Signals and Systems Group: NUIM/SS/2001/03

Necessary and Sufficient .....

Robert N. Shorten and K. S. Narendra

Necessary and Sufficient Conditions For The Existence Of a Common Quadratic Lyapunov Function For a Finite Number of Stable Second Order Linear Time-Invariant Systems

Robert N. Shorten and Kumpati S. Narendra
Center for Systems Science
Yale University

Abstract

In this paper, necessary and sufficient conditions are derived for the existence of a common quadratic Lyapunov function for a finite number of stable second order linear time-invariant systems.

1 Introduction

Recent years have witnessed an enormous growth in interest in dynamic systems that are characterised by a mixture of continuous dynamics and logic based switching. Such systems are referred to in the scientific literature as switching systems or hybrid systems, and are found in many industrial applications. In spite of the vast amount of interest in such systems, many of their fundamental properties are not yet understood. Perhaps the most important unresolved issues pertain to the stability of such systems. In the opinion of the authors, and other researchers in the field, the determination of tractable and interpretable necessary and sufficient conditions for the exponential stability of a system that switches arbitrarily between a number of asymptotically stable time invariant systems in $\mathbb{R}^n$ will have a profound impact on our understanding of the important issues related to the study of hybrid systems. However, at present there appears to be little hope of providing a complete solution to this problem. In view of this, a number of authors are focussing their attention on deriving sufficient conditions under which exponential stability will be assured for special classes of switching systems [1, 2, 3, 4, 5, 6, 7, 8].

In this paper we consider the quadratic stability of switching systems constructed by switching between a finite number of stable linear time-invariant systems. While a numerical solution to this problem using linear matrix inequalities has existed for some time [9], to the best of the author's knowledge, no general theoretical solution has yet been found which can be tested without resorting to numerical optimization methods. In this paper necessary and sufficient conditions are derived for the existence of a common quadratic Lyapunov function (CQLF) for a finite number of stable second order linear time-invariant systems. The conditions presented here are simple and interpretable, and are expressed as algebraic conditions that can be easily tested. Since this paper was first submitted for publication, it has been brought to the attention of the authors that similar conditions for two second order systems were presented in [10]. However, the work presented in this paper differs significantly from that presented in [10] in a number of crucial aspects. Our results are motivated by geometric rather than algebraic arguments. The resulting conditions are valid for not only two stable second order linear systems, but also for any finite number of such systems. The derived conditions are also applicable to the study of higher dimensional systems, as well as providing an exact indication of the relationship between the existence of a CQLF, and the stability of second order switching systems. Most importantly, a complete proof of these results is provided.

The paper is organised as follows. In Section 2, basic definitions are presented and preliminary results are derived which are useful in proving the main result of the paper. Necessary and sufficient conditions for the existence of a CQLF for two stable second order systems are presented in Section 3. These conditions are extended to an arbitrary number of matrices in Section 4, and numerical examples are presented in Section 5 to interpret and illustrate the main features of the proofs. Finally, the relevance of the results for the study of higher order systems is briefly discussed toward the end of the paper.

2 Mathematical Preliminaries

Consider the dynamical system

$$\dot{x} \;=\; A(t)x,\;\; A(t) \in \mathcal{A} \stackrel{\Delta}{=} \{A_1, A_2, ..., A_M\}, \tag{1}$$

where the A_i, $i \in \{1,2,...M\}$, are constant matrices in $\mathbb{R}^{2\times 2}$. The matrices A_i, $i \in \{1,2,...M\}$, are assumed to be Hurwitz (the eigenvalues of each A_i matrix lies in the open left half of the complex plane, denoted $\mathbb{C}^-$). An important problem is to determine necessary and sufficient conditions for the existence of a Lyapunov function $V(x) = x^T Px$, $P = P^T > 0$, $P \in \mathbb{R}^{2\times 2}$, such that its time derivative along any trajectory of the system (1) is negative definite, or alternatively that,

$$A_i^T P + PA_i \;=\; -Q_i,\; i \in \{1,2,...M\}, \tag{2}$$

where the matrices Q_i are symmetric and positive definite. The function $V(x)$ is a common quadratic Lyapunov function (CQLF) for the linear time-invariant (LTI) dynamic systems,

$$\Sigma_{A_i} : \dot{x} \;=\; A_i x,\; i \in \{1,2,...M\}, \tag{3}$$

where $A_i \in \mathbb{R}^{2\times 2}$. The existence of such a Lyapunov function is sufficient to guarantee the asymptotic stability of the switching system (1) (consequently, the CQLF existence problem has been the subject of much recent attention in the switching systems literature [1, 2, 3, 6, 4, 5]).

In this section we present several general results and definitions that are useful in proving the principal results of this paper. Where appropriate, the proofs are presented in the Appendix. We begin with the definition of a matrix pencil.

(i) <u>The matrix pencil $[\alpha A_1 + (1-\alpha)A_2]$:</u> The matrix pencil $\alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,1]$, is used frequently in the sequel. In view of this we shall use the notation

$$\alpha A_1 + (1-\alpha)A_2,\; \alpha \;\in\; [0,1] \;\;\stackrel{\Delta}{=}\;\; \sigma_\alpha[A_1, A_2], \tag{4}$$

and refer to $\sigma_\alpha[A_1, A_2]$ as a matrix pencil. If the eigenvalues of $\sigma_\alpha[A_1, A_2]$ are in $\mathbb{C}^-$ $\forall$ $\alpha \in [0,1]$, then we say that the pencil $\sigma_\alpha[A_1, A_2]$ is Hurwitz for $\alpha \in [0,1]$.

<u>Note :</u> For some discussions in the paper it is easier to consider the matrix $A_1 + \gamma A_2$ where $\gamma \in [0,\infty)$. For ease of exposition in such cases we shall also use the notation

$$A_1 + \gamma A_2,\; \gamma \;\in\; [0,\infty) \;\;\stackrel{\Delta}{=}\;\; \sigma_{\gamma[0,\infty)}[A_1, A_2], \tag{5}$$

and also refer to $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ as a matrix pencil.

(ii) <u>The stability of Σ_A and $\Sigma_{A^{-1}}$ [11, 12]:</u> Consider the linear time invariant systems

$$\begin{aligned} \Sigma_A : \dot{x} \;&=\; Ax, \\ \Sigma_{A^{-1}} : \dot{x} \;&=\; A^{-1}x, \end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$ is Hurwitz. Then, any quadratic Lyapunov function for Σ_A is also a Lyapunov function for $\Sigma_{A^{-1}}$.

(iii) **Lemma 2.1 :** Given two pencils $\sigma_\alpha[A_1, A_2]$, $\sigma_\alpha[A_1, A_2^{-1}]$, where $A_1, A_2 \in \mathbb{R}^{2\times 2}$, it follows that either one, or both, must have real eigenvalues for some $\alpha \in [0,1]$.

(iv) **Lemma 2.2** [13] **:** Let $A_1, A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz matrices, and let $\sigma_\alpha[A_1, A_2]$ be Hurwitz for $\alpha \in [0,1]$, and have real eigenvalues for some $\alpha \in [0,1]$. Then, positive constants d_1, d_2 exist such that the pencil $\sigma_\alpha[B_1, B_2]$ is Hurwitz, or has purely imaginary eigenvalues for $\alpha \in [0,1]$, and is singular for a single value of $\alpha = \alpha_0$, where $B_1 = A_1 + d_1 I$, $B_2 = A_2 + d_2 I$.

Lemma 2.1 and Lemma 2.2 are useful in the context of demonstrating the existence of a CQLF for exactly two stable second order linear systems. The extension of these results to an arbitrary number of systems requires the development of a number of seperate arguments. In this case it is convenient to associate a single second order system with a given quadratic Lyapunov function. This can be carried out as described in Lemma 2.3.

(v) **Lemma 2.3 :** Let $V(x) = x^T P x$ be a positive definite function where

$$P = \begin{bmatrix} 1 & p_{12} \\ p_{12} & p_{22} \end{bmatrix}, \; p_{22} > p_{12}^2 \geq 0.$$

Then, there exists a matrix $Z \in \mathbb{R}^{2\times 2}$,

$$Z = \begin{bmatrix} a & -b \\ c & -a \end{bmatrix}, bc - a^2 > 0,$$

such that

$$Z^T P + PZ = 0.$$

We also note that much of the discussion in Section 4 concerns sets of system matrices, $A_i \in \mathbb{R}^{2\times 2}$, that are Hurwitz, and have the property that $a_{21i} \neq 0$, for all matrices in the set ($i \in \{1, ..., M\}$). Without loss of generality, it is possible to show that, given any set of matrices, $A_i \in \mathbb{R}^{2\times 2}$, $i \in \{1, .., M\}$, a single transformation exists that transforms each of original matrices such that either the $(2,1)$ entry of the matrix A_i, denoted a_{21i}, is non-zero, or that $A_i = k_i I$, $k_i < 0$, for all $i \in \{1, .., M\}$. This is stated formally in Lemma 2.4.

(vi) **Lemma 2.4 :** Let $\mathcal{A} \triangleq \{A_1, ..., A_M\}$, $A_i \in \mathbb{R}^{2\times 2}$. Let $\bar{\mathcal{A}} \triangleq \{\bar{A}_1, ..., \bar{A}_M\}$, where $\bar{A}_i = G A_i G^{-1}$, with

$$G = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}.$$

Then, there exists an $\alpha \in \mathbb{R}$, such that for all $i \in \{1, ..., M\}$,

$$\bar{A}_i = \begin{bmatrix} k_i & 0 \\ 0 & k_i \end{bmatrix}, \quad k_i \in \mathbb{R}, \tag{6}$$

$$\text{or} \quad \bar{A}_i = \begin{bmatrix} \bar{a}_{11i} & \bar{a}_{12i} \\ \bar{a}_{21i} & \bar{a}_{22i} \end{bmatrix}, \tag{7}$$

where $\bar{a}_{21i} \neq 0$.

Finally, we note a well known theorem from convex set theory.

(vii) Helly's Theorem [14] : Let $\mathcal{F}$ be a family of a finite number of convex sets in an n-dimensional Minkowski space L_n, containing at least $n+1$ members. A necessary and sufficient condition that all of the members of $\mathcal{F}$ have a point in common, is that every $n+1$ members of $\mathcal{F}$ have a point in common.

3 Necessary and Sufficient conditions for two stable second order LTI systems

In this Section we derive necessary and sufficient conditions for the existence of a CQLF for two stable second order linear time-invariant systems via simple geometric observations in $\mathbb{R}^2$. While the statement of these results is based upon simple geometric observations, the algebra required for its proof is involved. Hence, for ease of exposition, we begin by presenting an overview of the main ideas involved in developing the main result of this Section, Theorem 3.1.

<u>Preamble to Theorem 3.1</u>

Consider the following switching system:

$$\dot{x} = A(t)x,\ A(t) \in \{A_1, A_2\},\ A_i \in \mathbb{R}^{2\times 2}, \tag{8}$$

where A_1 and A_2 are Hurwitz matrices. Let the matrix pencil $\sigma_\alpha[A_1, A_2]$ satisfy the following properties:

(a) $\sigma_\alpha[A_1, A_2]$ is Hurwitz $\forall\ \alpha \in [0, 1],\ \alpha \neq \alpha_0$;

(b) $\sigma_\alpha[A_1, A_2]$ is singular for $\alpha = \alpha_0$.

Hence, there exists a non-zero point x_0 in the state space , such that

$$\alpha_0 A_1 x_0 = -(1 - \alpha_0) A_2 x_0. \tag{9}$$

It follows using simple arguments (see proof of Theorem 3.1) that x_0 is not an eigenvector of

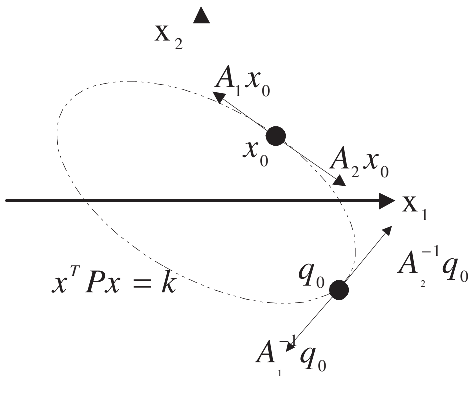


Figure 1: Ellipse constructed in the preamble

A_1 or of A_2, and that

$$\alpha_0 A_2^{-1} q_0 = -(1 - \alpha_0) A_1^{-1} q_0, \tag{10}$$

with $q_0 = A_2 x_0$. Equations (9) and (10) define two lines in $\mathbb{R}^2$. It will be shown in the sequel that it is possible to use Equation (9) (or Equation (10)) to define a family of elliptical curves $x^T P x = k,\ k \in \mathbb{R},\ k > 0, P = P^T$, with the property $(A_1^T P + PA_1)x_0 = 0$ and

$(A_2^TP + PA_2)x_0 = 0$. The lines (9) and (10) are both tangentential to any ellipse in this family, as depicted in in Figure 1. Further, for this P matrix it will be shown that

$$A_1^TP + PA_1 \leq 0, \quad (11)$$
$$A_2^TP + PA_2 \leq 0, \quad (12)$$

where the symbol $\leq$ denotes negative semi-definiteness. Hence, given a switching system, a positive-definite function can be found such that the stability (rather than asymptotic stability) of the original system can be demonstrated.

Surprisingly, the analysis of more general second order linear switching systems can be carried out in a similar manner to that described above. Here we attempt to transform the general system using Lemma 2.1 and Lemma 2.2, into the form considered above, or into a related form. The precise conditions under which such a transformation is possible are summarised as Throrem 3.1.

