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Abstract

In this paper we present a sufficient condition for global asymptotic stability of certain classes of second order time-varying systems. While the proof is based upon geometrical and non-quadratic Lyapunov arguments, the stability conditions are expressed as simple restrictions on a matrix pencil. These restrictions possess a simple geometrical interpretation and may generalize to higher order systems. The conditions are invariant under similarity transformations, and are readily verifiable via root-locus checking. Further, the conditions establish a direct connection between a matrix pencil and the generalized infinity norm, and can therefore be viewed as extending the link between known matrix pencil conditions and quadratic stability criteria.

1 Introduction

We consider a class of second order linear switching systems of the form

$$\dot{\boldsymbol{x}} \;=\; \boldsymbol{A}(t)\boldsymbol{x}, \qquad \boldsymbol{A}(t) \in \{\boldsymbol{A}_1, \boldsymbol{A}_2\} \tag{1}$$

where $\boldsymbol{A}_1, \boldsymbol{A}_2$ are constant Hurwitz matrices in $\mathbb{R}^{2\times 2}$. We refer to the system (1) as the switching system constructed from the composite linear time-invariant systems $\Sigma_{\boldsymbol{A}_i} : \dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t), i \in [1,2]$. Switching occurs according to an arbitrary switching sequence $\mathcal{S}$, defined by

$$\mathcal{S}: \quad (\boldsymbol{A}_1, t_0), (\boldsymbol{A}_2, t_1), (\boldsymbol{A}_1, t_2), (\boldsymbol{A}_2, t_3), \ldots \tag{2}$$

where t_j denotes the switching instant at which the dynamics change to those specified by the respective matrix $\boldsymbol{A}_i$. While systems of this form have been widely studied [1], many fundamental questions pertaining to their stability remain unanswered. Of these, questions pertaining to non-quadratic stability criteria have been almost entirely developed from a numerical viewpoint in recent literature. In this paper we present non-quadratic stability criteria for second order systems. These conditions can be simply verified by checking that the locus of eigenvalues of the matrix pencil $\alpha\boldsymbol{B}_1 + (1-\alpha)\boldsymbol{B}_2$ remains exclusively in a sector of the complex plane, where the matrices $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are simply related to $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ in Equation (1). This region is depicted in Figure 1, and is referred to as the 45^o*-region.* In this report we illustrate that this region arises naturally when discussing the stability of second order switching systems, where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have real distinct eigenvalues.

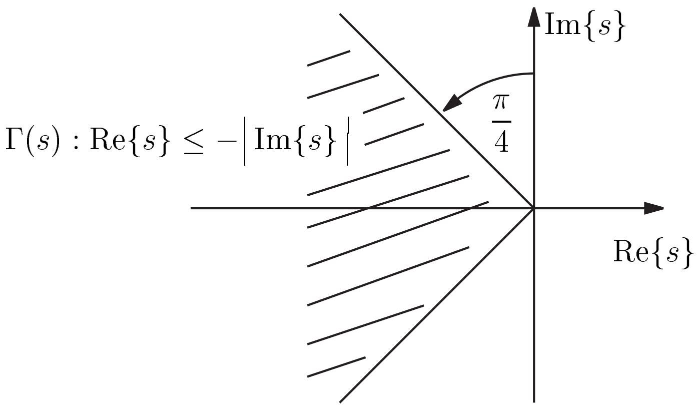

Figure 1: The 45^o-region.

The main results in this report are based upon the simple observation that the solution to the system (1) can be written

$$\boldsymbol{x}(t) \quad = \quad \boldsymbol{H}(t)\bar{\boldsymbol{x}}(t) \tag{3}$$

where $\lim_{t\to\infty} \boldsymbol{H}(t) = 0$ (c.f. Lemma 2.1). If $\bar{\boldsymbol{x}}$ is bounded, then the state vector $\boldsymbol{x}(t) \to 0$ as $t \to \infty$. Hence, a sufficient condition for the stability of system (1) is that the vector $\bar{\boldsymbol{x}}$ remains bounded; this implies that the solution of the switching system

$$\dot{\bar{\boldsymbol{x}}}(t) \quad = \quad \bar{\boldsymbol{A}}(t)\bar{\boldsymbol{x}}(t) \tag{4}$$

is bounded.

