On nonregular feedback linearization *

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Key Words— Nonlinear control systems; feedback linearization; linear systems; nonregular static feedback; differential geometric methods.

Abstract— This paper investigates the use of nonregular (not necessarily regular) static/dynamic state feedbacks to achieve feedback linearization of affine nonlinear systems. First, we provide an example which is nonregular static feedback linearizable but is not regular dynamic feedback linearizable. Then, we present some necessary conditions as well as sufficient conditions for nonregular feedback linearization. The sufficient conditions are checkable, and if they are verified, a linearing–feedback could be calculated following a recursive procedure, provided the integrations of a set of completely integrable systems are available.

1 Introduction

This paper addresses the problem of locally transforming an affine nonlinear system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + G(x)u \tag{1}$$

with $x \in R^n, u \in R^m, f(0) = 0$ and $rankG(0) = m$ into a controllable linear system

$$\dot{z} = Az + Bv, \quad z \in R^{n'}, v \in R^{m'}, \tag{2}$$

where $n' \geq n, m' \leq m$.

Since Krener(1973) this problem has been studied using increasingly more general transformations.State space diffeomorphisms

$$z = \phi(x), \qquad \phi(0) = 0 \tag{3}$$

were the first transformations considered.Brockett(1978) proposed to enlarge the class of transformations (3) by adding state feedback transformations

$$u = \alpha(x) + \beta v, \quad v \in R^m, \alpha(0) = 0, det\beta \neq 0. \tag{4}$$

*The original version of this paper was presented at the IFAC Conference on System Structure and Control, which was held in Nantes,France during 5–7th July,1995. Corresponding author Dr. Zhendong Sun. Tel:(86) 10 62782421; Fax:(86) 10 62015347; E–mail:zdsun.bbs@bbs.net.tsinghua.edu.cn

These feedbacks were generalized in Jakubczyk and Respondek(1980) and Hunt *et al.*(1983) by

$$u = \alpha(x) + \beta(x)v, \quad v \in R^m, det\beta(0) \neq 0 \tag{5}$$

where the nonsingular matrix $\beta(x)$ was allowed to depend on the states as well. A system (1) is said to be locally regular static feedback linearizable, if it can be transformed into (2) around the origin via (3) and (5). Gardner and Shadwick(1992) proposed an efficient algorithm to compute controllers which drive feedback linearizable systems to Brunovsky normal forms. For those systems that are not regular static feedback linearizable,the problem of partial feedback linearization was solved by Krener *et al.*(1983) and Marino(1986).

A more general class of state feedback transformations were given by dynamic state feedback transformations:

$$\begin{cases} \dot{w} = a(x,w) + b(x,w)v, & w \in R^q, \\ u = \alpha(x,w) + \beta(x,w)v, & v \in R^{m'} \end{cases} \tag{6}$$

with $a(0,0) = 0, \alpha(0,0) = 0$,and q being the order of the compensator. The extended system of (1) controlled by a dynamic compensator (6) can be written as

$$\dot{\bar{x}} = \begin{pmatrix} f + G\alpha \\ a \end{pmatrix}(\bar{x}) + \begin{pmatrix} G\beta \\ b \end{pmatrix}(\bar{x})v \overset{def}{=} \bar{f}(\bar{x}) + \sum_{i=1}^{m'} \bar{g}_i(\bar{x})v_i \tag{7}$$

with $\bar{x} = \begin{pmatrix} x \\ w \end{pmatrix}$ being the extended states.If $u = \alpha(\bar{x}) + \beta(\bar{x})v$ are viewed as outputs for system (7), one can define the corresponding differential output rank (see Di Benedetto *et al.*1989). A dynamic compensator (6) is said to be regular for system (1), if the corresponding differential output rank is m around the origin.

The problem of transforming the systems (1) into (2) via regular dynamic compensators and extended state space diffeomorphisms is called (locally) regular dynamic feedback linearization problem. It was studied by Cheng(1987) and Charlet *et al.*(1989,1991). Sluis(1993) and Rouchon(1994) presented two necessary conditions for dynamic feedback linearization, while Levine and Marino(1990), Pomet *et al.*(1992) and Pomet(1993) addressed the problem of regular dynamic feedback linearization for systems on R^4. More recent work in this area could be found in Fliess *et al.*(1995),Aranda *et al.*(1995) and the references therein.

The feedback transformations considered in the above cited papers all satisfy the regularity condition.The problem of transforming the systems (1) into (2) via regular feedback transformations (5) or (6) and (possibly extended) state space diffeomorphisms (3), will be called here regular feedback linearization problem.

The interest of this paper is to enlarge the class of feedback linearizable systems by exploiting more general state feedbacks, i.e. , feedbacks not necessarily satisfy the regularity condition. All of the static state feedback transformations

$$u = \alpha(x) + \beta(x)v, \quad \alpha(0) = 0, \beta(x) : m \times m', m' \leq m \tag{8}$$

and the dynamic state feedback transformations (6) are called nonregular. The idea of using nonregular state feedbacks in control system design could be traced back to the work of Heymann(1968) which showed that a multi–input controllable linear system can always be brought to a single–input controllable linear system via a nonregular static state feedback, thus enabling an easy proof of the pole assignment theorem for the multi–input case. This idea was generalized to nonlinear case by Tsinias and Kalouptsidis(1981,1987). Another implementation of nonregular state feedbacks could be found in the well–known Morgan's problem,see,for example,Morgan(1964). For exact linearization of nonlinear system, the idea of using nonregular state feedbacks was proposed by Charlet *et al.*(1989) and Marino(1990).However,so far no systematic research in this direction has been reported.

