A proof of global attractivity for a class of switching systems using a non-Lyapunov approach

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Abstract

A sufficient condition for the existence of a Lyapunov function of the form $V(x) = x^T Px,\ P = P^T > 0,\ P \in \mathbb{R}^{n\times n}$, for the stable linear time invariant systems $\dot{x} = A_i x,\ A_i \in \mathbb{R}^{n\times n},\ A_i \in \mathcal{A} \triangleq \{A_1, ..., A_m\}$, is that the matrices A_i are Hurwitz, and that a non-singular matrix T exists, such that $TA_iT^{-1}, i \in \{1, ..., m\}$, is upper triangular (Mori, Mori & Kuroe 1996, Mori, Mori & Kuroe 1997, Liberzon, Hespanha & Morse 1998, Shorten & Narendra 1998*b*). The existence of such a function, referred to as *a common quadratic Lyapunov function* (CQLF), is sufficient to guarantee the exponential stability of the switching system $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$. In this paper we investigate the stability properties of a related class of switching systems. We consider sets of matrices $\mathcal{A}$, where no single matrix T exists that simultaneously transforms each $A_i \in \mathcal{A}$ to upper triangular form, but where a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\},\ i, j \in \{1, ..., m\}$, are upper triangular. We show that, for a special class of such systems, the origin of the switching system $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$, is globally attractive. A novel technique is developed to derive this result, and the applicability of this technique to more general systems is discussed towards the end of the paper.

1 Introductory remarks

A sufficient condition for the existence of a common quadratic Lyapunov function (CQLF), $V(x) = x^T P x,\ P = P^T > 0,\ P \in \mathbb{R}^{n \times n}$, for the stable linear time invariant (LTI) systems $\dot{x} = A_i x,\ x \in \mathbb{R}^n, A(t) \in \mathcal{A} \triangleq \{A_1, ..., A_m\}, A_i \in \mathbb{R}^{n \times n}$, is that the matrices A_i are Hurwitz, and that a non-singular transformation T exists such that TA_iT^{-1} is upper triangular for all $i \in \{1, ..., m\}$. The existence of a CQLF is sufficient to guarantee the exponential stability of the switching system $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$. This result was first derived by Mori et al. (1997), and further discussed by Liberzon et al. (1998) and Shorten & Narendra (1998*b*). Unfortunately, from a practical viewpoint, the requirement of simultaneous triangularisability imposes unrealistic conditions on the matrices in the set $\mathcal{A}$. It is therefore of interest to extend the results derived by Mori et al. (1997) with a view to relaxing this requirement. In this context several authors have recently published new conditions which also guarantee exponential stability of the switching system. Typically, the approach adopted is to bound the maximum allowable perturbations of the matrix parameters from a nominal (triangularisable) set of matrices, thereby guaranteeing the existence of a CQLF; see Mori et al. (1997). In this paper we consider a class of switching systems that is closely related to those studied by Mori et al. (1997). However, rather than assuming maximum allowable perturbations from nominal matrix parameters, we explicitly assume that no single non-singular transformation T exists that simultaneously triangularises all of the matrices in $\mathcal{A}$. Rather, we assume that a number of non-singular matrices T_{ij} exist, such that for each pair of matrices in $\mathcal{A}$, $\{A_i, A_j\}$, the pair of matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\}$ are upper triangular. For a class of such systems, we demonstrate that the origin of the switching system, $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$, is globally attractive.

The results presented in this paper are important for a number of reasons. Primarily, they suggest that the requirement for simultaneous triangularisability may be relaxed considerably without the loss of asymptotic stability. Secondly, a novel technique, which does not utilise concepts from Lyapunov theory, is developed to demonstrate the global attractivity of the origin, for the system class considered. This technique may be of sufficient generality to be of use in the study of other classes of hybrid systems.

This paper is organised as follows. Preliminary definitions, and some useful results, are presented in Section 2. The main result of the paper is presented in Section 3. Finally, the relevance of the results for more general systems is discussed in Section 4.

2 Definitions

In this section we introduce some simple concepts and definitions (from Narendra & Annaswamy (1989)) which are useful in the remainder of the paper.

(i) The switching system : Consider the linear time-varying system

$$\dot{x} = A(t)x, \tag{1}$$

where $x \in \mathbb{R}^n$, and where the matrix switches between the matrices $A_i \in \mathbb{R}^{n \times n}$ belonging to the set $\mathcal{A} = \{A_1, ..., A_m\}$. We shall refer to this as *the switching system.* The time-invariant linear system $\dot{x} = A_i x$, denoted Σ_{A_i} is referred to as the i^{th} *constituent system.*

Suppose that (1) is described by the γ^{th} system $\dot{x} = A_\gamma x$ over a time interval $[t_\gamma, t_{\gamma+1}]$. By definition, the next system that we switch to, say the $(\gamma+1)^{th}$ system, starts at time $t_{\gamma+1}$ with initial conditions equal to the terminal conditions of the γ^{th} system at time $t_{\gamma+1}$.

