Stabilization with Disturbance Attenuation over a Gaussian Channel

J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky

***Abstract*— We propose a linear control and communication scheme for the purposes of stabilization and disturbance attenuation when a discrete Gaussian channel is present in the feedback loop. Specifically, the channel input is amplified by a constant gain before transmission and the channel output is processed through a linear time invariant filter to produce the control signal. We show how the gain and filter may be chosen to minimize the variance of the plant output. For an order one plant, our scheme achieves the theoretical minimum taken over a much broader class of compensators.**

I. Introduction

Many authors have studied the problem of controlling a linear system with a communication channel in the feedback loop (e.g., [1]–[9]). The most general framework for doing so allows compensation at the channel input and output that may be nonlinear, time-varying, and dynamical. In the present paper we shall consider the simpler communication and control scheme shown in Figure 1 wherein the channel precompensator (the "encoder") is assumed to be a constant gain, λ, and the postcompensator (the "decoder") is assumed to be a causal linear time-invariant filter. The plant is discrete, linear, and time-invariant, and the channel is Gaussian with input power limit P and noise variance σ_n^2. The purpose of control is to stabilize the plant, if necessary, and to minimize the variance of the plant output in response to a disturbance. It is the simple nature of the communication and control scheme in Figure 1 that motivates us to study its properties.

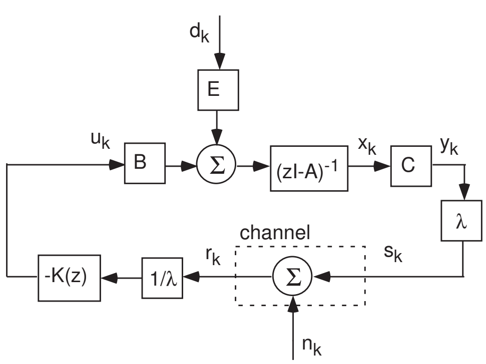

Fig. 1. Feedback control over a Gaussian communication channel with input power constraint $\mathcal{E}\{s_k^2\} < P$ and additive white noise of variance σ_n^2.

J. S. Freudenberg is with Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, MI, 48109-2122, USA jfr@eecs.umich.edu

R. H. Middleton and J. H. Braslavsky are with the Centre for Complex Dynamic Systems and Control, The University of Newcastle, 2308, Australia richard.middleton@newcastle.edu.au, julio.braslavsky@newcastle.edu.au

If the scalar amplifier at the channel input were a unity gain, and if the power constraint were not present, then the problem of minimizing the variance of the plant output would be a standard linear quadratic Gaussian (LQG) control problem, whose solution is state feedback applied to a state estimate obtained from a Kalman filter. With the pre-channel amplification and the power limit present, the problem no longer fits into the standard LQG framework. However, we shall show that the LQG results may be modified to yield the desired solution.

The remainder of this paper is outlined as follows. In Section II we provide a precise problem statement, review the results on stabilization over a Gaussian channel from [6], and develop necessary background on the discrete-time LQG problem. The latter results are applied in Section III for the case $\lambda = 1$. We show that, under appropriate hypotheses, the minimal output variance is equal to that of the minimal estimation error for the *predicting* version of the optimal estimator. This observation allows us to apply results from estimation theory to derive properties of the feedback system under optimal control. In Section IV we use the concept of entropy rate to develop a formula for the optimal estimation error and the minimal channel capacity. In Section V this formula is modified to include nonunity values of λ, and we show that the variance of the plant output is minimized by choosing λ so that the channel input satisfies the power limit arbitrarily closely. An example is also given in Section V. In Section VI, we show that, for a first order plant, our linear communication and control scheme achieves a theoretical lower bound on disturbance response that holds for general nonlinear, time-varying, and dynamical control and communication schemes. An alternate approach to this result is obtained in [21], based on an inequality applicable to open loop unstable systems derived in [16]. Conclusions and directions for further research are presented in Section VII.

II. Preliminaries

Denote a random sequence by $x = \{x_k\}$, and define the subsequence $x^k \triangleq \{x_\ell;\ \ell \le k\}$. Unless stated otherwise, all signals are assumed to have stationary distributions. Hence if x is a scalar valued sequence, the variance of x is given by $\sigma_x^2 = \mathcal{E}\{x_k^2\}$, and may be computed from its power spectral density $\mathcal{S}_x(\omega)$ by $\sigma_x^2 = (1/2\pi)\int_{-\pi}^{\pi} \mathcal{S}_x(\omega)d\omega$, if the spectral density is well-defined. The open and closed unit disks are denoted by $\mathbb{D}$ and $\bar{\mathbb{D}}$. A rational transfer function $G(z)$ is minimum phase if all its zeros lie in $\bar{\mathbb{D}}$, and is nonminimum phase (NMP) otherwise. We say that $G(z) \in H_2$ if $G(z)$ is strictly proper and all its poles lie in $\mathbb{D}$. The H_2 norm of $G(z) \in H_2$ is given by $\|G\|_{H_2}^2 = (1/2\pi)\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega$.

A. Problem Statement

Throughout the paper we consider the feedback system of Figure 1, which we now describe in detail. The plant to be controlled has state equations

$$x_{k+1} = Ax_k + Bu_k + Ed_k,\ x_k \in \mathbb{R}^n,\ u_k, d_k \in \mathbb{R}, \quad (1)$$

$$y_k = Cx_k, \qquad y_k \in \mathbb{R}, \quad (2)$$

where (A, B) and (A, E) are assumed controllable, (A, C) is assumed observable, and d is a zero mean Gaussian white noise sequence of variance σ_d^2. The transfer functions from control and disturbance inputs to the plant output are denoted by $G_u = C\Phi B$ and $G_d = C\Phi E$, respectively, where $\Phi(z) \triangleq (zI - A)^{-1}$. The output of the plant is multiplied by a scalar gain λ and transmitted over a communication channel whose input s and output r are related by

$$r_k = s_k + n_k, \quad (3)$$

where n is a zero mean Gaussian white noise sequence of variance σ_n^2. The channel input s is required to satisfy the power limit $\mathcal{E}\{s_k^2\} < P$, and thus the capacity of the channel is determined by the signal to noise ratio (SNR) P/σ_n^2 [10]:

$$\mathcal{C} = \frac{1}{2}\log_e(1 + P/\sigma_n^2) \quad \text{nats/transmission.} \quad (4)$$

The channel output is scaled by $1/\lambda$ and used as the input to a linear time invariant filter with transfer function $K(z)$, whose output is the control signal.

