplification along the string of the response to a disturbance to the lead vehicle. Different measures of disturbance amplification have been proposed in the literature. For example, [17] uses a frequency domain (effectively an H_∞) definition, whilst a more complete discussion in [23] gives more formal definitions and uses the norm induced by the L_∞ norm in the definition of string stability. Discussions in an H_∞ setting are presented in [10], where the authors use the terminology of *ill-conditioning* and *non-scalability*, used in more general settings of networked dynamic systems, to describe similar phenomena. In this paper, we use the term 'string stability' to denote the situation where an appropriately defined H_∞ norm is bounded independently of the string length.

Although the problem setting above has been described for formation control of autonomous vehicles, very similar network structures and dynamics have been described in other application domains. For example, in the area of irrigation flow control (see for example [13]), a series of 'pools' connected by gates with local control laws is studied with the same phenomena being present. Another example of closely related dynamics occurs in supply chain, or production and inventory control systems [18]. These are often modeled in discrete time (see for example [4, Figure 1]) and use feedforward to achieve the equivalent of type II servo response (zero steady state error to ramp references), in some cases[1]. In this context, concepts similar to string instability are sometimes known as the 'bullwhip effect', or the 'Forrester effect' [5].

String instability is clearly undesirable[2] and has lead to a number of analyzes of the difficulties and proposed solutions. Some of the main solutions proposed to string instability and non-scalability issues include:

[1]In terms of [4], type II servo response occurs when the average estimate of the production lead time is correct, which is required to ensure inventory levels 'lock-on' to their target values.

[2]However, in some applications, where it is known a-priori that the length of the string is bounded, it may be possible to tolerate string instability for sufficiently small rates of amplification.

- **Extended Information Flow** The most obvious examples of string instability occur when each vehicle only has access to its relative position error to the preceding vehicle. In [17], [19] and other related references, control laws are designed so that both separation from the preceding and succeeding vehicles (sometimes called 'bidirectional' control [10]) in the platoon are used in computing a vehicle's actions. One extension of this idea is 'multi-look ahead' control, see for example [3]. In other cases, transmission of some global information is used in individual control calculations. Such schemes include 'leader following' control [19] where each has access to information from the lead vehicle. Analysis and discussions in [8] also point to the need for some global information in the problem formulation and control.
- **Relaxing Formation Rigidity** It is also known that maintaining a strict formation position separation exacerbates string stability problems. For example, in [23], weak coupling (which relaxes formation rigidity) is shown to give string stability. Other approaches that relax formation rigidity include both a position headway, and a time headway in the tracking error definition (see for example [24], [2]).
- **Heterogeneous Controller Tuning** The concatenation of identical transfer functions, implicit in some homogeneous strings, implies that any magnitude peak above 0dB in the transfer function will result in unbounded amplification as the string length grows. This suggests that by having non-uniform controller tuning in different vehicles, it may be possible to avoid string stability issues. This approach is pursued in, for example, [2, Remark 4], [24, §3.E], [9], and [1].
- **Nonlinear Controllers** A range of different controller nonlinearities (for example [2], [24] including some cases of switched or hybrid elements [12]) have been proposed to improve stability properties in strings of vehicles.

A key question therefore is to analyze general underlying causes and remedies for string instability problems. We wish to extend the work of [19] where some specific classes of Linear Time Invariant (LTI) feedback control systems are analyzed. In particular, an analysis of the implications of the complementary sensitivity integral ([14]) can be used to establish conditions under which $\|T(.)\|_{H_\infty} > 1$ (where $T(.)$ denotes the closed loop transmission from one vehicle in the string to the next). Certain types of control strategies (namely homogeneous controller tuning with nearest neighbor only communication) give rise to these conditions on the transfer function $T(s)$ and thereby dictate that sequential disturbance amplification must occur. The authors in [16] analyze a generalization of the Bode Sensitivity integral to circulant systems of asymptotically infinite dimension, that satisfy a spatial invariance assumption. However, the authors of [16] do not analyze the specific impact on stability and scalability issues discussed in the present paper.

The particular extensions to [19] studied in the present paper are the analysis of cases including:

1) Heterogeneous individual feedback loop dynamics, that is, non-uniform vehicle or controller dynamics. Here, we require a uniform bound on the high frequency behavior (see Assumption 7) of the feedback loops, and a bound on the settling response behavior of the closed loop system (see Assumption 8).
2) Much more general information structures, with the only restriction being that communications are restricted to be between vehicles within a limited range of each other.
3) A slightly broader class of spacing policies. In particular, we show that relaxing the constant spacing policy to allow a sufficiently small time headway (semi-rigid formation) does not qualitatively alter the results.

The paper is organized as follows. Section II defines the system considered and presents some preliminary results. Section III presents the main result of the paper: a lower bound on the peak gain from a disturbance at the leading vehicle to the last vehicle in the platoon. Section IV provides interpretations and implications of the main result, which are illustrated by examples in Section V. Conclusions and final remarks are given in Section VI.

An earlier version of these results was presented in [15].

Terminology

Most of the notation used is fairly standard in the systems and control literature. the Laplace transform and inverse Laplace transform operators are denoted by $\mathcal{L}$ and $\mathcal{L}^{-1}$. The Laplace transform complex variable is $s \in \mathbb{C}$, and Laplace transforms will typically be denoted by an upper case letter, that is: $\mathcal{L}\left\{u(t)\right\} = U(s)$, and $\mathcal{L}^{-1}\left\{U(s)\right\} = u(t)$. The notation $P(s) * u(t)$ is used to denote the time response (with zero initial conditions) of a linear time invariant system with transfer function $P(s)$ and input $u(t)$. The relative degree r of a rational transfer function is the difference between the degrees of its denominator and numerator polynomials. A transfer function is proper if $r \geq 0$, and strictly proper if $r > 0$. A real scalar-valued function of time $x : \mathbb{R} \rightarrow \mathbb{R}$ is denoted $x(t) \in \mathbb{C}$. Similarly, a complex scalar-valued function of s $X : \mathbb{C} \rightarrow \mathbb{C}$ is denoted $X(s) \in \mathbb{C}$. Vector and matrix-valued functions are denoted $\underline{x}(t) \in \mathbb{R}^n$ and $C(s) \in \mathbb{C}^{n\times n}$. Given a number $x \in \mathbb{R}$, the notation $\lceil x \rceil$ represents the smallest integer no smaller than x. We extend the standard product notation $\prod$ to include matrices as follows: $\prod_{i=1}^{n} M_i \triangleq M_n M_{n-1} \dots M_2 M_1$. The imaginary unit is j, that is, $j^2 = -1$.

II. Preliminaries

A. System Definition

We consider a one-dimensional array of vehicles as depicted in Figure 1. In this diagram, each vehicle, $1, 2, \dots n$ is traveling in the positive X direction, and the i vehicle has x-coordinate denoted by $x_i(t) \in \mathbb{R}$.