This report is structured as follows. We present some mathematical preliminaries in Section 2. In Section 3 we present the main results of the paper. The use of the infinity norm in the derivation of the main results is discussed in Section 4. The relationship between the results presented in this paper and those discussed in [2, 3] are discussed briefly in the Section 5.

2 Mathematical Preliminaries

In this section we present general results and definitions that are useful in stating the main results of this paper. For ease of exposition the results are simply quoted; where appropriate proofs are presented in the Appendix.

Definition 2.1 (Matrix Pencil) Matrix pencils form the basis of much of the discussion in this paper. We use two forms of the matrix pencil interchangeably. These are defined as follows:

$$\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2] = \alpha \boldsymbol{A}_1 + (1-\alpha)\boldsymbol{A}_2, \quad \alpha \in [0,1], \tag{5}$$

and

$$\boldsymbol{P}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2] = \boldsymbol{A}_1 + \gamma \boldsymbol{A}_2, \quad \gamma \in [0,\infty) \tag{6}$$

Comment: Note that $\boldsymbol{P}_\alpha$ can be written as $\alpha \boldsymbol{P}_\gamma$, where $\gamma = \frac{1-\alpha}{\alpha}$ for $\alpha \neq 0$. Hence, for all $\alpha \neq 0$, the eigenvalues of $\boldsymbol{P}_\gamma$ and $\boldsymbol{P}_\alpha$ are identical up to a positive scaling factor.

Definition 2.2 (The Matrix $\bar{\boldsymbol{A}}$) The Hurwitz matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ with distinct eigenvalues d_1, d_2 can be decomposed as follows:

$$\boldsymbol{A} = \boldsymbol{M} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \boldsymbol{M}^{-1}$$

where $\boldsymbol{M}$ is the modal matrix associated with $\boldsymbol{A}$.

Without loss of generality we assume that $|\, d_2| > |\, d_1|$. We will refer to d_2 as the *fast* eigenvalue and to d_1 as the *slow* eigenvalue. Then,

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{M}\left[\begin{pmatrix} d_1 & 0 \\ 0 & d_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d_2 - d_1 \end{pmatrix}\right]\boldsymbol{M}^{-1}, \qquad |\, d_2 \,|>|\, d_1 \,| \\ &= \boldsymbol{M} d_1 \boldsymbol{I} \boldsymbol{M}^{-1} + \boldsymbol{M}\begin{pmatrix} 0 & 0 \\ 0 & d_2 - d_1 \end{pmatrix}\boldsymbol{M}^{-1} \\ &= d_1 \boldsymbol{I} + \bar{\boldsymbol{A}}. \end{aligned} \tag{7}$$

Lemma 2.1 (Sufficient Condition for Stability) Consider the switching system in Equation (1) with real negative eigenvalues. If the solution to the corresponding switching system

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}(t)\bar{\boldsymbol{x}}, \qquad \bar{\boldsymbol{A}}(t) \in \{\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2\} \tag{8}$$

is bounded, then the system (1) is globally asymptotically stable.

Proof: see Appendix.

Lemma 2.2 (Eigenvector Transformation) Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ be vectors in $\mathbb{R}^2$ where each pair of vectors $\{\boldsymbol{v}_i, \boldsymbol{v}_j\}$ are linearly independent. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be linearly independent vectors in $\mathbb{R}^2$ with $\boldsymbol{w}_3 = k_1\boldsymbol{w}_1 + k_2\boldsymbol{w}_2$, where $\boldsymbol{v}_3 = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2$. Then, there exists an invertible matrix $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{v}_1 = \boldsymbol{w}_1$, $\boldsymbol{T}\boldsymbol{v}_2 = \boldsymbol{w}_2$ and $\boldsymbol{T}\boldsymbol{v}_3 = \boldsymbol{w}_3$.

Proof: see Appendix.

