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On the relationship between matrix pencil eigenvalue criteria and the choice of Lyapunov function for the analysis of second order switching systems

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Abstract

In this report we present sufficient conditions for the stability of a class of switching systems. These results are of interest for two reasons. Firstly, the conditions presented are matrix pencil eigenvalue criteria, and are therefore co-ordinate independent and easily verifiable, and may generalise to high order switching systems. Secondly, and more importantly, we show a direct relationship between the form of our criteria, and the choice of Lyapunov function used to demonstrate stability (quadratic or otherwise). Extension of these statements to higher order systems are currently being investigated.

1 Introduction

Recent years have witnessed an enormous growth in stability questions generated by the study of hybrid dynamical systems [1, 2, 3]. While many conditions have been derived, relatively few of these can be checked without resorting to numerical optimization packages such as the MATLAB LMI toolbox [4, 5, 6, 7] (with the notable exceptions of [8, 9, 10, 11, 12, 13, 14, 15]). In this report we present sufficient conditions for the stability of second order switching systems. These conditions are related to various results and conjectures in the stability literature [16, 17]. We state our results in the

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form of matrix pencil eigenvalue criteria. Hence, the results presented are coordinate independent, are easily verifiable as they involve only the calculation of eigenvalues, and can be viewed as an extension of the results presented in [13] and [18]. More importantly, we show for the first time the direct relationship between the qualitative nature of the eigenvalues of a matrix pencil and the type of Lyapunov function used to prove stability. Roughly speaking, we show that the stability of switching systems, for which the eigenvalues of the matrix pencil is always real, can be demonstrated via a quadratic Lyapunov function. For matrix pencils whose root-locus bifurcate, i.e. become complex, stability of the associated switching system can be verified using a non-quadratic Lyapunov function of a special type (referred to in this report as a unic Lyapunov function). Work under progress is concerned with investigating whether these results generalise to special n-dimensional switching systems.

This report is organised as follows. We present a problem statement and several preliminary mathematical results and definitions in Section 2. The main results are stated in Section 3, and examples of their use are presented in Section 4. Our conclusions are presented in Section 5. All mathematical proofs are given in the Appendix.

2 Motivation and Mathematical Preliminaries

In this report we consider a class of second order linear switching systems of the form

$$\dot{\boldsymbol{x}}(t) \quad = \quad \boldsymbol{A}(t)\boldsymbol{x}(t), \qquad \boldsymbol{A}(t) \in \{\boldsymbol{A}_1, \boldsymbol{A}_2\} \tag{1}$$

where $\boldsymbol{A}_1, \boldsymbol{A}_2$ are constant Hurwitz matrices[1] in $\mathbb{R}^{2\times 2}$ with real eigenvalues. We refer to the system (1) as the switching system constructed from the constituent linear time-invariant systems $\Sigma_{\boldsymbol{A}_i}$: $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t),\ i \in \{1, 2\}$. Switching is allowed to occur according to arbitrary switching sequences. The overall objective of our work is to find easily verifiable and coordinate independent conditions that ensure (global asymptotic) stability for the switching system under arbitrary switching sequences.

In this section we present general results and definitions that are useful in stating and proving the main results of this report. For ease of exposition the results are simply quoted; where appropriate, proofs are presented in the Appendix.

We begin by defining a region in the complex plane depicted in Figure 1 that is of primary importance to our discussion. We will refer to it as the 45°-region.

Definition 2.1 (45°-Region) *The 45°-region is the closed subset Ω_{45} of the complex plane defined by*

$$\Omega_{45} = \left\{ \, s \;\middle|\; \mathrm{Re}\{s\} \leq -\Big|\mathrm{Im}\{s\}\Big| \, \right\}_{\mathbb{C}} \tag{2}$$

[1]We refer to "Hurwitz matrices" as matrices with eigenvalues located in the open left complex half-plane.

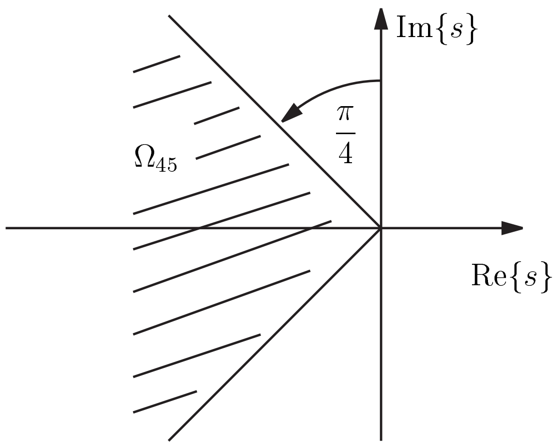

Figure 1: The 45°-region.

Our main results are derived by relating the evolution of the eigenvalues of matrix pencils, to the existence of specific types of Lyapunov functions. We define a matrix pencil as follows.

Definition 2.2 (Matrix Pencil) *The matrix pencil is a convex combination of matrices, defined as follows*

$$\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2] = \alpha \boldsymbol{A}_1 + (1-\alpha)\boldsymbol{A}_2, \quad \alpha \in [0,1]. \tag{3}$$

We consider two types of functions in this report. The first is the standard quadratic Lyapunov function and the second is a piecewise linear Lyapunov function that we term the unic Lyapunov function. Common quadratic Lyapunov functions (CQLF's) provide the basis for many of the known stability results in the switching literature, and are defined in the following manner.

Definition 2.3 (Common Quadratic Lyapunov Function) *Consider the linear, time invariant systems $\Sigma_{\boldsymbol{A}_i}$, $i \in \{1, ..., m\}$, where all the matrices A_i are Hurwitz. The quadratic function*

$$V(x) = \boldsymbol{x}^\mathsf{T} \boldsymbol{P} \boldsymbol{x}, \qquad \boldsymbol{P} = \boldsymbol{P}^\mathsf{T} > 0, \qquad \boldsymbol{P} \in \mathbb{R}^{n \times n}, \tag{4}$$

is said to be a common quadratic Lyapunov function for each of the constituent subsystems $\Sigma_{\boldsymbol{A}_i}$, $i \in \{1, ..., m\}$, if symmetric, positive definite matrices $\boldsymbol{Q}_i$, $i \in \{1, ..., m\}$, exist such that the matrix $\boldsymbol{P}$ is a solution of the matrix equations

$$\boldsymbol{A}_i^\mathsf{T} \boldsymbol{P} + \boldsymbol{P} \boldsymbol{A}_i = -\boldsymbol{Q}_i. \tag{5}$$

Comment: The existence of a common quadratic Lyapunov function implies the asymptotic stability of the switching system $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t), \quad \boldsymbol{A}(t) \in \{\boldsymbol{A}_1, ..., \boldsymbol{A}_m\}$, where m is finite.

Computationally nice necessary and sufficient conditions for the existence of a CQLF are known ([13] and [19]) when the systems $\Sigma_{\boldsymbol{A}_i}$, $i \in [1, 2]$, are second order. Here we only present a summary of the results that are of interest in our context.

Theorem 2.1 *A necessary and sufficient condition for the dynamic systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ to have a CQLF is that the pencils $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ have eigenvalues in the open left half of the complex plane for all $\alpha \in [0, 1]$ [13].*

Theorem 2.2 *A necessary and sufficient condition for the dynamic systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ to have a CQLF is that the matrices $\boldsymbol{A}_1\boldsymbol{A}_2$ and $\boldsymbol{A}_1\boldsymbol{A}_2^{-1}$ do not have real eigenvalues that are negative.*

Proof: see Appendix.

Theorem 2.3 *A sufficient condition for the dynamic systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ to have a CQLF is that the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has negative, real eigenvalues for all $\alpha \in [0, 1]$, [19] [2].*

The results of Theorem 2.1 and Theorem 2.3 relate the existence of CQLF's to the eigenvalues of matrix pencils and matrix products. The main object of this report is to relate the eigenvalues of matrix pencils to other types of common Lyapunov functions. In this context we introduce the unic Lyapunov function which has piecewise linear level curves in shape of a parallelogram and can be considered as a generalised l_1-norm.

