PAPER ID: ACC01-SHT1018

ON THE STABILITY OF PAIRWISE TRIANGULARISABLE AND RELATED SWITCHING SYSTEMS

Robert Shorten
Department of Computer Science
National University of Ireland
Maynooth, Co. Kildare
Ireland

Fiacre Ó Cairbre
Department of Mathematics
National University of Ireland
Maynooth, Co. Kildare
Ireland

ON THE STABILITY OF PAIRWISE TRIANGULARISABLE AND RELATED SWITCHING SYSTEMS

Robert Shorten
Department of Computer Science
National University of Ireland
Maynooth, Co. Kildare
Ireland

Fiacre Ó Cairbre
Department of Mathematics
National University of Ireland
Maynooth, Co. Kildare
Ireland

Abstract

It was conjectured in [1] that a sufficient condition for the asymptotic stability of the switching system $\dot{x} = A(t)x, \; A(t) \in \mathcal{A} \triangleq \{A_1, ..., A_m\}, \; A_i \in \mathbb{R}^{n \times n}$, is that the component systems $\Sigma_{A_i} : \dot{x} = A_i x, \; i \in \{1, ..., m\}$ are asymptotically stable (i.e. the A_i matrices are Hurwitz), and a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij} A_i T_{ij}^{-1}, T_{ij} A_j T_{ij}^{-1}\}, \; i, j \in \{1, ..., m\}$, are upper triangular. Switching systems that are constructed in this manner are referred to as pairwise triangularisable switching systems. It was shown in [1] that the origin is globally attractive for a special class of such systems. In this paper we show that the general condition of pairwise triangularisability is not sufficient to guarantee asymptotic stability. A simple example is given which exhibits instability for a given periodic switching sequence, thereby invalidating the conjecture made in [1]. However, we show that the ideas developed in [1] can be extended to demonstrate global attractivity of the origin for a related class of switching system. Global attractivity of the origin for this class of system is demonstrated using a novel technique.

1 Introductory remarks

We consider switching systems of the following form,

$$\dot{x} \;=\; A(t)x, \;\; A(t) \in \mathcal{A} \triangleq \{A_1, A_2, ..., A_m\}, \tag{1}$$

where the $A_i, \; i \in \{1, 2, ...m\}$, are constant matrices in $\mathbb{R}^{N \times N}$. The matrices $A_i, \; i \in \{1, 2, ...m\}$, are assumed to be Hurwitz (the eigenvalues of each A_i matrix lies in the open left half of the complex plane), and it is also assumed that a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij} A_i T_{ij}^{-1}, T_{ij} A_j T_{ij}^{-1}\}, \; i, j \in \{1, ..., m\}$, are upper triangular. We refer to switching systems that are constructed in this manner as pairwise triangularisable switching systems. The following properties of pairwise triangularisable switching systems are known.

(i) Sufficient conditions for existence of common quadratic Lyapunov function : A sufficient condition for the existence of a common quadratic Lyapunov function (CQLF), $V(x) = x^T P x,\ P = P^T > 0,\ P \in \mathbb{R}^{n \times n}$, for the stable linear time invariant (LTI) systems $\dot{x} = A_i x,\ x \in \mathbb{R}^n, A_i \in \mathcal{A} \triangleq \{A_1, ..., A_m\}, A_i \in \mathbb{R}^{n \times n}$, is that the matrices A_i are Hurwitz, and that a constant non-singular transformation $T_{ij} = T\ \forall\ i, j \in \{1, ..., m\}$, exists such that TA_iT^{-1} is upper triangular for all $i \in \{1, ..., m\}$ [2, 3, 4, 5, 6].

(ii) Non-existence of CQLF : The existence of a set of non-singular matrices T_{ij} such that the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\},\ i, j \in \{1, ..., m\}$, are upper triangular, is not sufficient to guarantee the existence of a CQLF for the dynamic systems $\dot{x} = A_i x,\ x \in \mathbb{R}^n, A_i \in \mathcal{A} \triangleq \{A_1, ..., A_m\},\ i \in \{1, ..., m\}$ [7].

(iii) Global attractivity of the origin of (1): A sufficient condition for the global attractivity of the origin of (1) is that there exists a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\},\ i, j \in \{1, ..., m\}$, are upper triangular, and that every pair of matrices in the set $\mathcal{A}$ have at least $n - 1$ common eigenvectors (and satisfy some additional technical conditions) [1, 7].

The conditions in item (i) are sufficient to guarantee the exponential stability of the switching system $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$. Unfortunately, from a practical viewpoint, the requirement of simultaneous triangularisability imposes unrealistic conditions on the matrices in the set $\mathcal{A}$. It is therefore of interest to extend the results in [3, 4] with a view to relaxing this requirement. The conditions specified in item (iii) were derived in this context [1]. In this paper we extend the work initially presented in [1], and consider general pairwise triangularisable systems. We show that while the condition of pairwise triangularisability is not sufficient to guarantee asymptotic stability, the techniques introduced for their study in [1, 7], are applicable to the study of a wide class of systems. These results are important for a number of reasons. Primarily, they conclusively invalidate the conjecture made in [1] that the condition of pairwise triangularisability alone is sufficient to guarantee asymptotic stability of (1) for arbitrary switching sequences. Secondly, a novel technique, first introduced in [1], is significantly developed to prove global attractivity of the origin for a related class of switching system. This technique does not utilise concepts from quadratic Lyapunov theory and may therefore be used in situations where such functions do not exist. This technique is also of sufficient generality to be of use in the study of other classes of switching systems. Finally, a new class of switching system is introduced whose origin is globally attractive irrespective of switching sequence.

This paper is organised as follows. Preliminary definitions and mathematical results are presented in Section 2. Pairwise triangular systems are considered in Section 3. The main result of the paper is presented in Section 4; namely, a new class of switching system is introduced whose origin is globally attractive irrespective of switching sequence. Finally, the relevance of the results for more general systems is briefly discussed in Section 5.

2 Definitions

In this section we introduce some simple concepts and definitions [8] which are useful in the remainder of the paper.

(i) The switching system : Consider the linear time-varying system

$$\dot{x} = A(t)x, \tag{2}$$

where $x(t) \in \mathbb{R}^n$, and where the matrix switches between the matrices $A_i \in \mathbb{R}^{n\times n}$ belonging to the set $\mathcal{A} = \{A_1, ..., A_m\}$. We shall refer to this as *the switching system*. The time-invariant linear system $\dot{x} = A_i x$, denoted Σ_{A_i} is referred to as the i^{th} *constituent system*.

Suppose that (1) is described by the γ^{th} system $\dot{x} = A_\gamma x$ over a time interval $[t_\gamma, t_{\gamma+1}]$. By definition, the next system that we switch to, say the $(\gamma+1)^{th}$ system, starts at time $t_{\gamma+1}$ with initial conditions equal to the terminal conditions of the γ^{th} system at time $t_{\gamma+1}$.

(ii) Stability of the origin : The equilibrium state $x = 0$ of Equation (2) is said to be stable if for every $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(\epsilon, t_0) > 0$ such that $\| x_0 \| < \delta(\epsilon, t_0)$ implies that $\| x(t; x_0, t_0) \| < \epsilon, \ \forall\, t \geq t_0$.