Corollary 2.1 Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{2\times 2}$, be invertible matrices with real eigenvalues and real eigenvectors. Let the eigenvectors of $\boldsymbol{A}_1$ be denoted $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, and those of $\boldsymbol{A}_2$ by $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$. Let every pair of vectors $\{\boldsymbol{v}_i, \boldsymbol{v}_j\} \in \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$, $i \neq j$ be linearly independent. Then, there exists an invertible matrix $\boldsymbol{T}$ such that three of the eigenvectors of the transformed matrices $\boldsymbol{T}\boldsymbol{A}_1\boldsymbol{T}^{-1}$ and $\boldsymbol{T}\boldsymbol{A}_2\boldsymbol{T}^{-1}$ are given by $\boldsymbol{w}_1 = [\,1,\ 1\,]^\mathsf{T}, \boldsymbol{w}_2 = [\,1,\ -1\,]^\mathsf{T}$ and $\boldsymbol{w}_3 = [\,2,\ 0\,]^\mathsf{T}$, respectively. The remaining forth eigenvector can be represented by $\boldsymbol{w}_4 = [\,y,\ 1\,]^\mathsf{T}$, where $y \in \mathbb{R}$.

3 Main result

In this section we will present a simple test for stability of second order switching systems.

Theorem 3.1 (45° Criterion) Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be two Hurwitz matrices with real distinct eigenvalues and let $\bar{\boldsymbol{A}}_i,\ i \in \{1, 2\}$ be defined as in Definition 2.2. A sufficient condition for the global asymptotic stability of the switching system defined in Equation (1), is that the eigenvalues of the matrix pencil $\boldsymbol{P}_\alpha[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ satisfy $\mathrm{Re}\{\lambda(\boldsymbol{P}_\alpha)\} \leq -|\,\mathrm{Im}\{\lambda(\boldsymbol{P}_\alpha)\}|$, i.e. the root locus lies within the closed 45^o-region (45^o-condition).

Proof

To prove the theorem we consider the two possible cases:

1. one or more eigenvectors of $\boldsymbol{A}_1$ can be written as a linear combination of the eigenvectors of $\boldsymbol{A}_2$, i.e. Lemma 2.2 is not applicable;
2. every pair of eigenvectors of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ is linear independent, i.e. Lemma 2.2 applies.

Case 1

In case 1 the two matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are simultaneously triangularisable, i.e. there exists a transformation matrix $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{A}_1\boldsymbol{T}^{-1}$ and $\boldsymbol{T}\boldsymbol{A}_2\boldsymbol{T}^{-1}$ are upper triangular. For such a set of systems a common quadratic Lyapunov function exists [4]. Elementary algebra reveals that the root locus of $\boldsymbol{P}_\alpha[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ lies within the 45^o-region. Therefore the 45^o Criterion holds for these systems.

Case 2

We now consider the case where the two matrices have pair-wise linear independent eigenvectors, i.e. Lemma 2.2 and Corollary 2.1 apply. Further we note that a constant positive factor in each of the matrices $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ neither effects the stability property of the switching system, nor changes the root locus of the matrix pencil $\boldsymbol{P}_\alpha[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ in respect of the 45^o-condition. Therefore, we assume without loss of generality that both fast eigenvalues of $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ are -1.

After transformation according to Corollary 2.1 we get

$$\bar{\boldsymbol{A}}_1 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \tag{9}$$

$$\bar{\boldsymbol{A}}_2 = \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{1-y} \begin{pmatrix} y & -y \\ 1 & -1 \end{pmatrix} \tag{10}$$

where $y \neq 1$ denotes the unknown component of the non-fixed eigenvector.[1]

We will now examine for which values of y the root-locus of the matrix pencil $\boldsymbol{P}_\alpha[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ stays within the 45^o-region. Then we will show the existence of a common Lyapunov function for these systems $\Sigma_{\bar{\boldsymbol{A}}_1}$ and $\Sigma_{\bar{\boldsymbol{A}}_2}$. According to Lemma 2.1 this proofs asymptotic stability of the corresponding switching system (1).

For ease of computation we consider the matrix pencil $\boldsymbol{P}_\gamma$ (rather than $\boldsymbol{P}_\alpha$). Since the eigenvalues of both are related by a positive constant factor, the root locus of $\boldsymbol{P}_\alpha$ lies within the 45^o-region if and only if the root locus of $\boldsymbol{P}_\gamma$ does.