Definition 2.4 *An unic Lyapunov function is defined as follows,*

$$V(\boldsymbol{x}) \quad = \quad \| \boldsymbol{M}\boldsymbol{x} \|_1 \tag{6}$$

with $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is non-singular.

We will be referring to the unic Lyapunov function throughout this report. In view of this we summarise the basic properties of such functions in the following.

[2] Originally, this theorem was restricted to matrices with distinct eigenvalues. However, it can be shown that the theorem extends to matrices with identical eigenvalues, by introducing ε as the difference between the eigenvalues and letting $\varepsilon \to 0$.

Comments on the unic Lyapunov function

i.) The unic Lyapunov function is positive definite and has a unique minimum. Therefore it is a proper Lyapunov function candidate [20].

ii.) Just as the quadratic Lyapunov function can be thought of as a generalised l_2-norm, the unic Lyapunov function can be considered as a generalised l_1-norm. This is illustrated in Figure 2.

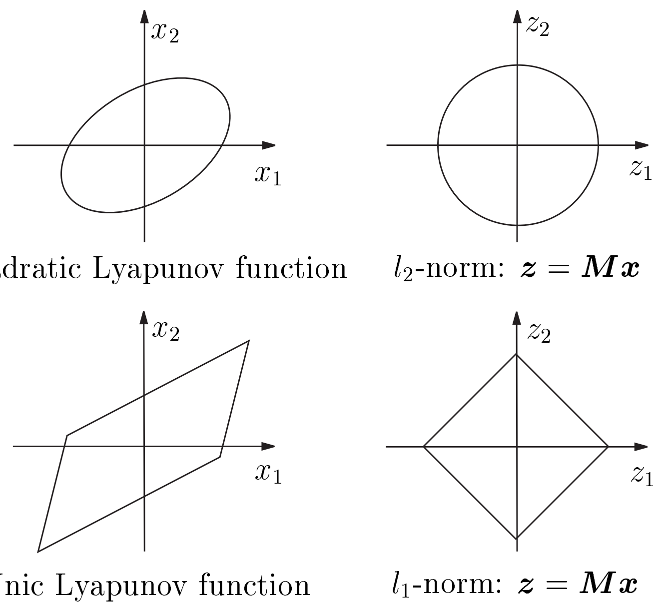

Figure 2: Motivation for unic Lyapunov function

iii.) Any unic Lyapunov function can be transformed into a diamond-shaped unic Lyapunov function. The main results in Section 3 are obtained by transforming the system matrices $\boldsymbol{A}_i$ such that the Lyapunov function is of diamond-shaped form:

$$V_d(\boldsymbol{x}) \;=\; \left\| \boldsymbol{M}_d\, \boldsymbol{x} \right\|_1 \qquad \text{with } \boldsymbol{M}_d \text{ diagonal.} \tag{7}$$

iv.) **Lemma 2.1** *Given the system matrix*

$$\boldsymbol{A} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}.$$

A necessary and sufficient condition for the system $\Sigma_{\boldsymbol{A}}$ to have a diamond-shaped unic Lyapunov function $V_d(\boldsymbol{x})$ is that

$$a^2 d^2 \;\geq\; b^2 c^2 \tag{8}$$

$$\textit{and} \qquad a, d \;<\; 0. \tag{9}$$

Possible matrices $\boldsymbol{M}_d$ are then given by

$$\boldsymbol{M}_d \;=\; \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \textit{with} \qquad m_1 \in [\,c, d\,], \quad m_2 \in [\,a, b\,]. \tag{10}$$

Proof: see Appendix.

v.) It follows from Lemma 2.1 that a necessary condition for the system $\Sigma_{\tilde{\boldsymbol{A}}}$ to have a diamond-shaped Lyapunov function $V_d(\boldsymbol{x})$ is that the eigenvalues of the system matrix $\tilde{\boldsymbol{A}}$ lie in the 45°-region Ω_{45}. Hence, any system $\Sigma_{\boldsymbol{A}}$ whose eigenvalues are outside Ω_{45} cannot be transformed via a similarity transformation such that a $V_d(\boldsymbol{x})$ exists. Hence, a necessary condition for the existence of an arbitrary l_1-LF is that the eigenvalues of $\boldsymbol{A}$ are in Ω_{45}.

Similarly, it can be shown that every Hurwitz matrix with eigenvalues in Ω_{45} can be transformed such that a diamond-shaped Lyapunov function $V_d(\boldsymbol{x})$ exists. These findings are summarised in Theorem 2.4.

Theorem 2.4 (Diamond-shaped Lyapunov Function) *A necessary and sufficient condition for the second order system $\Sigma_{\boldsymbol{A}}$ to have a unic Lyapunov function (l_1-LF) is that the eigenvalues of $\boldsymbol{A}$ lie within the closed 45°-region.*

Proof: see Appendix.

Finally we present a lemma, which ensures the existence of a specific transformation that is helpful for proving the main results.

Lemma 2.2 *For two second order matrices $\boldsymbol{A}_1, \boldsymbol{A}_2$ with negative, real eigenvalues and for which the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has complex eigenvalues for some $\alpha \in [0, 1]$, the matrix $\boldsymbol{S} = \boldsymbol{A}_1^{-1}\boldsymbol{A}_2$ exists and has positive, real eigenvalues.*

Proof: see Appendix.

Comment: This lemma implies that the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have points in the phase-plane where the flow is colinear, i.e.

$$\begin{aligned} \boldsymbol{A}_2\, \boldsymbol{v}_{\mathrm{cf},i} &= k_i \boldsymbol{A}_1 \boldsymbol{v}_{\mathrm{cf},i} \\ \boldsymbol{A}_1^{-1}\boldsymbol{A}_2\, \boldsymbol{v}_{\mathrm{cf},i} &= k_i \boldsymbol{v}_{\mathrm{cf},i} \qquad\qquad k_i > 0. \end{aligned}$$

The vectors $\boldsymbol{v}_{\mathrm{cf},i}$ of common flow are the eigenvectors of $\boldsymbol{S} = \boldsymbol{A}_1^{-1}\boldsymbol{A}_2$ and $k_i > 0$ are the eigenvalues of $\boldsymbol{S}$. The transformation $\boldsymbol{T} = [\boldsymbol{v}_{\mathrm{cf},1}\ \boldsymbol{v}_{\mathrm{cf},2}]$ provides that both transformed matrices $\tilde{\boldsymbol{A}}_i = \boldsymbol{T}^{-1}\boldsymbol{A}_i\boldsymbol{T}$ have colinear flow along the x_1- and x_2-axis. These observations are summarised in the following corollary.

Corollary 2.1 *Consider two second order Hurwitz matrices with real eigenvalues $\boldsymbol{A}_1, \boldsymbol{A}_2$. If the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has complex eigenvalues for some $\alpha \in [0,1]$ then there exists a transformation $\boldsymbol{T}$ such that*

$$\tilde{\boldsymbol{A}}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \tilde{\boldsymbol{A}}_2 = \begin{pmatrix} k_1 a & k_2 b \\ k_1 c & k_2 d \end{pmatrix}$$

with $a, b, c, d \in \mathbb{R}$ and $k_1, k_2 > 0$.

3 Sufficient Condition for Stability

In this section we present theorems on the existence of a common unic Lyapunov function. Combined with an existing theorem on common quadratic Lyapunov functions, they result in a sufficient condition on the stability of a class of second order switching systems.

We begin our discussion by noting that the existence of a common quadratic Lyapunov function (CQLF) for the system class considered is guaranteed by requiring that $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has negative, real eigenvalues for all $\alpha \in [0,1]$, c.f. Theorem 2.3. However, examples exist, which demonstrate that a CQLF may not exist when the eigenvalues of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ become complex for some $\alpha \in [0,1]$. In this case despite the non-existence of a CQLF, stability of the switching system (1) may be demonstrated via a non-quadratic Lyapunov function. Conditions guaranteeing the existence of one such Lyapunov function are summarised in the following theorem.