(iii) Attractivity of the origin : The equilibrium state $x = 0$ of Equation (2) is said to be attractive if for some $\rho > 0$, and for every $\theta > 0$ and t_0, there exists a number $T(\theta, x_0, t_0)$ such that $\| x_0 \| < \rho$ implies that $\| x(t; x_0, t_0) \| < \theta, \ \forall\, t \geq t_0 + T$.

(iv) Global attractivity of the origin : The equilibrium state $x = 0$ of Equation (2) is said to be globally attractive if $\lim_{t\to\infty} x(t; x_0, t_0) = 0$, for all initial conditions x_0 and for all $t_0 \geq 0$.

(v) Asymptotic stability : The equilibrium state of Equation (2) is said to be asymptotically stable if it is both stable and attractive.

(vi) Common quadratic Lyapunov function: In the following discussion we refer to common quadratic Lyapunov functions (CQLF's). A common quadratic Lyapunov function is defined as follows.

Consider the switching system defined in (2) where all the elements of $\mathcal{A}$ are Hurwitz. The quadratic function

$$V(x) = x^T P x, \quad P = P^T > 0, \quad P \in \mathbb{R}^{n\times n}, \tag{3}$$

is said to be a common quadratic Lyapunov function for each of the constituent subsystems $\Sigma_{A_i}, \ i \in \{1, ..., m\}$, if symmetric positive definite matrices $Q_i, \ i \in \{1, ..., m\}$, exist such that the matrix P is a solution of the matrix equations

$$A_i^T P + P A_i = -Q_i. \tag{4}$$

The existence of a common quadratic Lyapunov function implies the exponential stability of the switching system (2) [9].

Finally, we note the following important result.

(vii) **Theorem 2.1 : [10, 11, 12]** A sufficient condition for the existence of a switching sequence, such that the system (1) is unstable, is that there exist non-negative constants $\{\alpha_1, \alpha_2, ..., \alpha_M\}$, such that the matrix pencil $\sum_{i=1}^{M} \alpha_i A_i,\ \alpha_i \geq 0,\ \sum_{i=1}^{M} \alpha_i > 0$ has an eigenvalue with a positive real part.

Proof : In order to demonstrate the existence of an unstable switching sequence for Equation (1), it is sufficient to show the existence of a periodic switching sequence for which the state of the system becomes unbounded.

Consider the system (1) with $A(t) = A(t+T)$ for all $t > 0$, where T is the fixed period. Further, let the matrices $\{A_1, A_2\ , ..., A_M\}$ be indexed according to the order in which they describe the system dynamics, and let the M-tuple $\{t_1, t_2\ , ..., t_M\}$ describe the time for which each of the matrices describes the system dynamics, i.e., $\dot{x} = A_j x$, describes the evolution of the system dynamics for $(\sum_{i=1}^{j-1} t_i) \leq t \leq (\sum_{i=1}^{j} t_i)$. The solution to Equation (1) at time $T = \sum_{i=1}^{M} t_i$, is given by,

$$\begin{aligned} x(T) &= (e^{A_i t_1} e^{A_j t_2} \dots e^{A_M t_M}) x_0, \\ &= \Phi(T) x_0, \end{aligned} \tag{5}$$

where $x_0 \in \mathbb{R}^N$ is some initial condition. A sufficient condition for instability of (1) is that the matrix $\Phi(T)$ has an eigenvalue whose magnitude is greater than unity [13]. Equation (5) can be expanded as a power series,

$$\begin{aligned} \Phi(T) &= I + (\sum_{i=1}^{M} \alpha_i A_i) T + K_2 T^2 + ... \\ &= I + \sigma[A_i, M] T + ... \end{aligned} \tag{6}$$

where $\alpha_i = \frac{t_i}{T}$, $K_2 \in \mathbb{R}^{N \times N}$, and where we define $\sigma[A_i, M]. = \sum_{i=1}^{M} \alpha_i A_i$. The expansion (6) is the product of M absolutely convergent series, and is analytic in T. Hence, from Theorem 2.1 in [12] and [14] (page 81), it follows that, given any eigenvalue a_j of $\sigma[A_i, M]$, an eigenvalue of $\Phi(T)$ can be written,

$$\lambda_j(T) = 1 + a_j T + f(T), \tag{7}$$

where $f(T) = O(T^\beta)$, $\beta > 1$, as $T \to 0$. Thus, there exist constants C, ϵ such that

$$\mid f(T) \mid \leq C T^\beta, \text{ for } 0 < T < \epsilon. \tag{8}$$

Hence, $\mid Re(f(T)) \mid \leq C T^\beta$ for $0 < T < \epsilon$, where $Re(z)$ denotes the real part of a complex number z. Now consider the case when the j^{th} eigenvalue, a_j, of the matrix $\sigma[A_i, M]$, has a positive real part for some $\{\alpha_1, ..., \alpha_M\}$, $\alpha_i \geq 0$, $\sum_{i=1}^{M} \alpha_i > 0$. Hence, there exists $\delta < \epsilon$, such that for $0 < T < \delta$, we have the following:

$$\begin{aligned} C T^\beta &< Re(a_j) T, \\ \mid Re(f(T)) \mid &< Re(a_j) T, \\ Re(a_j) T + Re(f(T)) &> 0, \\ Re(1 + a_j T + f(T)) &> 1, \\ Re(\lambda_j(T)) &> 1. \ \textbf{Q.E.D.} \end{aligned}$$

3 Pairwise triangular systems: counter examples

Much of the discussion in [1] is motivated by the following example.

Example 3.1 : Consider the following stable LTI systems,

$$\Sigma_{A_i} : \dot{x} = A_i x, \ A_i \in \mathbb{R}^{2\times 2},$$

with,

$$A_1 \approx \begin{bmatrix} -1.0000 & 0.0998 \\ -0.9982 & 0.0980 \end{bmatrix}, \quad A_2 \approx \begin{bmatrix} -1.00000 & -0.0995 \\ -0.9945 & -0.1049 \end{bmatrix}, \quad A_3 \approx \begin{bmatrix} -1.0000 & -0.0818 \\ -8.1818 & -1.0000 \end{bmatrix}.$$

The set of matrices for which $A_i^T P + P A_i < 0, \ P = P^T > 0, \ P \in \mathbb{R}^{2\times 2}$, is given by

$$\mathcal{P}_{A_i} : det\{A_i^T P + P A_i\} > 0, \tag{9}$$

where

$$P = \begin{bmatrix} 1 & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

Equation (9) defines a convex set in (p_{12}, p_{22}) space. These sets are depicted in Figure 1 for each of the systems $\Sigma_{A_1}, \Sigma_{A_2}$, and Σ_{A_3}, as the interior of each of the respective elliptical regions. Clearly, no CQLF exists as the sets $\mathcal{P}_{A_i}, \ i \in [1, 2, 3]$, have no common point of intersection. However, the matrix pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_1, A_3\}$ are pairwise triangularisable with

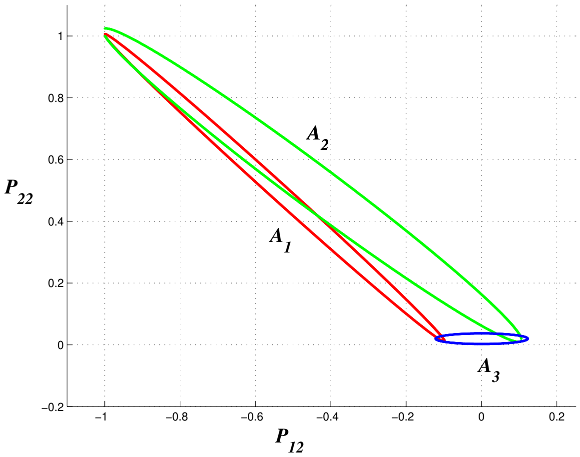

Figure 1: $\mathcal{P}_{A_i}$ for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

$$T_{12} = \begin{bmatrix} 1 & 0.1 \\ 1 & 1 \end{bmatrix}, \quad T_{23} = \begin{bmatrix} 1 & 1 \\ -10 & 1 \end{bmatrix}, \quad T_{13} = \begin{bmatrix} 0.1 & 1 \\ 1 & 1 \end{bmatrix}.$$

It follows that the condition of pairwise triangulisability of the system matrices does not necessarily imply the existence of a CQLF[1].