Theorem 3.1 [13, 10]: A necessary and sufficient condition for the dynamic systems Σ_{A_1} and Σ_{A_2} to have a CQLF is that the pencils $\sigma_\alpha[A_1, A_2]$, $\sigma_\alpha[A_1, A_2^{-1}]$ are both Hurwitz for $\alpha \in [0, 1]$.

Comment 1: The conditions of Theorem 3.1 can be checked by plotting two root-locii, or by checking equivalent algebraic conditions.

Comment 2: Consider $x \in \mathbb{R}^2$. Theorem 3.1 imples that for a CQLF to exist for Σ_{A_1} and Σ_{A_2}, the vectors A_2x, $A_2^{-1}x$, cannot lie in the cone spanned by positive combinations of the vectors A_1x, $A_1^{-1}x$.

Proof of Theorem 3.1

Proof of necessity : The necessity part of the proof follows directly. Let $V(x)$ be a quadratic Lyapunov function for Σ_{A_1} and Σ_{A_2}. Then, it automatically follows that $V(x)$ is a quadraic Lyapunov for Σ_A, where $A = \alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0, 1]$. However, this implies that the system Σ_A is stable, and that the matrix A is Hurwitz. If A is not Hurwitz, for some $\alpha \in [0, 1]$, then a CQLF cannot exist for Σ_{A_1} and Σ_{A_2}. Hence, a necessary condition for the existence of a CQLF for the dynamic systems Σ_{A_1} and Σ_{A_2} is that the pencil $\sigma_\alpha[A_1, A_2]$ is Hurwitz for $\alpha \in [0, 1]$. However, it follows from item (ii) in Section 2, that the existence of a CQLF for Σ_{A_1} and Σ_{A_2} implies the existence of a CQLF for Σ_{A_1} and $\Sigma_{A_2^{-1}}$. Hence, it is also necessary for the matrix pencil $\sigma_\alpha[A_1, A_2^{-1}]$ to be Hurwitz for $\alpha \in [0, 1]$. These conditions are illustrated by means of an Example in Section 5 where $\sigma_\alpha[A_1, A_2]$ is Hurwitz for $\alpha \in [0, 1]$, whereas $\sigma_\alpha[A_1, A_2^{-1}]$ is not.

Proof of sufficiency : To show sufficiency we note the following facts concerning the matrix pencils $\sigma_\alpha[A_1, A_2]$ and $\sigma_\alpha[A_1, A_2^{-1}]$.

1. It follows from Lemma 2.1, that one of the pencils $\sigma_\alpha[A_1, A_2]$ and $\sigma_\alpha[A_1, A_2^{-1}]$ has real eigenvalues for some $\alpha \in [0, 1]$. In the sequel it is assumed, without loss of generality that $\sigma_\alpha[A_1, A_2]$ has real eigenvalues for some $\alpha \in [0, 1]$

2. Let $\sigma_\alpha[A_1, A_2]$ have real eigenvalues for some $\alpha \in [0, 1]$. Then, using Lemma 2.2, positive

constants d_1 and d_2 can be found such that the pencil $\sigma_\alpha[B_1, B_2]$ is Hurwitz, or has purely imaginary eigenvalues for $\alpha \in [0,1]$, $\alpha \neq \alpha_0$, and is singular for $\alpha = \alpha_0$.

For convenience we consider the following two cases separately: Part (A) $\alpha_0 \in (0,1)$; Part (B) $\alpha_0 = 0$ or $\alpha_0 = 1$.

Part (A): $\alpha_0 \in (0,1)$

Let the matrix pencil $\sigma_\alpha[B_1, B_2]$ be Hurwitz for all $\alpha \in [0,1]$, $\alpha \neq \alpha_0$, and be singular for $\alpha = \alpha_0$. Since the matrix $B_1 + \gamma_0 B_2$ ($\gamma_0 = \frac{1-\alpha_0}{\alpha_0}$) is singular, a vector x_0 can be found such that

$$B_1 x_0 = -\gamma_0 B_2 x_0. \tag{13}$$

Let $q \in \mathbb{R}^2$ be a vector such that $q^T x_0 = 0$. Then it follows that:

1. x_0 cannot be an eigenvector of B_1 or of B_2 (since neither matrix has a real eigenvalue that is positive);
2. q cannot be an eigenvector of B_1^T since this would imply that $x_0^T B_1^T q = 0$, which would in turn imply that x_0 is an eigenvector of B_1.

Hence, q and $B_1^T q$ are linearly independent vectors, and the matrix $[q, B_1^T q]$ is of full rank. Further, since B_1 is Hurwitz, or has purely imaginary eigenvalues, it follows from [15] that there exists a positive definite symmetric matrix P which is the solution of the Lyapunov equation

$$B_1^T P + P B_1 \;\; = \;\; -k_1 q q^T, \; k_1 \geq 0 \tag{14}$$

Further, from the definition of B_1,

$$A_1^T P + P A_1 = -2 d_1 P - k_1 q q^T < 0, \tag{15}$$

or that $V(x) = x^T P x$ is a Lyapunov function for the system Σ_{A_1}.

Equation (13) and Equation (14) also imply that

$$\gamma_0 x_0^T \{B_2^T P + P B_2]\} x_0 \;\; = \;\; 0, \tag{16}$$

We then have that $[B_2^T P + P B_2] x_0$ is either zero or is orthogonal to x_0. The consequences of both of these two possibilities are discussed below in cases (i) and (ii).

Case (i): $[B_2^T P + P B_2] x_0 = 0$

$B_2^T P + P B_2$ is a symmetric matrix in $\mathbb{R}^{2\times 2}$ with an eigenvector x_0 corresponding to a zero eigenvalue. This implies that

$$B_2^T P + P B_2 = -k_2 q q^T. \tag{17}$$

If $k_2 < 0$, then the matrix B_2 is unstable ($V(x) = x^T P x$ and $\dot{V}(x)$ along any trajectory is $k_2 (x^T q)^2$) which contradicts the assumption that both B_1 and B_2 are Hurwitz matrics or have purely imaginary eigenvalues. Hence,

$$B_2^T P + P B_2 \;\; = \;\; -k_2 q q^T \leq 0, \; k_2 \geq 0. \tag{18}$$

and $V(x) = x^T P x$ is a CQLF for the systems Σ_{A_1} and Σ_{A_2}.

Case (ii): $[B_2^T P + PB_2]x_0$ is orthogonal to x_0

Let $\gamma_0[B_2^T P + PB_2]x_0$ be orthogonal to x_0. Then, since the non zero vectors $B_1^T Px_0$, PB_1x_0 are also orthogonal to x_0 (from Equation (14)), and all vectors are in $\mathbb{R}^2$, it follows that

$$\begin{aligned}
\gamma_0[B_2^T P + PB_2]x_0 &= kB_1^T Px_0,\ k \neq 0, &(19)\\
[\gamma_0 B_2^T P - PB_1]x_0 &= kB_1^T Px_0, \quad \text{[from equation (13)]} &(20)\\
[\gamma_0 B_2^T P + B_1^T P]x_0 &= kB_1^T Px_0, \quad \text{[from equation (14)]} &(21)\\
[\gamma_0 B_2^T + (1-k)B_1^T]Px_0 &= 0. &(22)
\end{aligned}$$

Let equation (22) be satisfied for some non-zero $k = \bar{k}$, $\bar{k} \in \mathbb{R}$, and let $\frac{\gamma_0}{1-\bar{k}} = \bar{\gamma}$. Since P is a nonsingular matrix, it follows that $[\bar{\gamma}B_2^T + B_1^T]$ is singular (its determinant is zero). Since B_1 and B_2 are Hurwitz, or have purely imaginary eigenvalues, the determinant of $[\gamma B_2^T + B_1^T]$ can be zero for at most two values of γ. Further, since the determinants of B_1 and B_2 are positive, both values of γ have to be greater than or less than zero. However, since it is assumed that $[\gamma_0 B_2^T + B_1^T]$ is singular for some $\gamma_0 > 0$, then $\bar{\gamma} = \frac{\gamma_0}{1-\bar{k}}$ must also be non-negative. This contradicts the assumption that $[\gamma B_2^T + B_1^T]$ is singular only for $\gamma = \gamma_0$ and is Hurwitz, or has purely imaginary eigenvalues, for all $\gamma \neq \gamma_0$, $\gamma > 0$. This implies that $\bar{\gamma} = \gamma_0$, $\bar{k} = 0$, and that Case (ii) reduces to Case (i).

Part (B) $\alpha_0 = 0$ or $\alpha_0 = 1$ (B_1 or B_2 singular) :

Here we develop a slightly different argument to that developed above. We show that the condition that $\alpha_0 = 1$ or $\alpha_0 = 0$ for both pencils imposes very strong conditions on the matrices A_1 and A_2; namely that:

(a) A_1 and A_2 are both upper triangular matrices,

(b) or that A_1 is a general matrix with $A_2 = -kI$, $k > 0$,

(c) or that A_1 is a diagonal matrix and the diagonal entries of A_2 are both negative.

In each of the above cases it follows that a CQLF exists [5, 4]. Hence, together with Part (A), the following discussion completes the proof for Theorem 3.1.

We assume, without loss of generality, that the matrices $\{A_1, A_2, A_1^{-1}, A_2^{-1}\}$ have been labelled such that the maximum real eigenvalue of A_1, denoted in Figure 10 by $-\lambda_{max}(A_1)$, is closer to (or as close to) the imaginary axis than (as) the maximum real eigenvalue of A_1^{-1}, or of A_2 and A_2^{-1} when the matrix A_2 has real eigenvalues. Furthermore, it is also assumed that the maximum real eigenvalue of $\sigma_\alpha[A_1, A_2]$, denoted $-d_m^1$, and the maximum real eigenvalue of $\sigma_\alpha[A_1, A_2^{-1}]$, denoted $-d_m^2$, are bounded above by $-\lambda_{max}(A_1)$. If this were not the case, i.e. if $d_m^{1,2} > \lambda_{max}(A_1)$, then at least one of the pencils, $\sigma_\alpha[A_1 + d_m^1 I, A_2 + d_m^1 I]$, $\sigma_\alpha[A_1 + d_m^2 I, A_2^{-1} + d_m^2 I]$, would satisfy the condition in case (a) above that $\alpha_0 \in (0, 1)$. Hence, both of the pencils

$$\sigma_\alpha[B_1, B_2] = \alpha(A_1 + d_m I) + (1-\alpha)(A_2 + d_2 I), \alpha \in [0, 1], \tag{23}$$

$$\sigma_\beta[B_1, B_3] = \beta(A_1 + d_m I) + (1-\beta)(A_2^{-1} + d_4 I), \beta \in [0, 1], \tag{24}$$

where $B_3 = A_2^{-1} + d_4 I$, and where $d_m = \lambda_{max}(A_1)$, $d_2 \leq \lambda_{max}(A_1)$, $d_4 \leq \lambda_{max}(A_1)$, $d_m > 0, d_2 > 0, d_4 > 0$, satisfy Lemma 2.2 and are singular for $\alpha_0 = 1$ and $\beta_0 = 1$ respectively.

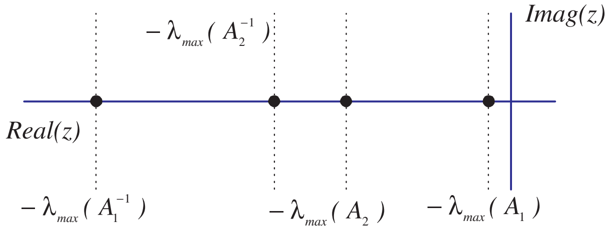


Figure 2: $\alpha_0 = 1$

Equations (23) and (24) imply that the pencils

$$B_1 + \theta B_2,\ \theta = \frac{1-\alpha}{\alpha},\ \alpha \in (0,1], \tag{25}$$

$$B_1 + \lambda B_3,\ \lambda = \frac{1-\beta}{\beta},\ \beta \in (0,1], \tag{26}$$

are singular for $\theta = 0$ and for $\lambda = 0$, and cannot be singular for a positive θ or for a positive λ.

Again, two cases need to be considered; (i) the case where B_1 has two eigenvalues at the origin; (ii) the case where B_1 has a single eigenvalue at the origin (these correspond to the statements (b) and (c) made earlier).

Case (i):

Let B_1 have two eigenvalues at the origin. Hence, we can assume that B_1 and A_2 are of the form:

$$B_1 = \begin{bmatrix} 0 & K \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, K \geq 0.$$

Straightforward algebraic manipulation reveals that $B_1 + \theta B_2$ and $B_1 + \lambda B_3$ are singular for the following values of the constants θ and λ:

$$\theta_0 = 0, \theta_0 \quad = \quad \frac{Ka_{21}}{(a_{11}+d_2)(a_{22}+d_2) - a_{12}a_{21}}, \tag{27}$$

$$\bar{\lambda}_0 = 0, \bar{\lambda}_0 \quad = \quad \frac{-Ka_{21}}{(a_{11}+Det\{A_2\}d_4)(a_{22}+Det\{A_2\}d_4) - a_{12}a_{21}}. \tag{28}$$

where $\bar{\lambda} = \frac{\lambda}{Det\{A_2\}}$. The constants d_2, d_4, are chosen such that the matrices $A_2 + d_2 I$, and $A_2^{-1} + d_4 I$ are both Hurwitz. Hence,

$$(a_{11}+d_2)(a_{22}+d_2) - a_{12}a_{21} \quad > \quad 0, \tag{29}$$

$$(a_{11}+Det\{A_2\}d_4)(a_{22}+Det\{A_2\}d_4) - a_{12}a_{21} \quad > \quad 0. \tag{30}$$

Therefore, one of the singular points occurs for some positive λ or θ. However, this contradicts the assumption that both pencils $\sigma_\alpha[B_1, B_2]$, $\sigma_\alpha[B_1, B_3]$ are non-singular for all $\alpha \in [0, 1)$.