Theorem 3.1 (Sufficient Condition for Common unic LF) *Two stable linear systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ with $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{2\times 2}$ having real eigenvalues, have a common unic Lyapunov function if the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has complex eigenvalues for some $\alpha \in [0,1]$ and its root-locus stays within the closed $45°$-region for all $\alpha \in [0,1]$.*

Outline of the proof: A full proof of Theorem 3.1 is given in the Appendix. Here, to aid exposition, we present an overview of the main steps involved.

Step 1: We use the fact that $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has complex eigenvalues for some $\alpha \in [0,1]$. According to Corollary 2.1 the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ can be transformed such that

$$\begin{aligned} \tilde{\boldsymbol{A}}_1 &= \boldsymbol{T}^{-1}\boldsymbol{A}_1\boldsymbol{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \tilde{\boldsymbol{A}}_2 &= \boldsymbol{T}^{-1}\boldsymbol{A}_2\boldsymbol{T} = \begin{pmatrix} k_1 a & k_2 b \\ k_1 c & k_2 d \end{pmatrix}, \qquad k_1, k_2 > 0. \end{aligned}$$

Step 2: We then show that if the eigenvalues of $\boldsymbol{\sigma}_{\alpha}[\tilde{\boldsymbol{A}}_1,\ \tilde{\boldsymbol{A}}_2]$ lie in the 45°-region, then

$$\begin{array}{rrcl} & a^2b^2 & \geq & b^2c^2 \\ \text{and} & a, d & < & 0. \end{array}$$

This proves the existence of a diamond-shaped unic Lyapunov function for the system $\Sigma_{\tilde{\boldsymbol{A}}_1}$, as shown in Lemma 2.1.

Step 3: Since $a^2d^2 \geq b^2c^2$ and $k_1, k_2 > 0$, it follows immediately that

$$\begin{array}{rrcl} & k_1k_2a^2d^2 & \geq & k_1k_2b^2c^2 \\ \text{and} & k_1a,\ k_2d & < & 0. \end{array}$$

Step 4: According to Lemma 2.1, $V(\boldsymbol{x}) = ||\, \boldsymbol{M}_d\boldsymbol{x}\, ||_1$, with

$$\boldsymbol{M}_d \;=\; \begin{pmatrix} c & 0 \\ 0 & a \end{pmatrix}$$

is a common diamond-shaped Lyapunov function for the systems $\Sigma_{\tilde{\boldsymbol{A}}_1}$ and $\Sigma_{\tilde{\boldsymbol{A}}_2}$ and hence, a common unic Lyapunov for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ exists.

From Theorem 2.3 and Theorem 3.1 it follows that either a common quadratic Lyapunov function or a common unic Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ exists if the root-locus of $\boldsymbol{\sigma}_{\alpha}[\boldsymbol{A}_1, \boldsymbol{A}_2]$ lies within the 45°-region, thereby implying the following condition for the global stability of second order switching systems.

Corollary 3.1 (45°**Criterion)** *The switching system defined in (1) is globally (asymptotically) stable if the root-locus of the matrix pencil* $\boldsymbol{\sigma}_{\alpha}\left[\boldsymbol{A}_1, \boldsymbol{A}_2\right],\ \alpha \in [0,1]$ *lies completely within the (open)* 45°*-region.*

<u>Comment:</u> This stability condition is easily evaluated by plotting the root-locus of the matrix pencil as a function of α, as illustrated in the examples in Section 4. Furthermore, the condition is coordinate independent, since it is expressed in terms of eigenvalue properties.

To conclude our discussion, we note that there is neither a common unic Lyapunov function nor a common quadratic Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ if the matrix pencil $\boldsymbol{\sigma}_{\alpha}[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has eigenvalues outside the 45°-region for some $\alpha \in [0,1]$.

Theorem 3.2 (Necessary Condition) *Let* $\Sigma_{\boldsymbol{A}_1}$ *and* $\Sigma_{\boldsymbol{A}_2}$ *be second order systems where* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *are Hurwitz matrices with real eigenvalues, and where the matrix pencil* $\boldsymbol{\sigma}_{\alpha}[\boldsymbol{A}_1, \boldsymbol{A}_2]$ *has eigenvalues outside the* 45°*-region for some* $\alpha \in [0,1]$*. Then:*

a) a common quadratic Lyapunov function cannot exist for $\Sigma_{\boldsymbol{A}_1}$ *and* $\Sigma_{\boldsymbol{A}_2}$*;*

b) a common unic Lyapunov function cannot exist for $\Sigma_{\boldsymbol{A}_1}$ *and* $\Sigma_{\boldsymbol{A}_2}$*.*

Proof: see Appendix.

Summarising the results, we can relate the different behaviour of the eigenvalues of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ to the existence of the two types of Lyapunov functions considered in this report. If the root-locus remains real and is inside the 45°-region for all $\alpha \in [0,1]$, then a common quadratic Lyapunov function exists (Theorem 2.3); if there are complex eigenvalues of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ for *some* values $\alpha \in [0,1]$, but the root-locus stays within the 45°-region for all $\alpha \in [0,1]$, then the existence of a common unic Lyapunov function is guaranteed (Theorem 3.1); and if the matrix pencil has eigenvalues outside the 45°-region for some $\alpha \in [0,1]$, then neither a common quadratic Lyapunov function nor a common unic Lyapunov function exists (Theorem 3.2). These results are summarised in Table 1.

Root-Locus of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$	**Common Quadratic LF**	**Common Unic LF**
within Ω_{45} and real $\forall\ \alpha \in [0,1]$	existence guaranteed	existence possible
within Ω_{45} $\forall\ \alpha \in [0,1]$, and complex for some $\alpha \in [0,1]$	existence possible	existence guaranteed
outside Ω_{45} for some $\alpha \in [0,1]$	non-existence	non-existence

Table 1: Overview of the main results

4 Numerical Examples

In this section we present a number of numerical examples to illustrate the main features of our results.

Example 4.1 (No common quadratic LF, but common unic LF) This is an example where a common quadratic Lyapunov function does not exist, but stability of the switching system can be shown by the existence of a common unic Lyapunov function.

Let

$$\boldsymbol{A}_1 = \begin{pmatrix} -3.0 & -2.8 \\ 0 & -0.2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}_2 = \begin{pmatrix} -0.2 & 0.1 \\ 0.8 & -0.9 \end{pmatrix} \tag{11}$$

It was shown in [13, 8] that a necessary condition for the existence of a common quadratic Lyapunov function (CQLF) for the systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ is, that the root-locii of the matrix pencils $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are both Hurwitz.

In Figure 3 the root-locii of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are depicted. Since there are real, positive values for the latter, it is not possible to determine stability for this switching system by using a common quadratic Lyapunov function.

An even simpler way to show that no CQLF exists is provided by Theorem 2.2: the eigenvalues of $\boldsymbol{A}_1\boldsymbol{A}_2$ are approximately: -1.42 and -0.04. Since they are both negative a CQLF cannot exist.