With $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ as defined in Equation (9) and (10) we get

$$\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_2, \bar{\boldsymbol{A}}_1] = \begin{pmatrix} -\gamma + \frac{y}{1-y} & -\gamma - \frac{y}{1-y} \\ \frac{1}{1-y} & \frac{-1}{1-y} \end{pmatrix}, \qquad \gamma \in [0, \infty) \tag{11}$$

[1] That representation of the eigenvector $\boldsymbol{v}_{22}$ covers all possible constellations after transformation except for $\boldsymbol{v}_{22} = [1, 0]^\mathsf{T} = v_{12}$. We considered this possibility in Case 1.

A quick overview of the location of the eigenvalues can be achieved by looking at the determinant and the trace of the pencil, since $\det \boldsymbol{P}_\gamma = \prod \lambda_i(\gamma)$ and $\operatorname{tr} \boldsymbol{P} = \sum \lambda_i(\gamma)$.

The determinant is

$$\begin{aligned}
\det \boldsymbol{P}_\gamma &= -\frac{1}{1-y}\left(-\gamma + \frac{y}{1-y}\right) + \frac{1}{1-y}\left(\gamma + \frac{y}{1-y}\right) \\
&= \frac{2\gamma}{1-y}\begin{cases} \leq 0 & \text{for} \quad y > 1 \\ \geq 0 & \text{for} \quad y < 1 \end{cases}
\end{aligned} \tag{12}$$

The trace is

$$\begin{aligned}
\operatorname{tr} \boldsymbol{P}_\gamma &= -\gamma + \frac{y}{1-y} + \frac{-1}{1-y} \\
&= -\gamma + \frac{y-1}{1-y} \quad \leq \quad 0 \quad \forall \quad y \neq 1
\end{aligned} \tag{13}$$

Equations (12) and (13) imply that there are positive real eigenvalues for $y > 1$ and that all eigenvalues have negative (or zero) real part for $y < 1$. Hence, all matrix pencils with $y > 1$ cannot meet the 45^o-condition and $\dot{\bar{\boldsymbol{x}}}(t) = \bar{\boldsymbol{A}}(t)\bar{\boldsymbol{x}}(t)$ is unstable [5]. For the remaining possibility ($y < 1$) we have to consider only matrix pencils where complex eigenvalues occur.

Consider the values γ_{im} for which $\lambda(\gamma_{\mathrm{im}})$ is complex (and $y < 1$). The eigenvalues are within the closed 45^o-region if and only if

$$\left|\operatorname{Re}\{\lambda(\gamma_{\mathrm{im}})\}\right| \quad \geq \quad \left|\operatorname{Im}\{\lambda(\gamma_{\mathrm{im}})\}\right|, \qquad \forall \quad \gamma_{\mathrm{im}} \geq 0.$$

Though,

$$\begin{aligned}
\left|\frac{1}{2}\left(\frac{y-1}{1-y} - \gamma_{\mathrm{im}}\right)\right| &\geq \left|\sqrt{-\frac{1}{4}\left(\frac{y-1}{1-y} - \gamma_{\mathrm{im}}\right)^2 + \frac{2\,\gamma_{\mathrm{im}}}{1-y}}\right| \\
\frac{1}{4}\left(\frac{y-1}{1-y} - \gamma_{\mathrm{im}}\right)^2 &\geq -\frac{1}{4}\left(\frac{y-1}{1-y} - \gamma_{\mathrm{im}}\right)^2 + \frac{2\,\gamma_{\mathrm{im}}}{1-y} \\
\left(\frac{y-1}{1-y} - \gamma_{\mathrm{im}}\right)^2 &\geq 2\,\frac{2\,\gamma_{\mathrm{im}}}{1-y} \\
\left(\frac{y-1}{1-y}\right)^2 - 2\,\frac{y-1}{1-y}\,\gamma_{\mathrm{im}} + \gamma_{\mathrm{im}}^2 - 2\,\frac{2\,\gamma_{\mathrm{im}}}{1-y} &\geq 0 \\
\gamma_{\mathrm{im}}^2 - 2\,\frac{y+1}{1-y}\,\gamma_{\mathrm{im}} + \left(\frac{y-1}{1-y}\right)^2 &\geq 0
\end{aligned}$$

The parabola on the left defined in terms of γ_{im} has a minimum. Though, the inequation holds if and only if the parabola has exactly one or no root.