(ii) Stability of the origin : The equilibrium state $x = 0$ of Equation (1) is said to be stable if for every $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(\epsilon, t_0) > 0$ such that $\| x_0 \| < \delta(\epsilon, t_0)$ implies that $\| x(t; x_0, t_0) \| < \epsilon, \ \forall \, t \geq t_0$.

(iii) Attractivity of the origin : The equilibrium state $x = 0$ of Equation (1) is said to be attractive if for some $\rho > 0$, and for every $\theta > 0$ and t_0, there exists a number $T(\theta, x_0, t_0)$ such that $\| x_0 \| < \rho$ implies that $\| x(t; x_0, t_0) \| < \theta, \ \forall \, t \geq t_0 + T$.

(iv) Global attractivity of the origin : The equilibrium state $x = 0$ of Equation (1) is said to be globally attractive if $\lim_{t \to \infty} x(t; x_0, t_0) = 0$, for all initial conditions x_0 and for all $t_0 \geq 0$.

(v) Asymptotic stability : The equilibrium state of Equation (1) is said to be asymptotically stable if it is both stable and attractive.

(vi) Common quadratic Lyapunov function: In the following discussion we refer to common quadratic Lyapunov functions (CQLF's). A common quadratic Lyapunov function is defined as follows.

Consider the switching system defined in (1) where all the elements of $\mathcal{A}$ are Hurwitz. The quadratic function

$$V(x) = x^T P x, \quad P = P^T > 0, \quad P \in \mathbb{R}^{n \times n}, \tag{2}$$

is said to be a common quadratic Lyapunov function for each of the constituent subsystems $\Sigma_{A_i}, \ i \in \{1, ..., m\}$, if symmetric positive definite matrices $Q_i, \ i \in \{1, ..., m\}$, exist such that the matrix P is a solution of the matrix equations

$$A_i^T P + P A_i = -Q_i. \tag{3}$$

The existence of a common quadratic Lyapunov function implies the exponential stability of the switching system (1) as discussed by Narendra & Balakrishnan (1994).

Common quadratic Lyapunov functions provide the basis for many known stability results; see for example Narendra & Taylor (1973) and Vidysagar (1993). It is therefore of interest to develop conditions which either guarantee the existence or non-existence of such a function. For completeness we include the following results which are useful in this context (see Cohen & Lewkowicz (1997) and Shorten & Narendra (1998a) for more extensive discussions).

Theorem 2.1 : (see Barker, Berman & Plemmons (1978) and Shorten & Narendra (1998a)) The stable dynamic systems Σ_A and $\Sigma_{A^{-1}}$, $A \in \mathbb{R}^{n \times n}$ share the same quadratic Lyapunov function $V(x) = x^T P x$, $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$.

Corollary 2.1 :

(a) Let $V(x)$ be a CQLF for Σ_{A_i}, $i \in \{1, ..., m\}$. Then $V(x)$ is also a Lyapunov function for

$$\dot{x} = (\sum_{i=1}^{m} \alpha_i A_i)x, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i > 0, \tag{4}$$

Therefore, a necessary condition for a CQLF to exist is that the matrix pencil $\sum_{i=1}^{m} \alpha_i A_i$, is also Hurwitz for all $\alpha_i \geq 0$, $\sum_{i=1}^{m} \alpha_i > 0$ Narendra & Balakrishnan (1994).

(b) From (a) and Theorem 2.1, a necessary condition for a CQLF to exist for the system (1), is that the matrix pencil

$$\sum_{i=1}^{m} \alpha_i A_i + \beta_i A_i^{-1}, \quad \forall \ \alpha_i \geq 0, \quad \beta_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i + \beta_i > 0, \tag{5}$$

is Hurwitz.

(c) Let $V(x)$ be a CQLF for Σ_{A_i}, $i \in \{1, ..., m\}$. Then, $V(x)$ is also a Lyapunov function for

$$\dot{x} = (\sum_{i=1}^{m} \alpha_i A_i + \beta_i A_i^{-1})x, \quad \alpha_i \geq 0, \quad \beta_i \geq 0, \quad \sum_{i=1}^{m} \alpha_i + \beta_i > 0, \tag{6}$$

and for

$$\dot{x} = \left[\sum_{i=1}^{m} \alpha_i A_i + \beta_i A_i^{-1} + (\sum_{i=1}^{m} \gamma_i A_i + \delta_i A_i^{-1})^{-1} \right] x, \tag{7}$$

where $\alpha_i \geq 0,\ \beta_i \geq 0,\ \gamma_i \geq 0,\ \delta_i \geq 0,\ \sum_{i=1}^{m} \alpha_i + \beta_i + \gamma_i + \delta_i > 0$. Hence, a necessary condition for the stable LTI systems, $\Sigma_{A_i},\ i \in \{1, ..., m\}$, to have a CQLF is that the matrix pencil

$$\sum_{i=1}^{m} \alpha_i A_i + \beta_i A_i^{-1} + (\sum_{i=1}^{m} \gamma_i A_i + \delta_i A_i^{-1})^{-1}, \tag{8}$$

is Hurwitz, where $\alpha_i \geq 0,\ \beta_i \geq 0,\ \gamma_i \geq 0,\ \delta_i \geq 0$, and where $\sum_{i=1}^{m} \alpha_i + \beta_i + \gamma_i + \delta_i > 0$.