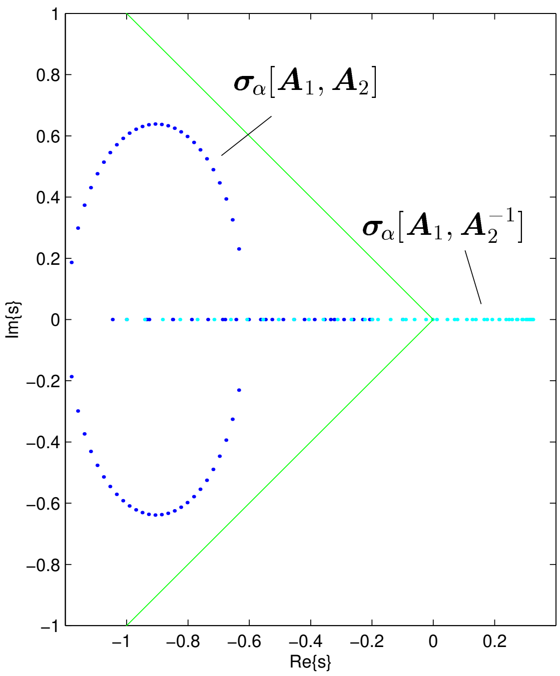

Figure 3: No CQLF but a common unic Lyapunov function exists

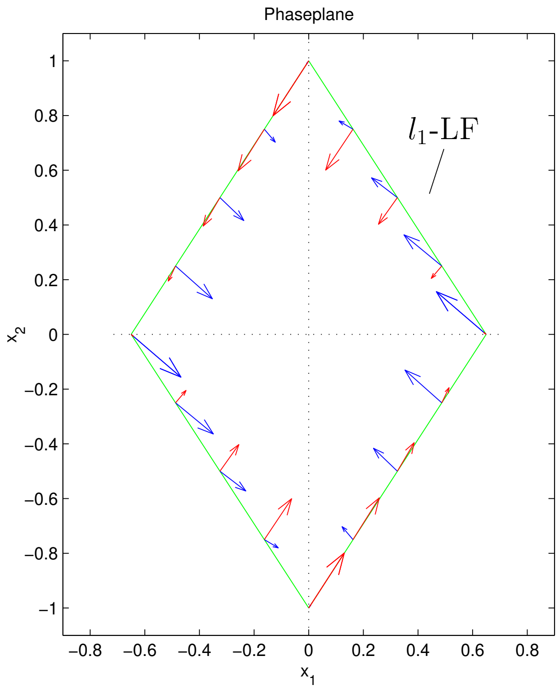

Figure 4: Common unic Lyapunov function

But the root locus of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ lies completely within the 45°-region and therefore meets the 45° Criterion, i.e. a common unic Lyapunov function exists and the switching system defined in (1) is stable for arbitrary switching sequences.

Example 4.2 (No common quadratic LF, no common unic LF) This example illustrates the necessity of the 45° condition. Here, neither a common unic nor a common quadratic Lyapunov function exists.

Let

$$\boldsymbol{A}_1 = \begin{pmatrix} 0.3 & 0.7 \\ -2.3 & -2.3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{A}_2 = \begin{pmatrix} -1.8 & 1.0 \\ -0.8 & 0.1 \end{pmatrix} \tag{12}$$

Figure 5 shows the root-locus of the matrix pencils $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$. As there are eigenvalues outside the 45°-region there is neither a common unic nor a common quadratic Lyapunov function. The latter is confirmed by the root-locus of the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ in Figure 6, where some eigenvalues are in the positive half-plane and hence, there is no common quadratic Lyapunov function. The same result is obtained by looking at the eigenvalues of $\boldsymbol{A}_1\boldsymbol{A}_2$, which are approximately -0.16 and -3.47. As both are negative, a common quadratic Lyapunov function cannot exist (Theorem 2.2).

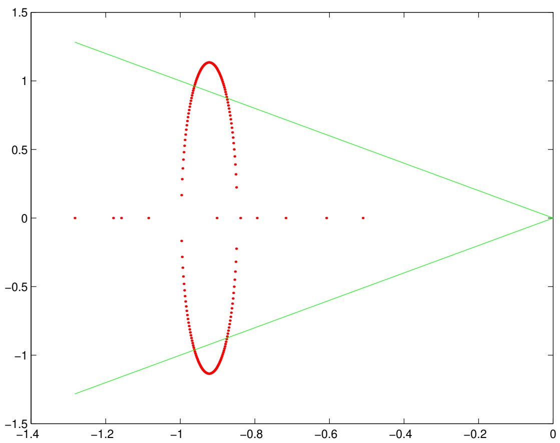

Figure 5: Root-locus of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$

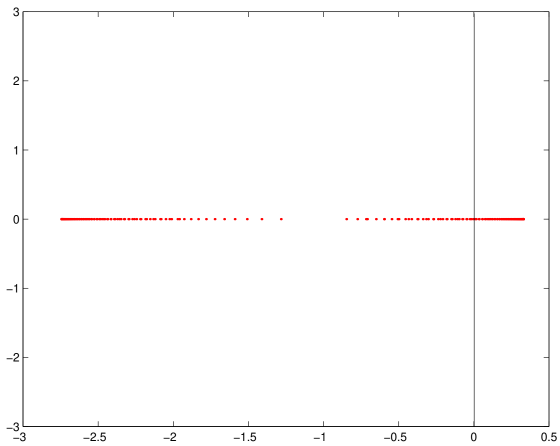

Figure 6: Root-locus of $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$

5 Concluding remarks

In this report we have studied the existence of two types of common Lyapunov functions for two linear second order systems with negative, real eigenvalues. One function under consideration has been the well know common quadratic Lyapunov function. Secondly, we introduced the unic Lyapunov function.

A relationship between the evolution of the eigenvalues of the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and the existence of the respective common Lyapunov functions has been derived. These results are summarised as follows:

- the root-locus of the matrix pencil lies completely within the 45°-region and is real for all $\alpha \in [0, 1]$: a common quadratic Lyapunov function exists;
- the root-locus of the matrix pencil lies completely within the 45°-region and is complex for some $\alpha \in [0, 1]$: a common unic Lyapunov function exists;
- the root-locus of the matrix pencil is outside the 45°-region: neither a common quadratic Lyapunov function nor a common unic Lyapunov function exists.

From the first two cases an easily verifiable and coordinate independent sufficient condition for the (asymptotic) stability of a class of second order switching systems has been derived. Future work will consider the extension of these statements to higher order systems, and to discrete-time systems.

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[3]This work is the sole responsibility of the authors and does not reflect the European Union's opinion

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Appendix

In the following we prove the lemmas and theorems presented in this report. The proofs are give in the order of appearance in the report.

Preliminary remarks

i.) The main results are given by considering the evolution of the root-locii of the matrix pencil $\boldsymbol{\sigma}_\alpha$ with regard to the 45°-region. However, throughout the proofs we will consider the related matrix pencil $\boldsymbol{\sigma}_\gamma$ defined by

$$\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2] = \boldsymbol{A}_1 + \gamma \boldsymbol{A}_2, \quad \gamma \in [0, \infty). \tag{13}$$

Note that $\boldsymbol{\sigma}_\alpha$ can be written as $\alpha\boldsymbol{\sigma}_\gamma$, where $\gamma = \frac{1-\alpha}{\alpha}$ for $\alpha \neq 0$. Hence, for all $\alpha \neq 0$, the eigenvalues of $\boldsymbol{\sigma}_\gamma$ and $\boldsymbol{\sigma}_\alpha$ are identical up to a positive scaling factor. For $\alpha = 0$ the pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has negative, real eigenvalues for the matrices $\boldsymbol{A}_1, \boldsymbol{A}_2$ considered in this report. Therefore, the eigenvalues of $\boldsymbol{\sigma}_\alpha$ are within the 45°-region if and only if the eigenvalues of $\boldsymbol{\sigma}_\gamma$ are.

ii.) We recall the following elementary fact about 2×2-matrices, which will be used frequently throughout the proofs.

The second order matrix $\boldsymbol{A}$ has real eigenvalues if and only if

$$\operatorname{tr}^2 \boldsymbol{A} \quad \geq \quad 4 \det \boldsymbol{A}. \tag{14}$$

The second order matrix $\boldsymbol{A}$ has eigenvalues within the 45°-region if and only if

$$\operatorname{tr}^2 \boldsymbol{A} \quad \geq \quad 2 \det \boldsymbol{A}. \tag{15}$$

Proof Theorem 2.2

According to Theorem 2.1 there exists a common quadratic Lyapunov function if and only if the matrix pencils $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are both Hurwitz for all $\alpha \in [0, 1]$. For ease of computation, we will use the matrix pencil $\boldsymbol{\sigma}_\gamma$ which has the same eigenvalue properties in view of the 45°-region as $\boldsymbol{\sigma}_\alpha$ (c.f. preliminary remarks). We will show that the matrix pencil $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has non-negative, real eigenvalues if and only if $\boldsymbol{A}_2^{-1}\boldsymbol{A}_1$ has a negative, real eigenvalue. Likewise it can be shown that $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ has non-negative, real eigenvalues if $\boldsymbol{A}_1\boldsymbol{A}_2$ has a negative, real eigenvalue.