Hence, either $K = 0$, $a_{21} = 0$, or both. The former condition, $K = 0$, implies that $A_1 = -\kappa I$, $\kappa > 0$. The existence of a CQLF for Σ_{A_1} and Σ_{A_2} follows using simple arguments. The condition that $a_{21} = 0$ implies that both B_1 and B_2, and hence A_1 and A_2, are upper triangular matrices. Therefore a CQLF exists from the arguments presented in [3, 5, 6].

Case (ii):

Let B_1 have a single eigenvalue at the origin. We assume that B_1 and A_2 are of the form:

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\mu \end{bmatrix}, A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \mu > 0.$$

Again, straight-forward algebraic manipulation reveals

$$\theta_0 = 0, \theta_0 = \frac{\mu(a_{11} + d_2)}{(a_{11} + d_2)(a_{22} + d_2) - a_{12}a_{21}}, \tag{31}$$

$$\bar{\lambda}_0 = 0, \bar{\lambda}_0 = \frac{\mu(a_{22} + Det\{A_2\}d_4)}{(a_{11} + Det\{A_2\}d_4)(a_{22} + Det\{A_2\}d_4) - a_{12}a_{21}}. \tag{32}$$

where $\bar{\lambda} = \frac{\lambda}{Det\{A_2\}}$. Using the same arguments as in (i) it can be deduced that a_{11} and a_{22} cannot be of opposite signs. Hence, a_{11} and a_{22} are both negative. This case was considered in [4] and a CQLF exists for Σ_{A_1} and Σ_{A_2} (a diagonal P matrix exists when A_1 is diagonal and A_2 is a matrix where a_{11} and a_{22} are both negative). Similar arguments can be developed for the case when B_1 is of the form,

$$B_1 = \begin{bmatrix} -\mu & 0 \\ 0 & 0 \end{bmatrix}, \mu > 0.$$

3.1 The consequences of the non-existence of a CQLF

Theorem 3.1 provides remarkable insights into the relationship between the non-existence of a CQLF for a pair of stable second order LTI systems, and the instability of an associated switching system. We note the following result.

Theorem 3.2 [16, 4, 17]: A sufficient condition for the existence of a switching sequence, such that the system (1) is unstable, is that the there exists non-negative constants $\{\alpha_1, \alpha_2, ..., \alpha_M\}$, such that the matrix $\sum_{i=1}^{M} \alpha_i A_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{M} \alpha_i > 0$, has an eigenvalue with a positive real part.

It follows from the arguments in Theorem 3.1 that the transition from existence to non-existence of a CQLF for a pair of second order LTI systems, corresponds to the transition from stability to instability of one of the dual switching systems

$$\dot{x} = A(t)x, \quad A(t) \triangleq \{A_1, A_2\}, \tag{33}$$

$$\dot{x} = A(t)x, \quad A(t) \triangleq \{A_1, A_2^{-1}\}. \tag{34}$$

Although this observation is not true for $M > 2$ systems [18], it is somewhat surprising since it implies a type of equivalence between the stability problem for (1), and the CQLF existence problem for simple second order switching systems. This is contrary to the commonly held view, even for simple second order systems, that the existence of a CQLF places unnecessarily conservative restrictions on the design of switching systems.

4 Necessary and Sufficient conditions for a finite number stable second order LTI systems

Despite the elegance of the conditions presented in Section 3, their usefulness is limited by the fact that they are valid for exactly two stable second order linear systems. Given more than two systems, the existence of a CQLF for each pair of systems, is generally not sufficient to guarantee the existence of a CQLF for all of the systems. This fact is illustrated by the following example.

Example 4.1: Consider the following stable LTI systems,

$$\Sigma_{A_i} : \dot{x} = A_i x, \ A_i \in \mathrm{I\!R}^{2\times 2},$$

with,

$$A_1 \approx \begin{bmatrix} -1.0000 & 0.0998 \\ -0.9982 & 0.0980 \end{bmatrix}, \quad A_2 \approx \begin{bmatrix} -1.00000 & -0.0995 \\ -0.9945 & -0.1049 \end{bmatrix}, \quad A_3 \approx \begin{bmatrix} -1.0000 & -0.0818 \\ -8.1818 & -1 \end{bmatrix}.$$

The set of matrices for which $A_i^T P + P A_i < 0, \ P = P^T > 0, \ P \in \mathrm{I\!R}^{2\times 2}$, A_i Hurwitz, is given by

$$\mathcal{P}_{A_i} : det\{A_i^T P + P A_i\} > 0, \tag{35}$$

where

$$P = \begin{bmatrix} 1 & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

Equation (35) defines a two-dimensional convex set in (p_{12}, p_{22}) space. These sets are depicted graphically for each of the systems $\Sigma_{A_1}, \Sigma_{A_2}$, and Σ_{A_3}, as the interior of the ellipses in Figure 1. Clearly, no CQLF exists as the sets $\mathcal{P}_{A_i}, \ i \in [1, 2, 3]$, have no common intersection [19].

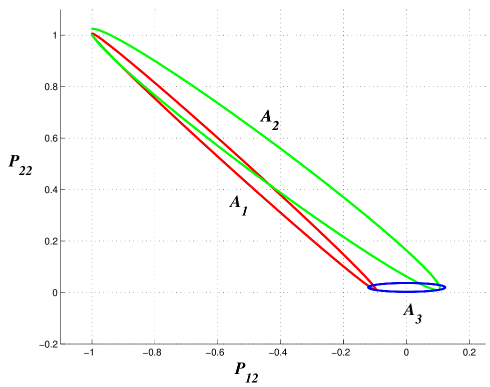


Figure 3: $\mathcal{P}_{A_i}$ for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

However, the dynamic systems $\{\Sigma_{A_1}, \Sigma_{A_2}\}$, $\{\Sigma_{A_2}, \Sigma_{A_3}\}$, $\{\Sigma_{A_1}, \Sigma_{A_3}\}$ each have a CQLF[1].

Example 4.1 illustrates that the solution of the CQLF existence problem for two stable second order LTI systems does not yield a complete solution to the CQLF existence problem for a finite number of second order LTI systems. From a mathematical viewpoint, this is not surprising since it is possible for a number of convex sets to intersect pairwise, without the existence of a point of intersection common to all of the sets. Fortunately, convex sets have been widely studied in the mathematics literature, and several mathematical results are of use in our context. Among these, Helly's theorem (see item (vii) in Section 2) is of most interest. We are concerned with two-dimensional convex sets defined in a normed linear space. Hence, from Helly's theorem, the interesction of a finite number of such sets is guaranteed provided that every 3-tuple of sets intersect. Hence, deriving conditions that guarantee the existence of a CQLF for <u>three</u> stable second order systems, constitutes a solution to the CQLF existence problem for an arbitrary number of systems. Full details of the solution to the CQLF existence problem for an arbitrary number of stable second order LTI systems can be found in [18]. Here, we present an outline of the work presented therein. As in Section 3, the problem of determining the existence of a CQLF for three stable second order LTI systems is made tractable via key geometric observations in $\mathbb{R}^2$. We note that the quadratic function $V(x) = x^T P x$, $P = P^T > 0$ can be parameterised indirectly via the dynamic system

$$\dot{x} = Zx, \; Z \in \mathbb{R}^{2\times 2}, \tag{36}$$

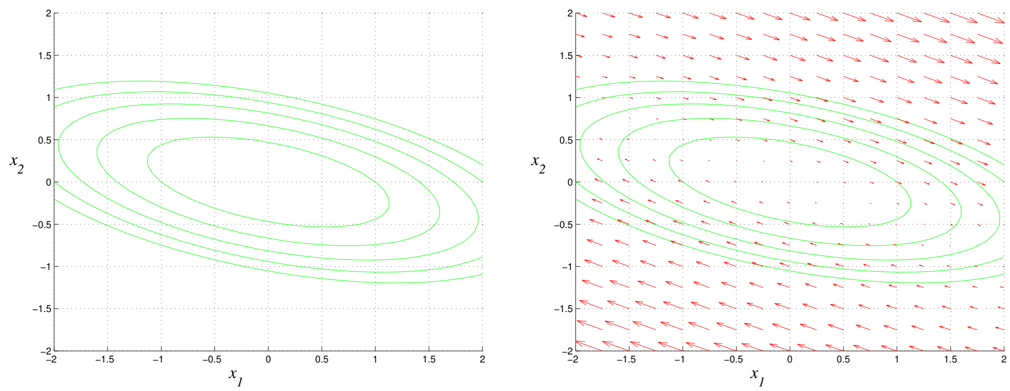
where the matrix Z is defined in Lemma 2.3, and $Z^T P + PZ = 0$. The equivalence between the Lyapunov function $V(x)$ and the dynamic system (36) is illustrated for a specific example in Figure (4). This unusual parameterisation of a positive definite function provides a basis for a simple solution to the CQLF existence problem for three stable second order LTI systems. In the remainder of this Section we present an overview of the key ideas in the evolution the results in [18]. We begin by reviewing a number of key facts concerning matrices in $\mathbb{R}^{2\times 2}$, convex sets, and quadratic Lyapunov functions for stable LTI systems. As in Section 2, the proofs of individual statements are provided in the Appendix where appropriate.

(a) Given any set of 2×2 matrices, $\mathcal{A}$, it follows from Lemma 2.4, that a similarity transformation exists such that $\bar{A}_i = GA_iG^{-1}$, $A_i \in \mathcal{A}$, $i \in \{1, ..., M\}$, is of the form of Equations (6) or (7).

  Let $\bar{\mathcal{A}}_I$ denote the set of matrices in $\bar{\mathcal{A}}$ that are of the form of Equation (6) (with $k_i < 0$ since the A_i are assumed to be Hurwitz). Let $\bar{\mathcal{A}}_a$ denote the set of matrices in $\bar{\mathcal{A}}$ that are of the form of Equation (7). For each $\bar{A}_i \in \bar{\mathcal{A}}_I$, it follows that any $V(x) = x^T P x$, $P = P^T > 0$, $P \in \mathbb{R}^{2\times 2}$, is a Lyapunov function for $\Sigma_{\bar{A}_i}$ ($\dot{x} = k_i I x$). Hence, the existence of a CQLF for $\Sigma_{\bar{A}_i}$, $\bar{A}_i \in \bar{\mathcal{A}}$, is shown by demonstrating the existence of a CQLF when $\bar{A}_i \in \bar{\mathcal{A}}_a$. In view of these facts, we assume in the remainder of this paper, that the matrices $A_i \in \mathcal{A}$, are of the form of $\bar{A}_i$ in Equation (7) with $\bar{a}_{21i} \neq 0$.

(b) Let $A \in \mathbb{R}^{2\times 2}$ be a Hurwitz matrix with $a_{21} \neq 0$. Then, the locus of points (p_{12}, p_{22}), defined by $det\{A^T P + PA\} = 0$, $P = P^T > 0$, describes an ellipse.

[1] Each pair of system matrices can be simultaneously triangularized. Hence, in general, for stable LTI systems, pairwise triangulizability of the system matrices does not imply the existence of a CQLF for all of the systems. Full details are given in [20].



(a) The level curves of the function $V(x) = x^T P x$, with $V(x) = k$, $k = \{1, 2, 3, 4, 5\}$.

(b) The flow associated with the system $\dot{x} = Zx$, and the level curves of the function $V(x) = k$, $k = \{1, 2, 3, 4, 5\}$.

Figure 4:

The set of symmetric positive definite matrices P, for which $det\{A^T P + PA\} = 0$, is denoted $\mathcal{Z}_{\mathcal{A}}$. We refer to the set of points, (p_{12}, p_{22}), for which $det\{A^T P + PA\} = 0$, $P = P^T > 0$, as the ellipse $\mathcal{E}_A$.

The interior of the ellipse, $det\{A^T P + PA\} = 0$, defines an open convex set for which $det\{A^T P + PA\} > 0$. The set of symmetric positive definite matrices P for which this inequality is true, is referred to, in this paper, as the Lyapunov set associated with A, and is denoted $\mathcal{P}_A$.

(c) Let A_1, $A_2 \in \mathbb{R}^{2\times 2}$ be Hurwitz. Furthermore, without loss of generality, let A_1 and A_2 be of the form of $\bar{A}_i$ in Lemma 2.4 with $a_{21i} \neq 0$, and $\bar{A}_i \neq -kI$, $k \in \mathbb{R}, k > 0$, $i \in \{1, 2\}$ (see item (a)). Let $\mathcal{P}_{A_1}$ denote the set of positive definite matrices for which $det\{A_1^T P_1 + P_1 A_1\} > 0$ and $\mathcal{P}_{A_2}$ denote the set of positive definite matrices for which $det\{A_2^T P_2 + P_2 A_2\} > 0$. Determining the existence of a CQLF is equivalent to showing that the set $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2} \neq \emptyset$. We distinguish the following two cases when $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2} \neq \emptyset$.

(a) $\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} \neq \emptyset$, $i, j \in \{1, 2\}$, $i \neq j$.

(b) $\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} = \emptyset$, $i, j \in \{1, 2\}$, $i \neq j$.