<u>Comment :</u> Let the matrix A_i be Hurwitz. Suppose that the pencil

$$\sum_{i=1}^{m} \alpha_i A_i + \beta_i A_i^{-1},\ \ \forall\ \ \alpha_i \geq 0,\ \ \beta_i \geq 0, \sum_{i=1}^{m} \alpha_i + \beta_i > 0, \tag{9}$$

has eigenvalues in the right half of the complex plane for some $\alpha_i \geq 0,\ \beta_i \geq 0, \sum_{i=1}^{m} \alpha_i + \beta_i > 0$. Then a switching sequence exists such that the solution of the differential equation

$$\dot{x} = A(t)x, A(t) \in \{A_1, A_1^{-1}, ..., A_m, A_m^{-1}\}, \tag{10}$$

is unbounded (Shorten 1996, Shorten, Ó Cairbre & Curran 2000).

3 Main result

The condition that every matrix in the set $\mathcal{A}$ can be simultaneously triangularised, and is Hurwitz, is sufficient to guarantee the existence of a CQLF for every Σ_{A_i}, $i \in \{1, ..., m\}$. Unfortunately, the weaker condition of pairwise triangularisability is not sufficient to guarantee the existence of a CQLF. This is illustrated by the following example.

<u>**Example 1:**</u> Consider the following stable LTI systems,

$$\Sigma_{A_i} : \dot{x} = A_i x,\ \ A_i \in \mathbb{R}^{2\times 2},$$

with,

$$A_1 = \begin{bmatrix} -1.0000 & 0.0998 \\ -0.9982 & 0.0980 \end{bmatrix},\ \ A_2 = \begin{bmatrix} -1.00000 & -0.0995 \\ -0.9945 & -0.1049 \end{bmatrix},\ \ A_3 = \begin{bmatrix} -1.0000 & -0.0818 \\ -8.1818 & -1 \end{bmatrix}.$$

The set of matrices for which $A_i^T P + PA_i < 0,\ P = P^T > 0,\ P \in \mathbb{R}^{2\times 2}$, is given by

$$\mathcal{P}_{A_i} : det\{A_i^T P + PA_i\} > 0, \tag{11}$$

where

$$P = \begin{bmatrix} 1 & p_{12} \\ p_{12} & p_{22} \end{bmatrix}. \tag{12}$$

Equation (11) defines a convex set in (p_{12}, p_{22}) space. These sets are depicted graphically for each of the systems $\Sigma_{A_1}, \Sigma_{A_2}$, and Σ_{A_3} in Figure 1.

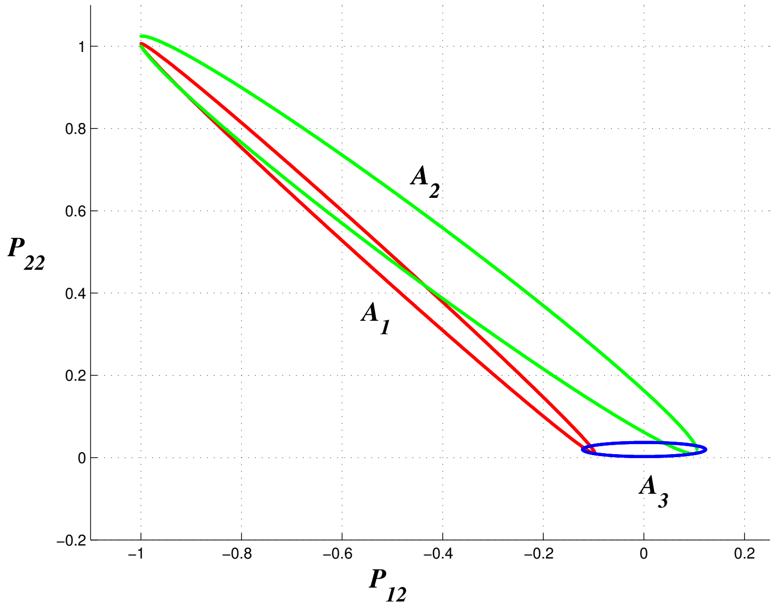

Figure 1: $\mathcal{P}_{A_i}$ for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

Clearly, no CQLF exists as the sets $\mathcal{P}_{A_i}$, $i \in [1, 2, 3]$, have no common intersection (Shorten & Narendra 2000). However, the matrix pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_1, A_3\}$ are pairwise triangularizable with

$$T_{12} = \begin{bmatrix} 1 & 0.1 \\ 1 & 1 \end{bmatrix},\ T_{23} = \begin{bmatrix} 1 & 1 \\ -10 & 1 \end{bmatrix},\ T_{13} = \begin{bmatrix} 0.1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Hence, in general, for stable LTI systems, pairwise triangulizability of the system matrices does not imply the existence of a CQLF.