According to the assumptions $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are both Hurwitz. Hence,

$$\operatorname{tr} \boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2] \;<\; 0, \qquad \text{for all} \quad \gamma \in [0, \infty). \tag{16}$$

It follows that $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2]$ cannot have purely imaginary eigenvalues. Thus, the matrix pencil $\boldsymbol{\sigma}_\gamma$ is non-Hurwitz if and only if there is one γ_0 for which one eigenvalue of $\boldsymbol{\sigma}_\gamma$ is zero. Hence,

$$\begin{aligned}
\Big(\boldsymbol{A}_1 + \gamma_0 \boldsymbol{A}_2\Big)\boldsymbol{x} &= 0 \\
\Big(\boldsymbol{A}_2^{-1}\boldsymbol{A}_1 + \gamma_0 \boldsymbol{I}_2\Big)\boldsymbol{x} &= 0 \\
\boldsymbol{A}_2^{-1}\boldsymbol{A}_1\boldsymbol{x} &= -\gamma_0\boldsymbol{x}, \qquad (17)
\end{aligned}$$

i.e. $\boldsymbol{A}_2^{-1}\boldsymbol{A}_1$ has a negative, real eigenvalue $-\gamma_0$.

□

Proof of Lemma 2.1 (Condition for diamond-shaped Lyapunov function)

Suppose a diamond-shaped unic Lyapunov function exists for a certain system described by the system matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R}. \tag{18}$$

Then there are two possible flow schemes along the axes of the phaseplane (c.f. Figure 7):

Case 1: the flow along both axes is clockwise (or anti-clockwise),

Case 2: the flow along both axes is opposite.

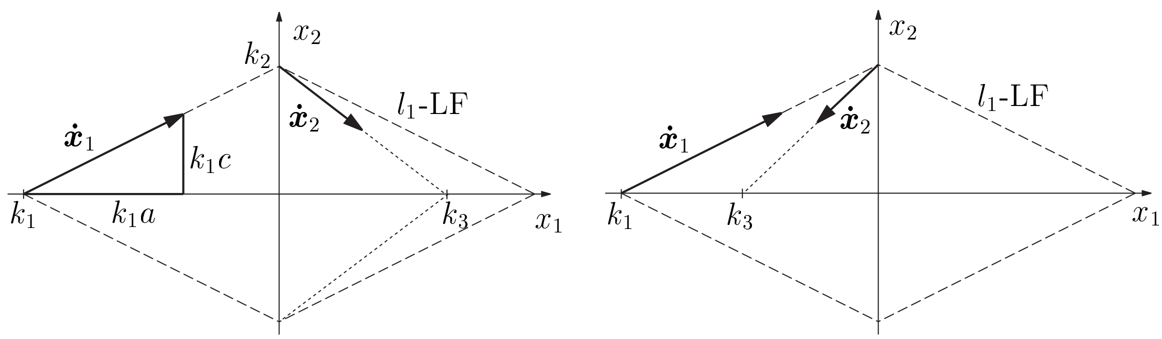

Figure 7: Two possible flow-schemes of systems with diamond-shaped Lyapunov function.

The flow vectors along the x_1- and x_2-axis are obtained by

$$\begin{aligned} \boldsymbol{A}\begin{pmatrix} n_1 \\ 0 \end{pmatrix} &= n_1 \begin{pmatrix} a \\ c \end{pmatrix}, \quad n_2 \in \mathbb{R}, \\ \boldsymbol{A}\begin{pmatrix} 0 \\ n_2 \end{pmatrix} &= n_2 \begin{pmatrix} b \\ d \end{pmatrix}, \quad n_2 \in \mathbb{R}. \end{aligned}$$

Geometrical considerations reveal immediately that $a, d < 0$ in every case. For Case 1 the entries b and c have opposite signs and in Case 2, b and c have the same sign. Therefore, we will find for a matrix for which a diamond-shaped Lyapunov function exists the following sign patterns:

- for Case 1: $\begin{pmatrix} - & - \\ + & - \end{pmatrix}$ or $\begin{pmatrix} - & + \\ - & - \end{pmatrix}$,

- for Case 2: $$\begin{pmatrix} - & + \\ + & - \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} - & - \\ - & - \end{pmatrix}.$$

Consider the trajectory starting at point $[k_1 \ 0]^{\mathsf{T}}$ following the flow at that point and going in a straight line towards the x_2-axis, as depicted in Figure 7. At point $[0 \ k_2]^{\mathsf{T}}$ we follow the direction of the flow back to the x_1-axis and end at the point $[k_3 \ 0]^{\mathsf{T}}$, where

$$k_3 = \frac{b}{d}\frac{c}{a} k_1.$$

For Case 1 we will find that a diamond-shaped Lyapunov function exists if and only if

$$\begin{aligned} \frac{k_3}{k_1} &= \frac{b}{d}\frac{c}{a} \geq -1 \\ bc &\geq -ad, \end{aligned} \tag{19}$$

and for Case 2

$$\begin{aligned} \frac{k_3}{k_1} &= \frac{b}{d}\frac{c}{a} \leq 1 \\ bc &\leq ad. \end{aligned} \tag{20}$$

According to the discussion on the possible sign patterns, both sides of each resulting inequality have the same sign: negative in Case 1 and positive Case 2.

By combining the conditions (19) and (20), the following necessary and sufficient conditions for the existence of a unic Lyapunov function result

$$a^2 d^2 \geq b^2 c^2 \tag{21}$$
$$a, d \leq 0. \tag{22}$$

From Figure 7 can be trivially derived that all possible diamond-shaped Lyapunov functions are given by

$$\boldsymbol{M}_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad \text{with} \qquad m_1 \in [\,c, d\,], \quad m_2 \in [\,a, b\,]. \tag{23}$$

□

Proof of Lemma 2.2

According to our assumption the matrix pencil $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has complex eigenvalues for some $\gamma > 0$. Therefore, there exist $\gamma_\mathrm{c} > 0$ for which

$$\begin{aligned} \mathrm{tr}^2 \boldsymbol{\sigma}_\gamma &< 4 \det \sigma_\gamma \\ \underbrace{(\mathrm{tr} \boldsymbol{A}_1 + \gamma_\mathrm{c} \, \mathrm{tr} \boldsymbol{A}_2)^2}_{p_1(\gamma)} &< \underbrace{4 \det \boldsymbol{A}_2 \, \gamma_\mathrm{c}^2 + 4 \, \mathrm{tr} \boldsymbol{S} \, \gamma_\mathrm{c} + 4 \det \boldsymbol{A}_1}_{p_2(\gamma)} . \end{aligned}$$

Consider the parabolas p_1 and p_2 defined in terms of γ. By assumption the traces of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are negative. Hence, $p_1(\gamma)$ has a unique negative zero. Further, the eigenvalues of $\boldsymbol{\sigma}_\gamma$ are real for $\gamma = 0$ and $\gamma \to \infty$. Hence, $p_1(0) > p_2(0)$ and there exists a $\gamma_{\mathrm{c}2}$ such that $p_1(\gamma) > p_2(\gamma)$ for all $\gamma > \gamma_{\mathrm{c}2}$, c.f. Figure 8.

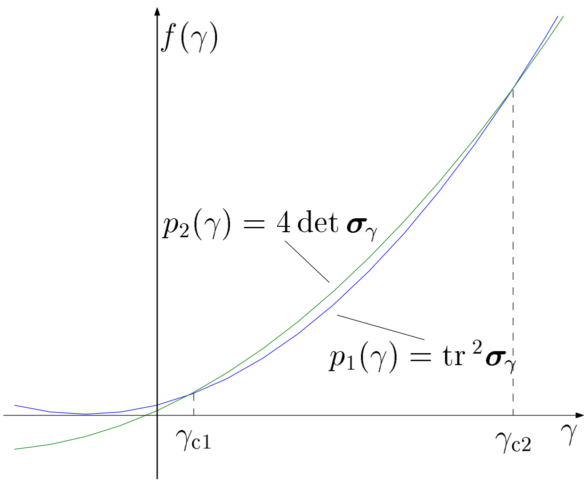

Figure 8:

Note that two parabolas can intersect at most twice and $p_2(\gamma)$ has at least one zero for $\gamma_0 < 0$. Hence, one eigenvalue of $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2]$ is zero for $\gamma_0 < 0$. Hence,

$$\begin{aligned} (\boldsymbol{A}_1 + \gamma_0 \boldsymbol{A}_2) \, \boldsymbol{v} &= 0 \\ \left(\boldsymbol{A}_1^{-1} \boldsymbol{A}_2 + \gamma_0 \boldsymbol{I} \right) \boldsymbol{v} &= 0 \end{aligned}$$

Since $\gamma_0 < 0$, the matrix $\boldsymbol{S} = \boldsymbol{A}_1^{-1} \boldsymbol{A}_2$ has at least one positive, real eigenvalue. However, the determinant of $\boldsymbol{S}$ is positive and therefore both eigenvalues are real, positive. $\square$

Proof of Theorem 2.4 (Diamond-shaped Lyapunov Function)

Sufficiency:

A modal transformation immediately reveals the existence of a diamond-shaped unic Lyapunov function for systems with negative, real eigenvalues.