Figure 5: Intersection of Lyapunov sets

(i) $\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} \neq \emptyset$, $i, j \in \{1, 2\}$, $i \neq j$: In this case, the ellipse $det\{A_1^T P + PA_1\} = 0$, and the ellipse $det\{A_2^T P + PA_2\} = 0$, intersect. This implies that there exists at least one $P(\in \mathcal{Z}_{A_1} \cap \mathcal{Z}_{A_2})$ matrix, such that $A_1^T P + PA_1$ and $A_2^T P + PA_2$ are simultaneously negative semi-definite.

(ii) $\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} = \emptyset,\ i,j \in \{1,2\},\ \ i \neq j$: In this case, the ellipse $det\{A_1^T P + PA_1\} = 0$, and the ellipse $det\{A_2^T P + PA_2\} = 0$ do not intersect.

(d) The set $\mathcal{P}_A$ is convex. Furthermore, for the class of problem considered in this paper, the set of matrices $\mathcal{P}_A$, can be represented as an open set in R^2 which is bounded by the ellipse $\mathcal{E}_A$. Hence, from Helly's theorem, determining the existence of a CQLF for the dynamic systems $\{\Sigma_{A_1}, ..., \Sigma_{A_M}\}$, $A_i \in \mathbb{R}^{2\times 2}$, is equivalent to ensuring that every 3-tuple of systems, $\{\Sigma_{A_i}, \Sigma_{A_j}, \Sigma_{A_k}\},\ \ i,j,k \in \{1,...,M\},\ \ i \neq j \neq k$, has a CQLF.

(e) Given two Hurwitz matrices, A_1, A_2, it is of interest in this paper to determine symmetric positive definite matrices P_i for which

$$\begin{aligned} A_1^T P_i + P_i A_1 &\leq 0, \\ A_2^T P_i + P_i A_2 &\leq 0. \end{aligned}$$

The existence of such P_i matrices can be determined algebraically by solving

$$det\{A_1 + \lambda Z\} = 0, \quad \text{and} \tag{37}$$

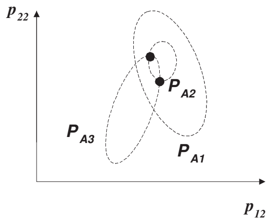
$$det\{A_2 + \gamma Z\} = 0, \tag{38}$$

simultaneously for exactly one $\lambda \in \mathbb{R}$ and exactly one $\gamma \in \mathbb{R}$, where Z is defined in Lemma 2.3. In practice this reduces to determining the existence of solutions to the above equations for either $a = 0$ or $a = 1$ (the parameter in the matrix Z). Note that Equations (37) and (38) follow directly from the following argument. Let P exist such that $A_1^T P + PA_1 \leq 0,\ P = P^T > 0$. This implies that for a Hurwitz matrix A_1, there exists exactly one vector $x_1 \in \mathbb{R}^2,\ x_1 \neq 0$ such that $g^T PA_1 g = 0$, with $g = kx_1,\ \ k \in \mathbb{R}, k \neq 0$. Let $x_1^T PA_1 x_1 = 0$. Further, it follows that for all x, $x^T PZx = 0$. Since, P, A_1, and Z are non-singular (i.e. $Px_1 \neq 0,\ \ Ax_1 \neq 0,\ \ Zx_1 \neq 0$), and since all vectors are in $\mathbb{R}^2$, it follows that,

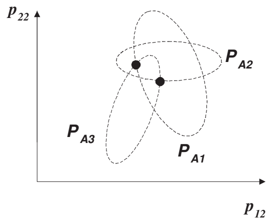
$$\begin{aligned} Ax_1 &= -\lambda Z x_1,\ \ \lambda \in \mathbb{R}, \\ \Rightarrow det\{A_1 + \lambda Z\} &= 0. \end{aligned}$$

Equation (38) can be derived in an identical fashion. Once the Z matrices have been determined, the associated P matrices can be deduced. For a given pair of matrices, the existence of a simultaneous solution to Equations (37) and (38) can be determined in a simple manner using elementary algebra.

Observations (a)-(e) lead directly to necessary and sufficient conditions for a CQLF for a finite number of stable second order LTI systems ($\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2} \cap \mathcal{P}_{A_3} \neq \emptyset$). Consider the following two distinct cases.



(a) Case (i)



(b) Case (ii)

Figure 6: Intersection of Lyapunov sets

Case (i) : Let the dynamic systems Σ_{A_1}, Σ_{A_2}, and Σ_{A_3}, satisfy the conditions of Theorem 3.1 pairwise. Furthermore, let at least one of the following be true,

$$\begin{aligned}\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} &= \emptyset, \ i,j \in \{1,2\}, i \neq j,\\ \mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} &= \emptyset, \ i,j \in \{1,3\}, i \neq j,\\ \mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} &= \emptyset, \ i,j \in \{2,3\}, i \neq j.\end{aligned}$$

Without loss of generality, let $\mathcal{E}_{A_2} \cap \mathcal{E}_{A_1} = \emptyset$ as depicted in Figure 3 (a). In this case every $V(x) = x^T P x$ that is a Lyapunov function for Σ_{A_2} is also a Lyapunov function for Σ_{A_1} ($\mathcal{P}_{A_2} \subset \mathcal{P}_{A_1}$). The existence of a CQLF for all three systems is therefore implied by the existence of a CQLF for Σ_{A_2} and Σ_{A_3}. Hence, a necessary and sufficent condition for the existence of a CQLF for Σ_{A_1}, Σ_{A_2}, and Σ_{A_3} in this case, is that the systems satisfy the conditions of Theorem 3.1 pairwise.

Case (ii) : Let the dynamic systems Σ_{A_1}, Σ_{A_2}, and Σ_{A_3}, satisfy the conditions of Theorem 3.1 pairwise. Furthermore, let all of the following be true

$$\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} \neq \emptyset, \ i,j \in \{1,2\}, i \neq j, \tag{39}$$
$$\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} \neq \emptyset, \ i,j \in \{1,3\}, i \neq j, \tag{40}$$
$$\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} \neq \emptyset, \ i,j \in \{2,3\}, i \neq j. \tag{41}$$

Consider the set ($\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}$). Since (39), (40) and (41), are true, it follows that the set $(\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}) \cap \mathcal{P}_{A_3}$ is non empty provided that the ellipse $\mathcal{E}_{A_3}$ encloses the open region that is common to the interior of the ellipse $\mathcal{E}_{A_1}$ and the interior of the ellipse $\mathcal{E}_{A_2}$, or that the ellipse $\mathcal{E}_{A_3}$ intersects the open region common to the interiors of $\mathcal{E}_{A_1}$ and $\mathcal{E}_{A_2}$ (corresponding to the set $\mathcal{P}_{A_1} \cap \mathcal{P}_{A_2}$). This implies the existence of at least two distinct symmetric positive definite matrices P such that the matrices $-Q_1 = A_1^T P + P A_1$, $-Q_2 = A_2^T P + P A_2$, $-Q_3 = A_3^T P + P A_3$ are simultaneously, either negative definite, or negative semi-definite.

Both of these observations are summarised formally in Lemma 4.1 and Lemma 4.2.

Lemma 4.1 [19]: Let Σ_{A_1}, Σ_{A_2}, and Σ_{A_3} be stable second order LTI systems with $a_{21i} \neq 0, \ i \in \{1,2,3\}$. Let $\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} = \emptyset$, for some $i,j \in \{1,2,3\}, \ i \neq j$. Then, a necessary and sufficient condition for the existence of a CQLF for all three systems is that the systems satisfy pairwise the conditions of Theorem 3.1.

Lemma 4.2 [19]: Let Σ_{A_1}, Σ_{A_2}, and Σ_{A_3} be stable second order LTI systems that pairwise satisfy the conditions of Theorem 3.1, with $a_{21i} \neq 0, \ i \in \{1,2,3\}$. Let $\mathcal{E}_{A_i} \cap \mathcal{E}_{A_j} \neq \emptyset, \ \forall \ i,j \in$

$\{1,2,3\}$, $\;i \neq j$. Let $\mathcal{Z}_{ij} = \mathcal{Z}_{A_i} \cap \mathcal{Z}_{A_j}$ denote the set of positive definite matrices P for which

$$\begin{aligned} A_i^T P + P A_i &= -Q_i \leq 0, \\ A_j^T P + P A_j &= -Q_j \leq 0, \end{aligned}$$

where $i,j \in \{1,2,3\}$, $\;i \neq j$. A necessary and sufficient condition for a CQLF to exist for Σ_{A_1}, Σ_{A_2}, and Σ_{A_3}, is that there exists at least two positive definite matrices, $P \in (\mathcal{Z}_{12} \cup \mathcal{Z}_{13} \cup \mathcal{Z}_{23})$, such that the matrices

$$\begin{aligned} A_1^T P + P A_1 &= -Q_1, \\ A_2^T P + P A_2 &= -Q_2, \\ A_3^T P + P A_3 &= -Q_3, \end{aligned}$$

are simultaneously, either negative definite, or negative semi-definite.

Lemmas 4.1 and 4.2 lead directly to the main results of this section.

Theorem 4.1 [19]: Let Σ_{A_1}, Σ_{A_2}, and Σ_{A_3} be stable second order LTI systems that pairwise satisfy the conditions of Theorem 3.1. Let $a_{21i} \neq 0$, $\; i \in \{1,2,3\}$. A necessary and sufficient condition for the existence of a CQLF is that the systems either satisfy the conditions specified in Lemma 4.1, or those specified Lemma 4.2.

Comment 3: Given two stable second order LTI systems that satisfy Theorem 3.1, Σ_{A_1} and Σ_{A_2}, it is readily established, from the details of Item (e) above, if $\mathcal{E}_{A_i} \cap \; \mathcal{E}_{A_j} = \emptyset$ for $i,j \in \{1,2\}, i \neq j$.

For second order systems, Theorem 4.1 provides the key to solving the general CQLF problem. From Helly's theorem, given any finite number of convex sets in $\mathbb{R}^2$, it follows that a common intersection exists, provided that every 3-tuple of sets has an intersection.

Theorem 4.2 [19] : Let Σ_{A_i}, $i \in \{1,...,M\}$, be stable second order LTI systems with $a_{21i} \neq 0$. A necessary and sufficient condition for the existence of a CQLF for Σ_{A_i}, $\; i \in \{1,...,M\}$, is that a CQLF exists for every 3-tuple of systems, $(\Sigma_{A_i}, \Sigma_{A_j}, \Sigma_{A_k})$, $\quad i \neq j \neq k$, for all $i,j,k \in \{1,...,M\}$.

Finally, we note that Theorem 4.1 and Theorem 4.2 can be summarised in matrix pencil form.

Theorem 4.3 [19] : A necessary and sufficient condition for the esistence of a CQLF for $\{\Sigma_{A_1}, \Sigma_{A_2}, ..., \Sigma_{A_M}\}$ is that every 3-tuple of systems $\{\Sigma_{A_i}, \Sigma_{A_i}, \Sigma_{A_i}\}$, $i,j,k \in \{1,...,M\}$, has a CQLF. A necessary and sufficient condition for the dynamic systems Σ_{A_i}, Σ_{A_j}, Σ_{A_k} to have a CQLF is that there exists a matrix Z, as defined in Lemma 2.3, such that

$$\begin{aligned} det\{A_i + \lambda Z\} &\neq 0, \\ det\{A_j + \gamma Z\} &\neq 0, \\ det\{A_k + \delta Z\} &\neq 0, \end{aligned}$$

for all $\lambda \in \mathbb{R}$, $\;\gamma \in \mathbb{R}, \delta \in \mathbb{R}$.

5 Numerical examples

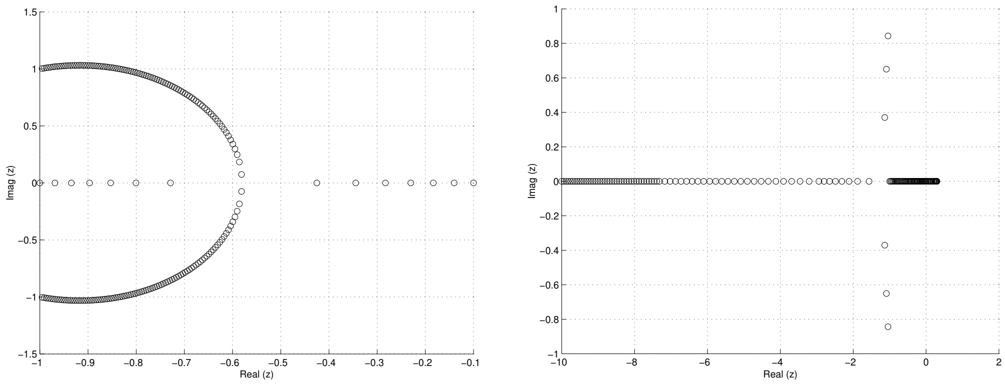
In this section several examples are presented that illustrate some of the main results discussed in the paper; namely, necessary and sufficient conditions for the existence of a CQLF for two stable second order LTI systems, and necessary and sufficient conditions for the existence of a CQLF for three stable second order LTI systems. The second of these results is sufficient to establish the existence of a CQLF for any finite number of stable second order LTI systems. Where appropriate, numerical values are given to four decimal places.

Example 5.1: The pencil $\sigma_\alpha[A_1, A_2]$ is Hurwitz for $\alpha \in [0,1]$, whereas $\sigma_\alpha[A_1, A_2^{-1}]$ is not.

Let

$$A_1 = \begin{bmatrix} 2 & -100 \\ 0.1 & -4 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

The root-locii associated with $\sigma_\alpha[A_1, A_2]$ and with $\sigma_\alpha[A_1, A_2^{-1}]$ are both depicted in Figure 7. $\sigma_\alpha[A_1, A_2]$ is Hurwitz for $\alpha \in [0,1]$, whereas $\sigma_\alpha[A_1, A_2^{-1}]$ is not. This implies, by Theorem 3.1, that a CQLF does not exist for the dynamic systems Σ_{A_1} and Σ_{A_2}.