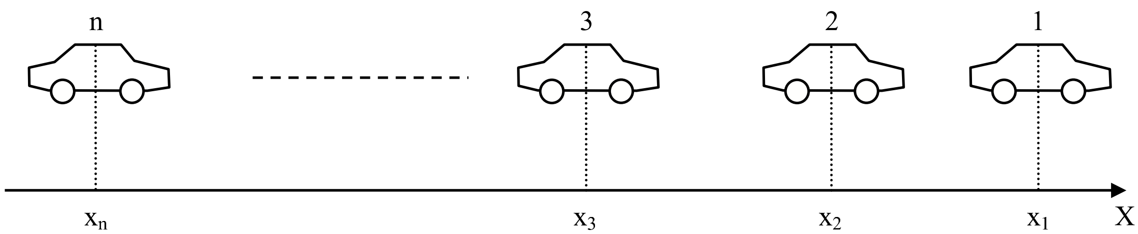

Fig. 1. Diagram depicting a vehicle platoon

The dynamics for the i-vehicle are assumed to be linear time invariant with a scalar transfer function $P_i(s) \in \mathbb{C}$, and scalar input $u_i(t) \in \mathbb{R}$. The vehicle dynamics are then given by

$$x_i(t) = P_i(s) * u_i(t); \quad i = 1, 2, \dots, n. \tag{1}$$

In vector form, let $\underline{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & \dots & x_n(t) \end{bmatrix}^T$, and similarly define the vector control variable $\underline{u}(t) = \begin{bmatrix} u_1(t) & u_2(t) & \dots & u_n(t) \end{bmatrix}^T$. We further define the multivariable plant transfer function, $P(s) = \mathrm{diag}\{P_i(s)\} \in \mathbb{C}^{n\times n}$, and therefore rewrite (1) as

$$\underline{x}(t) = P(s) * \underline{u}(t). \tag{2}$$

The vehicle dynamics are typically modeled as a second order system including damping (see for example [17], [8]), in which case $P_i(s) = g_i/s(s+\mu_i)$. Other references such as [2] use a double integrator model, sometimes augmented with first order actuator dynamics [19]. Here we shall not be concerned directly with the details of the vehicle dynamics and make the following initial assumption on the plant whilst later assumptions deal with the overall dynamics of the system.

Assumption 1 (Plant): Each of the n individual vehicle transfer functions, $P_i(s)$, for $i = 1, \dots n$, is strictly proper, has no unstable hidden modes, and has no zeros at $s = 0$. □

Key aspects of the performance of the platoon are to regulate the vehicles relative positions whilst maintaining a target velocity generated by the first vehicle. Therefore, we introduce as 'performance' variables the vehicle separations $e_i(t)$ defined for $i = 1, \dots, n$, by

$$e_i(t) \triangleq \begin{cases} d_1(t) - x_1(t), & \text{for } i = 1, \\ \delta_i(t) + x_{i-1}(t) - x_i(t), & \text{for } i = 2, 3, \dots, n, \end{cases} \tag{3}$$

where $d_1(t) \in \mathbb{R}$ denotes the desired position for the string lead vehicle, and $\delta_i(t) \in \mathbb{R}$ for $i > 1$ denotes the *target separation* (negative) for the i-vehicle. In vector form, using the notation $\underline{e}(t) = \begin{bmatrix} e_1(t) & e_2(t) & \dots & e_n(t) \end{bmatrix}^T$ and $\underline{\delta}(t) = \begin{bmatrix} 0 & \delta_2(t) & \dots & \delta_n(t) \end{bmatrix}^T$, we rewrite (3) as

$$\underline{e}(t) = \underline{\delta}(t) - M\underline{x}(t) + V_1^n d_1(t), \tag{4}$$

where $V_1^n \in \mathbb{R}^n$ is the first elementary basis vector, $V_1^n = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$, and $M \in \mathbb{R}^{n \times n}$ denotes the coupling matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}. \tag{5}$$

Note that in the most general case the vehicles separation $\underline{e}(t)$ could be permitted to be a general function of both position in the platoon, and also time (which permits constant time headway policies). Here, we restrict attention to the following class of separation policies.

Assumption 2 (Vehicle separation policy): The target vehicle separations $\delta_i(t)$, with $i = 2, 3, \dots, n$, are either constant or increase linearly with the vehicle's own velocity. That is,

$$\underline{\delta}(t) = \underline{\delta}_0 - H\frac{d\underline{x}(t)}{dt}, \tag{6}$$

where $H \in \mathbb{R}^{n \times n}$, defined as $H = \operatorname{diag}\{h_i\} \geq 0$, is the matrix of *time headways*, and $\underline{\delta}_0 \in \mathbb{R}^n$ is a vector of constant reference spacings. □

We shall see that, as might be expected, using negative time headways (equivalently, deliberately introducing 'negative damping') aggravates the frequency domain constraints described later. Thus, negative headways would seem to offer no benefit in the control design, and we therefore consider only the case of positive time headways.

Also, because of the above definitions, under normal circumstances $\underline{\delta}_0$ will consist of negative elements, and $h_i \in \mathbb{R}$ will be positive, indicating that at higher speeds increased spacing is desired. If $H = 0$, then the vehicle separation policy is termed a *constant spacing* policy; otherwise, (6) is referred to as a *constant time headway* policy.

Subject to Assumption 2, the vehicle separation vector $\underline{e}(t)$ from Equation (4) can be expressed as

$$\begin{aligned}\underline{e}(t) &= \underline{\delta}(t) - M\underline{x}(t) + V_1^n d_1(t) \\ &= -(M + sH) * \underline{x}(t) + V_1^n d_1(t) + \underline{\delta}_0. \end{aligned} \tag{7}$$

Assumption 3 (Control policy): We assume that the control is linear time invariant, possibly multivariable (depending on the communications range to be defined in Assumption 4), and based on error measurements, $\underline{e}(t)$ as defined by Equation (7). That is,

$$\underline{u}(t) = C(s) * \underline{e}(t), \tag{8}$$

where $C(s) \in \mathbb{C}^{n \times n}$. □

Remark 1 (On more general control structures): Note that more general linear time invariant control structures could be considered. For example, we could permit the local control actions, $u_i(t)$ to depend on the local state variable, $x_i(t)$ as well as the error variables in the form

$$\underline{u}(t) = C(s) * \left(\underline{e}(t) - C_X(s) * \underline{x}(t)\right), \tag{9}$$

where $C_X(s)$ is a diagonal transfer function matrix.

In the case of (9), the derivations below follow except that the loop transfer function becomes $L(s) = P(s)C(s)(M + sH + C_X(s))$. In this case, unless $C_X(0) = 0$, the closed loop system will have an unbounded steady state error in response to a ramp input. It would then seem reasonable to restrict attention to the case where we can factor $C_X(s) = sH_X(s)$, in which case the analysis that follows is identical except: (i) H is replaced by $H + H_X(s)$; and, (ii) the effective time headway for the ith vehicle is $h_i + H_{Xi}(0)$. Therefore, for simplicity of exposition, we restrict attention to control laws of the form (8). □

If the controller of each vehicle uses exclusively information about the separation from the vehicle immediately ahead, we say that the control communication range of the string is limited to one vehicle forward. This situation is represented by a diagonal control matrix $C(s)$.

To model communication range more generally, we shall use properties of *banded* matrices (see for example [6]): Given integers k, ℓ, a matrix $A \in \mathbb{R}^{n\times n}$ is called (k,ℓ)-*banded* if $a_{ij} = 0$ for all $i > j + k$ and $a_{ij} = 0$ for all $j > i + \ell$. For example, the coupling matrix M in (5) is $(1,0)$-banded, or bidiagonal matrix.

We make the following structural assumption on the controller $C(s)$, based on the banding inherent in providing for limited range communications.

Assumption 4 (Communication ranges): There are fixed natural numbers, $c_r, c_f \in \mathbb{N}$ (independent of the string length, n), with $c_f \geq 1$, which we term the *reverse* and *forward communication ranges*, such that the control transfer function matrix $C(s)$ is $(c_f - 1, c_r)$-banded. We will refer to the integer $\ell_r = \lceil c_r/c_f \rceil$ as the *communication range ratio*. □

The forward communication range c_f in Assumption 4 specifies the number of vehicles in front of the i-vehicle that are permitted to communicate with the i-vehicle. Conversely, the reverse communication range c_r specifies the number of vehicles behind the i-vehicle that are permitted to communicate with the i-vehicle.