The goal of feedback control is to stabilize the plant, if necessary, and to attenuate the response of the plant output y to the disturbance input d. Specifically, we seek λ and $K(z)$ to minimize a cost function equal to the variance of the plant output

$$J_y \triangleq \mathcal{E}\{y_k^2\} \quad (5)$$

under the assumptions that the feedback system is internally stable and the channel power limit P is satisfied. In addition, the controller must be causal, so that each value of the control signal u is allowed to depend only on the current and previous values of the channel output r.

Denote the optimal value of the cost function (5) by

$$J_y^* \triangleq \min_{K,\lambda} \mathcal{E}\{y_k^2\}. \quad (6)$$

We shall also consider the problem of minimizing $\mathcal{E}\{y_k^2\}$ for a fixed value of λ, and denote the optimal cost by

$$J_y^*(\lambda) \triangleq \min_{K} \mathcal{E}\{y_k^2\}\Big|_{\lambda}. \quad (7)$$

For a fixed value of λ, the problem of choosing a causal controller to minimize (7) is a cheap control LQG optimization problem. We now provide a frequency domain version of the cost function. Under the assumption that all signal distributions are stationary, the variance of the system output y may be computed from its power spectral density $\mathcal{S}_y(\omega)$, given by $\mathcal{S}_y(\omega) = |S(e^{j\omega})|^2|G_d(e^{j\omega})|^2\sigma_d^2 + |T(e^{j\omega})|^2\sigma_n^2/\lambda^2$, where S and T are the sensitivity and complementary sensitivity functions

$$S \triangleq 1/(1 + \lambda G_u K), \qquad T \triangleq 1 - S. \quad (8)$$

It follows that $J_y = \|SG_d\|_{H_2}^2\sigma_d^2 + \|T\|_{H_2}^2\sigma_n^2/\lambda^2$.

B. SNR Limited Stabilization

The authors of [6] consider the feedback system in Figure 1 with an unstable plant but *without* a plant disturbance, and determine the minimum value of P required to stabilize the plant. With no disturbance present, there is no loss of generality in assuming $\lambda = 1$, and the problem of minimizing (5) for a fixed noise variance is equivalent to that of minimizing the H_2 norm of T:

$$\mathcal{E}\{s_k^2\} = \|T\|_{H_2}^2\sigma_n^2. \quad (9)$$

The following result is obtained in [6].

Proposition II.1 *Consider the feedback system in Figure 1 with* no disturbance *and* $\lambda = 1$. *Assume that* G_u *has no nonminimum phase zeros and has relative degree equal to one. Suppose further that* G_u *has poles* $\phi_i, i = 1, \ldots, m$, *with* $|\phi_i| > 1$. *Then there exists a controller* $K(z)$ *that stabilizes the feedback system if and only if the channel power constraint* P *satisfies the lower bound*

$$P > J_y^*(1), \quad (10)$$

where

$$J_y^*(1) = \left(\prod_{i=1}^{m} |\phi_i|^2 - 1\right)\sigma_n^2. \quad (11)$$

■

It follows from Proposition II.1 and (4) that the minimal channel capacity required for stabilization with a linear time invariant controller is given by

$$\mathcal{C} > \sum_{i=1}^{m} \log_e |\phi_i| \quad \text{nats/transmission.} \quad (12)$$

The authors of [16] show that nonlinear, time-varying control cannot achieve a channel capacity lower that that obtained with linear time-invariant control for a minimum phase, relative degree one plant. If the plant G_u has nonminimum phase zeros and/or relative degree greater than one, then it is shown in [6] that the channel capacity required for stabilization with a linear time invariant controller is strictly greater than the bound given in (12).

The proof technique used to obtain Proposition II.1 in [6] involved a Youla parametrization of all stabilizing controllers together with an application of residue theory to determine the optimal value of $\|T\|_{H_2}$. An alternate approach, which we pursue in the present paper, is to directly apply LQG theory.

C. The Discrete Time LQG Control Problem

Consider the feedback system of Figure 1, and assume that $\lambda = 1$, so that the plant is described by (1)-(2), and (3) becomes $r_k = y_k + n_k$. Under the assumption that all signals are stationary, the "cheap control" LQG cost function is given by $J_{LQG} = \mathcal{E}\{y_k^2\}$. It is well known [11] that the problem of finding a control law to stabilize the system and to minimize J_{LQG} has a solution given by state feedback applied to a state estimate obtained from an optimal estimator that driven by the channel output. There are two possibilities for such an estimator, depending on whether or not the state estimate is allowed to depend on the current value of the channel output. We now review both versions of the estimator, as each plays a role in subsequent developments.

Consider first the state estimate of a *predicting* estimator, denoted $\hat{x}_{k|k-1}$, which depends only on previous values of the channel output. This estimate satisfies the state equations

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + L_p(r_k - C\hat{x}_{k|k-1}) \qquad (13)$$

where $L_p = AL_f$, $L_f = \Sigma C^T(C\Sigma C^T + \sigma_n^2)^{-1}$, and Σ is the stabilizing solution to the Riccati equation

$$\Sigma = A\Sigma A^T - A\Sigma C^T(C\Sigma C^T + \sigma_n^2)^{-1}C\Sigma A^T + \sigma_d^2 EE^T. \qquad (14)$$

Define the *output estimate* and *estimation error* by $\hat{y}_{k|k-1} = C\hat{x}_{k|k-1}$ and $\tilde{y}_{k|k-1} = y_k - \hat{y}_{k|k-1}$, respectively. Then the variance of the optimal predicting estimation error is given by

$$\mathcal{E}^*\{\tilde{y}_{k|k-1}^2\} = C\Sigma C^T. \qquad (15)$$

The state estimate of the *filtering* version of the optimal estimator, denoted $\hat{x}_{k|k}$, does depend on the current value of the channel output, and satisfies