(a) $\sigma_\alpha[A_1, A_2]$ (b) $\sigma_\alpha[A_1, A_2^{-1}]$

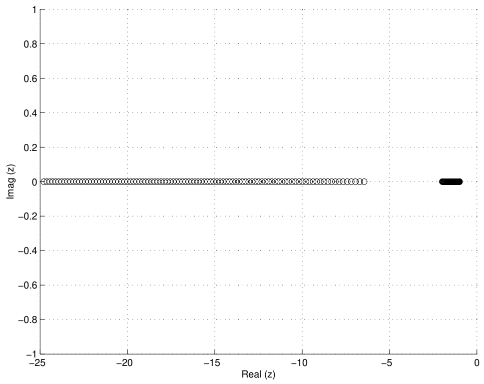
Figure 7: Root-locii: Example 5.1

Example 5.2: The pencils $\sigma_\alpha[A_1, A_2]$, $\sigma_\alpha[A_1, A_2^{-1}]$ are both Hurwitz for $\alpha \in [0,1]$.

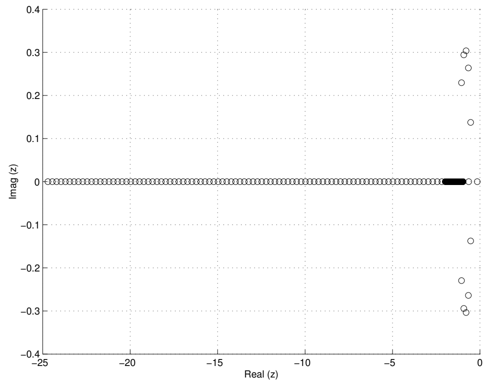
Let

$$A_1 = \begin{bmatrix} -2.0000 & 0 \\ 0 & -24.9545 \end{bmatrix}, \text{ and } A_2 = \begin{bmatrix} 0 & -2 \\ 5 & -8 \end{bmatrix}.$$

The root-locii associated with $\sigma_\alpha[A_1, A_2]$ and with $\sigma_\alpha[A_1, A_2^{-1}]$ are both depicted in Figure 8. Both pencils are Hurwitz for $\alpha \in [0,1]$, thereby guaranteeing the existence of a CQLF for Σ_{A_1} and Σ_{A_2}. In addition, by proceeding as in the proof of Theorem 3.1, it can be shown that



(a) $\sigma_\alpha[A_1, A_2]$



(b) $\sigma_\alpha[A_1, A_2^{-1}]$

Figure 8: Root-locii: Example 5.2

$\sigma_\alpha[A_1 + I, A_2 + I]$ satisfies Lemma 2.1, and is singular for $\alpha_0 = 0.2614$. For this value of α_0, the matrix $\alpha_0 B_1 + (1-\alpha_0)B_2$ is singular, the vector $x_0 \approx [0.9516 \;\; 0.3074]^T$ lies in kernel of $\alpha_0 B_1 + (1-\alpha_0)B_2$, and a vector $q \approx [-1.0000 \;\; 3.0956]^T$ exists such that $q^T x_0 = 0$.

Furthermore, a symmetric positive definite matrix P exists which is the solution to the matrix equations

$$[A_1 + I]^T P + P[A_1 + I] = -qq^T, \tag{42}$$

$$[A_2 + I]^T P + P[A_2 + I] = -kqq^T, \tag{43}$$

where

$$-qq^T \approx \begin{bmatrix} -1.0000 & 3.0956 \\ 3.0956 & -9.5825 \end{bmatrix}. \tag{44}$$

Equations (42) and (43) can be solved for for P and k. This yields a positive definite matrix P given by

$$P \approx \begin{bmatrix} 0.5000 & -0.1240 \\ -0.1240 & 0.2000 \end{bmatrix}, \tag{45}$$

and $k \approx 0.2404$. Note that the matrix $kB_1 - B_2$ has purely imaginary eigenvalues for $k \approx 0.2404$. In this case the matrix P satisfies Lemma 2.3 with $Z = kB_1 - B_2$. It follows that,

$$A_1^T P + PA_1 = -Q_1, \tag{46}$$

$$A_2^T P + PA_2 = -Q_2, \tag{47}$$

where

$$Q_1 \approx \begin{bmatrix} 2.0000 & -3.3437 \\ -3.3437 & 9.9825 \end{bmatrix} > 0, \tag{48}$$

$$Q_2 \approx \begin{bmatrix} 1.2405 & -0.9925 \\ -0.9925 & 2.7040 \end{bmatrix} > 0. \tag{49}$$

Example 5.3 : Three systems, CQLF exists.

Consider the following stable second order LTI systems Σ_{A_1}, Σ_{A_2} and Σ_{A_3} where,

$$A_1 = \begin{bmatrix} 0 & 5 \\ -30 & -1.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 5 \\ -26 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -6 & 27 \\ -150 & -1 \end{bmatrix}.$$

Each pair of systems satisfy Theorem 3.1. Hence, for any pair of systems, a CQLF exists.

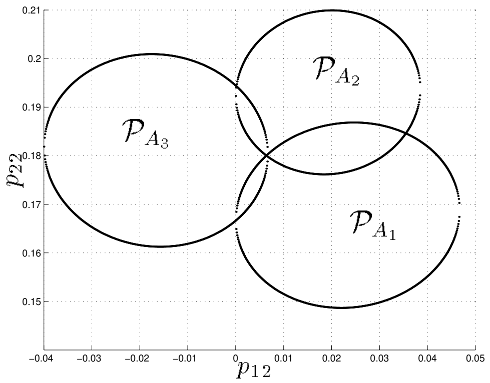


Figure 9: Lyapunov sets for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

From Figure 9 it can be observed that $\mathcal{P}_{A_i} \not\subset \mathcal{P}_{A_j}$, for all $i, j \in \{1,2,3\}$, $i \neq j$. Using the notation of Theorem 3.2, let $\mathcal{Z}_{12} = \{P_{121}, P_{122}\}$, $\mathcal{Z}_{13} = \{P_{131}, P_{132}\}$, $\mathcal{Z}_{23} = \{P_{231}, P_{232}\}$. The following matrices are determined by solving simultaneously Equations (37) and (38) for a symmetric positive definite matrix P,

$$P_{121} \approx \begin{bmatrix} 1.00000 & 0.00635 \\ 0.00635 & 0.18002 \end{bmatrix}, \quad P_{122} \approx \begin{bmatrix} 1.00000 & 0.03544 \\ 0.03544 & 0.18460 \end{bmatrix},$$
$$P_{131} \approx \begin{bmatrix} 1.00000 & 0.00667 \\ 0.00667 & 0.18031 \end{bmatrix}, \quad P_{132} \approx \begin{bmatrix} 1.00000 & 0.00000 \\ 0.00000 & 0.16667 \end{bmatrix},$$
$$P_{231} \approx \begin{bmatrix} 1.00000 & 0.00013 \\ 0.00013 & 0.19428 \end{bmatrix}, \quad P_{232} \approx \begin{bmatrix} 1.00000 & 0.00666 \\ 0.00666 & 0.17980 \end{bmatrix}.$$

However, for $P = P_{131}$ and $P = P_{232}$, the matrices

$$-Q_1 = A_1^T P + P A_1, \tag{50}$$
$$-Q_2 = A_2^T P + P A_2, \tag{51}$$
$$-Q_3 = A_3^T P + P A_3, \tag{52}$$

are simultaneously either negative definite, or negative semi-definite. Hence, from Theorem 4.2, a CQLF exists for the dynamic systems Σ_{A_1}, Σ_{A_2} and Σ_{A_3}. Any $V(x) = x^T P x$, with $P = \alpha P_{131} + (1-\alpha) P_{232}$, $\alpha \in (0,1)$, is a CQLF for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

Example 5.4 : Three systems, CQLF does not exist.

Consider the stable second order LTI systems from Example 3.1, with

$$A_1 = \begin{bmatrix} 0 & 5 \\ -30 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 5 \\ -26 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -6 & 27 \\ -150 & -1 \end{bmatrix}.$$

The Lyapunov sets associated with each system are depicted in Figure 10.

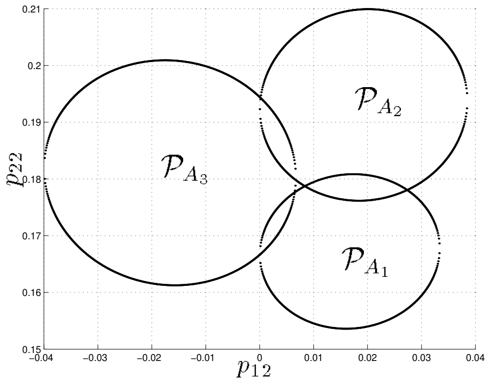


Figure 10: Lyapunov sets for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

It was already established that no CQLF exists for all three systems. Let $\mathcal{Z}_{12} = \{P_{121}, P_{122}\}$, $\mathcal{Z}_{13} = \{P_{131}, P_{132}\}$, $\mathcal{Z}_{23} = \{P_{231}, P_{232}\}$. The following matrices are determined by solving simultaneously Equations (37) and (38) for a symmetric positive definite matrix P,

$$P_{121} \approx \begin{bmatrix} 1.00000 & 0.00833 \\ 0.00833 & 0.17873 \end{bmatrix}, \quad P_{122} \approx \begin{bmatrix} 1.00000 & 0.02730 \\ 0.02730 & 0.17805 \end{bmatrix},$$

$$P_{131} \approx \begin{bmatrix} 1.00000 & 0.00645 \\ 0.00645 & 0.17763 \end{bmatrix}, \quad P_{132} \approx \begin{bmatrix} 1.00000 & 0.00000 \\ 0.00000 & 0.16667 \end{bmatrix},$$

$$P_{231} \approx \begin{bmatrix} 1.00000 & 0.00013 \\ 0.00013 & 0.19428 \end{bmatrix}, \quad P_{232} \approx \begin{bmatrix} 1.00000 & 0.00666 \\ 0.00666 & 0.17980 \end{bmatrix}.$$

It is readily verified that no P exists, $P \in (\mathcal{Z}_{12} \cup \mathcal{Z}_{13} \cup \mathcal{Z}_{23})$, which satisfies the conditions of Theorem 4.2.

6 Concluding remarks

In this paper we have considered the problem of determining the existence of a CQLF for a finite set of stable linear time-invariant second order systems. While the results presented are valid only for second order systems, many of the concepts and ideas are applicable for higher dimensional systems. The extension of the CQLF existence results presented here to higher dimensional systems will be reported in future publications.

Acknowledgements

This work was partially supported by the European Union funded research training network *Multi-Agent Control*, HPRN-CT-1999-00107. The first author gratefully acknowledges Professor A. O'Farrell and Dr. F. O'Cairbre, both with the Department of Mathematics, National University of Ireland, Maynooth, and Dr. P. Curran , National University of Ireland, Dublin, all of whom contributed to the work described in this paper.

References

[1] K. S. Narendra and J. Balakrishnan. A Common Lyapunov Function for Stable LTI Systems with Commuting $\mathcal{A}$-Matrices. *IEEE Transactions on Automatic Control*, 39(12):2469–2471, 1994.

[2] Y. Mori, T. Mori, and Y. Kuroe. Classes of Systems having Common Quadratic Lyapunov Functions: Triangular System Matrices. In *proceedings of Electronic Information and Systems Conference (in Japanese)*, 1996.

[3] Y. Mori, T. Mori, and Y. Kuroe. A Solution to the Common Lyapunov Function Problem for Continuous Time Systems. In *proceedings of 36th Conference on Decision and Control*, 1997.

[4] R. Shorten and K.S. Narendra. A Sufficient Condition for the Existence of a Common Lyapunov Function for Two Second Order Systems: Part 1. Technical report, Center for Systems Science, Yale University, 1997.

[5] R. Shorten and K.S. Narendra. On The Stability and Existence of Common Lyapunov Functions for Linear Stable Switching Systems. Technical report, Center for Systems Science, April 1998.

[6] D. Liberzon, J. P. Hespanha, and S. Morse. Stability of Switched Linear Systems: A Lie Algebraic Condition. Technical report, Laboratory for Control Science and Engineering, Yale University, 1998.

[7] D. Liberzon and A. S. Morse. Basic problems in stability and design of switched systems. *IEEE control systems magazine*, pages 59–70, 1999.

[8] N. E. Barabonov. Absolute characteristic exponent of a class of linear nonstationary systems of differential equations. *Siberian Mathematical Journal*, 29(4):12–22, 1986.

[9] S. Boyd and Q. Yang. Structured and simultaneous Lyapunov functions for system stability problems. *International Journal of Control*, 49(6):2215–2240, 1989.

[10] N. Cohen and I. Lewkowicz. A necessary and sufficient criterion for the stability of a convex set of matrices. *IEEE Transactions on Automatic Control*, 38(4):611–615, 1993.

[11] R. Loewy. On ranges of real Lyapunov transformations. *Linear Algebra and its Applications*, 13:79–89, 1976.

[12] G. P. Barker, A. Berman, and R. J. Plemmons. Positive Diagonal Solutions to the Lyapunov Equations. *Linear and Multilinear Algebra*, 5(3):249–256, 1978.

[13] R. Shorten and K. S. Narendra. On the Existence of a Common Quadratic Lyapunov Function for Linear Stable Switching Systems. In *proceedings of Tenth Yale Workshop on Adaptive and Learning Systems*, 1998.