In this paper, we present preliminary study on the nonregular feedback linearization problem. In the next section, we devote to formulating the problems of nonregular static/dynamic feedback linearization, and to providing an example demonstrating the independent theoretical value of our problems. Some necessary conditions for nonregular feedback linearization are given in Section 3. The main result, sufficient conditions for nonregular static feedback linearization, is stated and proved in Section 4. And lastly, Section 5 gives brief concluding remarks.

2 Problem formulation and preliminary results

We assume that all functions under consideration are defined and analytic in an open neighborhood of the origin. Our results will be stated in terms of an open neighborhood Γ of the origin in the respective Euclidean spaces, and we implicitly permit Γ to be made smaller to accommodate subsequent local argument.In what follows, unless otherwise stated, the same notations as in the textbook Isidori(1989) will be used.

Definition 2.1. A system (1) is said to be (locally) nonregular static feedback linearizable,if it can be transformed into (2) via a state feedback (8) and a state space diffeomorphism (3).The problem of linearization via (8) and (3) is called nonregular static feedback linearization problem. □

Definition 2.2. A system (1) is said to be (locally) nonregular dynamic feedback linearizable,if it can be transformed into (2) via a dynamic state feedback (6) and an extended state space diffeomorphism (3). The problem of linearization via (6) and (3) is called nonregular dynamic feedback linearization problem. □

The above two problems together will be generally called the problem of (locally) nonregular feedback linearization.

The following proposition gives an equivalent statement of nonregular feedback linearization problem.

Proposition 2.1. A system (1) is nonregular feedback linearizable if, and only if,the corresponding feedback gain matrix β of (8) (or (6)) can be chosen to be a column vector (i.e., $m' = 1$).

Proof. We need only to prove the necessity.

Suppose under some transformations (8) (or (6)) and (3) the system (1) is changed into (2). There must exists a linear feedback $v = Fz + Gv'$ such that the system (2) will be changed into a single-input controllable linear system. One may verify that under state feedback

$$u = \alpha(x) + \beta(x)F\phi(x) + \beta(x)Gv'$$

and (3), the system (1) reads to be a controllable linear system. □

Now we offer an example which is nonregular static feedback linearizable but is not regular dynamic feedback linearizable. This demonstrates that the problem of nonregular static feedback linearization is of independent theoretical interest.

Example 2.1. Consider the system

$$\dot{x} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_1 + \begin{pmatrix} 1 \\ x_2^2 x_4 \\ 1 + x_2 + x_1 x_5^2 \\ 0 \\ 0 \end{pmatrix} u_2. \tag{9}$$

Set $u_2 = 0$, then the system (9) reads to be a controllable linear system. Hence (9) is nonregular static feedback linearizable. Now we prove by contradiction that (9) is not regular feedback linearizable.

Suppose there exists a regular dynamic compensator

$$\begin{cases} \dot{w} = a(x,w) + b(x,w)v, & w \in R^q, \\ u = \alpha(x,w) + \beta(x,w)v, & v \in R^2 \end{cases} \tag{10}$$

such that the closed-loop system

$$\begin{aligned} \dot{\bar{x}} = & \begin{pmatrix} x_2 + \alpha_2 \\ x_3 + \alpha_2 x_2^2 x_4 \\ x_4 + \alpha_2(1 + x_2 + x_1 x_5^2) \\ x_5 \\ \alpha_1 \\ a \end{pmatrix} + \begin{pmatrix} \beta_{21} \\ \beta_{21} x_2^2 x_4 \\ \beta_{21}(1 + x_2 + x_1 x_5^2) \\ 0 \\ \beta_{11} \\ b_1 \end{pmatrix} v_1 \\ & + \begin{pmatrix} \beta_{22} \\ \beta_{22} x_2^2 x_4 \\ \beta_{22}(1 + x_2 + x_1 x_5^2) \\ 0 \\ \beta_{12} \\ b_2 \end{pmatrix} v_2 \stackrel{def}{=} f + g_1 v_1 + g_2 v_2 \end{aligned} \tag{11}$$

can be changed into a controllable linear system via a regular static state feedback and an extended state space diffeomorphism, here $a, b_1, b_2 \in R^q$.

If $u = \alpha(\bar{x}) + \beta(\bar{x})v$ are viewed as outputs for system (11), we may define the relative orders (Isidori 1989):

$$\gamma_j = \begin{cases} 0, & \exists i, 1 \le i \le 2, s.t. \beta_{j,i}(\bar{x}) \ne 0, \\ min\{r : \exists i, s.t. L_{g_i} L_f^{r-1} \alpha_j(\bar{x}) \ne 0\}, & \beta_{j,i}(\bar{x}) = 0, \forall i, \end{cases}$$

and decoupling matrix $D(x) = (\delta_{j,i})_{2\times 2}$, with

$$\delta_{j,i}(\bar{x}) = \begin{cases} \beta_{j,i}(\bar{x}), & \gamma_j = 0, \\ L_{g_i} L_f^{\gamma_j - 1} \alpha_j(\bar{x}), & \gamma_j > 0. \end{cases}$$

Denote $D_j(\bar{x}) = (\delta_{j,1}(\bar{x}), \delta_{j,2}(\bar{x}))$.

Applying Singh's Inversion Algorithm (Di Benedetto *et al.* 1989) to the system (11) with outputs $u = \alpha(\bar{x}) + \beta(\bar{x})v$, we obtain a full rank matrix $\tilde{B}_{n'+1}$(using the same notation as in Di Benedetto *et al.* 1989). The regularity of (10) means that $\tilde{B}_{n'+1}$ is square and hence is nonsingular at the origin. Therefore,γ_1 and γ_2 are finite. Note also that if $\gamma_1 \leq \gamma_2$ (respectively,$\gamma_1 \geq \gamma_2$), then the covector $D_1(\bar{x})$ (respectively, $D_2(\bar{x})$) is exactly the first row of the matrix $\tilde{B}_{n'+1}$, so $D_1(0) \neq 0$ (respectively,$D_2(0) \neq 0$). This observation is crucial to the following derivation.

