Nearly Triangularisable Systems: Part 1: Pairwise Triangularisability

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Abstract

A sufficient condition for the existence of a Lyapunov function of the form $V(x) = x^T P x,\ P = P^T > 0,\ P \in \mathbb{R}^{n\times n}$, for the stable linear time invariant systems $\dot{x} = A_i x,\ A_i \in \mathbb{R}^{n\times n},\ A_i \in \mathcal{A} \triangleq \{A_1, ..., A_m\}$, is that the matrices A_i are Hurwitz, and that a non-singular matrix T exists, such that $TA_iT^{-1}, i \in \{1, ..., m\}$, is upper triangular [1, 2, 3, 4]. The existence of such a function, referred to as *a common quadratic Lyapunov function* (CQLF), is sufficient to guarantee the exponential stability of the switching system $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$. In this paper we investigate the stability properties of related classes of switching systems. We consider sets of matrices $\mathcal{A}$, where no single matrix T exists that simultaneously transforms each $A_i \in \mathcal{A}$ to upper triangular form, but where a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\},\ i, j \in \{1, ..., m\}$, are upper triangular. We show that in general, this condition does not imply the existence of a common quadratic Lyapunov function (CQLF). Further, we also show by means of a simple example, that the condition of pairwise triangularisability is not sufficient to guarantee stability of an associated switching system. However, we show that, for special classes of related systems, the origin of the switching system. $\dot{x} = A(t)x,\ A(t) \in \mathcal{A}$, is globally attractive. A novel technique, referred to in this paper as *state-space-embedding*, is developed to derive this result. State-space-embedding is based upon the observation that the stability properties of a n-dimensional switching system may, on occasion, be analysed by embedding the n-dimensional system in a higher dimensional system. The efficacy of this technique is demonstrated by showing the stability of two distinct classes of switching systems, and by utilising these results to design a control system for a real industrial application.

1 Introductory remarks

We consider switching systems of the following form,

$$\dot{x} = A(t)x,\ A(t) \in \mathcal{A} \triangleq \{A_1, A_2, ..., A_m\} \quad (1)$$

where the $A_i,\ i \in \{1, 2, ...m\}$, are constant matrices in $\mathbb{R}^{N\times N}$. The matrices $A_i,\ i \in \{1, 2, ...m\}$, are assumed to be Hurwitz (the eigenvalues of each A_i matrix lies in the open left half of the complex plane). A sufficient condition for the exponential stability of Equation (1), and for the existence of a common quadratic Lyapunov function (CQLF), $V(x) = x^T Px,\ P = P^T > 0,\ P \in \mathbb{R}^{n\times n}$, for the stable linear time invariant (LTI) systems

$$\Sigma_{A_i} : \dot{x} = A_i x,\ x \in \mathbb{R}^n,\ A_i \in \mathcal{A},\ A_i \in \mathbb{R}^{n\times n}, \quad (2)$$

is that the matrices A_i are Hurwitz, and that a non-singular transformation T exists such that TA_iT^{-1} is upper triangular for all $i \in \{1, ..., m\}$. This result was first derived by [2], and further discussed by [3] and [4]. Unfortunately, from a practical viewpoint, the requirement of simultaneous triangularisability imposes unrealistic conditions on the matrices in the set $\mathcal{A}$. It is therefore of interest to extend the results derived by [2] with a view to relaxing this requirement. In this context several authors have recently published new conditions which also guarantee exponential stability of the switching system. Typically, the approach adopted is to bound the maximum allowable perturbations of the matrix parameters from a nominal (triangularisable) set of matrices, thereby guaranteeing the existence of a CQLF; see [2]. In this paper we consider classes of switching systems that are closely related to those studied by [2]. However, rather than assuming maximum allowable perturbations from nominal matrix parameters, we explicitly assume that no single non-singular transformation T exists that simultaneously triangularises all of the matrices in $\mathcal{A}$. Rather, we assume that a number of non-singular matrices T_{ij} exist, such that for each pair of matrices in $\mathcal{A}$, $\{A_i, A_j\}$, the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\}$ are upper triangular. We refer to switching systems that are constructed in this manner as *pairwise triangularisable* switching systems.