In cases when a CQLF does not exist, the stability properties of the system must be determined using non-CQLF techniques. In the remainder of this paper, we present a novel method for showing the global attractivity of the origin for a class of systems in the form of Equation (1). We consider systems where the A_i matrices in $\mathcal{A}$ are diagonalisable, and where any two of the A_i matrices have at least $n-1$ real linearly independent eigenvectors in common. For certain systems of this form, the origin of the switching system is globally attractive as verified in the following theorem.

Theorem 3.1.

Consider the switching system (1) with the set $\mathcal{A}$ defined as follows.

Let $\mathcal{V} = \{v_1, \ldots, v_{n+1}\}$ be a set of real vectors, where each $v_i \in \mathbb{R}^n$. Suppose any n vectors in $\mathcal{V}$ are linearly independent. For each $i \in \{1, 2, \ldots, n+1\}$, construct an $n \times n$ matrix M_i as follows: $M_1 = [v_1, v_2, \ldots, v_{i-1}, v_i, \ldots, v_{n-1}, v_n]$ and for $2 \leq i \leq n+1$ we define $M_i = [v_1, v_2, \ldots v_{n+1}, v_i, \ldots v_{n-1}, v_n]$, i.e. M_i is obtained by replacing the $(i-1)^{th}$ column in M_1 with the vector v_{n+1}. Suppose we also have p different diagonal matrices $D_1, D_2, \ldots, D_p$ with all diagonal entries negative. Define $A_{i,h} = M_i D_h M_i^{-1}$, for $1 \leq i \leq n+1$ and $1 \leq h \leq p$. Let $\mathcal{A}$ be a subset of $\{A_{i,h} : 1 \leq i \leq n+1,\ 1 \leq h \leq p\}$.

Then the origin of the system (1) is globally attractive.

Comment : The set $\mathcal{A}$ defined in Theorem 3.1 satisfies the following properties:

(a) Every matrix in $\mathcal{A}$ is Hurwitz and diagonalisable;

(b) The eigenvectors of any matrix in $\mathcal{A}$ are real;

(c) Every pair of matrices in $\mathcal{A}$ share at least $n-1$ linearly independent common eigenvectors.

(d) Every matrix pair in $\mathcal{A}$ can be simultaneously triangularised (Shorten 1996).

For ease of exposition we first present an outline of the main ideas. The proof is then developed by means of several key lemmas. Note that in the sequel we use row and column notation interchangeably to denote vectors.

Outline of Proof

Step 1: We replace each $n \times n$ matrix M_j by an $(n+1) \times (n+1)$ matrix $\bar{M}_j$. We then replace each $n \times n$ matrix $A_{i,h}$ in $\mathcal{A}$ by an $(n+1) \times (n+1)$ matrix $\bar{A}_{i,h}$. The matrices $\bar{A}_{i,h} \in \bar{\mathcal{A}} \triangleq \{\bar{A}_{i,h} : A_{i,h} \in \mathcal{A}\}$ are chosen such that there is at least one common eigenvector $\tau = (1, 0, 0, \ldots, 0)$ for all the matrices in $\bar{\mathcal{A}}$, and also such that the properties of the solutions of the dynamic system

$$\dot{\bar{x}} = \bar{A}(t)\bar{x}, \;\; \bar{A}(t) \in \bar{\mathcal{A}}, \tag{13}$$

will ultimately imply the global attractivity of the origin of the system

$$\dot{x} = A(t)x, \;\; A(t) \in \mathcal{A}, \tag{14}$$

where $x = (x_1, ..., x_n)$ and $\bar{x} = (x_{n+1}, x_1, ..., x_n)$.

Step 2: For a given $j \in \{1, 2, \ldots, n+1\}$ we consider the $n+1$ linearly independent columns of $\bar{M}_j$. These form an $n+1$ dimensional coordinate system which includes τ as one of the axes. We consider the projection of the state $\bar{x}(t)$ onto τ as the dynamics of the system (13) evolve. This projection is given by the first component of the vector

$$g_j(t) = \bar{M}_j^{-1} \bar{x}(t), \tag{15}$$

and is denoted by $[g_j]_1(t)$.

Step 3: We then show that $\lim_{t\to\infty} |[g_j]_1(t) - [g_i]_1(t)| = 0, \ \forall \, i, j \in \{1, ..., n+1\}$. From this fact we can deduce that $\lim_{t\to\infty}(x_1, \ldots, x_n) = 0$. This is sufficient to demonstrate the global attractivity of the origin of the system,

$$\dot{x} = A(t)x, \ \ A(t) \in \mathcal{A}. \tag{16}$$

Technical details of Proof

Lemma 3.1.