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_f(r_k - C\hat{x}_{k|k-1}), \qquad (16)$$

where $\hat{x}_{k|k-1}$ is given by (13). The output estimate and estimation error are given by $\hat{y}_{k|k} = C\hat{x}_{k|k}$ and $\tilde{y}_{k|k} = y_k - C\hat{x}_{k|k}$. The variance of the optimal filtering estimation error is equal to

$$\mathcal{E}^*\{\tilde{y}_{k|k}^2\} = \frac{\sigma_n^2 C\Sigma C^T}{\sigma_n^2 + C\Sigma C^T}. \qquad (17)$$

If the filtering estimator is used to minimize J_{LQG}, then the control law has the form

$$u_k = -K_c\hat{x}_{k|k}, \qquad (18)$$

where $\hat{x}_{k|k}$ is given by (16) and K_c is found by solving a Riccati equation. The transfer function of the resulting compensator is given by

$$K(z) = zK_c\left(zI - (I - L_fC)(A - BK_c)\right)^{-1} L_p \qquad (19)$$

A block diagram of the resulting feedback system is shown in Figure 2.

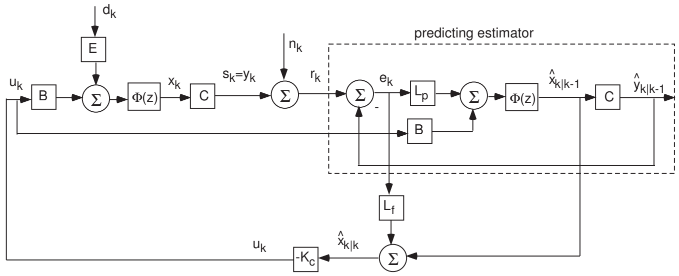

Fig. 2. Feedback system with state feedback based on a filtering estimator. The predicting estimator is in the dashed box.

Under appropriate hypotheses[1], the solution to the cheap control problem has appealing special properties.

Proposition II.2 *Consider the cheap control LQG problem. Assume that G_u has no nonminimum phase zeros and relative degree equal to one, and that the control signal is allowed to depend on both current and previous values of the channel output. Then the optimal controller, given by* (18)-(19)*, has the properties that*

$$K_c = (CB)^{-1}CA, \qquad (20)$$

and

$$K(z) = G_u^{-1}(z)C\Phi(z)L_p. \qquad (21)$$

Furthermore, the sensitivity and complementary sensitivity functions (8) *satisfy $S = S_{est}$ and $T = T_{est}$, where*

$$S_{est} = (1 + C\Phi L_p)^{-1}, \qquad T_{est} = 1 - S_{est}. \qquad (22)$$

Proof: The special structure of the state feedback gain (20) and the compensator (21) follows from equations (16) and (22) of [13], and the form of the sensitivity and complementary sensitivity functions follows by substituting (21) into (8) with $\lambda = 1$. ■

It is easy to verify that S_{est} and T_{est} are the sensitivity and complementary sensitivity functions of the feedback loop in a predicting estimator, and thus that the sensitivity and complementary sensitivity functions in Figure 2 are identical to those of a predicting estimator. The ability to make the sensitivity and complementary functions associated with an output feedback system match those of a predicting estimator feedback loop (or its dual, a state feedback loop) is termed "loop transfer recovery", and has received much discussion in the literature (e.g., [12]–[14]).

III. Structure of the Optimal Feedback System

In this section, we use Proposition II.2 to derive interesting properties of the feedback system in Figure 2 and $K(z)$ designed to minimize (5). Recall the output estimation error $\tilde{y}_{k|k-1}$ for the predicting estimator, define the associated *innovations sequence* by

$$e_k \triangleq r_k - \hat{y}_{k|k-1} = \tilde{y}_{k|k-1} + n_k, \qquad (23)$$

and denote the z-transforms of $\tilde{y}_{k|k-1}$ and e_k by $\tilde{Y}_p(z)$ and $E(z)$. Then

$$\tilde{Y}_p(z) = S_{est}(z)G_d(z)D(z) - T_{est}(z)N(z), \qquad (24)$$

$$E(z) = S_{est}(z)G_d(z)D(z) + S_{est}(z)N(z). \qquad (25)$$

Our next result shows that with optimal feedback control, the channel input and output in Figure 2 are identical to the estimation error and innovations sequence for a predicting estimator, and thus inherit special properties derived from those of the optimal predicting estimator.

[1]Expressions for the optimal state feedback gain, compensator, and sensitivity function that hold when the hypotheses of Proposition II.2 are violated may be found in [12].

Proposition III.1 *Consider the problem of minimizing* (7) *in the special case* $\lambda = 1$*. Assume that the hypotheses of Proposition II.2 are satisfied. Then the optimal control is state feedback* (18) *with* K_c *given by* (20) *and* $\hat{x}_{k|k}$ *given by* (16)*. Denote the optimal values of the channel input and output by* s_k^* *and* r_k^**, respectively. Then*

$$S^*(z) = S_{est}(z)G_d(z)D(z) - T_{est}(z)N(z), \qquad (26)$$

$$R^*(z) = S_{est}(z)G_d(z)D(z) + S_{est}(z)N(z), \qquad (27)$$

where S_{est} *and* T_{est} *are given by* (22)*. Furthermore, the channel power constraint must satisfy* $P > J_y^*(1)$*, where the optimal cost is equal to the variance of the optimal (predicting) estimation error*

$$J_y^*(1) = \mathcal{E}^*\{\tilde{y}_{k|k-1}^2\}, \qquad (28)$$

the optimal channel output r_k^* *is a white noise sequence, and the optimal channel input is orthogonal to the channel output.*

Proof: Since $\lambda = 1$, it follows that $Y(z) = S(z)G_d(z)D(z) - T(z)N(z)$, and thus (26) follows from the properties of S and T noted in Proposition II.2. The identity $r_k = y_k + n_k$ together with (26) implies that (27) holds. The fact that the channel output must be white follows since the innovations sequence for the optimal estimator is white [15]. Orthogonality of the input and output sequences follows from the orthogonality between an optimal estimate and the measurements upon which it is based [15]. ■

We also see that, with optimal control, the feedback system with filtering estimator depicted in Figure 2 is *equivalent* to the system in Figure 3 in the following sense. The responses of the channel input and output to the disturbance and noise in Figure 3 are identical to the responses of the estimation error and innovations sequence to the disturbance and noise in Figure 2. The feedback system with optimal control is thus equivalent to a communication channel with feedback to the channel input.