The matrices A_1, A_2 and A_3 in Example 3.1 satisfy the following properties.

(i) The matrices are Hurwitz.

(ii) Each pair of matrices can be simultaneously triangularised.

(iii) Each pair of matrices share $n-1$ common eigenvectors and satisfy some additional technical conditions specified in [16].

While these conditions are clearly not sufficient to guarantee the existence of a CQLF, it follows from the main result in [1] that all three properties are sufficient to guarantee global attractivity of the origin for switching systems constructed by switching between the flows of the dynamical systems $\Sigma_{A_1}, \Sigma_{A_2}$ and Σ_{A_3}. It further follows from the results in [17], that systems of this form are asymptotically stable. Based upon these observations, it was conjectured that the general condition of pairwise triangularisability is sufficient to guarantee asymptotic stability for the system (1) for arbitrary switching. This conjecture is false, as verified by the following example.

Example 3.2 : Consider the following switching system,

$$\dot{x} = A(t)x, \ A(t) \in \mathcal{A} \triangleq \{A_1, A_2, A_3\}, \tag{10}$$

with,

$$A_1 \approx \begin{bmatrix} -105.2607 & 33.7326 & 65.1648 \\ 8.3953 & -6.3633 & -8.3953 \\ -104.7741 & 33.7326 & 64.6782 \end{bmatrix},$$

$$A_2 \approx \begin{bmatrix} -19.4935 & 11.8692 & 18.7430 \\ 19.1542 & -12.6197 & -19.1542 \\ -19.0628 & 11.8692 & 18.3124 \end{bmatrix},$$

$$A_3 \approx \begin{bmatrix} 0.1333 & -0.0510 & -0.0921 \\ -1.3313 & 0.4788 & 0.8819 \\ 1.0659 & -0.3866 & -0.7101 \end{bmatrix}.$$

The matrix pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_1, A_3\}$ are pairwise triangularisable with,

$$T_{12} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & -11 \\ 2 & 1 & 9 \end{bmatrix},$$

$$T_{13} = \begin{bmatrix} 3 & 2 & 1 \\ -10 & -11 & -10 \\ 10 & 9 & 8 \end{bmatrix},$$

$$T_{23} = \begin{bmatrix} 2 & 1 & 3 \\ -11 & -10 & -10 \\ 9 & 8 & 10 \end{bmatrix},$$

[1] A complete solution to the CQLF existence problem for stable second order linear time-invariant systems is given in [15].

However, the matrix

$$\begin{aligned} A &= 0.36A_1 + 0.24A_2 + 0.2A_3, \\ &\approx \begin{bmatrix} -15.9127 & 4.7986 & 9.5435 \\ -258.6401 & 90.4471 & 168.7670 \\ 170.8861 & -62.3351 & -114.3442 \end{bmatrix}, \end{aligned}$$

has an eigenvalue with posive real part ($\lambda_i \approx \{-34.9906, 0.0169, -4.8360\}$). It follows from Theorem 2.1, that an unstable switching sequence exists for the system (10). Hence, the general condition of pairwise triangularsability is not a sufficient condition to guarantee asymptotic stability for the system (10). This observation is stated formally as follows.

Theorem 3.1 : Consider the system given by Equation (1). The condition that a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\}$, $i, j, \in \{1, ..., m\}$, are upper triangular, is not sufficient to guarantee the asymptotic stability of the system (1).

Proof : The proof is by contradiction and follows directly from Example 3.2.

4 Main result

The discussion in Section 3 clearly invalidates the conjecture made in [1]. However, in this Section we consider a natural extension of the system class studied in [1]; namely, switching systems that are constructed with the property that any pair of system matrices are Hurwitz and diagonalisable, and in addition share at least $n-1$ common conjugate pairs of complex eigenvectors. We find that the origin of such systems is globally attractive, as stated formally below. The proof of this theorem involves novel extensions to the methods first introduced in [1].

<u>**Preamble for Theorem 4.1:**</u> Let $\mathcal{V} = \{v_1, v_1^*, v_2, v_2^*, \ldots, v_{n+1}, v_{n+1}^*\}$ be a set of complex (non–real) vectors in $\mathbb{C}^{2n}$. Here $*$ denotes the complex conjugate. Suppose any $2n$ vectors of the form $v_1, v_1^*, \ldots, v_{i-1}, v_{i-1}^*, v_{i+1}, v_{i+1}^*, \ldots, v_{n+1}, v_{n+1}^*$, where both v_i and v_i^* are omitted, are linearly independent in $\mathbb{C}^{2n}$ (over $\mathbb{C}$), for any $i \in \{1, 2, \ldots, n+1\}$. For each $i \in \{1, 2, \ldots, n+1\}$, construct a $2n \times 2n$ matrix as follows: $M_1 = [v_1, v_1^*, v_2, v_2^*, \ldots, v_n, v_n^*]$ and for $2 \leq i \leq n+1$ we define $M_i = [v_1, v_1^*, \ldots, v_{n+1}, v_{n+1}^*, v_i, v_i^*, \ldots, v_n, v_n^*]$, i.e. M_i is obtained by replacing the v_{i-1} and v_{i-1}^* columns by v_{n+1} and v_{n+1}^* respectively. Thus, the columns of each M_j are linearly independent and so M_j is invertible. Suppose we also have p different $2n \times 2n$ diagonal matrices $D_1, D_2, \ldots, D_p$ with all diagonal entries negative. In each of these diagonal matrices we assume that the (k, k) entry equals the $(k-1, k-1)$ entry whenever k is even.

Suppose $A_{i,h} = M_i D_h M_i^{-1}$ has real entries for $1 \leq i \leq n+1$ and $1 \leq h \leq p$. In other words, $A_{i,h}$ is a real matrix with complex (non–real) eigenvectors (occurring in conjugate pairs) lying in $\mathcal{V}$ and negative eigenvalues. This gives us m different diagonalisable matrices $A_{i,h}$, where $m = p(n+1)$, because the eigenvectors of $A_{i,h}$ are the columns of M_i and the eigenvalues of $A_{i,h}$ are the diagonal entries of D_h. Let $\mathcal{A}$ be a subset of $\{A_{i,h} : 1 \leq i \leq n+1,\ 1 \leq h \leq p\}$. Thus, $\mathcal{A}$ is a set of at most m different diagonalisable matrices such that any two matrices in $\mathcal{A}$ have at least $n-1$ linearly independent common conjugate pairs of complex eigenvectors.

<u>**Statement of Theorem 4.1 :**</u> If we consider the switching system (1) with the set $\mathcal{A}$ defined as in the above preamble, then the origin is globally attractive.