System matrices $\boldsymbol{A}$ with complex eigenvalues $\lambda = p \pm \mathrm{j}q$ can be transformed into

$$\boldsymbol{A}^* = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}. \tag{24}$$

According to Lemma 2.1 a necessary and sufficient condition for the existence of a diamond-shaped unic Lyapunov function is, that $a^2d^2 \geq c^2b^2$ and $a, d < 0$. Hence, there exists a diamond-shaped Lyapunov function for the matrix $\boldsymbol{A}^*$ if and only if $|p| \geq |q|$ and $p < 0$, i.e. the eigenvalues lie within the 45°-region.

Necessity:

Suppose we have an appropriate transformed matrix

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{25}$$

with eigenvalues outside the 45°-region. Then,

$$\begin{aligned} \mathrm{tr}^2\boldsymbol{A} &< 2\det\boldsymbol{A} \\ (a+d)^2 &< 2ad - 2bc \\ a^2 + d^2 &< -2bc. \end{aligned}$$

Since $a^2 + d^2 \geq 0$, both sides are non-negative and we get after squaring

$$\left(a^2 + d^2\right)^2 < 4b^2c^2. \tag{26}$$

Assume that a diamond-shaped Lyapunov function for this system exists. Then

$$b^2c^2 \leq a^2d^2.$$

With inequality (26) we get

$$\left(a^2 + d^2\right)^2 < 4b^2c^2 \leq 4a^2d^2.$$

But this implies

$$\begin{aligned} \left(a^2 + d^2\right)^2 &< 4a^2d^2 \\ \left(a^2 - d^2\right)^2 &< 0. \end{aligned}$$

Since the left side is always non-negative, the inequality does not hold and thus the matrix $\boldsymbol{A}$ cannot have a diamond-shaped Lyapunov function.

□

Proof of Theorem 3.1 (Common unic Lyapunov function)

For ease of computation, we will use the matrix pencil $\boldsymbol{\sigma}_\gamma$ which has the same eigenvalue properties in view of the 45°-region as $\boldsymbol{\sigma}_\alpha$ (c.f. preliminary remarks).

Since the matrix pencil $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has complex eigenvalues for some $\gamma \in [0, \infty)$, there exists a non-singular matrix $\boldsymbol{T}$ (Corollary 2.1) such that

$$\tilde{\boldsymbol{A}}_1 = \boldsymbol{T}^{-1}\boldsymbol{A}_1\boldsymbol{T} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{R} \tag{27}$$

$$\text{and} \qquad \tilde{\boldsymbol{A}}_2 = \boldsymbol{T}^{-1}\boldsymbol{A}_2\boldsymbol{T} = \begin{pmatrix} k_1 a & k_2 b \\ k_1 c & k_2 d \end{pmatrix}, \qquad k_1, k_2 > 0. \tag{28}$$

The matrix pencil $\boldsymbol{\sigma}_\gamma[\boldsymbol{A}_1, \boldsymbol{A}_2]$ is then given by

$$\boldsymbol{\sigma}_\gamma = \left[(1 + k_1\gamma)\begin{pmatrix} a \\ c \end{pmatrix} \quad (1 + k_2\gamma)\begin{pmatrix} b \\ d \end{pmatrix} \right] \tag{29}$$

$$\boldsymbol{\sigma}_\gamma = (1 + k_1\gamma)\left[\begin{pmatrix} a \\ c \end{pmatrix} \quad \frac{1 + k_2\gamma}{1 + k_1\gamma}\begin{pmatrix} b \\ d \end{pmatrix} \right]. \tag{30}$$

Since the factor $1 + k_1\gamma$ is positive, it does not change the root-locus of $\boldsymbol{\sigma}_\gamma$ in respect of being within the 45°-region. For ease of computation we will therefore consider the matrix pencil

$$\boldsymbol{\sigma}_{\tilde{\gamma}} = \begin{pmatrix} a & \tilde{\gamma} b \\ c & \tilde{\gamma} d \end{pmatrix} \tag{31}$$

with

$$\tilde{\gamma} = \frac{1 + k_2\gamma}{1 + k_1\gamma}$$

where $\tilde{\gamma} \in \Gamma \equiv \left[1, \frac{k_2}{k_1}\right]$, if $k_2 > k_1$, and $\Gamma \equiv \left[\frac{k_2}{k_1}, 1\right]$, otherwise.

According to the assumptions the matrix pencil $\boldsymbol{\sigma}_{\tilde{\gamma}}$ has complex eigenvalues for some $\tilde{\gamma} \in \Gamma$, hence,

$$\mathrm{tr}^2 \boldsymbol{\sigma}_{\tilde{\gamma}} < 4 \det \boldsymbol{\sigma}_{\tilde{\gamma}} \tag{32}$$

holds for at least one $\tilde{\gamma}_c \in \Gamma_c \subset \Gamma$.[4]

[4]Note, that since the matrix pencil has real eigenvalues for the endpoints $\tilde{\gamma} = 1$ and $\tilde{\gamma} = \frac{k_2}{k_1}$, Γ_c is a proper subset of Γ.

For these $\tilde{\gamma}_c$ we get

$$\begin{aligned} \Big(a + \tilde{\gamma}_c d\Big)^2 &< 4\Big(ad - bc\Big)\tilde{\gamma}_c \\ d^2\tilde{\gamma}_c^2 + \Big(4bc - 2ad\Big)\tilde{\gamma}_c + a^2 &< 0 \end{aligned} \tag{33}$$

The parabola in terms of $\tilde{\gamma}_c$ has a minimum, hence there exist complex eigenvalues of $\boldsymbol{\sigma}_{\tilde{\gamma}}$ if and only if

$$\begin{aligned} \Big(2bc - ad\Big)^2 &> a^2d^2 \\ 4b^2c^2 - 4bc{\cdot}ad &> 0 \\ b^2c^2 &> bc{\cdot}ad. \end{aligned} \tag{34}$$

The zeros of the parabola are given by

$$\tilde{\gamma}_{c0_{1,2}} = \frac{ad - 2bc \pm \sqrt{4\big(\, b^2c^2 - ad{\cdot}bc\big)}}{d^2} \tag{35}$$

which define the endpoints of Γ_c.

Further we assumed that the complex eigenvalues of $\boldsymbol{\sigma}_{\tilde{\gamma}}$ are within the closed 45°-region, i.e.

$$\operatorname{tr}^2\boldsymbol{\sigma}_{\tilde{\gamma}} \;\geq\; 2\det\boldsymbol{\sigma}_{\tilde{\gamma}} \tag{36}$$

for all $\tilde{\gamma}_c \in \Gamma_c \equiv \big[\,\tilde{\gamma}_{0,1},\, \tilde{\gamma}_{c0,2}\,\big] \subset \Gamma$. Therefore,

$$\begin{aligned} (a + \tilde{\gamma}_c d)^2 &\geq 2(ad - bc)\tilde{\gamma}_c \\ a^2 + \tilde{\gamma}_c^2 d^2 &\geq -2bc\tilde{\gamma}_c \\ d^2\,\tilde{\gamma}_c^2 + 2bc\,\tilde{\gamma}_c + a^2 &\geq 0. \end{aligned} \tag{37}$$

The parabola $\mathcal{P}(\tilde{\gamma}_c) = d^2\,\tilde{\gamma}_c^2 + 2bc\,\tilde{\gamma}_c + a^2$ defined in terms of $\tilde{\gamma}_c$ is positive for all $\tilde{\gamma}_c \in \Gamma_c$. Hence, either the global minimum $\tilde{\gamma}_{\min}$ for $\tilde{\gamma}_c \in \mathbb{R}$ is positive, or the minimum is negative and "far enough" outside of Γ_c. These two possibilities are illustrated in Figure 9.