In this paper we establish a number of results concerning pairwise triangularisable switching systems. We show that while the condition of pairwise triangularisability is not sufficient to guarantee asymptotic stability, global attractivity of the origin can be established using state-space-embedding by making other assumptions. These results are important for a number of reasons. Primarily, they conclusively invalidate the conjecture made in [5] that the condition of pairwise triangularisability alone is sufficient to guarantee asymptotic stability of (1) for arbitrary switching sequences. Secondly, we show that *state-space-embedding* can be utilised to prove global attractivity of the origin for several classes of switching system. This technique does not utilise concepts from quadratic Lyapunov theory and may therefore be used in situations where such functions do not exist. Finally, these results are used to design a stable switching controller for a car speed control system.

This paper is organised as follows. Preliminary definitions and mathematical results are presented in Section 2. General pairwise triangularisable systems are discussed in Section 3. The method of *state-space-embedding* is formally introduced in Section 4, and the technique is applied to prove the attractivity of a class of switching systems related to those discussed in Section 3. These results are applied to the design of a stable speed control system for an automobile in a companion paper.

2 Comments on pairwise triangularisable systems

As discussed in the introduction, switching systems that are constructed from constituent dynamic systems whose system matrices are simultaneousy triangularisable have been the subject of much interest in the switching system community. The exponential stability of systems in this form can be readily established using standard arguments (an analytical expression for their solution is readily obtained). Nevertheless, the widespread interest in these systems is motivated by the fact that the condition of simultaneous triangularisability is one of the few known (simple) conditions that guarantees the existence of a CQLF for each of the constituent systems. While the theoretical interest in these systems is justified, practical applications motivate a number of open questions pertaining to related classes of switching systems. In this paper we consider switching systems where any two matrices in the switching set $\mathcal{A}$ are Hurwitz, and can be simultaneously triangularised. We refer to systems in this form as *pairwise triangularisable* switching systems. In this context we note the following important open questions.

(i) Is the condition of pairwise triangularisability sufficient guarantee the existence of a CQLF for each of the constituent sub-systems?

(ii) Is the condition of pairwise triangularisability sufficient to guarantee asymptotic stability of the origin of (1)?

(iii) In the event of negative answers to (i) and (ii) (which are not equivalent), is it possible to specify extra conditions in addition to pairwise tringularisability such that the origin of (1) is asymptotically stable?

In the sequel we show via a simple example that the condition of pairwise triangularisability is not sufficient to guarantee the existence of a CQLF for each of the constituent sub-systems. In fact, it is readily shown by means of another simple example, and Theorem 2.2, that this condition does not even guarantee asymptotic stability of the origin of Equation (1). However, despite these negative results, we show using state-space-embedding that a stable sub-class of pairwise traingularisable systems, of extreme practical importance, can be identified. The usefulness of this sub-class is illustrated in Section 6.

(i) Common quadratic Lyapunov functions

Example 3.1 : Consider the following stable LTI systems,

$$\Sigma_{A_i} : \dot{x} = A_i x,\ A_i \in \mathbb{R}^{2\times 2},$$

with,

$$A_1 \approx \begin{bmatrix} -1.0000 & 0.0998 \\ -0.9982 & 0.0980 \end{bmatrix},\ A_2 \approx \begin{bmatrix} -1.00000 & -0.0995 \\ -0.9945 & -0.1049 \end{bmatrix},\ A_3 \approx \begin{bmatrix} -1.0000 & -0.0818 \\ -8.1818 & -1.0000 \end{bmatrix}.$$

The set of matrices for which $A_i^T P + PA_i < 0,\ P = P^T > 0,\ P \in \mathbb{R}^{2\times 2}$, is given by

$$\mathcal{P}_{A_i} : det\{A_i^T P + PA_i\} > 0, \qquad (3)$$

where

$$P = \begin{bmatrix} 1 & p_{12} \\ p_{12} & p_{22} \end{bmatrix}.$$

Equation (3) defines a convex set in (p_{12}, p_{22}) space. These sets are depicted in Figure 1 for each of the systems $\Sigma_{A_1}, \Sigma_{A_2}$, and Σ_{A_3}, as the interior of each of the respective elliptical regions. Clearly, no