If $(\beta_{21}, \beta_{22}) \neq 0$, then without loss of generality, (up to some regular state feedback transformation) we can assume $\beta_{21} = 1$,and furthermore, $\alpha_2 = 0, \beta_{22} = 0$. Careful examination shows that $[g_1, [f, g_1]] \notin span\{g_1, g_2, [f, g_1], [f, g_2]\}$, which contradicts the fact that the distribution $span\{g_1, g_2, [f, g_1], [f, g_2]\}$ is involutive. Hence $\beta_{21} = \beta_{22} = 0$.

Similarly, if $(\beta_{11}, \beta_{12}) \neq 0$, then we assume $\beta_{11} = 1$, and furthermore, $\alpha_1 = 0,\ \beta_{12} = 0$. Routine computation shows that that the relationship $[[f, g_1], g_1] \in span\{g_1, g_2, [f, g_1], [f, g_2]\}$ implies

$$L_{g_1}^2 \alpha_2 = 0,\ \ \alpha_2 = -2x_5 L_{g_1}\alpha_2. \tag{12}$$

Therefore,

$$L_{g_1}\alpha_2 = -2(L_{g_1}x_5)L_{g_1}\alpha_2 - 2x_5 L_{g_1}^2\alpha_2 = -2(L_{g_1}x_5)L_{g_1}\alpha_2.$$

But $L_{g_1}x_5 = 1$, so $L_{g_1}\alpha_2 = 0$. It follows from (12) that $\alpha_2 = 0$,which, together with the fact $(\beta_{21}, \beta_{22}) = 0$, contradicts the regularity assumption.

Now we have shown that $\beta = 0$. After modifying a, b_1 and b_2 by a regular static feedback we may assume that the system (11) can be changed into a controllable linear system via a state space diffeomorphism. Then we can use Lie bracket identities of a linear system (Krener 1973):

$$[ad_f^i g_k, ad_f^j g_l](x) = 0,\ \ x \in \Gamma\ i, j = 0, 1, \cdots,\ \ k, l = 1, 2. \tag{13}$$

If $\gamma_1 \geq \gamma_2$, then by the regularity assumption we may assume $Lg_1 L_f^{\gamma_2 - 1}\alpha_2(0) \neq 0$. For simplicity, denote $Lg_1 L_f^{\gamma_2 - 1}\alpha_2$ by h, and $[ad_f^{\gamma_2} g_1, ad_f^{\gamma_2 + 1} g_1] = (\zeta_1, \cdots, \zeta_{q+5})^T$. Detail computation gives

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ 2h^2 + \psi_3 \end{pmatrix} + \psi_4 \begin{pmatrix} 1 \\ x_2^2 x_4 \\ 1 + x_2^2 + x_1 x_5^2 \end{pmatrix}.$$

Here $\psi_1, \cdots, \psi_4$ are certain real-valued functions of x, with $\psi_1(0) = \psi_3(0) = 0$. That $det \begin{pmatrix} \psi_1 & 1 \\ 2h^2 + \psi_2 & 1 + x_2^2 + x_1 x_5^2 \end{pmatrix}(0) \neq 0$ indicates that the two vectors $(\psi_1, \psi_2, 2h^2 + \psi_3)^T$ and $(1, x_2^2 x_4, 1 + x_2^2 + x_1 x_5^2)^T$ are independent at the origin, so $[ad_f^{\gamma_2} g_1, ad_f^{\gamma_2 + 1} g_1](0) \neq 0$, which contradicts (13).

Finally we suppose that $\gamma_1 < \gamma_2$. Without loss of generality, we assume that $\theta \stackrel{def}{=} L_{g_1} L_f^{\gamma_1 - 1}\alpha_1$ is not equal to zero at the origin. Denote $ad_f^{\gamma_1} g_1$ by g_0. Routine calculation shows that the equality $[ad_f^{\gamma_1} g_1, ad_f^{\gamma_1 + 1} g_1] = 0$ implies that

$$L_{g_0}^2 \alpha_2 = 0,\ \ \ L_{g_0}\theta = 0,\ \ \ 2x_5 L_{g_0}\alpha_2 + \alpha_2\theta = 0. \tag{14}$$

Therefore

$$(2L_{g_0}x_5 + \theta)L_{g_0}\alpha_2 = 0.$$

This , together with the fact $L_{g_0}x_5 = \theta$, yields $L_{g_0}\alpha_2 = 0$. It follows from (14) that $\alpha_2 = 0$, which contradicts the regularity assumption.

The above reasonings show that system (9) is not regular dynamic feedback linearizable. □

3 Some necessary conditions

Proposition 3.1. If a system (1) is locally nonregular dynamic feedback linearizable, then its linear approximation at the origin

$$\dot{x} = \frac{\partial f}{\partial x}|_{x=0}x + G(0)u \stackrel{def}{=} Fx + Gu$$

is controllable, i.e., $rank(G, FG, \cdots, F^{n-1}G) = n$.