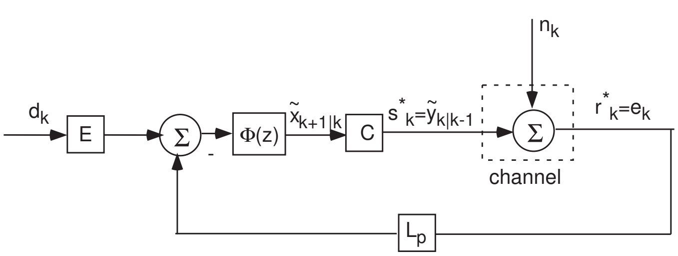

Fig. 3. Under optimal control, the feedback system in Figure 2 is input/output equivalent to a communication channel with feedback.

We have seen that the problem of power limited stabilization with a disturbance over a memoryless Gaussian channel has a solution with the following structure. First, an optimal estimator is applied to obtain the best estimate of the next value of the channel input (which is equal to the plant output) given the previous channel outputs. Second, a control signal is computed that inverts the plant and subtracts this estimate from the plant output. Because the estimate minimizes the mean square estimation error, it follows that the resulting control signal minimizes the variance of the system output, and thus also the power required at the channel input.

IV. Entropy Rate and the Optimal Estimation Error

We now derive an expression for the optimal prediction estimation error and provide an interpretation in terms of mutual information. The results below are an extension of those in [16], wherein the disturbance free stabilization problem was considered.

Given two random variables a and b, we denote the *mutual information* [10] by $I(a;b)$, and note that $I(a;b) = h(a) - h(a|b)$, where $h(a)$ and $h(a|b)$ denote the (differential) entropy of a and the conditional entropy of a given b, respectively. The (differential) *entropy rate* of a stationary, continuous-valued, discrete-time scalar random process a is given by [10] $h_\infty(a) = \lim_{k\to\infty} h(a_k|a^{k-1})$. The entropy rate of a stationary Gaussian random process a may be computed from its power spectral density $\mathcal{S}_a(\omega)$ [10] by the formula

$$h_\infty(a) = \frac{1}{2}\log_e 2\pi e + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_e \mathcal{S}_a(\omega) d\omega.$$

There is a connection between the entropy rate of a random sequence and the problem of estimating the next value of the sequence given all previous values. Denote such an estimate by $\hat{a}_{k|k-1}$, and the resulting estimation error by $\tilde{a}_{k|k-1} \triangleq a_k - \hat{a}_{k|k-1}$. Then the variance of the minimal mean square estimation error satisfies [10]

$$\mathcal{E}^*\{\tilde{a}_{k|k-1}^2\} = \frac{1}{2\pi e} e^{2h_\infty\{a\}}, \qquad (29)$$

and is thus completely determined by the entropy rate of a.

We now apply the connection between entropy rate and estimation error to the feedback system of Figure 1 with $\lambda = 1$. The power spectrum of the channel output may be written as

$$\mathcal{S}_r(\omega) = |S(e^{j\omega})|^2\sigma_n^2\left(1 + |G_d(e^{j\omega})|^2\frac{\sigma_d^2}{\sigma_n^2}\right). \qquad (30)$$

Since the exogenous inputs d and n are assumed Gaussian, the channel output is also Gaussian, with entropy rate given by

$$h_\infty(r) = \frac{1}{2}\log_e 2\pi e\sigma_n^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\log_e |S(e^{j\omega})| d\omega + \frac{1}{4\pi}\int_{-\pi}^{\pi}\log_e\left(1 + |G_d(e^{j\omega})|^2\frac{\sigma_d^2}{\sigma_n^2}\right) d\omega. \qquad (31)$$

Suppose that the plant G_u is strictly proper, has m anti-stable poles $|\phi_i| > 1$, and no nonminimum phase zeros. Then it is possible to stabilize the system using a controller with no anti-stable poles. With such a controller, S must satisfy the discrete Bode sensitivity integral [17]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log_e |S(e^{j\omega})| d\omega = \sum_{i=1}^{m}\log_e |\phi_i|. \qquad (32)$$

We now provide an interpretation of the third term on the right hand side of (31). Suppose, as shown in Figure 4, that state feedback $u_k = -K_{me}x_k$ is used to stabilize the plant and minimize the energy in the control signal, given by $\sum_{k=0}^{\infty} u_k^2$. The closed loop transfer function from d_k to r_k is given by $C\Phi E\left(1 + K_{me}\Phi E\right)^{-1}$. It may be shown from [18, Theorem 6.35 (d)] that $(1 + K_{me}\Phi E)^{-1}$ is allpass, and thus that the magnitude of the transfer function from r_k to d_k is identical to that of G_d. The mutual information rate

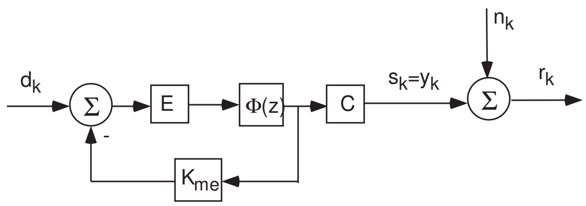

Fig. 4. System stabilized with minimal energy control has the same Bode gain plot as G_d.