Next, we will prove that the latter possibility cannot occur, by showing that if the minimum is negative, then $\tilde{\gamma}_{\min} \in \Gamma_c$.

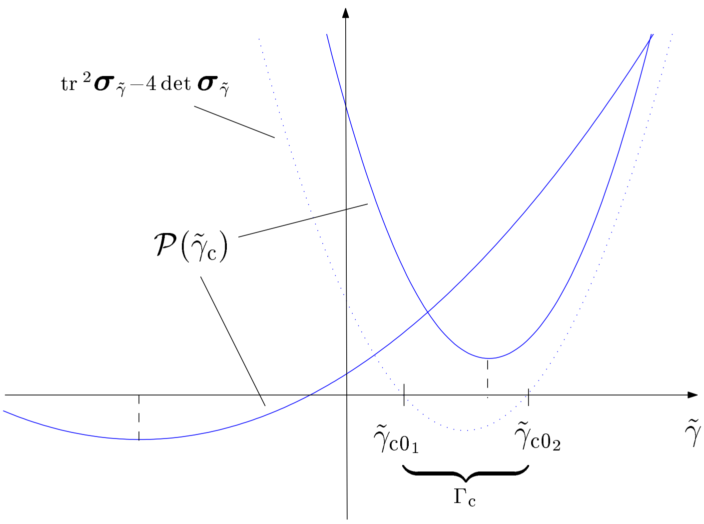

Figure 9:

The minimum of $\mathcal{P}$ is given by $\tilde{\gamma}_{\min} = \frac{-bc}{d^2}$ and lies within Γ_{c} if (and only if)

$$\tilde{\gamma}_{\mathrm{c}0_1} \quad\leq\quad \tilde{\gamma}_{\min} \quad\leq\quad \tilde{\gamma}_{\mathrm{c}0_2} \tag{38}$$

The upper bound gives

$$\begin{aligned} -bc &\leq ad - 2bc + \sqrt{4\big(b^2c^2 - ad{\cdot}bc\big)} \\ 0 &\leq ad - bc + \sqrt{4\big(b^2c^2 - ad{\cdot}bc\big)} \\ 0 &\leq \det\tilde{\boldsymbol{A}}_1 + \sqrt{4\big(b^2c^2 - ad{\cdot}bc\big)}. \end{aligned}$$

Since the determinant of $\tilde{\boldsymbol{A}}_1$ is non-negative, the inequality holds in any case.

From the lower bound we get

$$\begin{aligned} ad - 2bc - \sqrt{4\big(b^2c^2 - ad{\cdot}bc\big)} &\leq -bc \\ ab - bc &\leq \sqrt{4\big(b^2c^2 - ad{\cdot}bc\big)}. \end{aligned}$$

Both sides of the inequality are positive. Therefore squaring gives

$$\begin{aligned} (ab - bc)^2 &\leq 4b^2c^2 - 4ad{\cdot}bc \\ a^2d^2 - 2ad{\cdot}bc + b^2d^2 &\leq 4b^2c^2 - 4ad{\cdot}bc \\ a^2d^2 + 2ad{\cdot}bc - 3b^2d^2 &\leq 0. \end{aligned} \tag{39}$$

Assume now, that the minimum at $\tilde{\gamma}_{\min}$ is negative, i.e. there exist two zero-crossings for the parabola $\mathcal{P}$ and thus, some eigenvalues of $\boldsymbol{\sigma}_{\tilde{\gamma}}$ lie outside the 45°-region. Then,

$4b^2c^2 - 4a^2d^2 > 0$ and hence, $b^2c^2 > a^2d^2$. Using this, we can state

$$a^2d^2 + 2ad{\cdot}bc - 3b^2c^2 \;\; < \;\; b^2c^2 + 2ad{\cdot}bc - 3b^2c^2.$$

Therefore inequality (39) holds if

$$\begin{aligned} 2ad{\cdot}bc - 2b^2c^2 &\leq 0 \\ 2ad{\cdot}bc &\leq 2b^2c^2 \end{aligned}$$

But since the matrix pencil *does* have complex eigenvalues, this inequality holds, c.f. inequality (34). This means that if the minimum is negative, $\tilde{\gamma}_{\min}$ lies in Γ_{c}. But if the parabola is negative for $\tilde{\gamma}_{\mathrm{c}} \in \Gamma_{\mathrm{c}}$, the matrix pencil has eigenvalues outside the 45°-region, c.f. (36), which contradicts the assumption.

Hence, the minimum at $\tilde{\gamma}_{\min}$ of $\mathcal{P}$ is positive, i.e. there are no zeros. Hence,

$$\begin{aligned} 4b^2c^2 - 4a^2d^2 &\leq 0 \\ a^2d^2 &\geq b^2c^2. \end{aligned} \tag{40}$$

Thus, the first condition (8) for the diamond-shaped unic Lyapunov function is given. The second condition follows immediately. From the condition that the matrix pencil has complex eigenvalues we obtained

$$b^2c^2 > bc{\cdot}ad. \tag{41}$$

But the determinant of $\tilde{\boldsymbol{A}}_1$ is positive, i.e.

$$\begin{aligned} ad - bc &> 0 \\ ad &> bc. \end{aligned} \tag{42}$$

Therefore, $bc < 0$ if the matrix pencil has complex eigenvalues.

Assume now that $ad < 0$. Then both sides of (42) are negative and we get after squaring $a^2d^2 < b^2c^2$. But this contradicts the first condition. Hence, $ad > 0$, and together with the trace condition $a + d < 0$, we get $a, d < 0$.

□

Proof of Theorem 3.2 (Necessary Condition)

We will prove the necessity of the 45°-Condition separately for each type of Lyapunov function. Throughout the prove we will use the following abbreviations

$$\begin{aligned} \Delta_i &= \det A_i \\ \mathrm{tr}_i &= \mathrm{tr}\, A_i. \end{aligned}$$

For ease of computation, we will use the matrix pencil $\boldsymbol{\sigma}_\gamma$ which has the same eigenvalue properties in view of the 45°-region as $\boldsymbol{\sigma}_\alpha$ (c.f. preliminary remarks).

Part a: Common quadratic Lyapunov function

According to the assumptions the matrix-pencil $\sigma_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ has eigenvalues outside the 45°-region for some $\alpha \in [0,1]$. If there are positive eigenvalues, the conditions for the existence of a CQLF in Theorem 2.1 are violated and no CQLF exists. If the eigenvalues are outside the 45°-region but in the left half-plane, there exists a transformation (c.f. Corollary 2.1) such that

$$\tilde{\boldsymbol{A}}_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \tilde{\boldsymbol{A}}_2 = \begin{pmatrix} k_1 a & k_2 b \\ k_1 c & k_2 d \end{pmatrix}, \qquad a,b,c,d \in \mathbb{R} \quad \text{and} \quad k_1, k_2 > 0. \tag{43}$$

We will show that the matrix product $\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2$ has negative, real eigenvalues if the matrix pencil $\sigma_\alpha[\tilde{\boldsymbol{A}}_1, \tilde{\boldsymbol{A}}_2]$ has eigenvalues outside the 45°-region for some $\alpha \in [0,1]$; hence, no common quadratic Lyapunov function exists for the systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ (Theorem 2.2).

With Equations (43) the matrix product $\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2$ is

$$\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2 = \begin{pmatrix} k_1\big(a^2+bc\big) & k_2 b\big(a+d\big) \\ k_2 b\big(a+d\big) & k_2\big(d^2+bc\big) \end{pmatrix}$$

Since $\tilde{\boldsymbol{A}}_1$ and $\tilde{\boldsymbol{A}}_2$ have real negative eigenvalues, the determinant of the matrix product, given by $\Delta_1\Delta_2$, is positive. Hence, both eigenvalues are either positive or both are negative.