For ease of exposition we first present an outline of the main ideas. The proof is then developed by means of several key lemmas. Note that in the sequel we use row and column notation interchangeably to denote vectors. This is for convenience of notation.

Outline of Proof:

Step 1 : We replace each $2n \times 2n$ matrix M_i by a $(2n+2) \times (2n+2)$ matrix $\overline{M}_i$. We then replace each $2n \times 2n$ matrix $A_{i,h}$ in $\mathcal{A}$ by a $(2n+2) \times (2n+2)$ matrix $\overline{A}_{i,h}$. The matrices $\overline{A}_{i,h} \in \overline{\mathcal{A}} \triangleq \{\overline{A}_{i,h} : A_{i,h} \in \mathcal{A}\}$ are chosen such that there is at least one conjugate pair of common eigenvectors, $(a_1, a_2, 0, \ldots, 0), (a_1^*, a_2^*, 0, \ldots, 0)$, for all the matrices in $\overline{\mathcal{A}}$, and also such that the properties of the solutions of the dynamic system,

$$\dot{\overline{x}} = \overline{A}(t)\overline{x}, \quad \overline{A}(t) \in \overline{\mathcal{A}}, \tag{11}$$

will ultimately imply the global attractivity of the origin of the system,

$$\dot{x} = A(t)x, \quad A(t) \in \mathcal{A},$$

where $x = (x_1, x_2, \ldots, x_{2n})$ and $\overline{x} = (x_{2n+2}, x_{2n+1}, x_1, x_2, \ldots, x_{2n})$.

Step 2 : For a given $j \in \{1, 2, \ldots, n+1\}$ we consider the $2n+2$ linearly independent columns of $\overline{M}_j$. These form a $2n+2$ dimensional coordinate system which includes $(a_1, a_2, 0, \ldots, 0)$ as one of the axes. We consider the projection of the state $\overline{x}(t)$ onto $(a_1, a_2, 0, \ldots, 0)$ as the dynamics of the system (11) evolve. This projection is given by the first component of the vector,

$$g_j(t) = \overline{M}_j^{-1} x(t),$$

and is denoted by $[g_j]_1(t)$.

Step 3 : We then show that,

$$\lim_{t\to\infty} |Re[g_i]_1(t) - Re[g_j]_1(t)| = 0 = \lim_{t\to\infty} |Im[g_i]_1(t) - Im[g_j]_1(t)|,$$

for all $i, j \in \{1, 2, \ldots, n+1\}$. Note that Re denotes the real part of a complex number and Im denotes the imaginary part. From the above fact we can deduce that $\lim_{t\to\infty}(x_1, x_2, \ldots, x_{2n}) = 0$. This is sufficient to demonstrate the global attractivity of the origin of the system,

$$\dot{x} = A(t)x, \ A(t) \in \mathcal{A}.$$

Technical details of Proof:

Lemma 4.1 : Suppose $\delta_{2,1}, \delta_{2,2}, \delta_{3,1}, \delta_{3,2}, \ldots, \delta_{n+1,1}, \delta_{n+1,2}$ are any non–zero complex numbers. Then there exist non–zero complex numbers b_1, b_2 such that the set $W = \{(b_1, b_2.v_1)$, (b_1^*, b_2^*, v_1^*), $(\delta_{2,1},\delta_{2,2},v_2)$, $(\delta_{2,1}^*, \delta_{2,2}^*, v_2^*)$,$(\delta_{3,1}, \delta_{3,2}, v_3)$, $(\delta_{3,1}^*, \delta_{3,2}^*, v_3^*)$, $\ldots$, $(\delta_{n+1,1}^*, \delta_{n+1,2}^*, v_{n+1}^*)\}$ is linearly independent in $\mathbb{C}^{2n+2}$. Here, (b_1, b_2, v_1) denotes the vector whose first two coordinates are b_1, b_2 and the remaining $2n$ coordinates are the $2n$ coordinates of v_1.

Proof: We know that $\{v_2, v_2^*, \ldots, v_{n+1}, v_{n+1}^*\}$ is linearly independent in $\mathbb{C}^{2n}$ and so there exist unique $\alpha_j, \beta_j \in \mathbb{C}$ such that $v_1 = \sum_{j=2}^{n+1} \beta_j v_j + \alpha_j v_j^*$. Pick $b_2 \in \mathbb{C}$ such that $b_2 \neq 0$ and $b_2 \neq \sum_{j=2}^{n+1} \delta_{j,2}\beta_j + \delta_{j,2}^*\alpha_j$ We will first prove that the set $Z = \{(b_2, v_1), (\delta_{2,2}, v_2), (\delta_{2,2}^*, v_2^*), \ldots, (\delta_{n+1,2}^*, v_{n+1}^*)\}$ is linearly independent in $\mathbb{C}^{2n+1}$. Suppose Z is linearly dependent. Thus,

$$\gamma_1(b_2, v_1) + \sum_{j=2}^{n+1} \gamma_j(\delta_{j,2}, v_j) + \tau_j(\delta_{j,2}^*, v_j^*) = 0,$$

for some complex numbers $\gamma_1, \gamma_j, \tau_j,\ 2 \leq j \leq n+1$ that are not all zero. Hence, $\gamma_1 \neq 0$ since otherwise $\gamma_j = 0 = \tau_j$, for $2 \leq j \leq n+1$. Thus, $v_1 = -\frac{1}{\gamma_1}(\sum_{j=2}^{n+1} \gamma_j v_j + \tau_j v_j^*)$, and so $\beta_j = -\frac{\gamma_j}{\gamma_1}$ and $\alpha_j = -\frac{\tau_j}{\gamma_1}$. Also, $\gamma_1 b_2 + \sum_{j=2}^{n+1} \gamma_j \delta_{j,2} + \tau_j \delta_{j,2}^* = 0$. Thus,

$$b_2 = -\frac{1}{\gamma_1}(\sum_{j=2}^{n+1} \gamma_j \delta_{j,2} + \tau_j \delta_{j.2}^*) = \sum_{j=2}^{n+1} \beta_j \delta_{j,2} + \alpha_j \delta_{j,2}^*,$$

which is false. Hence, Z is linearly independent in $\mathbb{C}^{2n+1}$. Now we will show that W is linearly independent in $\mathbb{C}^{2n+2}$. First note that there exist unique $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n+1}, \omega_2, \omega_3, \ldots, \omega_{n+1} \in \mathbb{C}$ such that,

$$(b_2^*, v_1^*) = \epsilon_1(b_2, v_1) + \sum_{j=2}^{n+1} \epsilon_j(\delta_{j,2}, v_j) + \omega_j(\delta_{j,2}^*, v_j^*).$$