[19] between the signals r and d in Figure 4 is given by

$$I_\infty(r; d) \triangleq \limsup_{k\to\infty} \frac{I(r^{k-1}; d^{k-1})}{k}, \tag{33}$$

and is equal to [19]

$$I_\infty(r; d) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left(1 + |G_d(e^{j\omega})|^2 \frac{\sigma_d^2}{\sigma_n^2}\right) d\omega. \tag{34}$$

Substituting (32) and (34) into (31) and applying the formula (29) shows that the minimum mean square error in estimating r_k given r^{k-1} is given by

$$\mathcal{E}^*\{\tilde{r}_{k|k-1}^2\} = \sigma_n^2 \prod_{i=1}^{m} |\phi_i|^2 e^{2I_\infty(r;d)}. \tag{35}$$

Let us now relate the problem of estimating the current channel output r_k given previous outputs r^{k-1} to that of estimating the current channel input $s_k = y_k$ given r^{k-1}. Denote the estimation errors for r_k and y_k by $\tilde{r}_{k|k-1}$ and $\tilde{y}_{k|k-1}$. Then, since n is zero mean and white, it follows that

$$\mathcal{E}\{\tilde{r}_{k|k-1}^2\} = \mathcal{E}\{\tilde{y}_{k|k-1}^2\} + \sigma_n^2. \tag{36}$$

Combining (36) with (35) yields an expression for the minimal error in estimating the channel input y_k given previous values of the channel output r^{k-1} that provides an alternative expression for the minimal power required for stabilization.

Proposition IV.1 *Assume that the hypotheses of Proposition II.2 are satisfied and that $\lambda = 1$. Then the channel power limit must satisfy the lower bound $P > J_y^*(1)$, where*

$$J_y^*(1) = \left(\prod_{i=1}^{m} |\phi_i|^2 e^{2I_\infty(r;d)} - 1\right)\sigma_n^2. \tag{37}$$

■

As noted in Section III, under the hypotheses of Proposition II.2, the variance of the channel input will be equal to that of the optimal estimation error. The minimal channel capacity required for stabilization is thus obtained from (4), yielding

$$\mathcal{C} > \sum_{i=1}^{m} \log_e |\phi_i| + I_\infty(r; d) \quad \text{nats/transmission}. \tag{38}$$

We now provide an interpretation of the two terms that contribute to channel capacity in (38). First, it follows from [6] that the channel capacity required for stabilization alone is given by $\sum_{i=1}^{m} \log|\phi_i|$. Hence we see that the additional capacity required to stabilize in the presence of a disturbance depends on the mutual information between the disturbance and the channel output, once the plant has been stabilized. A similar discussion appears in [7].

V. Use of Channel Precompensation, $\lambda \neq 1$.

With $\lambda = 1$ the problem of minimizing the power required for stabilization is equivalent to that of minimizing the response of the plant output to the disturbance and channel noise. If other values of λ are allowed, then more flexibility is available with which to either achieve smaller variance in the plant output or satisfy a lower channel power requirement. The former scenario arises when a channel is given that has a greater power limit than the minimum calculated for $\lambda = 1$, and the latter scenario when the given channel has a lower power limit that satisfies the lower bound (10) required for stabilization.

To proceed, note that the feedback system of Figure 1 may be rearranged so that the plant has state equations

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + E\lambda d_k, \tag{39}$$
$$s_k = C\bar{x}_k. \tag{40}$$

It follows that *changing the parameter λ is equivalent to changing the variance of the disturbance input*. As a consequence, we may apply the results of Sections III and IV to minimize the power in the channel input simply by replacing the disturbance variance by $\lambda^2\sigma_d^2$ in all the respective formulas.

Consider the problem of minimizing the variance of s_k in Figure 1 for a fixed value of λ, and denote the optimal cost by

$$J_s^*(\lambda) \triangleq \min_K \mathcal{E}\{s_k^2\}\Big|_\lambda. \tag{41}$$

The value of the cost (7) for the plant output for the controller that achieves the minimum in (41) is equal to

$$J_y(\lambda) = \frac{J_s^*(\lambda)}{\lambda^2}. \tag{42}$$

Lemma V.1 *The variance of the plant output, given by the ratio* (42)*, is a monotonically decreasing function of* λ^2.

Proof: It follows by substituting $\lambda^2\sigma_d^2$ for the disturbance variance in (37) that

$$\frac{J_s^*(\lambda)}{\lambda^2} = \frac{\sigma_n^2}{\lambda^2}\left(\prod_{i=1}^{m} |\phi_i|^2 e^{2I_\infty(r;\lambda d)} - 1\right). \tag{43}$$

Taking the derivative with respect to λ^2 in (43) and simplifying yields

$$\frac{\lambda^4 e^{-2I_\infty(r;\lambda d)}}{\sigma_n^2}\frac{\partial J_s^*(\lambda)/\lambda^2}{\partial \lambda^2} =$$
$$\left(\prod_{i=1}^m |\phi_i|^2 \int_{-\pi}^{\pi} \frac{-1/2\pi}{1+|G_d(e^{j\omega})|^2\lambda^2\sigma_d^2/\sigma_n^2} d\omega + e^{-2I_\infty(r;\lambda d)}\right)$$

It follows from Jensen's inequality [20, p. 63] that

$$e^{-2I_\infty(r;\lambda d)} \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1+|G_d(e^{j\omega})|^2\lambda^2\sigma_d^2/\sigma_n^2} d\omega.$$

Hence

$$\frac{2\pi\lambda^4 e^{-2I_\infty(r;\lambda d)}}{\sigma_n^2}\frac{\partial J_s(\lambda)/\lambda^2}{\partial \lambda^2} \le$$
$$\int_{-\pi}^{\pi} \frac{1}{1+|G_d(e^{j\omega})|^2\lambda^2\sigma_d^2/\sigma_n^2} d\omega \left(-\prod_{i=1}^m |\phi_i|^2 + 1\right),$$

and the result follows because each ϕ_i satisfies $|\phi_i| > 1$. ■

Proposition V.2 *Assume that the channel power limit satisfies the lower bound in Proposition II.1 necessary for stabilization. Assume also that the hypotheses of Proposition II.2 are satisfied. Then the variance of the plant output* (5) *can be made arbitrarily close to the optimal cost*

$$J_y^* = \frac{P}{\lambda^2}, \tag{44}$$

where λ is chosen so that $J_s(\lambda) = P$.