Using Equations (1), (4), and (8), the state variables $\underline{x}$ may be related to the target separation variables $\underline{\delta}_0$ and the lead vehicle target position $d_1(t)$ by the expression

$$\underline{x}(t) = (I + L(s))^{-1} P(s) C(s) * (\underline{\delta}_0 + V_1^n d_1(t)), \tag{10}$$

where $L(s) \in \mathbb{C}^{n\times n}$ is the multivariable *loop transfer function matrix*

$$L(s) = P(s)C(s)(M + sH). \tag{11}$$

With the above notation, the vehicle string is represented by the multivariable feedback loop illustrated in Figure 2. For further reference, we also introduce from (10) the closed-loop multivariable transfer function matrix $H_{xd}(s) \in \mathbb{C}^{n\times n}$

$$\begin{aligned} H_{xd}(s) &= (I + L(s))^{-1} P(s) C(s) \\ &= \left(I - (I + L(s))^{-1}\right)(M + sH)^{-1}, \end{aligned} \tag{12}$$

which represents the frequency response from inputs $\underline{d}(t)$ to vehicle positions $\underline{x}(t)$ in Figure 2.

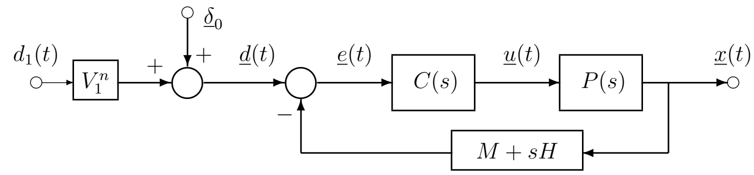

Fig. 2. Multivariable feedback loop representation of the vehicle string

In broad terms, we shall be interested in examining conditions on the feedback loop dynamics, and the communications structure, such that the closed loop behavior captured in H_{xd}, including that represented by Equation (10), cannot be made 'well behaved' for arbitrarily large n.

To ensure that we can achieve asymptotically zero tracking error with a constant speed reference signal $\delta_1(t) = \delta_x + \delta_v t$ in the case of a constant spacing policy ($H = 0$) we require a multivariable type-II servomechanism controller. This, along with other standing assumptions on the loop dynamics, are described in the following.

Assumption 5 (Feedback loop): The loop transfer function $L(s)$ in Equation (11) satisfies:

(a) $L(s)$ is strictly proper. In other words, every element of $L(s)$ has relative degree $r \geq 1$.

(b) $L(s)$ is free of unstable hidden modes.

(c) When restricted to a constant spacing policy, $H = 0$, $L(s)$ gives a multivariable type-II servomechanism. In other words, for general spacing policies of the form in Assumption 2, we can factor $P(s)C(s) = s^{-2}\bar{L}(s)$, where $\bar{L}(0)$ is non-singular. □

B. Basic Loop Properties

Assumption 5(c) allows us to establish some initial properties of the low frequency portion of the closed loop response matrix, H_{xd}.

Lemma 1 (Values of H_{xd} at $s = 0$): Consider H_{xd} as defined in (12). Then subject to Assumption 5 we have

$$H_{xd}(0) = M^{-1} \tag{13}$$

$$H_{xd}'(0) = -M^{-1}HM^{-1}. \tag{14}$$

Proof: From the definition of H_{xd} in (12) we have

$$\begin{aligned} H_{xd}(s) &= (I + P(s)C(s)(M + sH))^{-1} P(s)C(s) \\ &= \left(M^{-1}(P(s)C(s))^{-1} + I + sM^{-1}H\right)^{-1} M^{-1} \\ &= \left(I + sM^{-1}H + s^2 M^{-1}\bar{L}^{-1}(s)\right)^{-1} M^{-1} \end{aligned} \tag{15}$$

where the last line in (15) follows from Assumption 5(c). Evaluating (15) at $s = 0$ gives (13) since $\bar{L}(0)$ is assumed to be invertible. Similarly, differentiating (15) at $s = 0$ gives (14). ■

The analysis that follows makes use of the Bode Complementary Sensitivity integral (see for example [20, Theorem 3.1.5]) in a similar fashion to that in [19]. We restate this theorem here for completeness.

Lemma 2 (Bode integral for the Complementary Sensitivity Function): Let $T(s)$ be a real rational scalar function of the complex variable s. Suppose that $T(0) = 1$ and also that $T(s)$ is stable (that is, it is analytic in the closed right half complex plane). Then under these conditions:

$$\int_0^\infty \log_e |T(j\omega)| \frac{d\omega}{\omega^2} \geq \frac{\pi}{2} T'(0)\,. \tag{16}$$

Proof: This result follows immediately from [20, Theorem 3.1.5], where we have equality if $T(s)$ has no zeros in the closed right half complex plane. ■

The lower left element of H_{xd} describes the response of the state of the last vehicle to a disturbance at the first vehicle. 'String Instability' has been observed in this response, and we are therefore interested in analyzing this particular component[3] of the overall closed loop response. We shall be particularly interested in applying Lemma 2 to the lower left element of H_{xd}, namely the scalar transfer function

$$H_{x_n d_1}(s) = (V_n^n)^T H_{xd} V_1^n \tag{17}$$

where $V_1^n, V_n^n \in \mathbb{R}^n$ are the 1st and nth canonical basis vectors respectively. This application is described in the following result.

[3] Note that since we are considering only one component of the closed loop response, we can only make precise statements about conditions under which the overall response is not well behaved. The reverse implication, determining conditions under which the complete response is well behaved would require more extensive analysis.

Lemma 3 (Bode integral for H_{xd}): Consider $H_{x_n d_1}$ as defined in (17). Then subject to Assumption 5 we have

$$\int_0^\infty \log_e |H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \geq -\frac{n\pi}{2}\bar{h} \tag{18}$$

where $\bar{h}$ is the *average time headway*

$$\bar{h} = \frac{1}{n}\sum_{i=1}^{n} h_i. \tag{19}$$

Proof: Note from the definition of M in (5) that

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{bmatrix}. \tag{20}$$

Using (20) in Lemma 1 yields

$$\begin{aligned} H_{x_n d_1}(0) &= 1 \\ {H_{x_n d_1}}'(0) &= -n\bar{h}. \end{aligned} \tag{21}$$

The result then follows by using (21) in Lemma 2. ■

We now turn to the analysis of feedback systems of the form described above. In particular, we derive conditions for broad classes of systems under which it is not possible to retain certain 'well behaved' closed-loop properties for large platoon sizes. These are described in terms of a set of seemingly reasonable specifications and objectives that dictate a lower bound on the achievable performance. In addition, we will be able to show that some combinations of performance specifications and assumptions are infeasible.

III. Lower Bounds on Achievable Performance

From Assumption 4 and (11) we note that $L(s)$ is (c_f, c_r)-banded. For simplicity, consider n to be divisible by c_f, that is, $n = Nc_f$. Then $L(s)$ is a block matrix of the form

$$L(s) = \begin{bmatrix} L_{1,1}(s) & L_{1,2}(s) & L_{1,3}(s) & \dots & 0 \\ L_{2,1}(s) & L_{2,2}(s) & L_{2,3}(s) & \dots & 0 \\ 0 & L_{3,2}(s) & L_{3,3}(s) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & L_{N-1,N}(s) \\ 0 & \dots & 0 & L_{N,N-1}(s) & L_{N,N}(s) \end{bmatrix}, \tag{22}$$

where each block element $L_{i,j}(s)$ is a $c_f \times c_f$ dimensional transfer function matrix, and $L_{i,j}(s) = 0$ for $j > i + \ell_r$, where $\ell_r = \lceil c_r/c_f \rceil$ is the communication range ratio introduced in Assumption 4.

It follows that $I + L(s)$ can be conveniently factorized in a block LU form [6].