There exists a positive number a such that the set $W = \{(a, v_1), (1, v_2), (1, v_3), \ldots, (1, v_{n+1})\}$ is linearly independent in $\mathbb{R}^{n+1}$. Here (a, v_1) is the vector with $n+1$ coordinates, whose first coordinate is a and remaining n coordinates are the n coordinates of v_1.

Proof :

Let $\mathcal{V} = \{v_1, v_2, \ldots, v_{n+1}\}$ in $\mathbb{R}^n$. We know that any subset of $\mathcal{V}$ which contains n elements is linearly independent and thus forms a basis for $\mathbb{R}^n$. Consequently, $\{v_2, v_3, \ldots, v_{n+1}\}$ forms a basis for $\mathbb{R}^n$ and so there exist unique real numbers β_j such that $v_1 = \sum_{j=2}^{n+1} \beta_j v_j$. Pick a to be a positive number which is different from $\sum_{j=2}^{n+1} \beta_j$.

We now show that the set W is linearly independent in $\mathbb{R}^{n+1}$. Let $\bar{v}_1 = (a, v_1)$ and $\bar{v}_j = (1, v_j)$, for $2 \leq j \leq n+1$. Suppose $\sum_{i=1}^{n+1} \gamma_i \bar{v}_i = 0$ with at least one of the $\gamma_i^{'s}$ non–zero. We want to derive a contradiction. Note that $\gamma_1 a + \sum_{j=2}^{n+1} \gamma_j = 0$ and $\sum_{i=1}^{n+1} \gamma_i v_i = 0$. Also note that $\gamma_1 \neq 0$, because if $\gamma_1 = 0$ then we have $\sum_{j=2}^{n+1} \gamma_j v_j = 0$ and so $\gamma_i = 0$ for $1 \leq i \leq n+1$ and this is false. Consequently, we can write $v_1 = -\frac{1}{\gamma_1} \sum_{j=2}^{n+1} \gamma_j v_j$ and $a = -\frac{1}{\gamma_1} \sum_{j=2}^{n+1} \gamma_j$. Thus by uniqueness of β_j, we have $\beta_j = -\frac{\gamma_j}{\gamma_1}$, for $2 \leq j \leq n+1$ and so $a = \sum_{j=2}^{n+1} \beta_j$, which is false. Therefore W is linearly independent in $\mathbb{R}^{n+1}$. **Q.E.D.**

Define $\bar{M}_i$ to be the following $(n+1) \times (n+1)$ matrix:

$$\bar{M}_i = \begin{pmatrix} 1 & b & 1 & 1 & \ldots & \ldots 1 \\ 0 & & & & & \\ 0 & & & M_i & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix}$$

where $b = a$ (from Lemma 3.1), if $i \neq 2$ and $b = 1$, if $i = 2$. The change in the value of b is because v_1 only appears in M_i when $i \neq 2$. Note that the columns of $\bar{M}_i$, apart from the first column, are vectors from the set W in Lemma 3.1.

Note that

$$\bar{M}_i^{-1} = \begin{pmatrix} 1 & s_1 & s_2 & \ldots & s_n \\ 0 & & & & \\ 0 & & M_i^{-1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

for some real numbers $s_1, s_2, \ldots, s_n$ which depend on i.

Define $\bar{D}_h$ to be the following $(n+1) \times (n+1)$ diagonal matrix:

$$\bar{D}_h = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 0 & & & & \\ 0 & & D_h & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

Define $\bar{A}_{i,h} = \bar{M}_i \bar{D}_h \bar{M}_i^{-1}$. We then get

$$\bar{A}_{i,h} = \begin{pmatrix} 0 & c_1 & c_2 & \dots & c_n \\ 0 & & & & \\ 0 & & A_{i,h} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

for some real numbers $c_1, c_2, \dots, c_n$ which depend on i and h. Note that $(1, 0, 0, \dots, 0)$ is a common eigenvector for all the m matrices $\bar{A}_{i,h}$.

We then have that

$$\begin{pmatrix} \dot{x}_{n+1} \\ \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \bar{A}_{i,h} \begin{pmatrix} x_{n+1} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{17}$$

if and only if

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = A_{i,h} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \dot{x}_{n+1} = \sum_{i=1}^{n} c_i x_i$$

We will show that $\lim_{t\to\infty}(x_1, x_2, \dots, x_n) = 0$, for any solution $(x_{n+1}, x_1, x_2, \dots, x_n)$ to the switching system (13). By the above, this will then imply that $\lim_{t\to\infty}(x_1, x_2, \dots, x_n) = 0$, for any solution $(x_1, x_2, \ldots, x_n)$ to the switching system (1) and that will give us global attractivity of the origin in the switching system (1), and we will be done.

Let $\bar{x} = (x_{n+1}, x_1, x_2, \ldots, x_n)$. We consider the evolution of the system dynamics (13) in each of the coordinate systems

$$g_i = \bar{M}_i^{-1}\bar{x} \tag{18}$$

There are $n+1$ coordinate systems corresponding to $i \in \{1, 2, \ldots, n+1\}$. Let $G = \{[g_1]_1, [g_2]_1, \ldots, [g_{n+1}]_1\}$, where $[g_i]_1$ denotes the first component of the vector g_i.