Proof. By Proposition 2.1,there exists a dynamic compensation (6) in which $\beta(x, w)$ is a column vector,such that the extended system (7) is equivalent to (2) up to a state space change of coordinates. Set

$$F_0 = \frac{\partial \bar{f}}{\partial \bar{x}}|_{\bar{x}=0}, \quad b_0 = \bar{g}_1(0),$$

then (F_0, b_0) is controllable.It is easy to deduce that

$$\Im(b_0, F_0b_0, \cdots, F_0^{n'-1}b_0) \subseteq \Im(G, FG, \cdots, F^{n-1}G) \times R^q$$

where "$\Im M$" denotes the image space of matrix M. It follows easily that $rank(G, FG, \cdots, F^{n-1}G) = n$. □

Given a system (1), define the distributions

$$\mathcal{L}_0 = span\{g_1, \cdots, g_m\}, \ \mathcal{L}_{i+1} = \mathcal{L}_i^c + ad_f^{i+1}\mathcal{L}_0, \quad i = 0, 1, \cdots,$$

where $\mathcal{L}_i^c$ denotes the involutive closure of the distribution $\mathcal{L}_i$. Assume that $\mathcal{L}_i$ and $\mathcal{L}_i^c$ are of constant ranks around the origin.Define integers

$$r_0 = rank\mathcal{L}_0, \ r_i = rank\mathcal{L}_i - rank\mathcal{L}_{i-1}^c, \quad i = 1, 2, \cdots,$$

and furthermore

$$k_i = card\{j : r_j \geq i, j \geq 0\}, i = 1, \cdots, m,$$

where $card\{.\}$ denotes the number of elements of the set $\{.\}$. It was pointed out by Marino(1986) that there exist some regular static feedback (5) and state space diffeomorphism (3) under which the system (1) reads as

$$\begin{aligned} \dot{z}^{(1)} &= A_c z^{(1)} + B_c v, \\ \dot{z}^{(2)} &= a(z^{(1)}, z^{(2)}) + b(z^{(1)}, z^{(2)})v, \end{aligned} \tag{15}$$

where $z^{(1)} = (z_1, \cdots, z_\iota), z^{(2)} = (z_{\iota+1}, \cdots, z_n), \iota = \sum_{i=1}^m k_i$, and

$$A_c = blockdiag(A_1^c, \cdots, A_m^c), \quad B_c = blockdiag(B_1^c, \cdots, B_m^c),$$

with (A_i^c, B_i^c) being Brunovsky canonical pair of length k_i. The system (15) is said to be a pseudocanonical form for (1).

The following result characterizes a class of systems which cannot be linearizable via any dynamic compensators.

Proposition 3.2. Suppose the system (15) is a pseudocanonical form for the system (1). Define $j_\tau = \sum_{i=1}^\tau k_i$ for $1 \le \tau \le m$. If there exist integers $s, 1 \le s \le m$, $k_s > 1$ and $l, \sum_{i=1}^m k_i < l \le n$, such that $\dot{z}_l = a_l(z) = a_l(z_{j_{s-1}+1}, \cdots, z_{j_s})$ and $\frac{\partial^2 a_l}{\partial {z_{j_s}}^2} \neq 0$, then the system (1) is not nonregular dynamic feedback linearizable.

Proof. We prove by contradiction.

Suppose there exists some dynamic compensator (6) with $m' = 1$ such that the closed-loop system (7) can be transformed into a controllable linear one up to some change of coordinates.By Krener(1973), one has

$$[ad_{\bar{f}}^i \bar{g}_1, ad_{\bar{f}}^j \bar{g}_1] = 0, \ i, j = 0, 1, \cdots.$$

Denote $\zeta = (z_{j_{s-1}+1}, \cdots, z_{j_s})^T$, then we have

$$\begin{pmatrix} \dot{\zeta} \\ \\ \\ \\ \dot{z}_l \end{pmatrix} = \begin{pmatrix} z_{j_{s-1}+2} \\ \vdots \\ z_{j_s} \\ \alpha_s \\ a_l(\zeta) \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{s,j_s} \\ 0 \end{pmatrix} v_1$$

Note that there exists an integer $i_0, 0 \le i_0 \le n' - 1$, such that the $j_s th$ entry of $\vartheta \overset{def}{=} ad_{\bar{f}}^{i_0} \bar{g}_1$ is not equal to zero at the origin, while all ϑ's $(j_{s-1}+1)th \sim (j_s - 1)th$ entries and lth entry are zero. It can be verified that

$$[ad_{\bar{f}}^{i_0} \bar{g}_1, ad_{\bar{f}}^{i_0+1} \bar{g}_1]_l = \frac{\partial^2 a_l}{\partial z_{j_s}^2} \vartheta_{j_s} + \frac{\partial a_l}{\partial z_{j_s}} [ad_{\bar{f}}^{i_0} \bar{g}_1, ad_{\bar{f}}^{i_0+1} \bar{g}_1]_{j_s - 1}.$$

This,together with the assumption $\frac{\partial^2 a_l}{\partial z_{j_s}^2}(0) \neq 0$, implies that

$$[ad_{\bar{f}}^{i_0} \bar{g}_1, ad_{\bar{f}}^{i_0+1} \bar{g}_1](0) \neq 0,$$

which leads to contradiction. $\square$

Note that Proposition 3.2 relies on the availability of a pseudocanonical form. But this form is generally not available, so the conditions in Proposition 3.2 cannot be checked in general.

4 Sufficient conditions

In this section, we will present verifiable sufficient conditions for nonregular static feedback linearization. The result is motivated by Theorem 4.2 in Charlet et al.(1991).

Theorem 4.1. If,for a set of integers $\{\nu_1, \cdots, \nu_m\}, 0 \leq \nu_i \leq n$, the nested distributions

$$\begin{aligned} \Delta_0 &= span\{g_k : \nu_k = 0\}, \\ \Delta_{i+1} &= \Delta_i + ad_f \Delta_i + span\{g_k : \nu_k = i+1\}, \quad i \geq 0 \end{aligned}$$

are such that in Γ

(i) Δ_i is involutive and of constant rank for $0 \leq i \leq n-1$;

(ii) $rank \Delta_{n-1} = n$;

(iii) $[g_j, \Delta_i] \subseteq \Delta_{i+1}$ for all $j, 1 \leq j \leq m$,such that $1 \leq \nu_j \leq n-1$ and all $i, 0 \leq i \leq n-3$;

then the system (1) is locally nonregular static feedback linearizable.