Lemma 4 (Block LU factorisation of $L(s)$): Under Assumption 4, let $L(s)$ be the (c_f, c_r)-banded transfer function matrix defined in (22). Then, there exist block lower and upper triangular transfer function matrices $M_L(s)$ and $M_U(s)$ such that

$$(I + L(s)) \triangleq M_L(s) M_U(s)\,. \tag{23}$$

Proof: The proof is constructive. Define M_U as

$$M_U(s) = \begin{bmatrix} I & U_{1,2}(s) & U_{1,3}(s) & \dots & 0 \\ 0 & I & U_{2,3}(s) & \dots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & I \end{bmatrix}, \tag{24}$$

and $M_L(s)$ as

$$M_L(s) = \begin{bmatrix} \tilde{S}^{-1}_{1,1}(s) & 0 & 0 & \dots & 0 \\ L_{2,1}(s) & \tilde{S}^{-1}_{2,2}(s) & 0 & \dots & 0 \\ 0 & L_{3,2}(s) & \tilde{S}^{-1}_{3,3}(s) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & L_{N,N-1}(s) & \tilde{S}^{-1}_{N,N}(s) \end{bmatrix}, \tag{25}$$

where $\tilde{S}_{k,k}(s)$, $U_{k,j}$ are defined recursively by

$$\begin{aligned} \tilde{S}_{11}(s) &= (I + L_{11}(s))^{-1} \\ U_{1,j}(s) &= \tilde{S}_{11}(s) L_{1,j}(s) : j = 2, 3 \dots, N \\ \tilde{S}_{kk}(s) &= \left(I + L_{k,k}(s) - L_{k,k-1}(s) U_{k-1,k}(s)\right)^{-1} : k = 2, 3, \dots, N \\ U_{k,j}(s) &= \tilde{S}_{k,k}(s) \left(L_{k,j}(s) - L_{k,k-1}(s) U_{k-1,j}(s)\right) : 1 < k < j \le N. \end{aligned} \tag{26}$$

By computing the product $M_L(s) M_U(s)$ from (24) and (25), it can be verified after some algebra that (23) holds. ∎

Remark 2 (Forward communications case): Note that in the case where we permit only forward communications, the loop transfer function is lower triangular and therefore also lower block triangular, in which case, $M_U(s) = I$. This case is therefore a simpler special case of the general situation discussed below. □

From (22)-(26) it follows that the multivariable loop sensitivity function $S(s) = \big(I + L(s)\big)^{-1}$ can be written as product of upper and lower block triangular matrices

$$\begin{aligned} S(s) &= M_U^{-1}(s) M_L^{-1}(s) \\ &= \begin{bmatrix} I & * & \dots & * \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & I \end{bmatrix} \begin{bmatrix} \tilde{S}_{1,1}(s) & 0 & \dots & 0 \\ \tilde{S}_{2,1}(s) & \tilde{S}_{2,2}(s) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{S}_{N,1}(s) & \dots & \tilde{S}_{N,N-1}(s) & \tilde{S}_{N,N}(s) \end{bmatrix} \end{aligned} \tag{27}$$

where '*' denotes non-zero transfer function blocks within the matrix $M_U^{-1}(s)$ and from (25) we have that for all $j > i$

$$\tilde{S}_{j,i}(s) = -\tilde{S}_{j,j}(s) L_{j,j-1}(s) \tilde{S}_{j-1,i}(s) \tag{28}$$

Then selecting $j = N$ in (28), and recursively using this equation gives

$$\tilde{S}_{N,i}(s) = \tilde{S}_{N,N}(s) \prod_{k=i}^{N-1} (-L_{k+1,k}(s) \tilde{S}_{k,k}(s)) \tag{29}$$

In view of (27) and (29), the bottom block row of $S(s)$ satisfies

$$S_{N,i}(s) = \tilde{S}_{N,i}(s) = \tilde{S}_{N,N}(s) \prod_{k=i}^{N-1} (-L_{k+1,k}(s) \tilde{S}_{k,k}(s)) \tag{30}$$

Note from (30) that $\tilde{S}_{k,k}(s)$ is precisely the lower right hand block of the multivariable sensitivity function that results from a string of k vehicles. Therefore, if (for example) the individual controllers are not permitted to reconfigure themselves based on information about their position in the string, it would seem reasonable to restrict each $\tilde{S}_{k,k}(s)$ for all $k = 1, 2, \dots, N$ as in the following assumption.

Assumption 6 (Uniform bound on $\tilde{S}_{k,k}$): There exists a finite number $\sigma > 0$ such that

$$\prod_{k=1}^{N} \left\| \tilde{S}_{k,k}(s) \right\|_{\mathcal{H}_\infty} \leq \sigma^N . \tag{31}$$

□

We now proceed to give a tighter analysis of the high frequency response of the system.

A. High Frequency Bounds on $H_{x_n d_1}$

Using block notation, and letting $(H_{xd})_{N,1}$ denote the lower leftmost $c_f \times c_f$ block of H_{xd}, then we can also write the transfer function from lead disturbance to the nth vehicle as

$$H_{x_n d_1}(s) = (V_{c_f}^{c_f})^T \, (H_{xd})_{N,1} \, V_1^{c_f} \tag{32}$$

where $V_1^{c_f}, V_{c_f}^{c_f} \in \mathbb{R}^{c_f}$ are the 1st and c_fth elementary basis vectors. We then have the following result giving a bound on the frequency response of the state of the last vehicle in the platoon to a disturbance at the leading vehicle.

Lemma 5 (Pointwise high frequency bound on $H_{x_n d_1}$): Suppose that for some given $\omega \in \mathbb{R}^+$, that $\|L_{k,\ell}(j\omega)\| \leq \gamma < 1$ for all $k, \ell \in \{1 \ldots N\}$. Then subject to Assumption 6

$$|H_{x_n d_1}(j\omega)| \leq (\gamma\sigma)^N \, (1 + \ell_r)\sqrt{c_f} \tag{33}$$

Proof: Note from (32) that $H_{x_n d_1}$ is one element of the lower left block of the transfer function, H_{xd}. Then from (12), (23) and Assumption 5(c), $H_{x_n d_1}$ is one element of $(H_{xd}(s))_{N,1} V_1^{c_f} = (S(s)(PC)(s))_{N,1} V_1^{c_f}$, where $(PC)(s) \triangleq P(s)C(s)$.

Note that because of the banded structure of $(PC)(s)$ it follows that

$$\begin{aligned}
(S(s)(PC)(s))_{N,1} V_1^{c_f} &= (S_{N,1}(s)(PC)_{1,1}(s) + S_{N,2}(s)(PC)_{2,1}(s)) \, V_1^{c_f} \\
&= S_{N,1}(s)(PC)_{1,1}(s)V_1^{c_f} \\
&= \tilde{S}_{N,1}(s)(PC)_{1,1}(s)V_1^{c_f}
\end{aligned} \tag{34}$$

where the second equality in (34) follows since $(PC)_{2,1}$ is strictly upper triangular, and the third equality follows from (30).

Also, since $(PC)(s)(M + sH) = L(s)$ (see (11)), then

$$(PC)_{\ell,1}(s) = \sum_{k=1}^{N} L_{\ell,k}(s) \left((M + sH)^{-1}\right)_{k,1} \tag{35}$$

Note that each element of $(M + j\omega H)^{-1}$ is either zero or a term of the form $\prod_{\ell=i}^{j}(1 + j\omega h_\ell)^{-1}$ and therefore has magnitude less than unity. Therefore, since $L_{1,k}(s) = 0$ for $k > 1 + \ell_r$,

$$\left\|(PC)_{1,1}(j\omega)V_1^{c_f}\right\| \leq (1 + \ell_r)\gamma\sqrt{c_f} \tag{36}$$

Combining (34), (35) and (36) we obtain

$$\begin{aligned} |H_{x_n d_1}(j\omega)| &\leq \left\| \tilde{S}_{N,1}(j\omega)(PC)_{1,1}(j\omega)V_1^{c_f} \right\| \\ &\leq \left\| \tilde{S}_{N,1}(j\omega) \right\| (1+\ell_r)\gamma\sqrt{c_f} \\ &\leq (\gamma\sigma)^N(1+\ell_r)\sqrt{c_f} \end{aligned} \tag{37}$$

where we use (30) and various bounds in the last inequality in (37). ■

We make use of the results in Lemma 5 to bound the high frequency behavior of $H_{x_n d_1}$ in terms of the high frequency behavior of the loop transfer functions, $L_{k,\ell}$. To do so, we make the following assumption, which refines Assumption 5(a), on the behavior of the loop transfer functions.