The roots are given by

$$\gamma_{\mathrm{im}0_{1,2}} = \frac{y+1}{1-y} \pm \sqrt{\left(\frac{y+1}{1-y}\right)^2 - \left(\frac{y-1}{1-y}\right)^2}$$

Therefore, the parabola has exactly one or no root in $\mathbb{R}$ if and only if

$$\begin{aligned} (y+1)^2 - (y-1)^2 &\leq 0 \\ y^2 + 2y + 1 - (y^2 - 2y + 1) &\leq 0 \\ y &\leq 0 \end{aligned} \tag{14}$$

Consequently, the root locus of the matrix pencil (11) meets the 45^o-condition if and only if the parameter $y \leq 0$.

We have shown that the root locus of the matrix pencil $\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ stays within the closed 45^o-region if and only if $y \leq 0$. Since $\boldsymbol{P}_\alpha = \alpha \boldsymbol{P}_\gamma$ for $\alpha \in (0,1]$, the root locus of $\boldsymbol{P}_\alpha$ also remains in the 45^o-region. Moreover, $\boldsymbol{P}_\alpha = \bar{\boldsymbol{A}}_2$ for $\alpha = 0$ and according to the requirements, these eigenvalues are real negative.

We show (bounded) stability of the origin for systems that meet the 45^o-condition using a Lyapunov argument, namely we choose a Lyapunov function $V(\boldsymbol{x})$ such that $\dot{V}(\boldsymbol{x}) \leq 0$ along the trajectories of $\Sigma_{\bar{\boldsymbol{A}}_1}$ and $\Sigma_{\bar{\boldsymbol{A}}_2}$. This will guarantee asymptotic stability of system (1) for arbitrary switching between the vector fields defined by $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ (Lemma 2.1).

As Lyapunov function candidate we choose the L_∞-norm:

$$V(\boldsymbol{x}) = L_\infty(\boldsymbol{x}) = \max_i\{|x_i|\} \tag{15}$$

This function is positive definite. To show that this function is a common Lyapunov function (CLF) for both $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ we need to show that the time-derivative of the function along these trajectories is non-increasing.

Consider each side of the Lyapunov 'square' separately. Straightforward manipulation reveals that $\dot{V}(\boldsymbol{x}) \leq 0$ along the trajectories of $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ provided that y is non-positive. When y is positive, $\dot{V}(\boldsymbol{x})$ is positive along some of the trajectories of $\bar{\boldsymbol{A}}_2$, and instability of the system $\Sigma_{\bar{\boldsymbol{A}}(t)}$ (Equation 8) is possible by switching between the flows of $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$. Hence, $V(\boldsymbol{x})$ is a CLF provided that y is non-positive. A CLF for $\Sigma_{\bar{\boldsymbol{A}}_1}$ and $\Sigma_{\bar{\boldsymbol{A}}_2}$ cannot exist when y is positive.