Proof of Theorem 4.1. To clarify, we divide the proof into two separated parts. The main body contains overall analysis which depends upon three lemmas. The proofs of the lemmas will be given in the Appendix.

For a scalar real-valued function h defined on Γ and a distribution Δ, the dual product of dh and Δ is defined as $< dh, \Delta >= \{L_\psi h : \psi \in \Delta\}$, and we say $< dh, \Delta >$ is of constant rank one, if the space $< dh, \Delta > (0)$ is non–trivial. Given a smooth vector field $\underline{f}$,and smooth distributions $\underline{G}_0, \cdots, \underline{G}_\kappa$,,we say the multiplet $(\underline{f}; \underline{G}_1, \cdots, \underline{G}_\kappa)$ satisfies Condition($\aleph$),if the nested distributions

$$\underline{\Delta}_0 = \underline{G}_0, \ \underline{\Delta}_i = \begin{cases} \underline{\Delta}_{i-1} + ad_{\underline{f}} \underline{\Delta}_{i-1} + \underline{G}_i, \ i = 1, \cdots, \kappa, \\ \underline{\Delta}_{i-1} + ad_{\underline{f}} \underline{\Delta}_{i-1}, \ i = \kappa+1, \cdots \end{cases} \tag{16}$$

satisfy (a) $\underline{\Delta}_i$ is involutive and of constant rank for $0 \leq i \leq n-1$; (b) $rank \underline{\Delta}_{n-1} = n$; and (c) $[\underline{G}_j, \underline{\Delta}_i] \subseteq \underline{\Delta}_{i+1}, \ 1 \leq j \leq \kappa, 0 \leq i \leq n-3$.

Define integers $q = min\{j : rank \Delta_j = n\}, l = max\{i : i \leq q, rank(\Delta_{i-1} + ad_f \Delta_{i-1}) < rank \Delta_i\}$, and $m_0 = q - l$.

When $l = q$, we have the following lemma.

Lemma 4.2. Suppose $l = q$. If $(f; G_0, \cdots, G_{n-1})$ satisfies Condition($\aleph$), then there exist a vector field $\bar{f}, \bar{f}(0) = 0, \bar{f} - f \in span\{g_1, \cdots, g_m\}$, and smooth distributions $\bar{G}_0, \cdots, \bar{G}_k (k \leq n-1), \bar{G}_j \subseteq span\{g_1, \cdots, g_m\}, j = 1, \cdots, k$, such that

$$\sum_{j=0}^{k} rank(\bar{G}_j) < \sum_{j=0}^{n-1} rank(G_j) \tag{17}$$

and the multiplet $(\bar{f}; \bar{G}_0, \cdots, \bar{G}_k)$ satisfies Condition($\aleph$).

Now we assume that $l < q$. One has

$$\begin{aligned} \Delta_{q-1} &= \Delta_{l-1} + \cdots + ad_f^{m_0} \Delta_{l-1} + G_l + \cdots + ad_f^{m_0 - 1} G_l, \\ \Delta_q &= \Delta_{l-1} + \cdots + ad_f^{m_0+1} \Delta_{l-1} + G_l + \cdots + ad_f^{m_0} G_l. \end{aligned}$$

From the facts $\Delta_q = \Delta_{q-1} + ad_f^{m_0+1} \Delta_{l-1} + ad_f^{m_0} G_l$ and $rank \Delta_q > rank \Delta_{q-1}$ it follows that either

$$rank(\Delta_{q-1} + ad_f^{m_0} G_l) > rank \Delta_{q-1} \tag{18}$$

or

$$rank(\Delta_{q-1} + ad_f^{m_0+1}\Delta_{l-1}) = n. \tag{19}$$

To cope with these two cases, we need the following lemmas :

Lemma 4.3. Suppose $(f; G_0, \cdots, G_{n-1})$ satisfies Condition($\aleph$) and (18), then there exist a vector field $\bar{f}, \bar{f}(0) = 0, \bar{f} - f \in span\{g_1, \cdots, g_m\}$, and smooth distributions $\bar{G}_0, \cdots, \bar{G}_k (k \leq n-1), \bar{G}_j \subseteq span\{g_1, \cdots, g_m\}, j = 1, \cdots, k$, such that (17) holds and the multiplet $(\bar{f}; \bar{G}_0, \cdots, \bar{G}_k)$ satisfies Condition($\aleph$).

Lemma 4.4. Suppose $(f; G_0, \cdots, G_{n-1})$ satisfies Condition($\aleph$) and (19), then there exist a vector field $\bar{f}, \bar{f}(0) = 0, \bar{f} - f \in span\{g_1, \cdots, g_m\}$, and smooth distributions $\bar{G}_0, \cdots, \bar{G}_k (k \leq n-1), \bar{G}_j \subseteq span\{g_1, \cdots, g_m\}, j = 1, \cdots, k$, such that (17) holds and the multiplet $(\bar{f}; \bar{G}_0, \cdots, \bar{G}_k)$ also satisfies Condition($\aleph$).