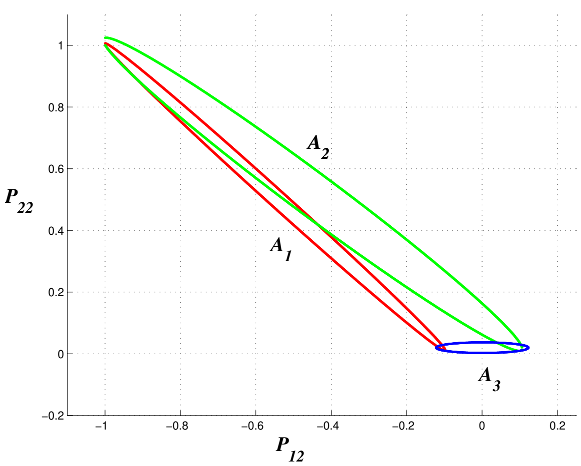

Figure 1: $\mathcal{P}_{A_i}$ for Σ_{A_1}, Σ_{A_2} and Σ_{A_3}.

CQLF exists as the sets $\mathcal{P}_{A_i}$, $i \in [1, 2, 3]$, have no common point of intersection. However, the matrix pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_1, A_3\}$ are pairwise triangularisable with

$$T_{12} = \begin{bmatrix} 1 & 0.1 \\ 1 & 1 \end{bmatrix}, \; T_{23} = \begin{bmatrix} 1 & 1 \\ -10 & 1 \end{bmatrix}, \; T_{13} = \begin{bmatrix} 0.1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(ii) Asymptotic stability of the origin of (1)

It follows from the above example that the condition of pairwise triangulisability of the system matrices does not necessarily imply the existence of a CQLF[1]. In fact, the following example demonstrates that the general condition of pairwise triangularisability is not sufficient to guarantee asymptotic stability for the system (1) for arbitrary switching.

Example 3.2 : Consider the following switching system,

$$\dot{x} = A(t)x, \; A(t) \in \mathcal{A} \triangleq \{A_1, A_2, A_3\}, \qquad (4)$$

[1] A complete solution to the CQLF existence problem for stable second order linear time-invariant systems is given in [?].

with,

$$A_1 \approx \begin{bmatrix} -105.2607 & 33.7326 & 65.1648 \\ 8.3953 & -6.3633 & -8.3953 \\ -104.7741 & 33.7326 & 64.6782 \end{bmatrix},$$

$$A_2 \approx \begin{bmatrix} -19.4935 & 11.8692 & 18.7430 \\ 19.1542 & -12.6197 & -19.1542 \\ -19.0628 & 11.8692 & 18.3124 \end{bmatrix},$$

$$A_3 \approx \begin{bmatrix} 0.1333 & -0.0510 & -0.0921 \\ -1.3313 & 0.4788 & 0.8819 \\ 1.0659 & -0.3866 & -0.7101 \end{bmatrix}.$$

The matrix pairs $\{A_1, A_2\}$, $\{A_2, A_3\}$, $\{A_1, A_3\}$ are pairwise triangularisable with,

$$T_{12} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & -11 \\ 2 & 1 & 9 \end{bmatrix},$$

$$T_{13} = \begin{bmatrix} 3 & 2 & 1 \\ -10 & -11 & -10 \\ 10 & 9 & 8 \end{bmatrix},$$

$$T_{23} = \begin{bmatrix} 2 & 1 & 3 \\ -11 & -10 & -10 \\ 9 & 8 & 10 \end{bmatrix},$$

However, the matrix

$$\begin{aligned} A &= 0.36A_1 + 0.24A_2 + 0.2A_3, \\ &\approx \begin{bmatrix} -15.9127 & 4.7986 & 9.5435 \\ -258.6401 & 90.4471 & 168.7670 \\ 170.8861 & -62.3351 & -114.3442 \end{bmatrix}, \end{aligned}$$

has an eigenvalue with posive real part ($\lambda_i \approx \{-34.9906, 0.0169, -4.8360\}$). It follows from Theorem 2.2, that an unstable switching sequence exists for the system (4). Hence, the general condition of pairwise triangularsability is not a sufficient condition to guarantee asymptotic stability for the system (4). This observation is stated formally as follows.

Theorem 3.1 : Consider the system given by Equation (1). The condition that a set of non-singular matrices T_{ij} exist such that the matrices $\{T_{ij}A_iT_{ij}^{-1}, T_{ij}A_jT_{ij}^{-1}\}$, $i, j, \in \{1, ..., m\}$, are upper triangular, is not sufficient to guarantee the asymptotic stability of the system (1).

Proof : The proof is by contradiction and follows directly from Example 3.2.