Pick $b_1 \in \mathbb{C}$ such that $b_1 \neq 0$ and $b_1^* - \epsilon_1 b_1 \neq \sum_{j=2}^{n+1} \epsilon_j \delta_{j,1} + \omega_j \delta_{j,1}^*$. Suppose W is linearly dependent. Thus,

$$\zeta_0(b_1^*, b_2^*, v_1^*) + \zeta_1(b_1, b_2, v_1) + \sum_{j=2}^{n+1} \zeta_j(\delta_{j,1}, \delta_{j,2}, v_j) + \mu_j(\delta_{j,1}^*, \delta_{j,2}^*, v_j^*) = 0,$$

for some complex numbers $\zeta_0, \zeta_1, \zeta_j, \mu_j,\ 2 \leq j \leq n+1$, that are not all zero. Hence, $\zeta_0 \neq 0$ since otherwise $\zeta_1 = \zeta_j = \mu_j = 0,\ 2 \leq j \leq n+1$. Hence,

$$(b_2^*, v_1^*) = -\frac{1}{\zeta_0}(\zeta_1(b_2, v_1) + \sum_{j=2}^{n+1} \zeta_j(\delta_{j,2}, v_j) + \mu_j(\delta_{j,2}^*, v_j^*)),$$

and so $\epsilon_1 = -\frac{\zeta_1}{\zeta_0},\ \epsilon_j = -\frac{\zeta_j}{\zeta_0},\ \omega_j = -\frac{\mu_j}{\zeta_0},\ 2 \leq j \leq n+1$. Also, $\zeta_0 b_1^* + \zeta_1 b_1 + \sum_{j=2}^{n+1} \zeta_j \delta_{j,1} + \mu_j \delta_{j,1}^* = 0$, which implies that $b_1^* - \epsilon_1 b_1 = \sum_{j=2}^{n+1} \epsilon_j \delta_{j,1} + \omega_j \delta_{j,1}^*$. This is false and so W is linearly independent. **Q.E.D.**

Define $\overline{M}_i$ to be the following $(2n+2) \times (2n+2)$ matrix:

$$\overline{M}_i = \begin{pmatrix} A & H_i \\ 0 & M_i \end{pmatrix},$$

where 0 is the $2n \times 2$ zero matrix, M_i is the $2n \times 2n$ matrix defined above, $A = \begin{pmatrix} a_1 & a_1^* \\ a_2 & a_2^* \end{pmatrix}$ is an invertible 2×2 matrix (and we choose $a_1,\ a_2$ to be non-real complex numbers) and finally H_i is a $2 \times 2n$ matrix constructed as follows: If v_k forms the j^{th} column of M_i, then the j^{th} column of H_i will comprise the first two coordinates of the vector, in W (from Lemma 4.1), containing v_k. The first coordinate will lie in the first row and the second coordinate will lie in the second row of H_i. Also, in a similar fashion, the $(j+1)^{st}$ column of H_i will comprise the first two coordinates of the vector, in W, containing v_k^*. Thus, for example,

$$H_1 = \begin{pmatrix} b_1 & b_1^* & \delta_{2,1} & \delta_{2,1}^* & \ldots & \delta_{n,1} & \delta_{n,1}^* \\ b_2 & b_2^* & \delta_{2,2} & \delta_{2,2}^* & \ldots & \delta_{n,2} & \delta_{n,2}^* \end{pmatrix},$$

$$H_2 = \begin{pmatrix} \delta_{n+1,1} & \delta_{n+1,1}^* & \delta_{2,1} & \delta_{2,1}^* & \ldots & \delta_{n,1} & \delta_{n,1}^* \\ \delta_{n+1,2} & \delta_{n+1,2}^* & \delta_{2,2} & \delta_{2,2}^* & \ldots & \delta_{n,2} & \delta_{n,2}^* \end{pmatrix},$$

$$H_3 = \begin{pmatrix} b_1 & b_1^* & \delta_{n+1,1} & \delta^*_{n+1,1} & \delta_{3,1} & \delta^*_{3,1} & \dots & \delta_{n,1} & \delta^*_{n,1} \\ b_2 & b_2^* & \delta_{n+1,2} & \delta^*_{n+1,2} & \delta_{3,2} & \delta^*_{3,2} & \dots & \delta_{n,2} & \delta^*_{n.2} \end{pmatrix}.$$

Note that,

$$\overline{M}_i^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}H_iM_i^{-1} \\ 0 & M_i^{-1} \end{pmatrix}.$$

Note that every even row in $\overline{M}_i^{-1}$ is the complex conjugate of the previous row. Define $\overline{D}_h$ to be the $(2n+2)\times(2n+2)$ matrix with zeros everywhere except in the bottom right $2n \times 2n$ corner which is occupied by D_h, i.e.

$$\overline{D}_h = \begin{pmatrix} 0 & 0 \\ 0 & D_h \end{pmatrix}.$$

Define $\overline{A}_{i,h} = \overline{M}_i\overline{D}_h\overline{M}_i^{-1}$. We then have that $\overline{A}_{i,h}$ is a $(2n+2)\times(2n+2)$ matrix of the form

$$\overline{A}_{i,h} = \begin{pmatrix} 0 & C_{i,h} \\ 0 & A_{i,h} \end{pmatrix},$$

where $C_{i,h}$ is a $2 \times 2n$ real matrix, the top left 0 is the 2×2 zero matrix and the bottom left 0 is the $2n \times 2$ zero matrix. Note that $(a_1, a_2, 0, \dots, 0)$, $(a_1^*, a_2^*, 0, \dots, 0)$ form a pair of common eigenvectors for all m matrices $A_{i,h}$. In the following we assume that $x_1, x_2, \dots, x_{2n}, x_{2n+1}, x_{2n+2}$ are all real valued functions of t.
We then have that,

$$\begin{pmatrix} \dot{x}_{2n+2} \\ \dot{x}_{2n+1} \\ \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{2n} \end{pmatrix} = \overline{A}_{i,h} \begin{pmatrix} x_{2n+2} \\ x_{2n+1} \\ x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix}, \tag{12}$$

if and only if,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{2n} \end{pmatrix} = A_{i,h} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix} \quad \text{and} \quad \dot{x}_{2n+2} = \sum_{j=1}^{2n} c_j x_j \quad \text{and} \quad \dot{x}_{2n+1} = \sum_{j=1}^{2n} d_j x_j,$$

for some scalars c_j, d_j. We will now show that $\lim_{t\to\infty}(x_1, x_2, \dots, x_{2n}) = 0$, for any solution $(x_{2n+2}, x_{2n+1}, x_1, \dots, x_{2n})$ to the switching system (11). By the above, this will imply that $\lim_{t\to\infty}(x_1, x_2, \dots, x_{2n}) = 0$, for any solution $(x_1, x_2, \dots, x_{2n})$ to the switching system (1) and that will give us global attractivity of the origin in the switching system (1), and we will be done.

Let $\overline{x} = (x_{2n+2}, x_{2n+1}, x_1, \dots, x_{2n})$. We consider the evolution of the system dynamics (11) in each of the coordinate systems,

$$g_i = \overline{M}_i^{-1}\overline{x}. \tag{13}$$

There are $n+1$ coordinate systems corresponding to $i \in \{1, 2, \ldots, n+1\}$. Each g_i is a vector valued function of t. Let $[g_i]_k$ denote the k^{th} component of g_i. Note that if k is even, then $[g_i]_k(t) = [g_i]^*_{k-1}(t)$. Let $G = \{Re[g_1]_1, Re[g_2]_1, \ldots, Re[g_{n+1}]_1\}$. Let the system dynamics be initially described by,

$$\dot{\overline{x}} = \overline{A}_{j,h}\overline{x}, \tag{14}$$

over some time interval $[t_1, t_2]$. Note that $\dot{g_j} = \overline{D}_h g_j$. Denote the (i,i) element in D_h by $\psi_{h,i}$. Hence, $\psi_{h,k} = \psi_{h.k-1}$, when k is even. Also, $\psi_{h,i}$ is the $(i+2, i+2)$ element in $\overline{D}_h$. Thus, $[\dot{g_j}]_m = \psi_{h.m-2}[g_j]_m$, for $3 \leq m \leq 2n+2$ and $[\dot{g_j}]_1 = 0 = [\dot{g_j}]_2$. Therefore, when we are in system (14), we have,

$$[g_j]_m(t) = [g_j]_m(t_1)e^{\psi_{h,m-2}(t-t_1)}, \text{ for } 3 \leq m \leq 2n+2, \tag{15}$$

and $[g_j]_1$, $[g_j]_2$ are constant functions of t. Consider the evolution of $Re[g_i]_1$ relative to $Re[g_j]_1$. This 'distance', denoted by $d_{i,j}(t)$, is given by,

$$d_{i,j}(t) = |Re[g_i]_1(t) - Re[g_j]_1(t)|, \tag{16}$$

and can be conveniently calculated from,

$$g_i = \overline{M}_i^{-1}\overline{M}_j g_j, \tag{17}$$

We now analyse the structure of the matrix $F_{i,j} = \overline{M}_i^{-1}\overline{M}_j$, for $i \neq j$.