Let the system dynamics be initially described by

$$\dot{\bar{x}} = \bar{A}_{j,h}\bar{x}, \tag{19}$$

over some time interval $[t_1, t_2]$. Note that $\dot{g_j} = \bar{D}_h g_j$. Let $[g_j]_m$ be the m^{th} component of the vector g_j and let $\lambda_{h,i}$ denote the $(i,i)^{th}$ diagonal entry in D_h and hence the $(i+1, i+1)^{th}$ diagonal entry in $\bar{D}_h$. Then we have that $[\dot{g_j}]_m = \lambda_{h,m-1}[g_j]_m$, for $m \neq 1$ and $[\dot{g_j}]_1 = 0$. Therefore, when we are in system (19), we have

$$[g_j]_m(t) = [g_j]_m(t_1)\ e^{\lambda_{h,m-1}(t-t_1)}, \quad \text{for} \quad m \neq 1, \tag{20}$$

and $[g_j]_1$ is a constant function of t.

The members of G, when we are in system (19), are illustrated in Figure 1. Note that $[g_j]_1$ is a constant function of time over $[t_1, t_2]$ while the other $[g_j]_m$'s vary with time according to (20).

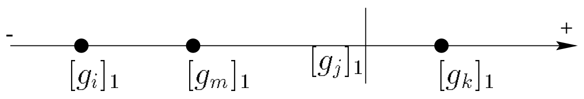

Figure 2: Members of the set G.

Consider the evolution of $[g_i]_1$ relative to $[g_j]_1$. This 'distance' denoted by $d_{i,j}(t)$, is given by

$$d_{i,j}(t) = |[g_i]_1(t) - [g_j]_1(t)| \tag{21}$$

and can be conveniently calculated from

$$g_i = \bar{M}_i^{-1} \bar{M}_j g_j \tag{22}$$

We now analyse the structure of the matrix $F_{i,j} = \bar{M}_i^{-1} \bar{M}_j$, for $i \neq j$. We see that 1 always appears in the first row first column entry of $F_{i,j}$. We claim that there is only one other non–zero entry in the first row.

Lemma 3.2.

If we exclude the first column of the matrix $F_{i,j}$, for $i \neq j$, then there is only one non–zero entry (denoted by $C_{i,j,k}$) in the first row. $C_{i,j,k}$ appears in the k^{th} column where $k = j$, when $i = 1$, and $k = i$, when $i \neq 1$. Note that k is never 1.

Proof :

Denote the first row of $\bar{M}_i^{-1}$ by $\vec{r}$. Suppose first that $i = 1$. We see that a basis for the orthogonal complement of $\vec{r}$ in $\mathbb{R}^{n+1}$ is given by $\{(a, v_1), (1, v_2), (1, v_3), \ldots, (1, v_n)\}$. Hence, using the result of Lemma 3.1, the only place (apart from the first column) in the first row of $F_{i,j}$ which is non–zero, is the k^{th} column where k is the number of the column in $\bar{M}_j$ which has $(1, v_{n+1})$. Thus $k = j$. Here $C_{i,j,k}$ is the dot product of $\vec{r}$ and $(1, v_{n+1})$.

Suppose next that $i = 2$. We see that a basis for the orthogonal complement of $\vec{r}$ in $\mathbb{R}^{n+1}$ is given by $\{(1, v_2), (1, v_3), \ldots, (1, v_{n+1})\}$. Hence, as above, the only place (apart from the first column) in the first row of $F_{i,j}$ which is non–zero, is the k^{th} column where k is the number of the column in $\bar{M}_j$ which has (a, v_1). Thus $k = 2$. Here $C_{i,j,k}$ is the dot product of $\vec{r}$ and (a, v_1).

Suppose finally that $i > 2$. We see that a basis for the orthogonal complement of $\vec{r}$ in $\mathbb{R}^{n+1}$ is obtained by deleting $(1, v_{i-1})$ from the set W in Lemma 3.1. Hence, as above, the only place (apart from the first column) in the first row of $F_{i,j}$ which is non–zero, is the k^{th} column where k is the number of the column in $\bar{M}_j$ which has $(1, v_{i-1})$. Thus $k = i$. Here $C_{i,j,k}$ is the dot product of $\vec{r}$ and $(1, v_{i-1})$. **Q.E.D.**

We combine Lemma 3.2 with (22) to obtain

$$[g_i]_1 = [g_j]_1 + C_{i,j,k}[g_j]_k \quad \text{for} \quad 1 \le i \le n+1 \quad \text{with} \quad i \ne j. \tag{23}$$

Note that (23) is true irrespective of what system we are in.