Assumption 7 (Loop high frequency bound): The loop block transfer functions, $L_{k,\ell}(s)$, with $k, \ell \in \{1, 2, \ldots, N\}$, obey the uniform high frequency bound

$$\|L_{k,\ell}(j\omega)\| \leq \left(\frac{\omega_H}{\omega}\right)^r, \quad \text{for all } \omega > \omega_H, \tag{38}$$

for some $\omega_H > 0$ and $r \geq 1$ and all $k, \ell \in \{1, 2, \ldots, N\}$. □

We then have the corollary below.

Corollary 6 (High frequency bound on $H_{x_n d_1}$): Under Assumption 7, we have for all $\omega \geq \omega_H$:

$$|H_{x_n d_1}(j\omega)| \leq \left(\frac{\omega_H \sigma}{\omega}\right)^{rN}(1+\ell_r)\sqrt{c_f}. \tag{39}$$

Furthermore, let $\tilde{\omega}_H \triangleq \omega_H\sigma$. Then,

$$\int_{\tilde{\omega}_H}^{\infty} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \leq \frac{1}{\tilde{\omega}_H}\left(\log((1+\ell_r)\sqrt{c_f}) - nr/c_f\right). \tag{40}$$

Proof: The bound (39) follows immediately by using (38) in Lemma 5. Then using (39) we have

$$\begin{aligned} \int_{\tilde{\omega}_H}^{\infty} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} &\leq \int_{\tilde{\omega}_H}^{\infty} \left(\log((1+\ell_r)\sqrt{c_f}) + Nr\log\left(\frac{\tilde{\omega}_H}{\omega}\right)\right)\frac{d\omega}{\omega^2} \\ &= \frac{\log((1+\ell_r)\sqrt{c_f})}{\tilde{\omega}_H} + \left[\frac{Nr}{\omega}\left(1 - \log\frac{\tilde{\omega}_H}{\omega}\right)\right]_{\tilde{\omega}_H}^{\infty} \\ &= \frac{\log((1+\ell_r)\sqrt{c_f})}{\tilde{\omega}_H} - \frac{Nr}{\tilde{\omega}_H} \end{aligned}$$

and (40) follows. ■

B. Time Domain Performance Specifications

We now consider a set specifications based on the response at the last vehicle to a ramp input as the target trajectory for the first vehicle. This is governed by the transfer function $H_{x_n d_1}(s)$ as defined previously in (17). We impose an integral absolute error (IAE) specification on the response of the vehicle separations to a ramp disturbance to the first vehicle.

Assumption 8 (IAE specification on transient response): For $k = 1, 2, \dots, n$, let $\xi_k(t)$ be the separation response of the kth vehicle to a ramp $d_1(t) = t$. We assume that for all n the average integral of the absolute value of $\xi_k(t)$ is bounded as

$$\sum_{k=1}^{n} \int_0^\infty |\xi_k(t)| dt \leq n\bar{\alpha}(n) =: \alpha(n), \tag{41}$$

for some positive function $\bar{\alpha}(n)$. $\square$

Clearly, if possible we would prefer a uniform bound in Assumption 8, that is, $\bar{\alpha}$ constant. However, this may be incompatible with other performance and robustness as we shall see later in Section IV. The following example illustrates the bound (41).

Example 1 (Transient behavior): Consider the simple case of a non-type II, homogeneous control policy with a communication scheme that uses only preceding vehicle information. Let the time headway be $H = 2I$ and $P(s)C(s) = I/s^2$. In this case,

$$\begin{aligned} H_{xd}(s) &= \left((P(s)C(s))^{-1} + M + sH\right)^{-1} \\ &= \left(s^2 I + M + 2sI\right)^{-1} \\ &= \begin{bmatrix} \frac{1}{(s+1)^2} & 0 & 0 & \dots & 0 \\ \frac{1}{(s+1)^4} & \frac{1}{(s+1)^2} & 0 & \dots & 0 \\ \frac{1}{(s+1)^6} & \frac{1}{(s+1)^4} & \frac{1}{(s+1)^2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{1}{(s+1)^{2n}} & \dots & \dots & \frac{1}{(s+1)^4} & \frac{1}{(s+1)^2} \end{bmatrix}. \end{aligned} \tag{42}$$

From (42) it follows that the two lower left hand corner elements are

$$H_{x_{n-2} d_1}(s) = \frac{1}{(s+1)^{2n-2}}, \qquad H_{x_n d_1}(s) = \frac{1}{(s+1)^{2n}}. \tag{43}$$

From (3) and (6) we have (assuming the response is due only to $d_1(t) = t$, and $\underline{\delta}_0 = 0$)

$$\begin{aligned}\int_0^\infty e^{-st}\xi_n(t)dt = \mathcal{L}\left\{\xi_n(t)\right\} &= \mathcal{L}\left\{-2\frac{dx_n(t)}{dt} + x_{n-1}(t) - x_n(t)\right\} \\ &= \left(\frac{-2s}{(s+1)^{2n}} + \frac{1}{(s+1)^{2n-2}} - \frac{1}{(s+1)^{2n}}\right)\frac{1}{s^2} \\ &= \frac{1}{(s+1)^{2n}}. \qquad (44)\end{aligned}$$

From (44), $e_n(t) = \frac{t^{2n-1}}{(2n-1)!}e^{-t}$ and $\int_0^\infty |\xi_n(t)|dt = 1$ for all n, which implies that Assumption 8 is satisfied with $\bar{\alpha}(n) = 1$. The time responses $\xi_k(t)$ for $k = 1, \ldots, 10$ are shown in Figure 3.

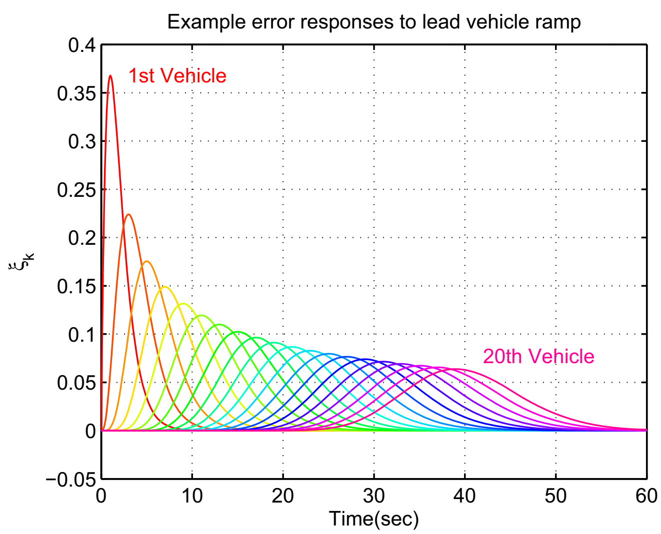

Fig. 3. Illustration of transient behavior for the system in Example 1

□

One immediate consequence of Assumption 8 follows.