Lemma 4.2 : For $i \neq j$, the first two rows of $F_{i,j}$ are of the form,

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & \alpha & \beta & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & \beta^* & \alpha^* & 0 & \ldots & 0 \end{pmatrix},$$

where α, β depend on i, j. At least one of α, β is non-zero. α appears in the k^{th} column, where $k = 2j-1$, if $i = 1$, and $k = 2i-1$, if $i > 1$. Note also that k is always odd and is never 1.

Proof : Denote the first row of $\overline{M}_i^{-1}$ by $\vec{r_1}$ and so the second row is $\vec{r_1}^*$. Suppose first that $i = 1$. We see that a basis for the orthogonal complement of $\vec{r_1}$ in $\mathbb{C}^{2n+2}$ is given by $\{(a_1, a_2, 0),(b_1, b_2, v_1),(b_1^*, b_2^*, v_1^*),(\delta_{2,1}, \delta_{2,2}, v_2), (\delta^*_{2,1},\delta^*_{2,2}, v_2^*), \ldots,(\delta^*_{n,1},\delta^*_{n,2}, v_n^*)\}$. Hence, by Lemma 4.1, the only places (apart from the first column) in the first row of $F_{i,j}$, which are potentially non–zero, are the k^{th} and $(k+1)^{st}$ columns where k is the number of the column in $\overline{M}_j$ which has $(\delta_{n+1,1}, \delta_{n+1,2}, v_{n+1})$. Thus, $k = 2j-1$. Here, α is the row–column product of $\vec{r_1}$ and $(\delta_{n+1,1}, \delta_{n+1,2}, v_{n+1})$. β is the row–column product of $\vec{r_1}$ and $(\delta^*_{n+1,1}, \delta^*_{n+1,2}, v^*_{n+1})$. Note that at least one of α, β is non–zero. Using the fact that the second row of $\overline{M}_i^{-1}$ is $\vec{r_1}^*$, and proceeding as above, we obtain the appropriate second row for $F_{i,j}$.

Suppose next that $i = 2$. We see that a basis for the orthogonal complement of $\vec{r_1}$ in $\mathbb{C}^{2n+2}$ is given by $\{(a_1, a_2, 0),(\delta_{n+1,1},\delta_{n+1,2},v_{n+1}),(\delta^*_{n+1,1}, \delta^*_{n+1,2}, v^*_{n+1}), (\delta_{2,1}, \delta_{2,2}, v_2), (\delta^*_{2,1},\delta^*_{2,2}, v_2^*), \ldots,(\delta^*_{n,1},\delta^*_{n,2}, v_n^*)\}$. Hence, by Lemma 4.1, the only places (apart from the first column) in the first row of $F_{i,j}$, which are potentially non–zero, are the k^{th} and $(k+1)^{st}$ columns where k is the number of the column in $\overline{M}_j$ which has (b_1, b_2, v_1). Thus, $k = 3 = 2i-1$. Here, α is the row–column product of $\vec{r_1}$ and (b_1, b_2, v_1). β is the row–column product of $\vec{r_1}$

and (b_1^*, b_2^*, v_1^*). Note that at least one of α, β is non–zero. Using the fact that the second row of $\overline{M}_i^{-1}$ is $\vec{r_1}^*$, and proceeding as above, we obtain the appropriate second row for $F_{i,j}$.

Suppose finally that $i \geq 3$. We see that a basis for the orthogonal complement of $\vec{r_1}$ in $\mathbb{C}^{2n+2}$ is given by deleting $(\delta_{i-1,1}, \delta_{i-1,2}, v_{i-1}), (\delta_{i-1,1}^*, \delta_{i-1,2}^*, v_{i-1}^*)$ from $W \cup \{(a_1^*, a_2^*, 0)\}$ where W is the set defined in Lemma 4.1. Hence, by Lemma 4.1 the only places (apart from the first column) in the first row of $F_{i,j}$, which are potentially non–zero, are the k^{th} and $(k+1)^{st}$ columns where k is the number of the column in $\overline{M}_j$ which has $(\delta_{i-1,1}, \delta_{i-1,2}, v_{i-1})$. Thus, $k = 2i-1$. Here, α is the row–column product of $\vec{r_1}$ and $(\delta_{i-1,1}, \delta_{i-1,2}, v_{i-1})$. β is the row–column product of $\vec{r_1}$ and $(\delta_{i-1,1}^*, \delta_{i-1,2}^*, v_{i-1}^*)$. Note that at least one of α, β is non–zero. Using the fact that the second tow of $\overline{M}_i^{-1}$ is $\vec{r_1}^*$, and proceeding as above, we obtain the appropriate second row for $F_{i,j}$. **Q.E.D.**

Since α, β, in the above Lemma, depend on i, j, we will denote them by $\alpha_{i,j}, \beta_{i,j}$.
We combine Lemma 4.2 with (17) to obtain,

$$\begin{aligned} [g_i]_1 &= [g_j]_1 + \alpha_{i,j}[g_j]_k + \beta_{i,j}[g_j]_{k+1}, \\ &= [g_j]_1 + \alpha_{i,j}[g_j]_k + \beta_{i,j}[g_j]_k^*, \quad \text{for} \quad 1 \leq i \leq n+1 \quad \text{with} \quad i \neq j. \end{aligned} \tag{18}$$

Note that (18) is true independent of what system we are in. We combine (15) and (18) to obtain,

$$Re[g_i]_1 - Re[g_j]_1 = e^{\psi_{h,k-2}(t-t_1)}(Re(\alpha_{i,j}[g_j]_k(t_1)) + Re(\beta_{i,j}[g_j]_k^*(t_1))), \quad \text{for} \quad i \neq j, \tag{19}$$

whenever we are in system (14). Hence,

$$d_{i,j}(t) = e^{\psi_{h,k-2}(t-t_1)}|Re(\alpha_{i,j}[g_j]_k(t_1)) + Re(\beta_{i,j}[g_j]_k^*(t_1))|.$$

Consequently, $\frac{dd_{i,j}(t)}{dt} < 0$, or else $Re[g_i]_1$ and $Re[g_j]_1$ both agree over the time interval $[t_1, t_2]$. Thus, the distance between $Re[g_i]_1(t)$ and the constant $Re[g_j]_1(t)$ is either getting smaller or always zero over the time interval $[t_1, t_2]$, when we are in the system described by $\dot{\overline{x}} = \overline{A}_{j,h}\overline{x}$.