□

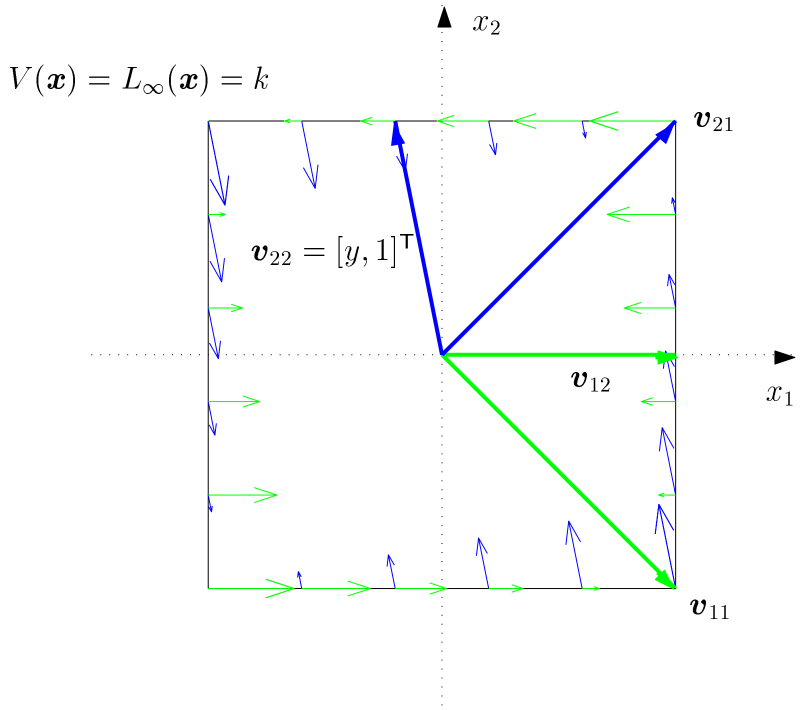

Figure 2: Proof of Stability

4 The infinity norm

In view of the central role played by the L_∞-norm in the discussion in Section 3, we comment here on its significance in the context of existing results. Matrix pencils arise naturally in the context of quadratic stability criteria. A quadratic Lyapuonov function for the system $\Sigma_{\boldsymbol{A}_1}$ is defined as a function $V(\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T}\boldsymbol{P}\boldsymbol{x}$, $\boldsymbol{P} = \boldsymbol{P}^\mathsf{T} > 0$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ such that $\dot{V}(\boldsymbol{x}) < 0$ along the trajectories of the system. This can be expressed mathematically by requiring that $\boldsymbol{A}^\mathsf{T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} < 0$. When $V(\boldsymbol{x})$ is a quadratic Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ *and* $\Sigma_{\boldsymbol{A}_2}$ then $V(\boldsymbol{x})$ is said to be a common quadratic Lyapunov function (CQLF) for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$. Necessary and sufficient conditions for the existence of a CQLF for $\boldsymbol{A}_1$, $\boldsymbol{A}_2 \in \mathbb{R}^{2\times 2}$ are known and are given by the requirement that the matrix pencils $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are Hurwitz [6, 7].

An alternative viewpoint on the CQLF existence problem is obtained by writing $V(\boldsymbol{x})$ in the following manner:

$$\begin{aligned} V(\boldsymbol{x}) &= \boldsymbol{x}^\mathsf{T}\boldsymbol{P}\boldsymbol{x}, \\ &= \boldsymbol{x}^\mathsf{T}\boldsymbol{H}^\mathsf{T}\boldsymbol{H}\boldsymbol{x}, \\ &= \boldsymbol{z}^\mathsf{T}\boldsymbol{z}, \end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{H}^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ is the square root of the matrix $\boldsymbol{P}$. Hence, the existence of a CQLF for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ is equivalent to the existence of a coordinate transformation $\boldsymbol{H}$, $\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x}$ such that the Euclidean norm (L_2) is a Lyapunov function for $\Sigma_{\boldsymbol{H}^{-1}\boldsymbol{A}_1\boldsymbol{H}}$ and $\Sigma_{\boldsymbol{H}^{-1}\boldsymbol{A}_2\boldsymbol{H}}$.

The discussion in Section 3 is concerned with establishing the existence of a non-quadratic CLF such that $V(\boldsymbol{x}) = \max_i\{|x_i|\}$ (L_∞-norm). As in the quadratic case, conditions for the existence of such a function can be expressed in matrix pencil form. The conservatism of the quadratic function with respect to the infinity norm is evident from Figure 3.

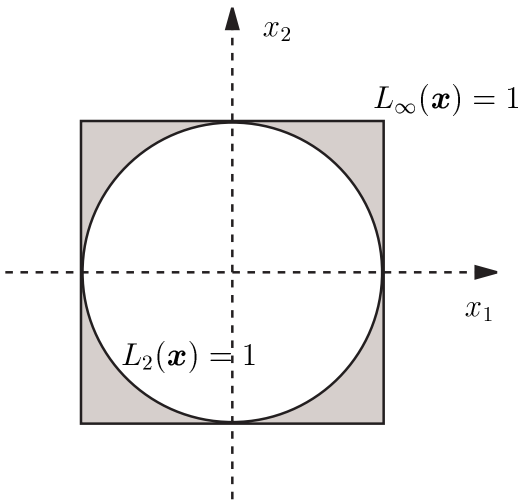

Figure 3: Conservatism of quadratic Lyapunov function

4.1 Numerical example

We now present a numerical example to illustrate the elegance of the 45^o Criterion.