Lemma 7 (Low frequency bound on $H_{x_n d_1}$): Let Assumption 8 hold. Then, for all $\omega \in \mathbb{R}$

$$|H_{x_n d_1}(j\omega)| \le 1 + \alpha(n)\omega^2. \qquad (45)$$

Furthermore, for any $\omega_L > 0$,

$$\begin{aligned}\int_0^{\omega_L} \log|H_{x_n d_1}(j\omega)|\frac{d\omega}{\omega^2} &\le -\frac{1}{\omega_L}\log(1+\alpha(n)\omega_L^2) + 2\sqrt{\alpha(n)}\tan^{-1}(\sqrt{\alpha(n)}\omega_L) \\ &\triangleq \eta(\alpha(n), \omega_L) \qquad (46)\end{aligned}$$

Proof: From the definition of vehicle separations (7) and the expression (12) for $H_{x_n d_1}(s)$, we have

$$\underline{e}(t) = (I - (M + sH)H_{xd})\, V_1^n * d_1(t). \tag{47}$$

By letting $d_1(t) = t$ (that is, $\mathcal{L}\{d_1(t)\} = 1/s^2$), and taking the kth row of (47) we obtain

$$\begin{aligned}\int_0^\infty e^{-st}\xi_k(t)dt &= V_k^{nT}(I - (M + sH)H_{xd})\, V_1^n \frac{1}{s^2} \\ &= \big((1 + h_k s)H_{x_k d_1}(s) - H_{x_{k-1} d_1}(s)\big)\frac{1}{s^2}.\end{aligned} \tag{48}$$

Following the recursion in k in (48) yields

$$H_{x_n d_1}(s) = \prod_{\ell=1}^{n}(1 + h_\ell s)^{-1} + \sum_{k=1}^{n}\left(\prod_{\ell=k}^{n}(1 + h_\ell s)^{-1}\right)\left(s^2\int_0^\infty e^{-st}\xi_k(t)dt\right). \tag{49}$$

By evaluating (49) at $s = j\omega$ we obtain the bound

$$|H_{x_n d_1}(j\omega)| \le 1 + \omega^2 \sum_{k=1}^{n}\int_0^\infty |e_k(t)|dt,$$

from which (45) follows.

Finally, using (45) we have

$$\begin{aligned}\int_0^{\omega_L} \log|H_{x_n d_1}(j\omega)|\frac{d\omega}{\omega^2} &\le \int_0^{\omega_L} \log(1 + \alpha(n)\omega^2)\frac{d\omega}{\omega^2} \\ &= \left[-\frac{1}{\omega}\log(1 + \alpha(n)\omega^2) + 2\sqrt{\alpha(n)}\tan^{-1}\big(\sqrt{\alpha(n)}\omega\big)\right]_0^{\omega_L}\end{aligned} \tag{50}$$

and (46) follows immediately. ■

We are then in a position to establish our main theorem. This result gives a lower bound on the worst-case disturbance amplification in terms of the communication constraints, high frequency behavior and transient behavior.

C. Main Theorem - Lower Bound on Disturbance Amplification

Theorem 8: Consider a system subject to Assumptions 1 4, 7 and 8. Then for any $\omega_L \in (0, \tilde{\omega}_H)$,

$$\max_{\omega\in[\omega_L,\tilde{\omega}_H]} \log|H_{x_n d_1}(j\omega)| \ge \left(\frac{\omega_L\tilde{\omega}_H}{\tilde{\omega}_H - \omega_L}\right) \times \big(\beta(n) - \eta\big(\alpha(n), \omega_L\big)\big), \tag{51}$$

with $\alpha(n)$ as in Assumption 8, $\tilde{\omega}_H = \omega_H\sigma$ as in Corollary 6, and

$$\beta(n) = n\left(\frac{r}{\tilde{\omega}_H c_f} - \frac{\pi\bar{h}}{2}\right) - \frac{\log((1 + \ell_r)\sqrt{c_f})}{\tilde{\omega}_H}. \tag{52}$$

Proof: We establish this result by splitting the interval of integration in Lemma 3. In particular, from (18)

$$\int_{\omega_L}^{\tilde{\omega}_H} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \geq -\int_0^{\omega_L} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} - \int_{\tilde{\omega}_H}^{\infty} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} - \frac{n\pi}{2}\bar{h} \quad (53)$$

Then using (53) together with Lemma 7 and Corollary 9 we obtain

$$\int_{\omega_L}^{\tilde{\omega}_H} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} \geq -\eta(\alpha(n), \omega_L) + \frac{1}{\tilde{\omega}_H}\left(\frac{nr}{c_f} - \log((1+\ell_r)\sqrt{c_f})\right) - \frac{n\pi}{2}\bar{h}. \quad (54)$$

Also, we can derive the following inequality for the left hand side of (54)

$$\begin{aligned}\int_{\omega_L}^{\tilde{\omega}_H} \log|H_{x_n d_1}(j\omega)| \frac{d\omega}{\omega^2} &\leq \max_{\omega\in[\omega_L,\tilde{\omega}_H]} \{\log|H_{x_n d_1}(j\omega)|\} \int_{\omega_L}^{\tilde{\omega}_H} \frac{d\omega}{\omega^2} \\ &= \left(\frac{\tilde{\omega}_H - \omega_L}{\omega_L \tilde{\omega}_H}\right) \max_{\omega\in[\omega_L,\tilde{\omega}_H]} \log|H_{x_n d_1}(j\omega)|. \end{aligned} \quad (55)$$

The result then follows by combining (54) and (55). ■

We now turn to consider various consequences and interpretations of Theorem 8.

IV. Consequences of Theorem 8

A. Sufficient Conditions for Exponential Growth in Disturbance Amplification

Exponential growth in disturbance amplification for some classes of distributed control problems has been observed by a number of authors (e.g. [17], [19]), though this has generally been restricted to homogeneous platoons, with nearest neighbor communications, and no time headway. Here we extend these results by obtaining sufficient conditions for exponential growth that include heterogeneous strings, with limited range communications (not just nearest neighbor), and with sufficiently small time headway.

Corollary 9: Suppose in Theorem 8 that $\tilde{\omega}_H \alpha(n) > \beta(n) > 0$. Then

$$\sup_{\omega} |H_{x_n d_1}(j\omega)| \geq \exp\left(\frac{\beta^2(n)}{2\alpha(n)}\right). \quad (56)$$

Furthermore, if $\bar{\alpha} \leq \alpha_1$, with $\alpha_1 > \left(\frac{r}{\tilde{\omega}_H c_f} - \frac{\pi\bar{h}}{2}\right)$ then

$$\sup_{\omega} |H_{x_n d_1}(j\omega)| \geq \exp\left(n\left(\frac{\rho_1}{2}\right)\left(\frac{r}{\tilde{\omega}_H c_f} - \frac{\pi\bar{h}}{2}\right)\right), \quad (57)$$

where $\rho_1 \triangleq \left(\frac{r}{\tilde{\omega}_H c_f} - \frac{\pi\bar{h}}{2}\right)\alpha_1^{-1} < 1$.

Proof: Firstly, we note from the definition of η (see (46)) that

$$\eta(\alpha(n), \omega_L) \leq \alpha(n)\omega_L. \tag{58}$$

Using (58) and (52) in Theorem 8 gives

$$\sup_{\omega} \log |H_{x_n d_1}(j\omega)| \geq \left(\frac{\omega_L \tilde{\omega}_H}{\tilde{\omega}_H - \omega_L}\right)(\beta(n) - \alpha(n)\omega_L) \tag{59}$$

for all $\omega_L \in (0, \tilde{\omega}_H)$. Under the conditions of Corollary 9, we can substitute $\omega_L = \frac{\beta}{2\alpha} < \frac{\tilde{\omega}_H}{2}$ in (59) and with some simple algebra obtain (56). ■

The case where the condition $\tilde{\omega}_H \alpha(n) > \beta(n) > 0$ is not satisfied is covered later in Corollary 10.

Remark 3 (Alternative bounds): Other bounds may be obtained with differing degrees of complexity in the expressions. For example, given the inequality (58), the tightest bound may be found as $\log |H_{x_n d_1}(j\omega)| \geq \alpha(n)\tilde{\omega}_H^2 \left(1 - \sqrt{1 - \beta(n)\alpha^{-1}(n)\tilde{\omega}_H^{-1}}\right)^2$ by taking $\omega_L = \tilde{\omega}_H \left(1 - \sqrt{1 - \beta(n)\alpha^{-1}(n)\tilde{\omega}_H^{-1}}\right)$. □

B. Infeasible Specifications

It turns out that in some cases, demands for certain types of high frequency and transient performance may be incompatible with communication and time headway constraints. This incompatibility can be demonstrated by proving that in certain cases, the lower bound on the frequency response peak is infinite. We refer to cases where the performance specifications in Assumptions 1, 4, 7 and 8, are sufficient to guarantee an unbounded peak in the closed loop transfer function, $H_{x_n d_1}(s)$, as *infeasible* specifications. The following corollary examines this situation.