Now we are ready to draw the conclusion. For this purpose, note that so far we have established that, if $(f; G_0, \cdots, G_\tau)$ satisfies Condition($\aleph$),then there exist a vector field $\bar{f}$ on $R^n, \bar{f}(0) = 0, \bar{f} - f \in span\{g_1, \cdots, g_m\}$, and $\bar{G}_0, \cdots, \bar{G}_k (k \leq \tau), \bar{G}_j \subseteq span\{g_1, \cdots, g_m\}, j = 1, \cdots, k$, such that

$$\sum_{j=0}^{k} rank(\bar{G}_j) < \sum_{j=0}^{\tau} rank(G_j)$$

and $(\bar{f}; \bar{G}_0, \cdots, \bar{G}_k)$ still satisfies Condition($\aleph$). By recurrence, eventually we can construct a vector field f^* on $R^n, f^*(0) = 0, f^* - f \in span\{g_1, \cdots, g_m\}$, and a smooth distribution $G_0^* \subseteq span\{g_1, \cdots, g_m\}$, such that $(f^*; G_0^*)$ satisfies Condition($\aleph$). By Theorem 5.2.4 of Isidori(1989) and Definition 2.1, the system (1) is nonregular static feedback linearizable. □

The proof indicates a recursive design procedure to compute desired feedback laws, provided the integrations of a set of completely integrable systems are available. We illustrate by an example that how to utilize this procedure to design a nonregular–feedback–linearizing controller.

Example 4.1. Consider the following system

$$\dot{x} = \begin{pmatrix} x_4 + x_5^2 + (x_4x_5 + x_2x_6)u_4 \\ x_3 + x_3u_3 + x_3^2u_4 \\ x_6 + u_4 \\ u_3 \\ u_2 \\ u_1 \end{pmatrix}. \tag{20}$$

Routine examination shows that this system verifies Theorem 4.1 (with $\nu_1 = 0, \nu_2 = 2, \nu_3 = 2, \nu_4 = 6$). Now we compute a linearizing controller following the procedure provided in the proof of Theorem 4.1.

Simple calculation shows that Lemma 4.3 is applicable. Routine calculation shows that

$$\bar{f} = (x_4 + x_5^2, x_3 + x_2x_3, x_6, x_2, 0, 0)^T, g_2' = (0, -2x_3x_5, 0, -2x_5, 1, 0)^T.$$

Denote $f^1 = \bar{f}, g_1 = (0, 0, 0, 0, 0, 1)^T$.

It can be checked that $(f^1; span\{g_1\}, span\{g_2'\})$ verifies (19). Applying the procedure presented in the proof of Lemma 4.4 gives

$$f^2 = f^1 + x_1 g_2', \quad G_0^2 = span\{g_1\}.$$

Note that the system

$$\dot{x} = f^2 + g_1 v^1 \tag{21}$$

is (regular static) feedback linearizable. Standard analysis exhibits that,under the feedback

$$v^1 = \frac{1}{1 + x_2 - 2x_1 x_5}(v + \rho(x)) \tag{22}$$

and state space change of coordinates

$$z = (x_5, x_1, x_4 + x_5^2, x_2, x_3 + x_2 x_3 - 2x_1 x_3 x_5, \sigma(x)), \tag{23}$$

the system (21) will be changed into a controllable linear system (here ρ and σ are certain tedious but easily computed functions of x).

Therefore,under the feedback law

$$u = \begin{pmatrix} \frac{1}{1+x_2-2x_1x_5}\rho \\ x_1 \\ x_2 - 2x_1 x_5 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{1+x_2-2x_1x_5} \\ 0 \\ 0 \\ 0 \end{pmatrix} v \tag{24}$$

and state space diffeomorphism (23), the system (20) reads to be a single–input controllable linear system. □

5 Conclusions

In this paper,we have introduced the problem of nonregular feedback linearization. This problem has been proved to be of independent interest. We also presented some elementary results with emphasis on the power and limitation of nonregular static feedbacks in the context of exact linearization. Theorem 4.1 stated checkable sufficient conditions for nonregular static feedback linearization, its proof indicated a design procedure to calculate desired control laws.

One drawback of Theorem 4.1 is its conditions are coordinate–dependent. It is of interest to improve Theorem 4.1 such that the conditions are invariant under regular static feedbacks. This is a topic of our future work. A clearer understanding of the relationships between the nonregular dynamic feedback linearization problem and the nonregular static feedback linearization problem also requires further investigation.

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Appendix

A. Proof of Lemma 4.2.

Without loss of generality,we assume that $G_q = span\{g_1, \cdots, g_s\}$ and $rank(\Delta_{q-1} + ad_f\Delta_{q-1} + span\{g_s\}) > rank(\Delta_{q-1} + ad_f\Delta_{q-1})$.By Frobenius Theorem,there exists a real–valued function (on R^n) $\lambda, \lambda(0) = 0$,such that

$$d\lambda \perp \Delta_{q-1},\ < d\lambda, g_s > (0) \neq 0,$$

where $< d\lambda, g_s > \stackrel{def}{=} L_{g_s}\lambda$. Let

$$g'_s = \frac{1}{< d\lambda, g_s >} g_s, f' = f - (L_f\lambda)g'_s, {g'}_i = g_i - (L_{g_i}\lambda)g'_s, i = 1, \cdots, s-1,$$

it is easy to show that $(f'; G_0, \cdots, G_q)$ satisfies Condition($\aleph$) and still verifies this lemma.
Simple computation shows that

$$d\lambda \perp ad_{f'}\Delta_{q-1}, d\lambda \perp span\{{g'}_1, \cdots, {g'}_{s-1}\},\ \Delta_{q-1} + ad_{f'}\Delta_{q-1} + span\{{g'}_1, \cdots, {g'}_s\} = \Delta_q.$$