Let

$$\boldsymbol{A}_1 = \begin{pmatrix} -3.0 & -2.8 \\ 0 & -0.2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}_2 = \begin{pmatrix} -0.2 & 0.1 \\ 0.8 & -0.9 \end{pmatrix} \tag{16}$$

It was shown in [6, 7] that a necessary condition for the existence of a common quadratic Lyapunov function (CQLF) for the systems $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ is, that the root-locii of the matrix pencils $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are both Hurwitz.

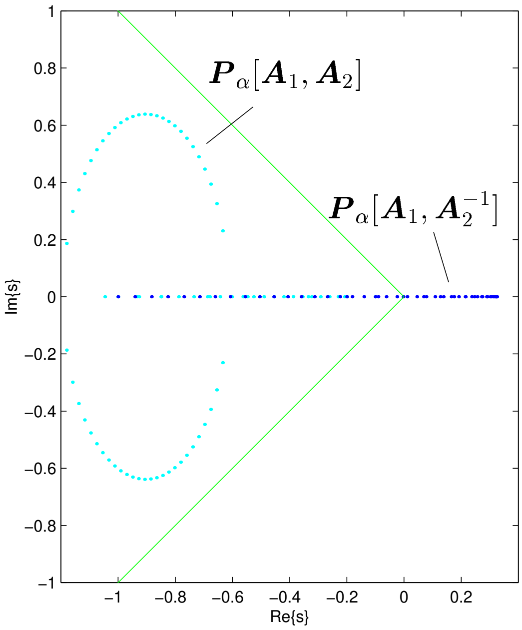

Figure 4: No CQLF exists ...

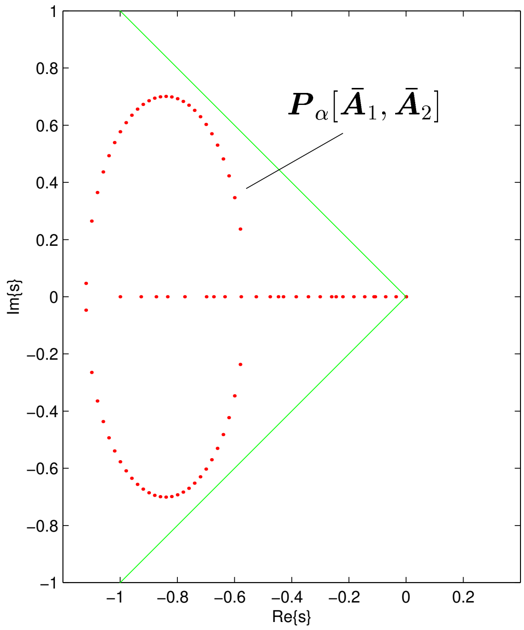

Figure 5: But a CILF exists

In Figure 4 the root-locii of $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are depicted. Since there are real positive values for the latter, it is not possible to determine stability for this switching system by using a common quadratic Lyapunov function.

The root locus associated with $\boldsymbol{P}_\alpha[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ is depicted in Figure 5. It is obvious, that the root locus meets the 45^o-condition. Therefore a common Infinity Lyapunov function (CILF) exists for the systems $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ and the switching system defined in (1) is asymptotically stable for arbitrary switching sequences.

5 Concluding remarks

In this paper we have presented sufficient conditions for second order switching systems. These conditions are related to various results and conjectures in the stability literature [2, 3, 8]. Work under progress is concerned with investigating whether these results generalize to n-dimensional systems.

Acknowledgments

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[2]This work is the sole responsibility of the authors and does not reflect the European Union's opinion. The EU is not responsible for any use of data appearing in this publication.