Proof: For a given value of λ, the problem of minimizing $J(s_k, \lambda)$ can be solved by applying Proposition III.1 with σ_d^2 replaced by $\lambda^2\sigma_d^2$, and it follows from (37) that

$$J_s^*(\lambda) = \sigma_n^2\left(\prod_{i=1}^m |\phi_i|^2 e^{2I_\infty(r;\lambda d)} - 1\right). \tag{45}$$

It is clear by inspection of (45) that $J_s^*(\lambda)$ is a monotonically increasing function of λ. Furthermore, as $\lambda \to 0$, $J_s^*(\lambda)$ approaches the limit (11), and as $\lambda \to \infty$, $J_s^*(\lambda) \to \infty$. By continuity, there exists a value of λ for which the variance of the channel input is equal to P, which is assumed to be greater than the bound (11). The optimal controller $K(z)$ has the form (21), where L_p is obtained from the Riccati equation (14) with σ_d^2 replaced by $\lambda^2\sigma_d^2$. ■

It follows immediately from Proposition V.2 and Lemma V.1 that if $P > J_y^*(1)$, then $\lambda > 1$ and $J_y^* < J_y^*(1)$. We thus see that if a given channel has a power limit *greater than* that required in the case $\lambda = 1$, then the optimal cost J_y^* is less than that for $\lambda = 1$. Similarly, if the channel has a power limit *less than* that required in the case $\lambda = 1$ (but greater than the limit (10) required for stabilization), then the optimal cost J_y^* is greater than that for $\lambda = 1$.

Example V.3 Consider the system (1)-(2) with

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.2 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and transfer function $G_d(z) = 1.5(z-0.1)/(z^2 - 2.3z + 1.32)$. Assume the disturbance and noise have variances $\sigma_d^2 = 1$ and $\sigma_n^2 = 0.1$, respectively. Plots of $J_s^*(\lambda)$ and $J_s^*(\lambda)/\lambda^2$ are depicted in Figure 5. Note that the former is monotonically increasing with λ and the latter, as proven in Lemma V.1, is monotonically decreasing; of course these plots intersect for $\lambda = 1$. For a given power limit, say $P = 10$, one finds the value of λ for which $J_s^*(\lambda) = 10$, and then corresponding value of $J_s^*(\lambda)/\lambda^2$ is equal to J_y^*, the optimal disturbance response. For the example $P = 10$, these values work out to be $\lambda \approx 1.817$ and $J_y^* \approx 3.029$. ■

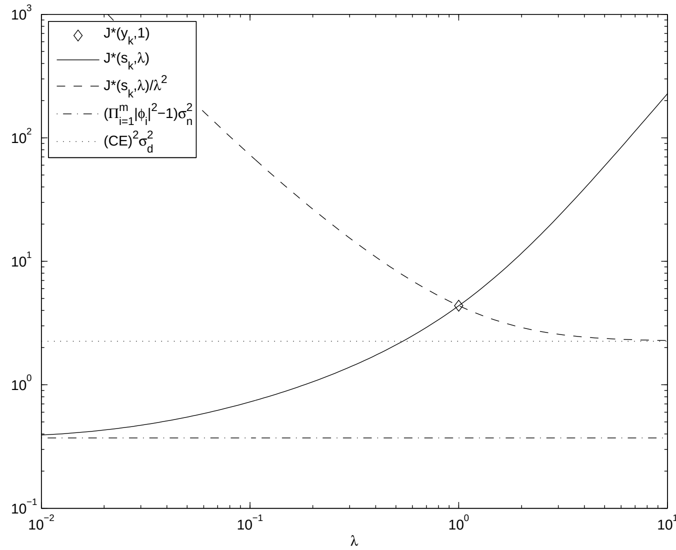

Fig. 5. As λ increases, the variance of the channel input increases and that of the plant output decreases.

Proposition V.4 *Assume that G_d is minimum phase and has relative degree equal to one. Then, in the limit as $\lambda \to \infty$,*

$$J_s(\lambda)/\lambda^2 \to \sigma_d^2(CE)^2. \tag{46}$$

Proof: The Riccati equation (14) with σ_d^2 replaced by $\lambda^2\sigma_d^2$ may be rearranged into the form

$$\tilde{\Sigma} = A\tilde{\Sigma}A^T - \frac{A\tilde{\Sigma}C^TC\tilde{\Sigma}A^T}{C\tilde{\Sigma}C^T + \sigma_n^2/\lambda^2} + \sigma_d^2 EE^T, \tag{47}$$

where $\tilde{\Sigma} = \Sigma/\lambda^2$. Hence, as $\lambda \to \infty$, the solution $\tilde{\Sigma}$ behaves like the solution to an estimation problem as the measurement noise approaches zero. The dual version of this problem is the state regulation problem in the case that the control cost approaches zero; i.e., "cheap control". Asymptotic properties of a feedback system under cheap control are described in Theorem 6.37 of [18]. Since G_d is assumed to have relative degree one and to be minimum phase, all the closed loop eigenvalues will be placed at the locations of these zeros, with the exception of one which will be at the origin. It may be shown that the closed loop transfer function from d to y satisfies $S(z)G_d(z) = z^{-1}CE$, and the result follows. ■

VI. OPTIMALITY OF LINEAR COMMUNICATION AND CONTROL: THE SCALAR CASE

In this section we assume that the plant (1)-(2) is first order ($n = 1$). We also suppose that the channel input and control signal are the outputs of nonlinear, time-varying, and dynamical systems:

$$s_k = f_k(y^k, s^{k-1}), \tag{48}$$
$$u_k = g_k(r^k). \tag{49}$$

We derive a lower bound on the disturbance attenuation achievable with the general communication and control scheme (48)-(49), and show that this lower bound is obtained using the *linear* compensation scheme of Figure 1. An alternate approach to this problem is found in [21], who apply the results of [16] to prove a similar result for a first order unstable plant. Our approach is more general in that the plant is allowed to be stable.

Our first result is applicable to plants of arbitrary order, and provides a lower bound on the reduction in the variance of y_k due to the channel output r_k.