We combine (20) and (24) to obtain

$$[g_i]_1(t) - [g_j]_1(t) = C_{i,j,k}\ [g_j]_k(t_1)\ e^{\lambda_{h,k-1}(t-t_1)} \quad \text{for} \quad i \ne j, \tag{24}$$

whenever we are in system (19). Hence

$$d_{i,j}(t) = |C_{i,j,k}||[g_j]_k(t_1)|\ e^{\lambda_{h,k-1}(t-t_1)}, \text{ for } t_1 \le t \le t_2 \text{ and } i \ne j.$$

Consequently $\frac{dd_{i,j}(t)}{dt} < 0$, or else $[g_i]_1$ and $[g_j]_1$ both agree over the time interval $[t_1, t_2]$. Thus the distance between $[g_i]_1(t)$ and the constant $[g_j]_1(t)$ is either getting smaller or always zero over the time interval $[t_1, t_2]$, when we are in the system described by $\dot{\bar{x}} = \bar{A}_{j,h}\bar{x}$.

Proof of Theorem 3.1 :

We will now prove that $\lim_{t\to\infty}(x_1, x_2, \ldots, x_n) = 0$, for any solution $(x_1, x_2, \ldots, x_n)$ to the system (1) with the set $\mathcal{A}$ defined as in statement of Theorem 3.1, and then we will be done. Denote the maximum value (minimum value) of $G(t)$, for a time t in the time interval $[t_1, t_2]$, by $\max^1 G(t)$ ($\min^1 G(t)$). Recall that we are in system (19) when $t \in [t_1, t_2]$. Then

$$\begin{aligned}
\max{}^1\ G(t) - \min{}^1\ G(t) &= [g_i]_1(t) - [g_r]_1(t), \text{ for some } i, r \in \{1, 2, \ldots, n+1\} \\
&= [g_i]_1(t) - [g_j]_1(t) + [g_j]_1(t) - [g_r]_1(t) \\
&= |C_{i,j,k}|\ |[g_j]_k(t_1)|\ e^{\lambda_{h,k-1}(t-t_1)} + |C_{r,j,q}|\ |[g_j]_q(t_1)|\ e^{\lambda_{h,q-1}(t-t_1)}
\end{aligned}$$

where, as in Lemma 3.2, $k = j$, if $i = 1$, and $k = i$, if $i \ne 1$. Similarly $q = j$, if $r = 1$, and $q = r$, if $r \ne 1$. Note that if $[g_j]_1$ is a maximum value (or minimum value) of $G(t)$, then the last line above collapses to just one term instead of two, and in this case the following arguments will also work. Now let $B_{i,j,t_1} = |C_{i,j,k}|\ |[g_j]_k(t_1)| =$ distance between $[g_i]_1(t_1)$ and $[g_j]_1(t_1)$. Let $B_{r,j,t_1} = |C_{r,j,q}|\ |[g_j]_q(t_1)| =$ distance between $[g_j]_1(t_1)$ and $[g_r]_1(t_1)$. Also let

$\lambda = \max\{\lambda_{\alpha,\beta} : 1 \leq \alpha \leq p,\ 1 \leq \beta \leq n\}$. Note that $\lambda < 0$. Then

$$\begin{aligned} \max{}^1\ G(t) - \min{}^1\ G(t) &\leq (B_{i,j,t_1} + B_{r,j,t_1})\ e^{\lambda(t-t_1)} \\ &\leq (\max{}^1\ G(t_1) - \min{}^1\ G(t_1))\ e^{\lambda(t-t_1)} \end{aligned} \tag{25}$$

The last inequality follows from the fact that, over the time interval $[t_1, t_2]$, $[g_i]_1(t)$ remains on the same side of the constant $[g_j]_1(t)$ and $[g_r]_1(t)$ remains on the other side of $[g_j]_1(t)$. This is because the right hand side of (24) does not change sign as time changes over the time interval $[t_1, t_2]$. Note that i and r may change with time and so $\max^1 G(t_1)$ may not correspond to $[g_i]_1(t_1)$, and $\min^1\ G(t_1)$ may not correspond to $[g_r]_1(t_1)$.

Now suppose we switch to the next (second) system described by

$$\dot{\bar{x}} = \bar{A}_{c,w}\bar{x} \tag{26}$$

over the time interval $[t_2, t_3]$. Denote the maximum value (minimum value) of $G(t)$, for some time $t \in [t_2, t_3]$, by $\max^2 G(t)$ ($\min^2\ G(t)$). Then as above we get

$$\begin{aligned} \max{}^2\ G(t) - \min{}^2\ G(t) &\leq (\max{}^2\ G(t_2) - \min{}^2\ G(t_2))\ e^{\lambda(t-t_2)} \\ &\leq (\max{}^1\ G(t_1) - \min{}^1\ G(t_1))\ e^{\lambda(t_2-t_1)}\ e^{\lambda(t-t_2)} \\ &\leq (\max{}^1\ G(t_1) - \min{}^1\ G(t_1))\ e^{\lambda(t-t_1)} \end{aligned}$$

The second inequality above follows from the fact that we start the second system (26) at time t_2 when we stop the first system (19), and the initial conditions for the second system are the terminal conditions for the first system at time t_2. Thus $\max^2 G(t_2) - \min^2\ G(t_2) \leq (\max^1 G(t_1) - \min^1\ G(t_1))\ e^{\lambda(t_2-t_1)}$ from (25).