Corollary 10: Suppose that

$$\beta(n) > \eta(\alpha(n), \tilde{\omega}_H). \tag{60}$$

Then the performance specifications in Assumptions 1, 4, 7 and 8 are infeasible in the sense that any closed loop stable system subject to the assumptions of Theorem 8 satisfies

$$\sup_{\omega} |H_{x_n d_1}(j\omega)| = +\infty. \tag{61}$$

Proof: Note that (51) applies for all $\omega_L \in (0, \tilde{\omega}_H)$. Under the condition (60) it follows that

$$\lim_{\omega_L \to \tilde{\omega}_H^-} \left\{ \left(\frac{\omega_L \tilde{\omega}_H}{\tilde{\omega}_H - \omega_L}\right) \times (\beta(n) - \eta(\alpha(n), \omega_L)) \right\} = +\infty \tag{62}$$

and (61) follows. ■

We can add further interpretations on this result in special cases, as indicated in the following corollary.

Corollary 11: Suppose that for all n the average IAE transient specification (41) satisfies

$$\bar{\alpha}(n) \leq \alpha_2 n + \alpha_1 \tag{63}$$

for some non-negative constants α_1 and α_2 where

$$\sqrt{\alpha_2} < \left(\frac{r}{\pi \tilde{\omega}_H c_f} - \frac{\bar{h}}{2} \right). \tag{64}$$

Then for sufficiently large n, the closed loop specifications are infeasible in the sense of Corollary 10.

Proof: Note that from the definition of $\eta(\alpha(n), \omega_L)$ in (46) that for any ω_L

$$\eta(\alpha(n), \omega_L) < \pi \sqrt{\alpha(n)} \tag{65}$$

where we have used the assumption that $\bar{h} = 0$. Then using (65) and (63) we obtain

$$(\beta(n) - \eta(\alpha(n), \tilde{\omega}_H)) > \left(n \left(\frac{r}{\tilde{\omega}_H c_f} - \frac{\pi \bar{h}}{2} \right) - n\pi\sqrt{\alpha_2} - \frac{\log n}{\tilde{\omega}_H} \right) \tag{66}$$

Under the condition (64) the RHS of (66) is positive for sufficiently large n and therefore by Corollary 10 the specifications are infeasible. ■

Thus we see from Corollary 11 that for a constant spacing policy the transient specification of Assumption 8 must grow at least quadratically with n, with minimum rate determined by the high frequency limit, ω_H, the permitted sensitivity peak, $\tilde{s}_{max}$, and the communications range, c_f, if infeasibility for sufficiently large n is to be avoided. Note that qualitatively similar though algebraically much more complicated results can be derived in the case where a small time headway, $\bar{h}$, is permitted.

C. *Interpretations and Discussion*

In this section we discuss several interpretations that may be drawn from the analysis above.

1) Sufficient time headway may avert string instability: Note that regardless of the transient specification, if the time headway satisfies

$$\bar{h} > \frac{2r}{\pi \tilde{\omega}_H c_f} \tag{67}$$

then none of the results above demand growth in the disturbance response (or infeasibility of the specifications) with large n. In particular, (67) guarantees that the condition $\beta(n) > 0$ in Corollary 9 is false. Similarly, the condition (60) in Corollary 10 is false regardless of $\eta > 0$. Furthermore, the condition (64) in Corollary 11 is never satisfied.

Therefore, (67) is an important benchmark for time headway allowance in the design of distributed controllers for strings of dynamics systems.

2) With zero time headway, the average transient performance specification, $\bar{\alpha}(n)$ may need to grow at least linearly with n: Clearly from Corollary 11, under the other assumptions, to avoid string stability problems, we will require that for large n

$$\bar{\alpha}(n) \geq \left(\frac{r}{\pi \tilde{\omega}_H c_f}\right)^2 n + O(1)\,. \tag{68}$$

One of the implications of this result is that requiring a uniform bound on the average IAE of the error response to a ramp may not be feasible for large strings, under the conditions discussed in Theorem 8. Note also that a qualitatively similar conclusion holds for small time headways.

3) Factors that may improve string stability properties: Apart from increasing the time headway as noted above, we can also identify a range of factors that the analysis indicates may allow improved string stability properties. These include:

(a) *Improved Communications Range - Particularly Forward Communications*. Note that larger forward communications, c_f, directly reduces the rate of exponential growth (see for example Corollary 9) and indeed, may avert conditions that guarantee string instability. Reverse communications, on the other hand, appear to have a much weaker, if any, direct affect on the results. Note however, that in some cases, reverse communications have been observed to give rise to very long transient responses and in such cases, would demand relaxed time domain specifications, and therefore may indirectly avoid string stability problems.

(b) *Increased loop high frequency response*. Improvements in both the high frequency roll-off, $\tilde{\omega}_H$, and reductions in the loop relative degree, r, are both seen to be beneficial in reducing the lower bound on performance.

(c) *Relaxing the requirements on the low frequency transient performance*. Increasing the permissible integral absolute error specification on transient performance, that is, increasing α, reduces the demands imposed by the 'waterbed effect' and thereby may permit improved string stability properties.

(d) *System Nonlinearities*. The analysis above requires the formation of a single closed loop transfer function, and this is clearly not possible for many systems incorporating nonlinear control elements. Several schemes proposed in practice for distributed vehicle control incorporate a number of non-linear elements (see for example [24]) which may circumvent some of the difficulties described above. Of course, if the nonlinearities are sufficiently smooth, then local approximation by linear behavior may predict small-signal string instability using the above analysis.

Note however, that there is no indication in any of the results presented above that heterogeneous system design is advantageous. This is in contrast to some earlier results, where in the context of the results here, the heterogeneous designs demand gains that increase (in fact exponentially) with string length, and thereby demand rapidly increasing high frequency response.

V. Examples

A. String Instability with Decentralized Control

We present a simple example illustrating string instability in a string of vehicles with constant spacing policy ($H = 0$ in Assumption 2) and homogeneous, fully decentralized (no communication between vehicles) control.

Consider a string of N identical vehicles defined by the plant transfer functions

$$P_i(s) = \frac{100}{s(s+100)}, \quad \text{for all } i = 1, \dots, N. \tag{69}$$

We assume a separation policy with no time headway; that is, $H = 0$ in Assumption 2. We select $\phi = 1$ and the homogeneous PI control

$$C_i(s) = \frac{5s+1}{4s}, \quad \text{for all } i = 1, \dots, N. \tag{70}$$

The control policy for each vehicle is given fully decentralized,

$$u_i(t) = C_i(s) * e_i(t), \quad \text{for } i = 2, \dots, N, \tag{71}$$

that is, $C(s) = C_i(s)I$.

Figure 4 (top) shows the evolution of the peak in sensitivity to a disturbance at the lead vehicle with respect to the position in the string. Clearly the peak grows fast with i, indicating string instability. Figure 4 (bottom) shows the step transient response, which also deteriorates very fast with i.