[14] F. A. Valentine. *Convex Sets*. McGraw Hill, 1965.

[15] K.S Narendra and A.M. Annaswamy. *Stable Adaptive Systems*. Prentice-Hall, 1989.

[16] R. Shorten. *A Study of Hybrid Dynamical Systems with Application to Automotive Control*. PhD thesis, Department of Electrical and Electronic Engineering, University College Dublin, Republic of Ireland, 1996.

[17] R. N. Shorten, F. Ó Cairbre, and P. F. Curran. On the dynamic instability of a class of switching systems. In *Proceedings of IFAC conference on Artificial Intelligence in Real Time Control*, 2000.

[18] R. Shorten and K Narendra. Necessary and Sufficient Conditions For The Existence Of A Common Quadratic Lyapunov Function For M-Stable Linear Second Order Systems. In *Proceedings of American Control Conference*, 2000.

[19] R. Shorten and K.S. Narendra. Necessary and Sufficient Conditions for the Existence of a Common Lyapunov Function for M Second Order Systems. Technical report, Center for Systems Science (in preparation), 2000.

[20] R. Shorten and F. Ó Cairbre. A Proof Of Global Attractivity for a Class of Switching Systems using a non-Lyapunov Approach. Accepted for publication by *Institute of Mathematics and its Applications :* Journal of Mathematical Control and Information.

[21] G. B. Thomas and R. L. Finney. *Calculus and Analytic Geometry*. Addison-Wesley, 1984.

Appendix

Proof of Lemma 2.1: The proof is trivial when $A_2 = kA_1,\ k \in \mathbb{R},\ k \neq 0$. In the case when $A_2 \neq kA_1$ the following argument is presented. Let the matrix pencil $\sigma_\alpha[A_1, A_2]$ have non-real (imaginary) eigenvalues for $\alpha \in [0,1]$. Since $A_2 \in \mathbb{R}^{2\times 2}$ is Hurwitz, its characteristic equation has the form

$$\lambda^2 - [Trace\{A_2\}]\lambda + Det\{A_2\} \;=\; 0. \tag{53}$$

By the Cayley-Hamilton theorem, A_2 satisfies its own characteristic equation and hence

$$A_2^2 - [Trace\{A_2\}]A_2 + Det\{A_2\}I \;=\; 0. \tag{54}$$

Since A_2 is non-singular, multiplying the left hand side of the above equation by A_2^{-1} we obtain

$$A_2 - [Trace\{A_2\}]I + Det\{A_2\}A_2^{-1} \;=\; 0. \tag{55}$$

Now consider some vector $x_1 \in \mathbb{R}^2$ such that $A_1x_1 \neq kA_2x_1$, where $k \in \mathbb{R}$. These vectors span $\mathbb{R}^2$. Hence, we can find non-zero constants θ, β such that

$$\theta A_1 x_1 + \beta A_2 x_1 \;=\; x_1, \tag{56}$$

$$(A_1 + \frac{\beta}{\theta}A_2)x_1 \;=\; \frac{1}{\theta}x_1. \tag{57}$$

However, by the above assumptions, the matrix pencil $\sigma_\alpha[A_1, A_2]$ is complex for $\alpha \in [0,1]$. Therefore, the constant $\frac{\beta}{\theta}$ in equation (57) must be negative. Substituting for A_2 in (57) from (55) yields

$$A_1x_1 + \frac{\beta}{\theta}([Trace\{A_2\}]I - Det\{A_2\}A_2^{-1})x_1 \;=\; \frac{1}{\theta}x_1, \tag{58}$$

$$A_1x_1 - \frac{\beta}{\theta}Det\{A_2\}A_2^{-1}x_1 \;=\; (\frac{1}{\theta} - \frac{\beta}{\theta}[Trace\{A_2\}])x_1. \tag{59}$$

Since the matrix A_2 is Hurwitz it follows that $Det\{A_2\}$ is positive and that the constant $-\frac{\beta}{\theta}Det\{A_2\}$ is positive. Hence, the matrix pencil $\sigma_\alpha[A_1, A_2^{-1}]$ is real for the value $\alpha = \frac{1}{1-\frac{\beta}{\theta}Det\{A_2\}}$. The value α is less than unity since the constant $-\frac{\beta}{\theta}Det\{A_2\}$ is positive. **Q.E.D**

Proof of Lemma 2.2 : The following three cases are considered separately; (a) both A_1 and A_2 have real eigenvalues; (b) A_1 has real eigenvalues and A_2 has complex eigenvalues; (c) both A_1 and A_2 have complex eigenvalues.

Part (a) Both matrices have real eigenvalues

In this section the following notation is adopted.

$-\mu_1$: The maximum eigenvalue(s) of the matrix A_1: $-\mu_1 = -\lambda_{max}(A_1)$.

$-\mu_2$: The maximum eigenvalue(s) of the matrix A_2: $-\mu_2 = -\lambda_{max}(A_2)$.

$-d_m$: The maximum real eigenvalue of the pencil $\sigma_\alpha[A_1, A_2]$ $(-\lambda_R(\alpha_0) = -d_m)$.

To prove the lemma for the case when both A_1 and A_2 have real eigenvalues the following two sub-cases need to be considered; (i) $d_m \leq min[\mu_1, \mu_2],\ \mu_1 \neq \mu_2$; (ii) $d_m \leq \mu_1,\ \mu_1 = \mu_2$.

Case (i) : $d_m \leq min[\mu_1, \mu_2],\ \mu_1 \neq \mu_2$ *(Figure 11)*

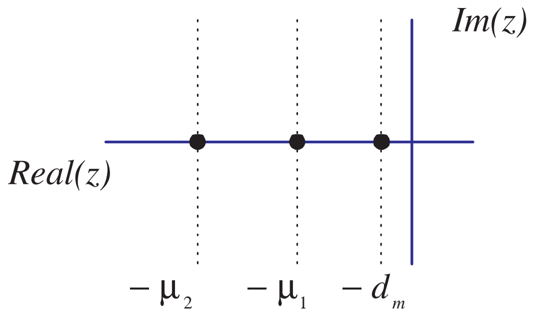


Figure 11: A_1 and A_2 have real eigenvalues.

Consider the pencil

$$\sigma_\alpha[B_1, B_2] = \alpha B_1 + (1-\alpha)B_2, \alpha \in [0,1], \tag{60}$$

where $B_1 = A_1 + d_m I$, $B_2 = A_2 + d_m I$. Then, it follows that

$$\begin{aligned} \sigma_\alpha[B_1, B_2] &= \alpha(A_1 + d_m I) + (1-\alpha)(A_2 + d_m I), & (61)\\ &= \alpha A_1 + (1-\alpha)A_2 + d_m I, & (62) \end{aligned}$$

where $\alpha \in [0,1]$. The pencil defined by equation (62) satisfies the following properties.

1. $Trace\{\sigma_\alpha[B_1, B_2]\} \leq 0$ for $\alpha \in [0,1]$.
2. $Det\{\sigma_\alpha[B_1, B_2]\}$ is either a constant, or is linear or quadratic in α.
3. $Det\{\sigma_\alpha[B_1, B_2]\} = 0$ for $\alpha = \alpha_0$.

In follows from 1 that $\sigma_\alpha[B_1, B_2]$ cannot have complex eigenvalues in the open right half of the complex plane, denoted $\mathbb{C}^+$. Similarly, since $-d_m$ is the maximum real eigenvalue of $\sigma_\alpha[A_1, A_2]$, it also follows from equation (62) that $\sigma_\alpha[B_1, B_2]$ cannot have real eigenvalues in $\mathbb{C}^+$. Hence, the pencil $\sigma_\alpha[B_1, B_2]$ is Hurwitz, or has purely imaginary eigenvalues on the imaginary axis, or is singular for $\alpha = \alpha_0,\ \alpha \in [0,1]$.

To show that the pencil $\sigma_\alpha[B_1, B_2]$ is singular for exactly one value of $\alpha = \alpha_0$ we note the following. Item 2 implies that $Det\{\sigma_\alpha[B_1, B_2]\}$ is a constant, or is linear or quadratic in α. However, the fact that $\mu_1 \neq \mu_2$ implies that either

$$Det\{B_1\} = Det\{\sigma_\alpha[B_1, B_2]\}|_{\alpha=1} \geq 0 \text{ and/or} \tag{63}$$
$$Det\{B_2\} = Det\{\sigma_\alpha[B_1, B_2]\}|_{\alpha=0} \geq 0. \tag{64}$$

However, $Det\{\sigma_\alpha[B_1, B_2]\} = 0$ for $\alpha = \alpha_0$. Hence, $Det\{\sigma_\alpha[B_1, B_2]\}$ cannot be a constant, and must be linear or quadratic in α. It therefore follows that $Det\{\sigma_\alpha[B_1, B_2]\} = 0$ for a single value of $\alpha = \alpha_0$ in the interval $[0,1]$, and hence that the pencil is singular for exactly one value of α in this interval, namely $\alpha = \alpha_0$.

Case (ii) : $d_m \leq \mu_1,\ \mu_1 = \mu_2$ (Figure 12)

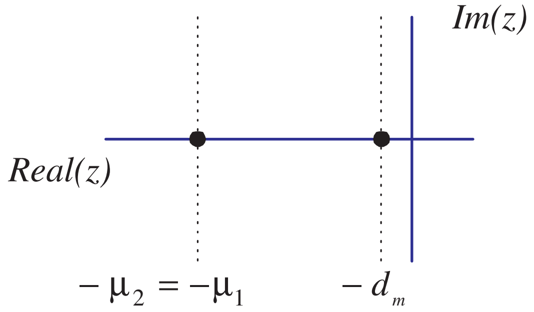


Figure 12: A_1 and A_2 have real eigenvalues.

In the case where $d_m < \mu_1,\ \mu_1 = \mu_2$, the pencil $\sigma_\alpha[B_1, B_2]$ can be constructed as in Case (i). Otherwise, when $d_m = \mu_1 = \mu_2$, the following procedure is followed. Since $\sigma_\alpha[A_1, A_2]$ is Hurwitz for all $\alpha \in [0,1]$, there exists a positive constant ϵ such that the pencil

$$\sigma_\alpha[A_1, \bar{A}_2] = \alpha A_1 + (1-\alpha)(A_2 + \epsilon I), \tag{65}$$

is also Hurwitz for all $\alpha \in [0,1]$, where $\bar{A}_2 = A_2 + \epsilon I$. Therefore, for this pencil, $\mu_1 > \bar{\mu}_2$, where $-\bar{\mu}_2 = -\mu_2 + \epsilon$. Further, it follows that $\bar{d}_m \leq \bar{\mu}_2 < \mu_1$, where $-\bar{d}_m$ is the maximum real eigenvalue of the pencil $\sigma_\alpha[A_1, \bar{A}_2]$. Hence, this pencil satisfies the conditions of Case (i), and matrices, $B_1 = A_1 + \bar{d}_m I$ and $B_2 = A_2 + \bar{d}_m I + \epsilon I$ can be constructed in the same manner as in Case (i) such that $\sigma_\alpha[B_1, B_2]$ satisfies the conditions of Lemma 2.1.

Part (b) A_1 has real eigenvalues and A_2 has complex eigenvalues

In this section the following notation is adopted.

$-\mu_1$: The maximum eigenvalue(s) of the matrix A_1: $-\mu_1 = -\lambda_{max}(A_1)$.

$-\mu_2$: The real part of the complex conjugate eigenvalues of A_2.

$-d_m$: The maximum real eigenvalue of the pencil $\sigma_\alpha[A_1, A_2]$ $(-\lambda_R(\alpha_0) = -d_m)$.

To prove the lemma the following two cases need to be considered; (i) $\mu_1 \leq \mu_2$; (ii) $\mu_2 < \mu_1$.

Case (i) : $\mu_1 \leq \mu_2$ *(Figure 13)*

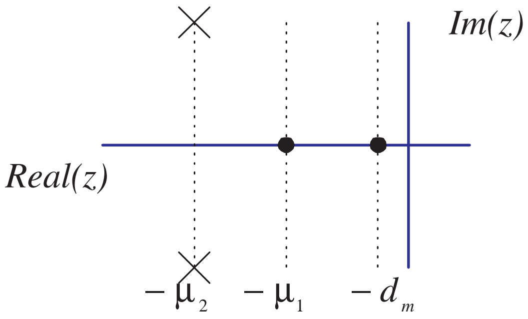


Figure 13:

Consider the pencil

$$\sigma_\alpha[B_1, B_2] = \alpha B_1 + (1-\alpha)B_2,\ \alpha \in [0,1], \tag{66}$$

where $B_1 = A_1 + d_m I$, $B_2 = A_2 + d_m I$. It follows that

$$\sigma_\alpha[B_1, B_2] = \alpha(A_1 + d_m I) + (1-\alpha)(A_2 + d_m I), \tag{67}$$

$$= \alpha A_1 + (1-\alpha)A_2 + d_m I, \tag{68}$$

where $\alpha \in [0,1]$. By definition $d_m \leq \mu_1 \leq \mu_2$. Hence, using identical arguments to those developed in Part (a) of this proof, it can be deduced that the pencil $\sigma_\alpha[B_1, B_2]$ satisfies the conditions of the lemma.