Proof of Theorem 4.1 : We will now prove that $\lim_{t\to\infty}(x_1, x_2, \ldots, x_{2n}) = 0$, for any solution $(x_1, x_2, \ldots, x_{2n})$ to the system (1) with the set $\mathcal{A}$ defined as in the preamble for Theorem 4.1, and then we will be done. Note that,

$$\begin{aligned} |Re[g_i]_1(t) - Re[g_j]_1)(t)| &= \frac{1}{2}|\alpha_{i,j}[g_j]_k(t) + \alpha_{i,j}^*[g_j]_k^*(t) + \beta_{i,j}[g_j]_k^*(t) + \beta_{i,j}^*[g_j]_k(t)|, \\ &= \frac{1}{2}|(\alpha_{i,j} + \beta_{i,j}^*)[g_j]_k(t) + (\alpha_{i,j}^* + \beta_{i,j})[g_j]_k^*(t)|, \\ &= \frac{1}{2}(r_{i,j}e^{i\theta_{i,j}}[g_j]_k(t_1)e^{\psi_{h,k-2}(t-t_1)} + r_{i,j}e^{-i\theta_{i,j}}[g_j]_k^*(t_1)e^{\psi_{h,k-2}(t-t_1)}), \\ &= |r_{i,j}\cos(\theta_{i,j} + \tau_{j,k,t_1})||[g_j]_k(t)|, \end{aligned}$$

where $\alpha_{i,j} + \beta_{i,j}^* = r_{i,j}e^{i\theta_{i,j}}$ and $[g_j]_k(t_1) = s_{j,k,t_1}e^{i\tau_{j,k,t_1}}$ are polar representations of the appropriate complex numbers. Denote $|r_{i,j}\cos(\theta_{i,j} + \tau_{j,k,t_1})|$ by Q_{i,j,k,t_1}.

Denote the maximum value (minimum value) of $G(t)$, for any time t in the time interval $[t_1, t_2]$, by $\max^1 G(t)$ ($\min^1 G(t)$). Recall that we are in system (14) when $t \in [t_1, t_2]$. Then,

$$\begin{aligned}
\max\nolimits^1 G(t) - \min\nolimits^1 G(t) &= Re[g_i]_1(t) - Re[g_b]_1(t), \text{ for some } i, b \in \{1, 2, \ldots, n+1\}, \\
&= Re[g_i]_1(t) - Re[g_j]_1(t) + Re[g_j]_1(t) - Re[g_b]_1(t), \\
&= Q_{i,j,k,t_1} |[g_j]_k(t_1)| e^{\psi_{h,k-2}(t-t_1)} + Q_{b,j,q,t_1} |[g_j]_q(t_1)| e^{\psi_{h,q-2}(t-t_1)},
\end{aligned}$$

where, as in Lemma 4.2, $k = 2j-1$, if $i = 1$, and $k = 2i-1$, if $i > 1$. Similarly, $q = 2j-1$, if $b = 1$, and $q = 2b-1$, if $b > 1$. Note that if $Re[g_j]_1$ is a maximum value (or minimum value) of $G(t)$, then the last line above collapses to just one term instead of two, and in this case the following arguments will also work. Now let $B_{i,j,t_1} = Q_{i,j,k,t_1}|[g_j]_k(t_1)| =$ distance between $Re[g_i]_1(t_1)$ and $Re[g_j]_1(t_1)$. Let $B_{b,j,t_1} = Q_{b,j,q,t_1}|[g_j]_q(t_1) =$ distance between $Re[g_j]_1(t_1)$ and $Re[g_b]_1(t_1)$. Also, let $\lambda = \max\{\psi_{h,s} : 1 \le h \le p,\ 1 \le s \le 2n\}$. Note that $\lambda < 0$. Then,

$$\begin{aligned}
\max\nolimits^1 G(t) - \min\nolimits^1 G(t) &\le (B_{i,j,t_1} + B_{b,j,t_1}) e^{\lambda(t-t_1)}, \\
&\le (\max\nolimits^1 G(t_1) - \min\nolimits^1 G(t_1)) e^{\lambda(t-t_1)}.
\end{aligned} \tag{20}$$

The last inequality follows from the fact that, over the time interval $[t_1, t_2]$, $Re[g_i]_1(t)$ remains on the same side of the constant $Re[g_j]_1(t)$, and $Re[g_b]_1(t)$ remains on the other side of $Re[g_j]_1(t)$. This is because the right hand side of (19) does not change sign as time changes over the time interval $[t_1, t_2]$. Note that i and b may change with time and so $\max^1 G(t_1)$ may not correspond to $Re[g_i]_1(t_1)$, and $\min^1 G(t_1)$ may not correspond to $Re[g_b]_1(t_1)$.
Now suppose we switch to the next (second) system described by,

$$\dot{\overline{x}} = \overline{A}_{c,w} \overline{x}, \tag{21}$$

over the time interval $[t_2, t_3]$. Denote the maximum value (minimum value) of $G(t)$, for any time t in $[t_2, t_3]$, by $\max^2 G(t)$ ($\min^2 G(t)$). Then, as above we get,

$$\begin{aligned}
\max\nolimits^2 G(t) - \min\nolimits^2 G(t) &\le (\max\nolimits^2 G(t_2) - \min\nolimits^2 G(t_2))\ e^{\lambda(t-t_2)}, \\
&\le (\max\nolimits^1 G(t_1) - \min\nolimits^1 G(t_1))\ e^{\lambda(t_2-t_1)}\ e^{\lambda(t-t_2)}, \\
&= (\max\nolimits^1 G(t_1) - \min\nolimits^1 G(t_1)) e^{\lambda(t-t_1)}.
\end{aligned}$$

The second inequality above follows from the fact that we start the second system (21) at the time t_2 when we stop the first system (14), and the initial conditions for the second system are the terminal conditions for the first system at time t_2. Thus, $\max^2 G(t_2) - \min^2 G(t_2) \le (\max^1 G(t_1) - \min^1 G(t_1))\ e^{\lambda(t_2-t_1)}$ from (20).

Now suppose we switch to the next (third) system described by,

$$\dot{\overline{x}} = \overline{A}_{d,u} \overline{x}, \tag{22}$$

over the time interval $[t_3, t_4]$. Denote the maximum value (minimum value) of $G(t)$, for any time t in $[t_3, t_4]$, by $\max^3 G(t)$ ($\min^3 G(t)$). Then, as above we get,

$$\max\nolimits^3 G(t) - \min\nolimits^3 G(t) \le (\max\nolimits^1 G(t_1) - \min\nolimits^1 G(t_1))\ e^{\lambda(t-t_1)}.$$

For the general situation, when we have switched for the m^{th} time, we are in the system described by $\dot{\overline{x}} = \overline{A}_{z,f} \overline{x}$ over the time interval $[t_m, t_{m+1}]$. Again, we denote the maximum value (minimum value) of $G(t)$, for any time t in $[t_m, t_{m+1}]$, by $\max^m G(t)$ ($\min^m G(t)$. Then, as above we get,

$$\max^m G(t) - \min^m G(t) \leq (\max^1 G(t_1) - \min^1 G(t_1))e^{\lambda(t-t_1)}.$$

Therefore, since $\lambda < 0$, we have $\lim_{t\to\infty}(\max G(t) - \min G(t)) = 0$, where $\max G(t)$ ($\min G(t)$) denotes the maximum value (minimum value) of $G(t)$ for any time $t \geq t_1$. Thus,

$$\lim_{t\to\infty} |Re[g_i]_1(t) - Re[g_j]_1(t)| = 0, \text{ for all } i, j \in \{1, 2, \ldots, n+1\}.$$