Hence the distribution $\Delta_{q-1} + ad_{f'}\Delta_{q-1} + span\{{g'}_1, \cdots, {g'}_{s-1}\}$ has constant rank $n-1$ and is involutive. By $rank(\Delta_{q-1} + ad_{f'}\Delta_{q-1}) > rank(\Delta_{q-1})$, there exists a real–valued function $\varphi, \varphi(0) = 0$ such that $d\varphi \perp \Delta_{q-1}$ and $< d\varphi, ad_{f'}\Delta_{q-1} >$ has constant rank one. Set

$$\bar{f} = f' + \varphi g'_s, \bar{G}_q = span\{{g'}_1, \cdots, {g'}_{s-1}\}.$$

Applying the algorithm (16) to $(\bar{f}; G_1, \cdots, G_{q-1}, \bar{G}_q)$ gives

$$\bar{\Delta}_q = \Delta_{q-1} + ad_{f'}\Delta_{q-1} + \bar{G}_q,\quad \bar{\Delta}_{q+1} = \bar{\Delta}_q + ad_{\bar{f}}\bar{\Delta}_q.$$

We may verify that $< d\lambda, \bar{\Delta}_{q+1} >$ has constant rank one,hence $rank(\bar{\Delta}_{q+1}) = n$.
So the multiplet $(\bar{f}; G_0, \cdots, G_{q-1}, \bar{G}_q)$ satisfies the properties as claimed. □

B. Proof of Lemma 4.3.

We assume that $G_l = span\{g_1, \cdots, g_s\}$ and $rank(\Delta_{q-1} + span\{ad_f^{m_0}g_s\}) = rank\Delta_{q-1} + 1$. By Frobenius Theorem, there exists a real–valued function $\mu_1,\ \mu_1(0) = 0$, such that

$$d\mu_1 \perp \Delta_{q-1},\ < d\mu_1, ad_f^{m_0}g_s > (0) \neq 0.$$

Define

$$g'_s = \frac{1}{< d\mu_1, ad_f^{m_0}g_s >} g_s,\quad f' = f - (L_f^{m_0+1}\mu_1)g'_s,$$

and

$$\mu_2 = L_f\mu_1, \cdots, \mu_{m_0+1} = L_f\mu_{m_0}.$$

It can be verified that $(f'; G_0, \cdots, G_l)$ still satisfies Condition($\aleph$) and verifies (18). For simplicity,here we will denote f' by f.
Let $p_0 = min\{j - l : j \geq l, rank(\Delta_{l-1} + \cdots + ad_f^{j-l}\Delta_{l-1} + G_l + \cdots + ad_f^{j-l}G_l) = rank\Delta_j\}$. There exists a vector field $g_0 \in \Delta_{l-1}$, and a real–valued function $\varphi, \varphi(0) = 0$, such that

$$ad_f^{p_0}g_0 \notin \Delta_{l-1} + \cdots + ad_f^{p_0-1}\Delta_{l-1} + G_l + \cdots + ad_f^{p_0-1}G_l$$

and

$$d\varphi \perp \Delta_{l+p_0-2},\ < d\varphi, ad_f^{p_0}g_0 >= 1.$$

Set

$$\bar{f} = f + \varphi g'_s,\ \ {g'}_i = g_i - (L_{g_i}\mu_{m_0+1})g'_s, \bar{G}_l = \{{g'}_1, \cdots, {g'}_{s-1}\}.$$

Applying the algorithm (16) to $(\bar{f}; G_0, \cdots, G_{l-1}, \bar{G}_l)$ gives

$$\begin{aligned}
&\bar{\Delta}_j = \Delta_j, \quad j = 1, \cdots, l-1,\\
&\bar{\Delta}_l = \Delta_{l-1} + ad_{\bar{f}}\Delta_{l-1} + \bar{G}_l = \Delta_{l-1} + ad_f\Delta_{l-1} + \bar{G}_l,\\
&\vdots\\
&\bar{\Delta}_{p_0+l-1} = \Delta_{l-1} + \cdots + ad_f^{p_0}\Delta_{l-1} + \bar{G}_l + \cdots + ad_f^{p_0-1}\bar{G}_l.
\end{aligned}$$

Using the relationships $d\mu_{m_0+1} \perp \bar{\Delta}_l, \bar{\Delta}_l + span\{g'_s\} = \Delta_l,$ and $< d\mu_{m_0+1}, g'_s >= 1$, we see that $\bar{\Delta}_l$ is involutive and of constant rank. Similarly, we can derive that $\bar{\Delta}_{l+1}, \cdots, \bar{\Delta}_{p_0+l-1}$ are involutive and of constant ranks.

Next we consider the distribution $\bar{\Delta}_{p_0+l}$. It follows from $\bar{\Delta}_{p_0+l-1} \subseteq \Delta_{p_0+l-1}$ that $\bar{\Delta}_{p_0+l} \subseteq \Delta_{p_0+l}$. It can be verified that

$$\Delta_{p_0+l} = \Delta_{p_0+l-1} + ad_{\bar{f}}^{p_0}\bar{G}_l + span\{ad_f^{p_0}g'_s\} = \bar{\Delta}_{p_0+l-1} + ad_{\bar{f}}^{p_0}\bar{G}_l + span\{g'_s, \cdots, ad_f^{p_0}g'_s\}. \quad (A.1)$$

Simple computation yields

$$d\mu_{m_0+1} \perp (\bar{\Delta}_{p_0+l-1} + ad_{\bar{f}}^{p_0}\bar{G}_l),\ < d\mu_{m_0+1}, ad_{\bar{f}}^{p_0+1}g_0 >= 1, \quad (A.2)$$

and

$$d\mu_i \perp ad_{\bar{f}}^{p_0+1}g_0, \quad i = 1, \cdots, m_0. \quad (A.3)$$

By (A.1), (A.2) and (A.3), we can find a vector field $h \in (\bar{\Delta}_{p_0+l-1} + ad_{\bar{f}}^{p_0}\bar{G}_l)$ such that

$$ad_{\bar{f}}^{p_0+1}g_0 = h + g'_s.$$

It follows from (A.1) that

$$\bar{\Delta}_{p_0+l} \supseteq \bar{\Delta}_{p_0+l-1} + ad_{\bar{f}}^{p_0}\bar{G}_l + span\{g'_s\},\ rank(\bar{\Delta}_{p_0+l}) \geq rank(\Delta_{p_0+l}) - p_0.$$