Now suppose we switch to the next (third) system described by

$$\dot{\bar{x}} = \bar{A}_{d,u}\bar{x} \tag{27}$$

over the time interval $[t_3, t_4]$. Denote the maximum value (minimum value) of $G(t)$, for some time $t \in [t_3, t_4]$, by $\max^3 G(t)$ ($\min^3\ G(t)$). Then as above we get

$$\max{}^3\ G(t) - \min{}^3\ G(t) \leq (\max{}^1\ G(t_1) - \min{}^1\ G(t_1))\ e^{\lambda(t-t_1)}$$

For the general suituation, when we have switched for the m^{th} time, we are in the system described by $\dot{\bar{x}} = \bar{A}_{z,l}\bar{x}$ over the time interval $[t_m, t_{m+1}]$. Again we denote the maximum value (minimum value) of $G(t)$, for some time $t \in [t_m, t_{m+1}]$, by $\max^m G(t)$ ($\min^m\ G(t)$). Then as above we get

$$\max{}^m\ G(t) - \min{}^m\ G(t) \leq (\max{}^1\ G(t_1) - \min{}^1\ G(t_1))\ e^{\lambda(t-t_1)}$$

Therefore, since $\lambda < 0$, we have $\lim_{t\to\infty}(\max G(t) - \min G(t)) = 0$, where $\max G(t)$ ($\min G(t)$) denotes the maximum value (minimum value) of $G(t)$ for any time $t \geq t_1$. Thus

$$\begin{aligned}
\lim_{t\to\infty} |[g_i]_1(t) - [g_j]_1(t)| &= 0, \text{ for all } i, j \in \{1, 2, \ldots, n+1\}. \\
\lim_{t\to\infty} |C_{i,j,k}||[g_j]_k(t)| &= 0, \text{ where } k = j \text{ if } i = 1, \text{ and } k = i, \text{ if } i \neq 1, \text{ and } i \neq j. \\
\lim_{t\to\infty} |[g_j]_k(t)| &= 0, \text{ for } j \in \{1, 2, \ldots, n+1\}, \text{ and } k \in \{2, 3, \ldots, n+1\}.
\end{aligned}$$

The second line above follows from (23), which is independent of what system we are in. The last line above follows from the fact that the $C_{i,j,k}^{'s}$ form a finite collection of non–zero numbers when $i \neq j$. Also note that the last line above might not hold for $k = 1$ because then $i = j = 1$. Therefore, since $\lim_{t\to\infty} \bar{x}(t) = \lim_{t\to\infty} \bar{M}_j g_j(t)$, we get

$$\lim_{t\to\infty} \begin{pmatrix} x_{n+1} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lim_{t\to\infty} \begin{pmatrix} 1 & b & 1 & 1 & \ldots & \ldots 1 \\ 0 & & & & & \\ 0 & & & M_j & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{pmatrix} \begin{pmatrix} [g_j]_1(t) \\ [g_j]_2(t) \\ \vdots \\ [g_j]_{n+1}(t) \end{pmatrix}$$

$$\lim_{t\to\infty} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lim_{t\to\infty} \begin{pmatrix} 0 & & \\ 0 & M_j & \\ \vdots & & \\ 0 & & \end{pmatrix} \begin{pmatrix} [g_j]_1(t) \\ [g_j]_2(t) \\ \vdots \\ [g_j]_{n+1}(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Thus

$$\lim_{t \to \infty} (x_1, x_2, \ldots, x_n) = 0,$$

and we have global attractivity of the origin in the switching system (1). **Q.E.D.**

4 Concluding remarks

In this paper we have shown global attractivity for a class of switching systems. This result is important for a number of reasons. Primarily, it can be readily shown from Example 1, or by using examples generated from standard convex optimization packages, such as the MATLAB LMI toolbox (Gahinet, Nemirovski, Laub & Chilali 1995), that the condition of pairwise triangularisability is not sufficient to guarantee the existence of a CQLF. Hence, in cases where no CQLF exists, qualitative statements concerning system stability, must be validated using other techniques. One such technique is presented in this paper; namely, a technique which proves global attractivity by embedding the original (n-dimensional) state space in a higher $(n + 1)$ dimensional state space. This technique can be used to prove the global attractivity in cases when a CQLF does not exist, and it is likely that derived methodology is applicable to a wide class of related switching systems.

The derived results also suggest that the condition of simultaneous triangularisability may be relaxed significantly without the loss of asymptotic stability[1]. This observation will form the basis of future work. It is hoped that more results will be reported at a later date.

[1]Note that the concepts of stability and attractivity are not equivalent; see Narendra & Annaswamy (1989).

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