Case (ii) : $\mu_2 < \mu_1$

In the case when $d_m \leq \mu_2$ (Figure 14 (a)) it follows that $\sigma_\alpha[B_1, B_2]$ can be constructed as above:

$$\sigma_\alpha[B_1, B_2] = \alpha(A_1 + d_m I) + (1-\alpha)(A_2 + d_m I),\ \alpha \in [0,1], \tag{69}$$

where $B_1 = A_1 + d_m I$, $B_2 = A_2 + d_m I$. In the case where $d_m > \mu_2$ (Figure 14 (b)) the pencil $\sigma_\alpha[B_1, B_2]$ is constructed in the following manner. Let,

$$\sigma_\alpha[B_1, B_2] = \alpha(A_1 + dI) + (1-\alpha)(A_2 + \mu_2 I),\ \alpha \in [0,1], \tag{70}$$

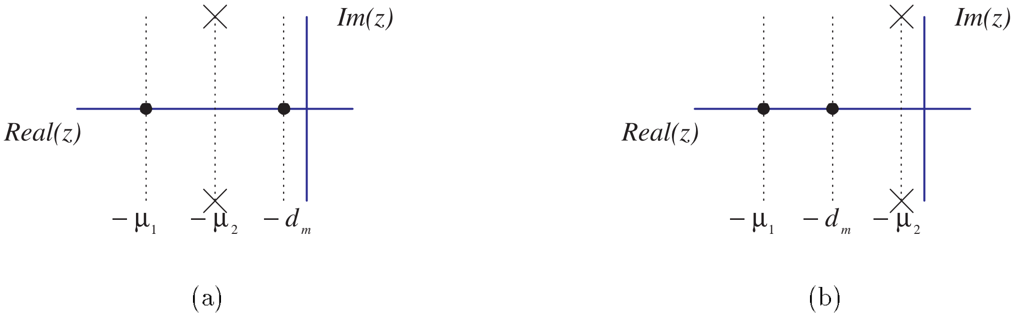


Figure 14:

where $B_1 = A_1 + dI$, $B_2 = A_2 + \mu_2 I$, and where $d \leq \mu_1$ is some positive constant. It is also assumed that,

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, A_2 + \mu_2 I = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}, w > 0.$$

The pencil $\sigma_\alpha[B_1, B_2]$ cannot be singular when $\alpha = 0$ (since the matrix $B_2 = A_2 + \mu_2 I$ has complex roots). Hence, for convenience, we consider in the following discussion the matrix pencil,

$$\sigma_{\gamma[0,\infty)}[B_1, B_2] = A_1 + dI + \gamma(A_2 + \mu_2 I),\ \gamma \in [0, \infty), \tag{71}$$

where $\gamma = \frac{1-\alpha}{\alpha}$, $\alpha \in (0, 1]$. A necessary and sufficient condition for $\sigma_{\gamma[0,\infty)}[B_1, B_2]$ to be singular is that its determinant is zero for some positive γ,

$$\begin{aligned} Det\{\sigma_{\gamma[0,\infty)}[B_1, B_2]\} &= \gamma^2 w^2 - \gamma w(a_{12} - a_{21}) + Det\{A_1 + dI\}, & (72)\\ &= 0. & (73) \end{aligned}$$

We wish to determine conditions under which the right hand side of equation (72) is identically zero. Two cases need to be considered; $(a_{12} - a_{21}) \leq 0; (a_{12} - a_{21}) > 0$. Consider the case when $(a_{12} - a_{21}) \leq 0$. It follows from the fact that $Det(A_1 + dI) \geq 0, d \in [0, \mu_1]$, that no positive γ exists for which

$$Det\{\sigma_{\gamma[0,\infty)}[B_1, B_2]\} = 0. \tag{74}$$

Hence, a positive α, $\alpha \in (0, 1)$, cannot exist which results in the pencil $\sigma_\alpha[B_1, B_2]$ being singular. However, for $d = \mu_1$, it follows that

$$\begin{aligned} Det\{\sigma_{\gamma[0,\infty)}[B_1, B_2]\} &\geq 0,\ \forall\, \gamma \in (0, \infty), & (75)\\ &= 0,\ \text{when } \gamma = 0. & (76) \end{aligned}$$

Therefore, the pencil $\sigma_{\gamma[0,\infty)}[B_1, B_2]$ is singular for $\gamma = 0$ when $d = \mu_1$. In the case when $(a_{12} - a_{21}) > 0$ a sufficient condition for equation (72) to be zero for exactly one γ is given by

$$\begin{aligned} (a_{12} - a_{21})^2 &= 4Det\{A_1 + dI\}, & (77)\\ &= 4\{(a_{11} + d)(a_{22} + d) - a_{12}a_{21}\}, & (78)\\ \Rightarrow 4Det\{A_1 + dI\} - (a_{12} - a_{21})^2 &= 0. & (79) \end{aligned}$$

We note that, when $d = 0$, the pencil described by equation (71) is Hurwitz, or has purely imaginary eigenvalues for all $\gamma \in [0, 1)$. This follows from the fact that its trace is always negative for $\alpha \in [0, 1]$, and from the fact that $d_m > \mu_2$. Hence,

$$(a_{12} - a_{21})^2 < 4(a_{11}a_{22} - a_{12}a_{21}). \tag{80}$$

Furthermore,

$$(a_{12} - a_{21})^2 \geq 0. \tag{81}$$

Equation (79) is quadratic in the variable d. However, from equation (80) it can be deduced that the left-hand side of equation (79) is positive when $d = 0$. Similarly, equation (81) implies that the left-hand side of equation (79) is negative or zero when $d = \mu_1$. Hence, a positive d, $d \in [0, \mu_1]$, must exist such that the pencil $\sigma_{\gamma[0,\infty)}[B_1, B_2]$ is singular for exactly one value of $\gamma = \gamma_0$ in the interval $[0, \infty)$, and consequently that the pencil $\sigma_\alpha[B_1, B_2]$ is singular for exactly one $\alpha = \alpha_0$ in the interval $[0, 1]$.

In both of the above cases (i.e. $a_{12} - a_{21} > 0$, $a_{12} - a_{21} \leq 0$) it follows that $Det\{\sigma_\alpha[B_1, B_2]\} \geq 0$ for all $\alpha \in [0, 1]$, and is zero for exactly one value of $\alpha = \alpha_0$, $\alpha_0 \in [0, 1]$. Furthermore,

$$Trace\{\sigma_\alpha[B_1, B_2]\} \leq 0, \ \alpha \in [0, 1]. \tag{82}$$

Hence, it follows that the pencil $\sigma_\alpha[B_1, B_2]$ is Hurwitz, or has purely imaginary eigenvalues for all $\alpha \in [0, 1]$, except at a single value of $\alpha = \alpha_0$ for which the pencil is singular.

Similar arguments can be developed for the case when $A_2 + \mu_2 I$ is of the form,

$$B_2 = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}, w > 0.$$

Part (c) A_1 has complex eigenvalues and A_2 has complex eigenvalues

In this section the following notation is adopted.

$-\mu_1$: The real part of the complex conjugate eigenvalues of A_1.

$-\mu_2$: The real part of the complex conjugate eigenvalues of A_2.

$-d_m$: The maximum real eigenvalue of the pencil $\sigma_\alpha[A_1, A_2]$ $(-\lambda_R(\alpha_0) = -d_m)$.

In the following discussion it is also assumed that $\mu_1 \geq \mu_2$. We again consider two cases; (i) $d_m \leq \mu_2$; (ii) $d_m > \mu_2$.

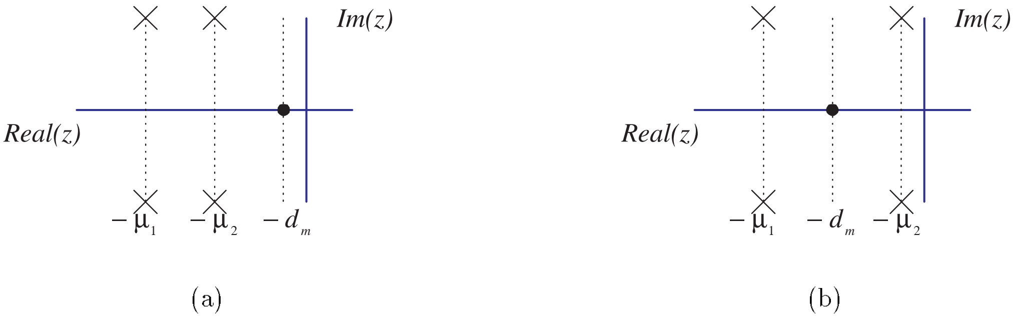


Figure 15:

Case (i): $d_m \leq \mu_2$ (Figure 15 (a))

Consider the pencil

$$\sigma_\alpha[B_1, B_2] = \alpha B_1 + (1-\alpha)B_2, \ \alpha \in [0, 1], \tag{83}$$

where $B_1 = A_1 + d_m I$, $B_2 = A_2 + d_m I$. It follows that

$$\begin{aligned} \sigma_\alpha[B_1, B_2] &= \alpha(A_1 + d_m I) + (1-\alpha)(A_2 + d_m I), & (84)\\ &= \alpha A_1 + (1-\alpha)A_2 + d_m I, & (85) \end{aligned}$$

where $\alpha \in [0, 1]$. By definition $d_m \leq min[\mu_1, \mu_2]$. Hence, using identical arguments to those developed in Part (a) of this proof, it can be deduced that the pencil $\sigma_\alpha[B_1, B_2]$ satisfies the conditions of the lemma.

Case (ii) : $d_m > \mu_2$ (Figure 15 (b))

Consider the following pencil:

$$\sigma_\alpha[B_1, B_2] \;=\; \alpha(A_1 + dI) + (1-\alpha)(A_2 + \mu_2 I),\ \alpha \in [0,1], \tag{86}$$

where $B_1 = A_1 + dI$, $B_2 = A_2 + \mu_2 I$, and where $d \le \mu_1$ is some positive constant. It is assumed that,

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, A_2 + \mu_2 I = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}.$$

The pencil $\sigma_\alpha[B_1, B_2]$ cannot be singular when $\alpha = 0$ or when $\alpha = 1$ (since the matrices B_2 and B_1 have complex roots). Hence, for convenience, we consider in the following discussion the matrix pencil,

$$\sigma_{\gamma[0,\infty)}[B_1, B_2] = A_1 + dI + \gamma(A_2 + \mu_2 I),\ \gamma \in [0, \infty), \tag{87}$$

where $\gamma = \frac{1-\alpha}{\alpha}$, $\alpha \in (0,1]$. A necessary and sufficient condition for $\sigma_{\gamma[0,\infty)}[B_1, B_2]$ to be singular that its determinant is zero for some positive γ,

$$Det\{\sigma_{\gamma[0,\infty)}[B_1, B_2]\} \;=\; \gamma^2 w^2 - \gamma w(a_{12} - a_{21}) + Det\{A_1 + dI\}, \tag{88}$$

$$\;=\; 0. \tag{89}$$

We wish to determine conditions under which the right hand side of equation (88) is identically zero. The matrix pencil $\sigma_\alpha[A_1, A_2]$ has real roots for some α in the interval $\alpha \in [0,1]$. Hence, both of the pencils $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ and $\sigma_{\gamma[0,\infty)}[B_1, B_2]$ also have real roots for some positive γ. It follows that $a_{12} - a_{21} > 0$. Therefore, a sufficient condition for equation (88) to be zero for exactly one γ is given by

$$(a_{12} - a_{21})^2 \;=\; 4Det\{A_1 + dI\}, \tag{90}$$

$$\;=\; 4\{(a_{11} + d)(a_{22} + d) - a_{12}a_{21}\}, \tag{91}$$

$$\Rightarrow (a_{12} + a_{21})^2 \;=\; 4\{(a_{11} + d)(a_{22} + d)\}. \tag{92}$$

However, as in Part (b) it can be shown that,

$$(a_{12} - a_{21})^2 \;<\; 4(a_{11}a_{22} - a_{12}a_{21}), \tag{93}$$

$$\Rightarrow (a_{12} + a_{21})^2 - 4a_{11}a_{22} \;<\; 0, \tag{94}$$

since the pencil $\sigma_\alpha[B_1, B_2]$ is non-singular for all $\alpha \in [0,1]$ when $d = 0$. Furthermore, when $d = \mu_1$, the right hand side of equation (92) is negative or zero (since $Trace\{A_1 + \mu_1 I\} = 0$). Hence,

$$(a_{12} + a_{21})^2 - 4\{(a_{11} + \mu_1)(a_{22} + \mu_1)\} \;\ge\; 0. \tag{95}$$

Therefore, as in Part (b), it can be deduced that a positive $d \in [0, \mu_1]$ exists such that the pencil $\sigma_\alpha[B_1, B_2]$ is singular for exactly one α in the interval $\alpha \in (0,1)$. Furthermore, for this choice of d the following statements are true:

$$Det\{\sigma_\alpha[B_1, B_2]\} \;\ge\; 0, \text{and} \tag{96}$$

$$Trace\{\sigma_\alpha[B_1, B_2]\} \;\le\; 0,\ \alpha \in [0,1]. \tag{97}$$

Therefore, the pencil $\sigma_\alpha[B_1, B_2]$ is Hurwitz, or has purely imaginary eigenvalues for all $\alpha \in [0,1]$, except at a single value of $\alpha = \alpha_0$, for which the pencil is singular. Similar arguments can be developed for the case when $A_2 + \mu_2 I$ is of the form,

$$B_2 = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}, w > 0.$$

Q.E.D.