These, together with the fact that Δ_{p_0+l} being involutive and the relationships

$$d\mu_i \perp \bar{\Delta}_{p_0+l},\ < d\mu_i, ad_f^{m_0+1-i}g'_s >= 1, \qquad i = m_0, \cdots, m_0 - p_0 + 1,$$

imply that $rank(\bar{\Delta}_{p_0+l}) = rank(\Delta_{p_0+l}) - p_0$ and $\bar{\Delta}_{p_0+l}$ is involutive and of constant rank. Therefore,

$$\bar{\Delta}_{p_0+l} = \bar{\Delta}_{p_0+l-1} + ad_{\bar{f}}^{p_0}\bar{G}_l + span\{g'_s\}.$$

Similar computation shows that $\bar{\Delta}_{p_0+l+1}, \cdots, \bar{\Delta}_{p_0+q-1}$ are involutive and of constant ranks. Hence $(\bar{f}; G_0, \cdots, G_{l-1}, \bar{G}_l)$ satisfies Condition(ℵ). □

C. Proof of Lemma 4.4.

Suppose the vector field $g_0 \in \Delta_{l-1}$ is such that

$$rank(\Delta_{q-1} + ad_f^{m_0}G_l + span\{ad_f^{m_0+1}g_0\}) = rank(\Delta_{q-1} + ad_f^{m_0}G_l) + 1.$$

Define an integer

$$\begin{aligned}
p_0 = min\{j - l - 1 : j \geq l + 1, rank(&\Delta_{l-1} + \cdots + ad_f^{j-l+1}\Delta_{l-1}\\
&+ G_l + \cdots + ad_f^{j-l-1}G_l) = rank(\Delta_j)\}.
\end{aligned}$$

Without loss of generality,we assume

$$ad_f^{p_0}g_s \notin \Delta_{l-1} + \cdots + ad_f^{p_0+1}\Delta_{l-1} + G_l + \cdots + ad_f^{p_0-1}G_l.$$

So there exists a real-valued function $\lambda_1, \lambda_1(0) = 0$, such that

$$d\lambda_1 \perp \Delta_{p_0+l-1},\ < d\lambda_1, ad_f^{p_0}g_s > (0) \neq 0.$$

Set

$$g_s' = \frac{1}{< d\lambda_1, ad_f^{p_0}g_s >}g_s,\ f' = f - (L_f^{p_0+1}\lambda_1)g_s'.$$

It can be easily verified that $(f'; G_0, \cdots, G_l)$ still satisfies Condition($\aleph$) and $L_{f'}^{p_0+1}\lambda_1 = 0$. Suppose $(f'; G_0, \cdots, G_l)$ still verify (19) ,then define

$$\lambda_2 = L_f\lambda_1, \cdots, \lambda_{p_0+1} = L_f\lambda_{p_0}.$$

Simple computation gives

$$d\lambda_i \perp \Delta_{p_0+l-i}, < d\lambda_i, ad_f^{p_0+1-i}g_s' >= 1, i = 1, \cdots, p_0 + 1. \qquad (A.4)$$

So there exists a real–valued function $\mu, \mu(0) = 0$, such that

$$d\mu \perp \Delta_{q-1},\ < d\mu, ad_f^{m_0+1}g_0 > (0) \neq 0.$$

Set

$$\bar{f} = f' + \mu g_s', {g'}_i = g_i - (L_{g_i}\lambda_{p_0+1})g_s', i = 1, \cdots, s-1, \bar{G}_l = span\{{g'}_i : 1 \leq i \leq s-1\}.$$

Apply the algorithm (16) to $(\bar{f}; G_0, \cdots, G_{l-1}, \bar{G}_l)$, denote the resulted sequence of distributions by $\bar{\Delta}_i, i = 1, 2, \cdots$. Simple computation shows that

$$d\lambda_i \perp \bar{\Delta}_j,\ i = p_0 + 1, \cdots, 1,\ q \geq j \geq p_0 + l + 1 - i.$$

From (A.4) it follows that the distributions $\bar{\Delta}_i,\ i = l, \cdots, q$ are involutive and of constant ranks. Next we turn to the distribution $\bar{\Delta}_{q+1}$. It can be verified that

$$< d\lambda_{p_0+1}, ad_f^{m_0+2}g_s' >= 1,\ d\lambda_i \perp \bar{\Delta}_{q+1},\ i = p_0, \cdots, 1. \qquad (A.5)$$

Note that $\Delta_q = \bar{\Delta}_q + span\{g_s', \cdots, ad_f^{p_0}g_s'\}$, this gives

$$\bar{\Delta}_{q+1} \supseteq \bar{\Delta}_q + span\{g_s'\}.$$

By (A.5),we obtain that

$$rank(\bar{\Delta}_{q+1}) \leq n - p_0 = rank(\bar{\Delta}_q + span\{g_s'\}).$$

Thus $\bar{\Delta}_{q+1} = \bar{\Delta}_q + span\{g_s'\}$ and is involutive and of constant rank.
In the same way,we can reach the conclusion that the distributions $\bar{\Delta}_{q+2}, \cdots, \bar{\Delta}_{q+p_0} = \Delta_q$ are involutive and of constant ranks. Hence $(\bar{f}; G_0, \cdots, G_{l-1}, \bar{G}_l)$ satisfies Condition($\aleph$). $\square$