Proposition VI.1 *Consider the linear system* (1)-(2)*, channel* (3)*, and the general communication and control scheme described by* (48)-(49)*. Then*

$$\mathcal{E}\{\tilde{y}_{k|k}^2\} \geq \frac{\sigma_n^2}{\sigma_n^2 + P}\mathcal{E}\{\tilde{y}_{k|k-1}^2\}. \tag{50}$$

Proof: We first show that

$$I(y_k; \hat{y}_{k|k}|r^{k-1}) \geq \frac{1}{2}\log_e\left(\frac{\mathcal{E}\{\tilde{y}_{k|k-1}^2\}}{\mathcal{E}\{\tilde{y}_{k|k}^2\}}\right). \tag{51}$$

To do so, we apply an argument similar to that on [10, p. 345]:

$$\begin{aligned}
I(y_k; \hat{y}_{k|k}|r^{k-1}) &\overset{(a)}{=} h(y_k|r^{k-1}) - h(y_k|\hat{y}_{k|k}, r^{k-1}) \\
&\overset{(b)}{=} h(\tilde{y}_{k|k-1}|r^{k-1}) - h(y_k|\hat{y}_{k|k}, r^{k-1}) \\
&\overset{(c)}{=} \frac{1}{2}\log_e 2\pi e\mathcal{E}\{\tilde{y}_{k|k-1}^2\} - h(y_k|\hat{y}_{k|k}, r^{k-1}) \\
&\overset{(d)}{=} \frac{1}{2}\log_e 2\pi e\mathcal{E}\{\tilde{y}_{k|k-1}^2\} - h(\tilde{y}_{k|k}|\hat{y}_{k|k}, r^{k-1}) \\
&\overset{(e)}{\geq} \frac{1}{2}\log_e 2\pi e\mathcal{E}\{\tilde{y}_{k|k-1}^2\} - h(\tilde{y}_{k|k}) \\
&\overset{(f)}{\geq} \frac{1}{2}\log_e 2\pi e\mathcal{E}\{\tilde{y}_{k|k-1}^2\} - \frac{1}{2}\log_e 2\pi e\mathcal{E}\{\tilde{y}_{k|k}^2\},
\end{aligned}$$

where (a) follows by definition, (b) follows since $\hat{y}_{k|k-1}$ is determined from r^{k-1}, (c) follows since $\tilde{y}_{k|k-1}$ is Gaussian when conditioned on r^{k-1}, (d) follows since $\hat{y}_{k|k}$ is given, (e) follows since conditioning reduces entropy, and (f) follows since the normal distribution maximizes the entropy for a given second moment.

The data processing inequality [10] implies that

$$I(y_k; \hat{y}_{k|k}|r^{k-1}) \leq I(r_k; s_k|r^{k-1}), \tag{52}$$

and the expression for channel capacity (4) implies that

$$I(r_k; s_k|r^{k-1}) \leq \frac{1}{2}\log_e\left(1 + P/\sigma_n^2\right). \tag{53}$$

Taken together, (51)-(53) yield (50). ■

We next apply Proposition VI.1 to derive a lower bound on the output variance in the special case that the plant is first order.

Proposition VI.2 *Consider the linear system* (1)-(2)*, channel* (3)*, and the general communication and control scheme described by* (48)-(49)*. Suppose further that the plant* (1)-(2) *is* first order*, and assume that the power limit satisfies*

$$P > (A^2 - 1)\sigma_n^2. \tag{54}$$

Then communication and control schemes exists for which $\max_k \mathcal{E}\{\tilde{y}_{k|k-1}^2\}$ *is finite and satisfies the lower bound*

$$\sup_k \mathcal{E}\{\tilde{y}_{k|k-1}^2\} \geq \frac{(1 + P/\sigma_n^2)\sigma_d^2 C^2 E^2}{(1 - A^2) + P/\sigma_n^2} \tag{55}$$

Proof: It follows from (1)-(2) that

$$y_k = CAx_{k-1} + CBu_{k-1} + CEd_{k-1}.$$

If the sequence of channel outputs r^{k-1} is given, then u_{k-1} is determined, and it follows that

$$\begin{aligned}
\mathcal{E}\{\tilde{y}_{k|k-1}^2\} &= \mathcal{E}\{\left(CA\tilde{x}_{k-1|k-1} + CEd_{k-1}\right)^2\} \\
&= \mathcal{E}\{\left(CA\tilde{x}_{k-1|k-1}\right)^2\} + (CE)^2\sigma_d^2
\end{aligned}$$

since $\tilde{x}_{k-1|k-1}$ and d_{k-1} are independent. The assumption of a first order plant implies that

$$\mathcal{E}\{\tilde{y}_{k|k-1}^2\} = A^2\mathcal{E}\{\tilde{y}_{k-1|k-1}^2\} + (CE)^2\sigma_d^2 \tag{56}$$

Hence, by (50), we have that

$$\mathcal{E}\{\tilde{y}_{k|k-1}^2\} \geq A^2\mathcal{E}\{\tilde{y}_{k-1|k-2}^2\}\frac{\sigma_n^2}{P + \sigma_n^2} + (CE)^2\sigma_d^2,$$

or

$$\begin{aligned}
\sup_k \mathcal{E}\{\tilde{y}_{k|k-1}^2\} &\geq A^2 \sup_k \mathcal{E}\{\tilde{y}_{k-1|k-2}^2\}\frac{\sigma_n^2}{P + \sigma_n^2} + (CE)^2\sigma_d^2 \\
&= A^2 \sup_k \mathcal{E}\{\tilde{y}_{k|k-1}^2\}\frac{\sigma_n^2}{P + \sigma_n^2} + (CE)^2\sigma_d^2,
\end{aligned}$$

and rearranging yields (55). ■

Our next result shows that the lower bound (50) may be satisfied with identity using linear control.