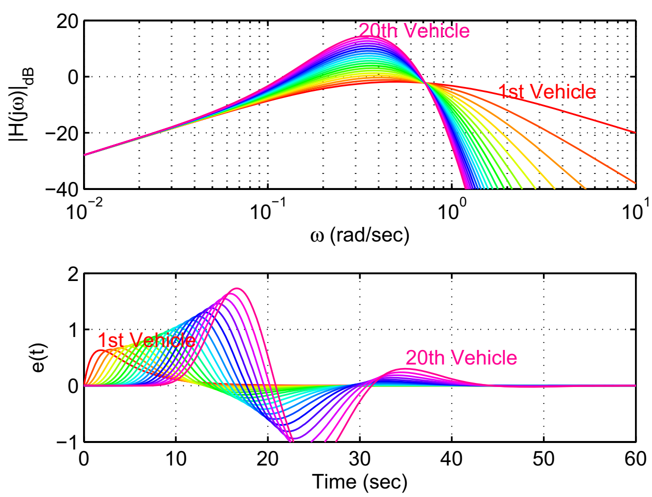

Fig. 4. String sensitivity peak (top) and transient performance (bottom), decentralized control, no time headway

B. Forward Communications

We now examine the effect of allowing forward communication between controllers in the string introduced in Section V-A. The example shows that forward communication can reduce the exponential growth of the disturbance sensitivity peak with N, although in this example string instability still arises.

Consider again the string defined in Section V-A, but this time the control communication ranges are $c_f = 2$ and $c_r = 0$. That is, we allow communication from one vehicle forward; no reverse communication. The control policy for each vehicle is

$$u_i(t) = C_i(s) * e_i(t) + \phi * e_{i-1}, \quad \text{for } i = 2, \ldots, N, \tag{72}$$

which means that the separation of the two vehicles immediately forward is used (although without integral action, which proved in simulations to be the best choice).

Figure 5 (top) shows the frequency response magnitude from a disturbance to the leading vehicle to the i-vehicle position, with $i = 1, \ldots, N = 30$. Note that the growth of the peak with i is reduced with respect to decentralized case in V-A, but string instability is still evident. Figure 5 (bottom) shows the transient step responses, which display peaks that also increase with i.

Table I compares the peak growth with i in Bode and transients for the decentralized control string in Section V-A ($c_f = 1$) and the 1-step forward control string in Section V-B.

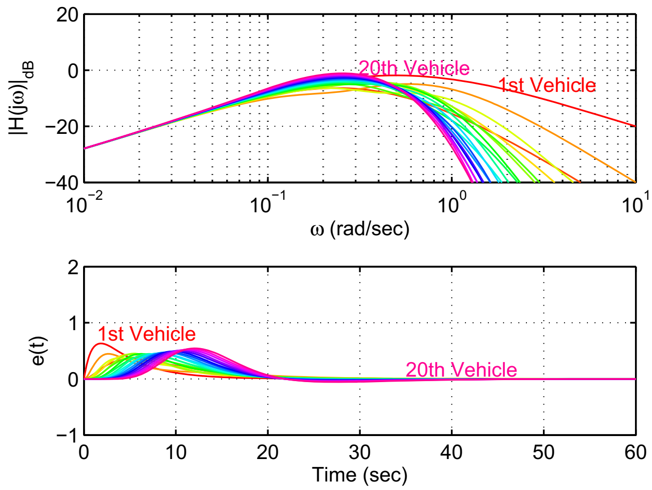

Fig. 5. String sensitivity peak (top) and transient performance (bottom), forward communication, no time headway

TABLE I

PEAK GROWTH IN THE EXAMPLES IN SECTIONS V-A,V-B

Forward range c_f	Peak growth in Bode	Peak growth in transients
1	0.8827 dB	1.0576
2	0.3090 dB	1.0214

C. Bidirectional Control

In this section, we consider an example where the ith control, u_i, is based on both the predecessor error e_i and the successor error e_{i+1}. In this case, it does make sense to include integral action on the successor error, and the control policy for each vehicle can be represented as

$$u_i(t) = C_i(s) * (e_i(t) - e_{i+1}), \quad \text{for } i = 1, \ldots, N-1. \tag{73}$$

The behavior of this scheme for a fixed number of vehicles is illustrated in Figure 6. Note that in this case, although not obvious from the figures, the responses do *not* show unbounded growth with string length. This can be attributed to a rapid increase in the integral absolute error of the transient response, described in Assumption 8. This increase happens as a result of both larger, and much longer transients in the string dynamics, as illustrated in Figure 7.

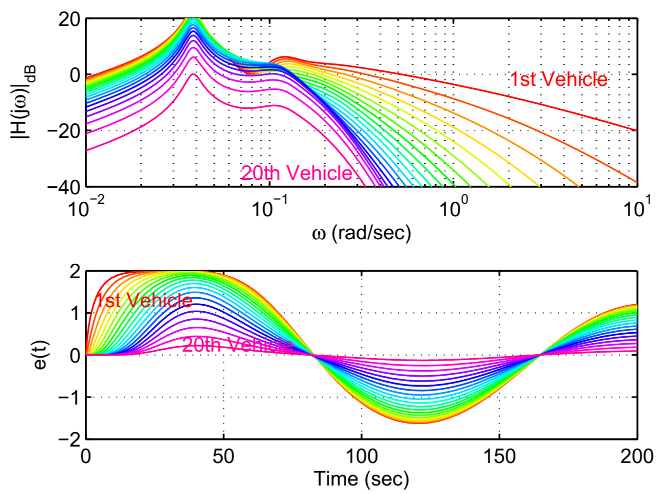

Fig. 6. String sensitivity peak (top) and transient performance (bottom), no time headway, $N = 20$, for bidirectional control

Note that other bidirectional control schemes have been examined, though clearly from the analysis presented in Section IV, it is not possible to both have well behaved string dynamics, and a transient response that has an average transient integral absolute error performance that is uniformly bounded with n.

VI. Conclusions

This paper reexamines and expands the string instability analysis presented in [19]. The analysis in the present paper includes heterogeneous, non-zero time headway, limited communication range systems, and shows that:

1) System heterogeneity, within reasonable confines of bounded high frequency response and integral absolute error, does not circumvent string stability problems.
2) Extra, though limited, forward communication range does not avoid string stability problems in a qualitative sense. It does, however, significantly reduce the rate of growth of transient disturbances.

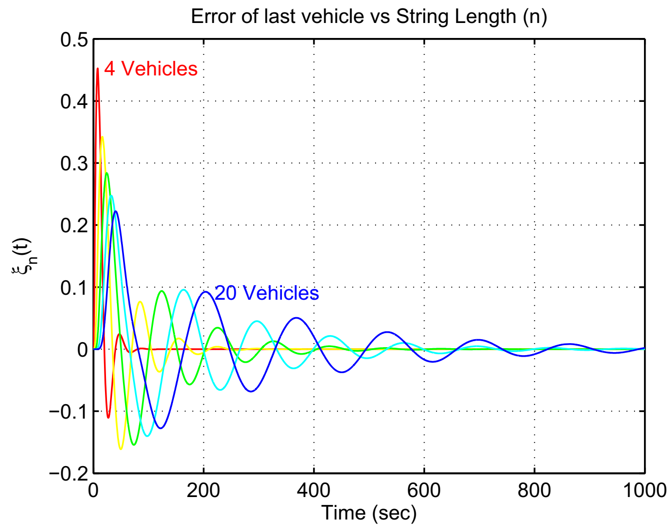

Fig. 7. Transient error performance of last vehicle, $\xi_n(t)$, versus string length, n for bidirectional control

3) Relaxing a rigid formation control policy, to allowing a small time headway, does not qualitatively alter the string instability results, though it does reduce the rate of growth. A sufficiently large time headway may permit string stability.
4) Bidirectional control, or reverse communication, appears to offer an advantage in terms of string stability mainly by virtue of the fact that it can generate very long transients as string length grows. More specifically, all else being equal, the average integral absolute value of the error in response to a ramp grows at least linearly with string length, if bidirectional control is to be used to avoid string instability.

These conclusions naturally raise a number of questions for future research, including potential advantages of non-linear and/or time-varying control schemes. In addition, there is a need to extend the analysis to much more general graph structures, with high order dynamics at each node.

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