Appendix

Proof of Lemma 2.1 (Sufficient Condition for Stability) Given the switching second order system given by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}, \qquad \boldsymbol{A}(t) \in \{\boldsymbol{A}_1, \boldsymbol{A}_2\} \tag{17}$$

where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have distinct real eigenvalues in the open left have plane $\mathbb{C}^-$. The solution of that differential equation for an arbitrary switching sequence $\mathcal{S}$ is given by

$$\boldsymbol{x}(t) = \left[e^{\boldsymbol{A}_1(t_1 - t_0)} \cdot e^{\boldsymbol{A}_2(t_2 - t_1)} \cdot e^{\boldsymbol{A}_1(t_3 - t_2)} \cdots\right] \boldsymbol{x}_0 \tag{18}$$

With $\Delta t_k = t_{k+1} - t_k$ we get

$$\boldsymbol{x}(t) = \left[e^{\boldsymbol{A}_1 \Delta t_0} \cdot e^{\boldsymbol{A}_2 \Delta t_1} \cdot e^{\boldsymbol{A}_1 \Delta t_2} \cdots\right] \boldsymbol{x}_0$$

Using the substitution $\boldsymbol{A}_i = d_{i1}\boldsymbol{I} + \bar{\boldsymbol{A}}_i$ get we

$$\boldsymbol{x}(t) = \left[e^{\left(d_{11}\boldsymbol{I} + \bar{\boldsymbol{A}}_{\boldsymbol{1}}\right)\Delta t_0} \cdot e^{\left(d_{21}\boldsymbol{I} + \bar{\boldsymbol{A}}_{\boldsymbol{2}}\right)\Delta t_1} \cdot e^{\left(d_{11}\boldsymbol{I} + \bar{\boldsymbol{A}}_{\boldsymbol{1}}\right)\Delta t_2} \cdots\right] \boldsymbol{x}_0$$

$$\boldsymbol{x}(t) = \left[e^{\boldsymbol{I}\,(d_{11}\Delta t_0 + d_{21}\Delta t_1 + \ldots)} e^{\left(\bar{\boldsymbol{A}}_{\boldsymbol{1}}\Delta t_0 + \bar{\boldsymbol{A}}_{\boldsymbol{2}}\Delta t_1 + \bar{\boldsymbol{A}}_{\boldsymbol{1}}\Delta t_2 + \ldots\right)}\right] \boldsymbol{x}_0 \tag{19}$$

The first part of Equation (19) is stable as the eigenvalues d_{11}, d_{21} are real negative. The second part is the solution of the switching system $\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}(t)\bar{\boldsymbol{x}}$. Therefore it follows that if the latter is stable then the switching system described by Equation (17) is stable.

□

Proof of Lemma 2.2 (Eigenvector Transformation) Every two eigenvectors form a basis in $\mathbb{R}^2$. Hence,

$$\boldsymbol{v}_k = k_i \boldsymbol{v}_i + k_j \boldsymbol{v}_j \tag{20}$$

As both pairs, $\{\boldsymbol{v}_i, \boldsymbol{v}_k\}$ and $\{\boldsymbol{v}_j, \boldsymbol{v}_k\}$, form as well a basis for $\mathbb{R}^2$, it follows that $k_i \neq 0$ and $k_j \neq 0$.

It is possible to find a transformation matrix $\boldsymbol{T}$ such that

$$\boldsymbol{T}\boldsymbol{v}_i = \zeta \boldsymbol{w}_1 \tag{21}$$

$$\boldsymbol{T}\boldsymbol{v}_j = \xi \boldsymbol{w}_2 \tag{22}$$

where $\xi, \zeta \in \mathbb{R} \setminus \{0\}$.From Equation (20) we get

$$\boldsymbol{T}\boldsymbol{v}_k = k_i \boldsymbol{T}\boldsymbol{v}_j + k_j \boldsymbol{T}\boldsymbol{v}_k$$

and with (21) and (22) we get

$$\boldsymbol{T}\boldsymbol{v}_k = k_i \zeta \boldsymbol{w}_1 + k_j \xi \boldsymbol{w}_2$$

If ζ and ξ in (21) and (22) are chosen as

$$\begin{aligned} \zeta &= \frac{1}{k_i} \\ \xi &= \frac{1}{k_j} \end{aligned}$$

the requirement $\boldsymbol{T}\boldsymbol{v}_k = \boldsymbol{w}_1 + \boldsymbol{w}_2 = \boldsymbol{w}_3$ holds.

$\square$