Proposition VI.3 *Consider the linear system* (1)-(2)*, channel* (3)*, and the linear communication and control scheme depicted in Figure 1. Assume that the hypotheses of Proposition II.2 and the bound* (54) *are satisfied, and that* λ *and* $K(z)$ *are chosen as in Proposition V.2. Then*

$$\mathcal{E}\{\tilde{y}_{k|k}^2\} = \frac{\sigma_n^2}{P + \sigma_n^2}\mathcal{E}\{\tilde{y}_{k|k-1}^2\}. \tag{57}$$

Proof: For a fixed value of λ, the optimal controller is found by minimizing $\mathcal{E}\{\tilde{s}_{k|k-1}^2\}$, and λ is chosen so that

$\mathcal{E}^*\{\tilde{s}^2_{k|k-1}\} = P$. Such a value of λ exists because the stabilization bound (54) is assumed to be satisfied, and may be found by replacing σ_d^2 with $\lambda^2\sigma_d^2$ in the scalar version of the Riccati equation (14), and multiplying by C^2 to obtain

$$C^2\Sigma = A^2C\Sigma - C^4\Sigma^2/(C^2\Sigma + \sigma_n^2) + \lambda^2\sigma_d^2C^2E^2. \quad (58)$$

Solving (58) for λ^2 yields

$$\lambda^2 = \frac{P(P + \sigma_n^2(1 - A^2))}{\sigma_d^2C^2E^2}, \quad (59)$$

which by the assumption (54) is guaranteed to be positive. It follows from (17) that $\mathcal{E}^*\{\tilde{s}^2_{k|k}\} = \sigma_n^2P/(\sigma_n^2 + P)$. The result (57) follows by noting that estimates for y_k may be obtained from those for s_k by dividing by λ. ■

Our final result shows that, for a scalar plant, the bound from Proposition VI.2 may be satisfied with equality using linear control.

Proposition VI.4 *In addition to the hypotheses of Proposition VI.3, assume that the plant is first order, and that the power limit satisfies* (54)*. Then choosing* λ *and* $K(z)$ *as in Proposition V.2 yields*

$$\mathcal{E}\{\tilde{y}^2_{k|k-1}\} = \frac{(1 + P/\sigma_n^2)\sigma_d^2C^2E^2}{(1 - A^2) + P/\sigma_n^2} \quad (60)$$

Proof: The assumption of stationarity, together with (56) and (57), imply that

$$\begin{aligned}\mathcal{E}\{\tilde{y}^2_{k|k-1}\} &= A^2\mathcal{E}\{\tilde{y}^2_{k|k}\} + (CE)^2\sigma_d^2 \\ &= A^2\mathcal{E}\{\tilde{y}^2_{k|k-1}\}\frac{\sigma_n^2}{\sigma_n^2 + P} + (CE)^2\sigma_d^2,\end{aligned}$$

and the result follows by rearranging. ■

VII. Conclusions and Directions for Further Research

In this paper we have assumed a specific and simple communication and control scheme and shown how to use this scheme to stabilize the plant and minimize the variance of the plant output in the case that the plant is minimum phase, relative degree one, and a filtering estimator is used. Extensions to cases where these assumptions fail to hold remain to be worked out. We also showed that no more general control scheme can achieve a lower variance than our linear scheme for a first order plant. Optimality properties of our linear scheme for higher order plants remain to be explored.

References

[1] P. Antsaklis and J. Baillieul. Special issue on networked control systems. *IEEE Transactions on Automatic Control*, 49(9), September 2004.

[2] G. N. Nair and R. J. Evans. Stabilizability of stochastic linear systems with finite feedback data rates. *SIAM Journal of Control and Optimization*, 43(2):413–436, July 2004.

[3] G. N. Nair and R. J. Evans. Exponential stabilisability of finite-dimensional linear systems with limited data rates. *Automatica*, 39(4):585–593, April 2003.

[4] S. Tatikonda and S. M. Mitter. Control under communication constraints. 49(7):1056–1068, July 2004.

[5] N. C. Martins and M. A. Dahleh. Feedback control in the presence of noisy channels: "Bode-like" fundamental limitations of performance. *IEEE Transactions on Automatic Control*, to appear, 2007.

[6] J. H. Braslavsky, R. H. Middleton, and J. Freudenberg. Feedback stabilization over signal-to-noise ratio constrained channels. *IEEE Transactions on Automatic Control*, to appear, 2007.

[7] N. Elia. The information cost of loop shaping over Gaussian channels. In *Proceedings of the Mediterranean Conference on Control and Automation*, June 2005.

[8] A. S. Matveev and A. V. Savkin. An analogue for Shannon information theory for networked control systems. Stabilization via a noisy discrete channel. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, The Bahamas, December 2004.

[9] A. S. Yüksel and T. Başar. Coding and control over discrete forward and feedback channels. In *Proceedings of the 44th IEEE Conference on Decision and Control*, Seville, Spain, December 2005.

[10] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley and Sons, New York, 1991.

[11] B. D. O. Anderson and J. B. Moore. *Optimal Control: Linear Quadratic Methods*. Prentice Hall, Englewood Cliffs, NJ, 1990.

[12] Z. Zhang and J. S. Freudenberg. Discrete-time loop transfer recovery for systems with nonminimum phase zeros and time delays. *Automatica*, 29(2):351–363, 1993.

[13] J. M. Maciejowski. Asymptotic recovery for discrete-time systems. *IEEE Transactions on Automatic Control*, AC-30(6):602–605, June 1985.

[14] M. Kinnaert and Y. Peng. Discrete-time LQG/LTR technique for systems with time delays. *Systems & Control Letters*, 15:303–311, 1990.

[15] B. D. O. Anderson and J. B. Moore. *Optimal Filtering*. Dover, Mineola, NY, 2005.

[16] J. S. Freudenberg, R. H. Middleton, and V. Solo. The minimal signal-to-noise ratio required to stabilize over a noisy channel. In *Proceedings of the 2006 American Control Conference*, pages 650–655, June 2006.

[17] M. M. Seron, J. H. Braslavsky, and G. C. Goodwin. *Fundamental Limitations in Filtering and Control*. Springer, 1997.

[18] H. Kwakernaak and R. Sivan. *Linear Optimal Control Systems*. Wiley-Interscience, 1972.

[19] M. S. Pinsker. *Information and Information Stability of Random Variables and Processes*. Holden Day, 1968.

[20] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, 2nd edition, 1974.

[21] A. J. Rojas, R. H. Middleton, J. S. Freudenberg, and J. H. Braslavsky. Scalar optimal signal to noise ratio feedback capacity and performance. submitted to the 2007 Conference on Decision and Control, March 2007.