The trace of the matrix product is

$$\mathrm{tr}\Big\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\Big\} = k_1\big(a^2+bc\big) + k_2\big(d^2+bc\big). \tag{44}$$

We have to show that

$$\begin{aligned} & \mathrm{tr}\Big\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\Big\} < 0, && \text{(eigenvalues are negative)} \\ \text{and} \qquad & \mathrm{tr}^2\Big\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\Big\} > 4\det\Big\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\Big\}, && \text{(eigenvalues are real).} \end{aligned}$$

The matrix pencil $\sigma_\gamma[\tilde{\boldsymbol{A}}_1, \tilde{\boldsymbol{A}}_2]$ given by

$$\sigma_\gamma[\tilde{\boldsymbol{A}}_1, \tilde{\boldsymbol{A}}_2] \;=\; \begin{pmatrix} a\,(1+k_1\gamma) & b\,(1+k_2\gamma) \\ c\,(1+k_1\gamma) & d\,(1+k_2\gamma) \end{pmatrix}$$

has eigenvalues outside the 45°-region (and in the left half-plane) if its trace squared is strictly less than twice its determinant.

Therefore,

$$\Big[a+d\;+\;\gamma_{\mathrm{o}}(k_1 a + k_2 d)\Big]^2 \;<\; 2(1+k_1\gamma_{\mathrm{o}})(1+k_1\gamma_{\mathrm{o}})\Delta_1$$

for some values of $\gamma_{\mathrm{o}} > 0$. Basic algebraic manipulation reveals

$$\big(\mathrm{tr}\,_2^2 - 2\Delta_2\big)\gamma_{\mathrm{o}}^2 + 2\big[\mathrm{tr}\,_1\mathrm{tr}\,_2 - (k_1+k_2)\Delta_1\big]\gamma_{\mathrm{o}} + \mathrm{tr}\,_1^2 - 2\Delta_1 \;<\; 0 \qquad (45)$$

Since the eigenvalues of $\tilde{\boldsymbol{A}}_1$ are real, the parabola in terms of γ_{o} is positive for $\gamma = 0$. Hence, there are two positive roots if the matrix pencil has complex eigenvalues outside the 45°-region. Hence,

$$\begin{aligned} \big[\mathrm{tr}\,_1\mathrm{tr}\,_2 - (k_1+k_2)\Delta_1\big]^2 &\;>\; \big(\mathrm{tr}\,_2^2 - 2\Delta_2\big)\big(\mathrm{tr}\,_1^2 - 2\Delta_1\big) \\ \big[k_1\big(a^2+bc\big) + k_2\big(d^2+bc\big)\big]^2 &\;>\; \mathrm{tr}\,_1^2\mathrm{tr}\,_2^2 - 2\Delta_1\mathrm{tr}\,_2^2 - 2\Delta_2\mathrm{tr}\,_1^2 + 4\Delta_1\Delta_2 \end{aligned}$$

Since

$$\mathrm{tr}\,_1^2\mathrm{tr}\,_2^2 - 2\Delta_1\mathrm{tr}\,_2^2 - 2\Delta_2\mathrm{tr}\,_1^2 = \frac{1}{2}\mathrm{tr}\,_2^2\left(\mathrm{tr}\,_1^2 - 4\Delta_1\right) + \frac{1}{2}\mathrm{tr}\,_1^2\left(\mathrm{tr}\,_2^2 - 4\Delta_2\right) > 0$$

we get

$$\begin{aligned} \big[k_1\big(a^2+bc\big) + k_2\big(d^2+bc\big)\big]^2 &\;>\; 4\Delta_1\Delta_2 \\ \mathrm{tr}^{\,2}\left\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\right\} &\;>\; 4\det\left\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\right\}. \end{aligned}$$

Further, both of the roots in inequality (45) are positive. Hence,

$$\begin{aligned} -2\big[k_1\big(a^2+bc\big) + k_2\big(d^2+bc\big)\big] &\;>\; 0 \\ \mathrm{tr}\,\left\{\tilde{\boldsymbol{A}}_1\tilde{\boldsymbol{A}}_2\right\} &\;<\; 0. \end{aligned}$$

Hence, a common quadratic Lyapunov function does not exist, if the matrix-pencils has eigenvalues outside the 45°-region.

Part b: Common unic Lyapunov function

First, we note that if (and only if) $V(\boldsymbol{x})$ is a common Lyapunov function for the second order systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$, then $V(\boldsymbol{x})$ is a Lyapunov function for all systems $\boldsymbol{\Sigma}_{\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1,\boldsymbol{A}_2]}$ for $\alpha \in [0,1]$. This follows from

$$\dot{V}(\boldsymbol{x}) \;=\; \nabla V \dot{\boldsymbol{x}}$$

and therefore

$$\nabla V \Big[\alpha \boldsymbol{A}_1 + \big(1-\alpha\big)\boldsymbol{A}_2\Big]\boldsymbol{x} \;\leq\; 0 \qquad \text{for all} \quad \alpha \in [0,1]$$

if and only if

$$\nabla V \boldsymbol{A}_i \boldsymbol{x} \;\leq\; 0 \qquad i \in \{1,2\}.$$

A geometrical interpretation is given in Figure 10: in part (a) it is illustrated that if $V(\boldsymbol{x})$ is a Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ then $V(\boldsymbol{x})$ is a Lyapunov function for all $\boldsymbol{\Sigma}_{\boldsymbol{P}_\alpha[\boldsymbol{A}_1,\boldsymbol{A}_2]}, \alpha \in [0,1]$. Part (b) shows that if there exists no Lyapunov function of a certain kind for a particular single system $\boldsymbol{\Sigma}_{\boldsymbol{P}_{\alpha_0}}$, $\alpha_0 \in [0,1]$ then any Lyapunov function (of that kind) for $\Sigma_{\boldsymbol{A}_2}$ cannot be a Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ and vice versa.

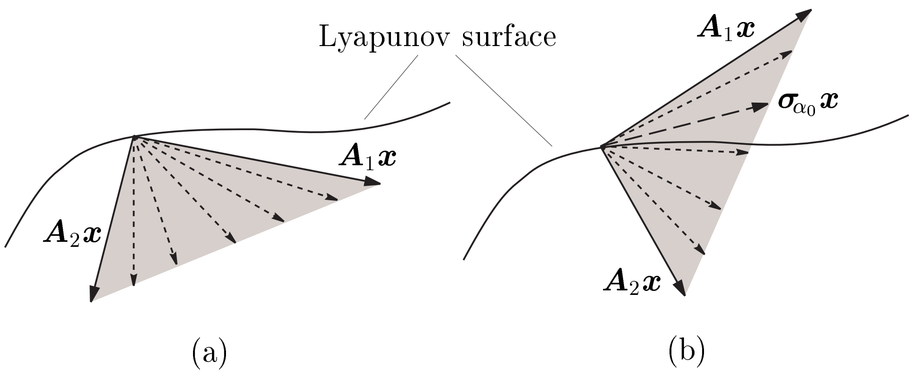

Figure 10: $V(\boldsymbol{x})$ is a common Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ if and only if $V(\boldsymbol{x})$ is a Lyapunov function for $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1,\boldsymbol{A}_2]$ for all $\alpha \in [0,1]$.

The proof for Theorem 3.2a follows immediately: if there is one $\alpha_0 \in [0,1]$ for which no unic Lyapunov function exists for $\boldsymbol{\sigma}_{\alpha_0}[\boldsymbol{A}_1,\boldsymbol{A}_2]$, then the systems $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ cannot have a *common* unic Lyapunov function. According to Theorem 2.4 there exists no unic Lyapunov function for a system $\Sigma_{\boldsymbol{A}}$ if the eigenvalues of $\boldsymbol{A}$ lie outside the 45°-region Ω_{45}. Hence, $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ cannot have a common unic Lyapunov function if the matrix pencil $\boldsymbol{\sigma}_\alpha[\boldsymbol{A}_1,\boldsymbol{A}_2]$ has eigenvalues outside Ω_{45} for some $\alpha \in [0,1]$.

□