<u>Proof of Lemma 2.3 :</u> Consider two cases: (a) $p_{12} = 0$; (b) $p_{12} \neq 0$. In case (a) the lemma follows trivially with $a = 0, c = 1$ and $b = p_{22}$. In case (b) we let

$$Z = \begin{bmatrix} 1 & \frac{p_{22}}{p_{12}} \\ -\frac{1}{p_{12}} & -1 \end{bmatrix},$$

It follows automatically that,

$$Z^T P + PZ \;\; = \;\; 0,$$

and that

$$\begin{aligned} det\{Z\} &= bc - 1, \\ &= -1 + \frac{p_{22}}{p_{12}^2}, \\ &> 0, \end{aligned}$$

as $p_{22} > p_{12}^2 > 0$. **Q.E.D.**

<u>Proof of Lemma 2.4 :</u> Let

$$A_i = \begin{bmatrix} a_{11i} & a_{12i} \\ a_{21i} & a_{22i} \end{bmatrix}.$$

Then,

$$\begin{aligned} \bar{A}_i &= GA_iG^{-1} \\ &= \begin{bmatrix} a_{11i} - \alpha a_{12i} & a_{12i} \\ -\alpha^2 a_{12} + \alpha(a_{11} - a_{22}) + a_{21i} & \alpha a_{12} + a_{22i} \end{bmatrix}. \end{aligned}$$

The $(2,1)$ entry of $\bar{A}_i$ is given by,

$$\bar{a}_{21i} = -\alpha^2 a_{12i} + \alpha(a_{11i} - a_{22i}) + a_{21i}. \tag{98}$$

This equation is either identically 0, linear in α, or quadratic in α. Two cases need to be considered, the case when Equation (98) is identically 0, and the case when Equation (98) is linear or quadratic in α.

Consider the case when Equation (98) is identically 0, i.e. $a_{12i} = 0,\ a_{11i} - a_{22i} = 0$, and $a_{21i} = 0$. Then $GA_iG^{-1} = k_iI$.

Now consider the case where at least one of the coefficients, $a_{12i} \neq 0,\ a_{11i} - a_{22i} \neq 0, a_{21i} \neq 0$. Then, Equation (98) is linear or quadratic in α. Hence, there exists α_{ki}, such that

$$\bar{a}_{21i} \neq 0,\ \forall\, \alpha > \alpha_{ki}.$$

It immediately follows that for all $\alpha > max\{\alpha_{k1}, ..., \alpha_{kM}\}$,

$$\begin{aligned} \bar{A}_i &= k_iI, \text{ or} \\ \bar{A}_i &= \begin{bmatrix} \bar{a}_{11i} & \bar{a}_{12i} \\ \bar{a}_{21i} & \bar{a}_{22i} \end{bmatrix} \end{aligned}$$

with $\bar{a}_{21i} \neq 0$ where $i \in \{1, ..., M\}$. **Q.E.D.**

Proof of item (b) in Section 4 : The general equation for a quadratic curve in R^2 is given by,

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \zeta y + \tau = 0, \tag{99}$$

where $\alpha,\ \beta,\ \gamma,\ \delta,\ \zeta,\ \tau \in \mathbb{R}$. This equation describes an ellipse if and only if [21]

$$\alpha\gamma > 0, \tag{100}$$

$$\beta^2 - 4\alpha\gamma < 0. \tag{101}$$

Therefore, to demonstrate that $det\{A^T P + PA\} = 0,\ A, P \in \mathbb{R}^{2\times 2},\ P = P^T > 0$, is an ellipse, we need to show that the respective coefficients of $p_{12}^2,\ p_{22}^2,$ and $p_{12}p_{22}$ (i.e. $\bar{\alpha}, \bar{\gamma}, \bar{\beta}$) satisfy Equations (100) and (101). Straightforward multiplication reveals that

$$\begin{aligned}
\bar{\alpha} &= 4a_{21}a_{12} - (a_{11} + a_{22})^2, \\
\bar{\gamma} &= -a_{21}^2, \\
\bar{\beta} &= 2a_{21}(a_{22} - a_{11}),
\end{aligned}$$

and that $\bar{\beta}^2 - 4\bar{\alpha}\bar{\gamma} = 4(a_{12}a_{21} - a_{11}a_{22})$. But the matrix A is Hurwitz. Hence, $det\{A\} = -a_{12}a_{21} + a_{11}a_{22}$. Further, $\bar{\alpha} \neq 0$, as this would imply that $(a_{11} - a_{22})^2 < 0$. It immediately follows that

$$\bar{\beta}^2 - 4\bar{\alpha}\bar{\gamma} < 0,$$

and that the equation, $det\{A^T P + PA\} = 0$, describes an ellipse, provided that $a_{21} \neq 0$. **Q.E.D.**

<u>Proof of Lemma 4.1 :</u> A necessary condition for all three systems Σ_{A_1}, Σ_{A_2} and Σ_{A_3} to share a CQLF, is that the systems, when taken pairwise, satisfy Theorem 3.1. However, by assumption, $\mathcal{P}_{A_i} \subset \mathcal{P}_{A_j}$ for some $i, j \in \{1, 2, 3\}$, $\quad i \neq j$. Let $\mathcal{P}_{A_2} \subset \mathcal{P}_{A_1}$. It therefore follows that any $V(x) = x^T P x$, $P = P^T > 0$, that satisfies,

$$A_2^T P + P A_2 < 0,$$

also satisfies

$$A_1^T P + P A_1 < 0.$$

However, Σ_{A_2} and Σ_{A_3} satisfy Theorem 3.1. Hence, there exists a symmetric positive definite $P_\gamma \in (\mathcal{P}_{A_2} \cap \mathcal{P}_{A_3})$ such that

$$\begin{aligned} A_2^T P_\gamma + P_\gamma A_2 &< 0, \\ A_3^T P_\gamma + P_\gamma A_3 &< 0. \end{aligned}$$

But $(\mathcal{P}_{A_2} \cap \mathcal{P}_{A_3}) \subset \mathcal{P}_{A_2} \subset \mathcal{P}_{A_1}$. It follows that

$$\begin{aligned} A_1^T P_\gamma + P_\gamma A_1 &< 0, \\ A_2^T P_\gamma + P_\gamma A_2 &< 0, \\ A_3^T P_\gamma + P_\gamma A_3 &< 0. \end{aligned}$$

Hence, in this case, the condition that the systems, when taken pairwise, satisfy Theorem 3.1, is both necessary and sufficient for the existence of a CQLF. **Q.E.D.**

Proof of Lemma 4.2 :

Necessity : Let a CQLF exist for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}. This implies that $\mathcal{P}_{A_3} \cap (\mathcal{P}_{A_2} \cap \mathcal{P}_{A_1}) \neq \emptyset$. As in the case of two dynamic systems, two types of intersection is possible: (i) $\mathcal{E}_{A_3}$ encloses the region that is common to the interior of the ellipse $\mathcal{E}_{A_1}$ and the interior of the ellipse $\mathcal{E}_{A_2}$, and (ii) the ellipse $\mathcal{E}_{A_3}$ intersects the region that is common to the interior of the ellipse $\mathcal{E}_{A_1}$ and the interior of the ellipse $\mathcal{E}_{A_2}$.

Case (i) : It follows that for any $P_\gamma \in \mathcal{Z}_{12}$,

$$
\begin{aligned}
A_1^T P_\gamma + P_\gamma A_1 &\leq 0, \\
A_2^T P_\gamma + P_\gamma A_2 &\leq 0, \\
A_3^T P_\gamma + P_\gamma A_3 &= -Q_3,
\end{aligned}
$$

where $-Q_3$ is either negative definite or negative semi-definite.

Case (ii) : Since $(\mathcal{P}_{A_2} \cap \mathcal{P}_{A_1}) \cap \mathcal{P}_{A_3} \neq \emptyset$, and since $\mathcal{P}_{A_i} \not\subset \mathcal{P}_{A_j}$ for all $i, j \in \{1, 2, 3\}, i \neq j$, it follows that the ellipse $\mathcal{E}_{A_3}$, intersects at least twice, the boundary of the region $\mathcal{P}_{A_2} \cap \mathcal{P}_{A_1}$. However, this boundary is defined piecewise by segments of the ellipse $\mathcal{E}_{A_1}$ and the ellipse $\mathcal{E}_{A_2}$. Hence, it follows that there exists at least two $P_\gamma \in (\mathcal{Z}_{12} \cup \mathcal{Z}_{13} \cup \mathcal{Z}_{23})$, such that

$$
\begin{aligned}
A_1^T P_\gamma + P_\gamma A_1 &= -Q_1, \\
A_2^T P_\gamma + P_\gamma A_2 &= -Q_2, \\
A_3^T P_\gamma + P_\gamma A_3 &= -Q_3,
\end{aligned}
$$

where $-Q_1, -Q_2$ and $-Q_3$ are either negative definite, or negative semi-definite matrices.

Sufficiency : Let P_i and P_j be two positive definite symmetric matrices that satisfy the conditions of the lemma. Then, it follows that

$$\begin{aligned} A_1^T P_i + P_i A_1 &= -Q_{1i}, && (102) \\ A_1^T P_j + P_j A_1 &= -Q_{2j}, && (103) \end{aligned}$$

where $-Q_{1i}$ and $-Q_{1j}$ are either negative definite or negative semi-definite. In the case that either $-Q_{1i}$ or $-Q_{1j}$ is negative definite, it follows immediately that for $P = \alpha P_i + (1-\alpha)P_j$, $\alpha \in (0,1)$,

$$A_1^T P + P A_1 < 0. \tag{104}$$

Now let both $-Q_{1i}$ and $-Q_{1j}$ be both negative semi-definite. In this case it follows that the function,

$$\begin{aligned} f(\alpha, x) &= -\alpha x^T (Q_{1i}) x - (1-\alpha) x^T (Q_{1j}) x, \\ &\leq 0, \end{aligned}$$

for all $x \in \mathbb{R}^2$ where $\alpha \in [0,1]$. Suppose that $f(\alpha, x_0) = 0$ for some $\alpha \in (0,1)$, $x_0 \in \mathbb{R}^2$. This implies that for some $x = x_0$,

$$\begin{aligned} x_0^T \{A_1^T P_{1i} + P_{1i} A_1\} x_0 &= 0, \\ x_0^T \{A_1^T P_{1j} + P_{1j} A_1\} x_0 &= 0. \end{aligned}$$

Hence, for every $P = \alpha P_i + (1-\alpha) P_j$,

$$\begin{aligned} A_1^T P_i + P_i A_1 &\leq 0, \\ A_1^T P_j + P_j A_1 &\leq 0. \end{aligned}$$

But this contradicts the fact that the set of P matrices for which $A_1^T P + P A_1 \leq 0$, is defined by the ellipse $det\{A_1^T P + P A_1\} = 0$ (since $P = \alpha P_i + (1-\alpha) P_j$, $\alpha \in (0,1)$, can be represented as a line segment in $\mathbb{R}^2$). Further, P_{1i} and P_{1j} satisfy $det\{A_1^T P + P A_1\} = 0$. Hence, for $P = \alpha P_{1i} + (1-\alpha) P_{1j}$, $\alpha \in (0,1)$, it follows that $P \in \mathcal{P}_{A_1}$. Hence,

$$f(\alpha, x) < 0, \ \forall \, x \in \mathbb{R}^2,$$

and that,

$$A_1^T P + P A_1 < 0.$$

But similar arguments can be applied for the matrices A_2 and A_3. Hence, given any two positive definite matrices P_i and P_j that satisfy the condition of the lemma, it follows that $P = \alpha P_i + (1-\alpha) P_j$, $\alpha \in (0,1)$, satisfies

$$\begin{aligned} A_1^T P + P A_1 &< 0, \\ A_2^T P + P A_2 &< 0, \\ A_3^T P + P A_3 &< 0. \end{aligned}$$

Q.E.D.

Proof of Theorem 4.3 :

Necessity : Let a CQLF exist for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}. Then, there exists a matrix $P = P^T > 0$ such that $x^T(A_i^T P + PA_i)x < 0$ for all $x \in \mathbb{R}^2$, $i \in \{1, 2, 3\}$. Hence, $x^T PA_i x < 0$ for all $x \in \mathbb{R}^2$, $i \in \{1, 2, 3\}$. However, from lemma 2.3 we have that $x^T PZx = 0$ for all $x \in \mathbb{R}^2$. Hence, since all vectors are in $\mathbb{R}^2$ it is not possible for $A_i x = kZx$ for some $x \in \mathbb{R}^2$ and some non zero k, $i \in \{1, 2, 3\}$. Hence, a necessary condition for the existence of a CQLF is that there exists at least one Z matrix that satisfies the conditions of Theorem 4.3. **Q.E.D.**

Sufficiency : Suppose that a Z matrix exists that satisfies the conditions of Theorem 4.3. Then, with P and Z as defined in Lemma 2.3, it follows from the above discussion that the conditions of the theorem imply that for all $x \in \mathbb{R}^2$, $-x^T Q_i x \neq 0$, with $-Q_i = A_i^T P + PA_i$. Hence, it follows for each $x \in \mathbb{R}^2$, that $-x^T Q_i x > 0$ or that $-x^T Q_i x < 0$. Note that $-x^T Qx > 0$ for all $x \in \mathbb{R}^2$ is not possible since Σ_{A_i} is a stable system.

Suppose that there exists vectors x_1, x_2, $x_1 \neq x_2$, such that,

$$x_1^T Q_i x_1 > 0, \tag{105}$$

$$x_2^T Q_i x_2 < 0. \tag{106}$$

However, since $-x^T Qx$ is a quadratic form, Equations (105) and (106) imply the existence of a vector $x_3 \in \mathbb{R}^2$, such that $x_3^T Q_i x_3 = 0$. However, this is not possible by the assumptions of the theorem, and it follows that $-x^T Q_i x < 0$, $\forall x \in \mathbb{R}^2$, $i \in \{1, 2, 3\}$. **Q.E.D.**