Similarly, we can show that,

$$\lim_{t\to\infty} |Im[g_i]_1(t) - Im[g_j]_1(t)| = 0, \text{ for all } i, j \in \{1, 2, \ldots, n+1\}.$$

Note that,

$$\begin{pmatrix} Re[g_i]_1(t) - Re[g_j]_1(t) \\ Im[g_i]_1(t) - Im[g_j]_1(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \alpha_{i,j} + \beta^*_{i,j} & \alpha^*_{i,j} + \beta_{i,j} \\ -i(\alpha_{i,j} - \beta^*_{i,j}) & -i(\beta_{i,j} - \alpha^*_{i,j}) \end{pmatrix}\begin{pmatrix} [g_j]_k(t) \\ [g_j]^*_k(t) \end{pmatrix}.$$

where the $-i$ is the complex square root of -1. Denote the above vector on the left hand side by $\Delta_{i,j}(t)$ and denote the above 2×2 matrix on the right hand side by $X_{i,j}$. Thus,

$$\Delta_{i,j}(t) = X_{i,j}\begin{pmatrix} [g_j]_k(t) \\ [g_j]^*_k(t) \end{pmatrix}.$$

By noting that the matrices $F_{i,j}$ are invertible, one can show that $X_{i,1}$ is invertible for $2 \leq i \leq n+1$. Hence, $\lim_{t\to\infty}[g_1]_k(t) = 0$, for $3 \leq k \leq 2n+2$, because $\lim_{t\to\infty}\Delta_{i,1}(t) = 0$ for $2 \leq i \leq n+1$.

Therefore, since $\lim_{t\to\infty}\overline{x}(t) = \lim_{t\to\infty}\overline{M}_1 g_1(t)$, we get,

$$\lim_{t\to\infty}\begin{pmatrix} x_{2n+2} \\ x_{2n+1} \\ x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix} = \lim_{t\to\infty}\begin{pmatrix} A & H_1 \\ 0 & M_1 \end{pmatrix}\begin{pmatrix} [g_1]_1(t) \\ [g_1]_2(t) \\ \vdots \\ [g_1]_{2n+2}(t) \end{pmatrix},$$

$$\lim_{t\to\infty}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n} \end{pmatrix} = \lim_{t\to\infty}\begin{pmatrix} 0 & M_1 \end{pmatrix}\begin{pmatrix} [g_1]_1(t) \\ [g_1]_2(t) \\ \vdots \\ [g_1]_{2n+2}(t) \end{pmatrix},$$

$$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus,

$$\lim_{t\to\infty}(x_1, x_2, \ldots, x_{2n}) = 0,$$

and we have global attractivity of the origin in the switching system (1). **Q.E.D.**

5 Concluding remarks

In this paper we have considered two classes of switching system. The first class of system is constructed by switching between component systems that are stable, and whose system matrices can be pairwise triangularised. The results of this paper clearly demonstrate that this condition neither implies the existence of a CQLF for the component systems, nor asymptotic stability of (1). The second class of system is related to pairwise triangularisable systems, and can be considered as a general form of the system class studied in [1]. The tools developed for the study of pairwise triangularisable swtching systems are useful for the study of this class of system, and are used to prove global attractivity of the origin. It can be readily shown for this class of system (using examples generated from standard convex optimization packages, such as the MATLAB LMI toolbox [18]), that these tools can be used to demonstrate asymptotic stability for switching systems of the form of (1), where common quadratic Lyapunov functions do not exist. In such cases, where no CQLF exists, qualitative statements concerning system stability, must be validated using non-Lyapunov techniques. The technique that is utilised in this paper involves proving global attractivity by embedding the original (n-dimensional) state space in a higher dimensional state space. It is likely that this methodology is applicable of other classes of switching system.

Acknowledgments

This work was partially supported by the European Union funded research training network *Multi-Agent Control*, HPRN-CT-1999-00107. The first author gratefully acknowledges many discussions with Professor K.S. Narendra relating to the material presented in this paper.

References

[1] R. N. Shorten and F. Ó Cairbre. On the global attractivity of a class of switching systems. In *proceedings of the American Control Conference*, 2000.

[2] N. Cohen and I. Lewkowicz. Convex Invertible Cones and the Lyapunov Equation. *Linear Algebra and its Applications*, 250(1):105–131, 1997.

[3] Y. Mori, T. Mori, and Y. Kuroe. Classes of Systems having Common Quadratic Lyapunov Functions: Triangular System Matrices. In *proceedings of Electronic Information and Systems Conference (in Japanese)*, 1996.

[4] Y. Mori, T. Mori, and Y. Kuroe. A Solution to the Common Lyapunov Function Problem for Continuous Time Systems. In *proceedings of 36th Conference on Decision and Control*, 1997.

[5] D. Liberzon, J. P. Hespanha, and S. Morse. Stability of Switched Linear Systems: A Lie Algebraic Condition. Technical report, Laboratory for Control Science and Engineering, Yale University, 1998.

[6] R Shorten and K.S. Narendra. On the Stability and Existence of Common Lyapunov Functions for Linear Stable Switching Systems. In *proceedings of Conference on Decision and Control*, 1998.

[7] R. Shorten and F. Ó Cairbre. A Proof Of Global Attractivity for a Class of Switching Systems using a non-Lyapunov Approach. Accepted for publication by *Institute of Mathematics and its Applications :* Journal of Mathematical Control and Information.

[8] K.S Narendra and A.M. Annaswamy. *Stable Adaptive Systems*. Prentice-Hall, 1989.

[9] K. S. Narendra and J. Balakrishnan. A Common Lyapunov Function for Stable LTI Systems with Commuting $\mathcal{A}$-Matrices. *IEEE Transactions on Automatic Control*, 39(12):2469–2471, 1994.

[10] R. Shorten. *A Study of Hybrid Dynamical Systems with Application to Automotive Control*. PhD thesis, Department of Electrical and Electronic Engineering, University College Dublin, Republic of Ireland, 1996.

[11] R. Shorten and K.S. Narendra. A Sufficient Condition for the Existence of a Common Lyapunov Function for Two Second Order Systems: Part 1. Technical report, Center for Systems Science, Yale University, 1997.

[12] R. N. Shorten, F. Ó Cairbre, and P. F. Curran. On the dynamic instability of a class of switching systems. In *Proceedings of IFAC conference on Artificial Intelligence in Real Time Control*, 2000.

[13] R.R. Mohler. *Nonlinear Systems: Dynamics and Control*. Prentice Hall, 1991.

[14] T. Kato. *Perturbation theory for linear operators*. Springer, 1980.

[15] R. Shorten and K.S. Narendra. Necessary and Sufficient Conditions for the Existence of a Common Lyapunov Function for M Second Order Systems. Technical report, Center for Systems Science (in preparation), 2000.

[16] R. Shorten and F. Ó Cairbre. A Proof Of Global Attractivity for a Class of Switching Systems using a non-Lyapunov Approach. Submitted to IMA Journal of Mathematical Control and Information.

[17] D. Angeli. A note on the stability of arbitrarily switched homogeneous systems. Submitted to systems and control letters, 1999.

[18] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali. *LMI Control Toolbox for use with MATLAB*. The Mathworks Inc., 1995.