(iii) Sufficient conditions for stability

A frequent practical requirement in the design of switching systems is that the nominal closed loop switching-system has real eigenvalues. This requirement ensures that the reponse of the system does not *overshoot*. In this context it is of interest to examine pairwise triangularisable systems with the characteristic that all of the matrices in the set $\mathcal{A}$ have real eigenvalues and real eigenvectors. A simple, but non-trivial, pairwise triangularisable system with this property is constructed by chosing $\mathcal{A}$ such that the matrices A_i are Hurwitz, and that each pair of matrices share $n-1$ common eigenvectors. The matrices specified in Example 3.1 have this property. While these conditions are clearly not sufficient to guarantee the existence of a CQLF, as demonstrated by Example 3.1, it follows via state-space-embedding arguments [5], and by specifying additional minor technical specifications, that these properties are indeed sufficient to guarantee global attractivity of the origin for switching systems of the form of (1). This observation leads to a powerful design technique based upon the following theorem.

Theorem 3.1. Consider the switching system (1) with the set $\mathcal{A}$ defined as follows.

Let $\mathcal{V} = \{v_1, \dots, v_{n+1}\}$ be a set of real vectors, where each $v_i \in \mathrm{I\!R}^n$. Suppose any n vectors in $\mathcal{V}$ are linearly independent. For each $i \in \{1, 2, \dots, n+1\}$, construct an $n \times n$ matrix M_i as follows: $M_1 = [v_1, v_2, \dots, v_{i-1}, v_i, \dots, v_{n-1}, v_n]$ and for $2 \le i \le n+1$ we define $M_i = [v_1, v_2, \dots v_{n+1}, v_i, \dots v_{n-1}, v_n]$, i.e. M_i is obtained by replacing the $(i-1)^{th}$ column in M_1 with the vector v_{n+1}. Suppose we also have p different diagonal matrices $D_1, D_2, \dots, D_p$ with all diagonal entries negative. Define $A_{i,h} = M_i D_h M_i^{-1}$, for $1 \le i \le n+1$ and $1 \le h \le p$. Let $\mathcal{A}$ be a subset of $\{A_{i,h} : 1 \le i \le n+1,\ 1 \le h \le p\}$. Then the origin of the system (1) is globally attractive. It further follows from the results in [6], that systems of the form specified in Theorem 3.1 are asymptotically stable.

<u>Comment :</u> The set $\mathcal{A}$ defined in Theorem 3.1 satisfies the following properties:

(a) Every matrix in $\mathcal{A}$ is Hurwitz and diagonalisable;

(b) The eigenvectors of any matrix in $\mathcal{A}$ are real;

(c) Every pair of matrices in $\mathcal{A}$ share at least $n-1$ linearly independent common eigenvectors and can be simultaneously triangularised [7].

Proof of Theorem 3.1: To aid clarity, and to present the arguments in as concise a manner as possible, the reader is now referred to Section 4, where the complete proof of global attractivity of the origin of (1) for a more general class of switching system, is presented. The proof of Theorem 3.1 uses state-space-embedding and is presented in full-detail the Appendix.

3 State-space-embedding

Before proceeding with the main result of this section, we first give an overview of *state-space-embedding*. This technique does not involve a quadratic Lyapunov function approach and consequently may be employed in problems where such functions do not exist. State-space embedding essentially comprises the following three steps:

(i) Denote our original switching system by $\dot{x} = A_i x$, where $1 \le i \le k$. We replace each matrix A_i by a matrix $\bar{A}_i$ of larger size. The $\bar{A}_i^{'s}$ are selected so that they all share some common eigenvectors. These $\bar{A}_i^{'s}$ generate a new switching system which has higher dimension than the original switching system. The new system is denoted by $\dot{\bar{x}} = \bar{A}_i \bar{x}$, where $1 \le i \le k$.

(ii) We then construct a collection of coordinate systems from various combinations of the eigenvectors of the $\bar{A}_i^{'s}$. One particular common eigenvector τ (of all the $\bar{A}_i^{'s}$) will appear as an axis in each one of these coordinate systems. We then consider, in each coordinate system, the projection of the state $\bar{x}(t)$ onto τ as the dynamics of the new switching system evolves.

(iii) We then consider the convergence of these projections in the long term. For many systems, concrete statements of global attractivity of the origin can be made by considering these projections.

4 Main results

<u>Preamble for Theorem:</u>

Let $\mathcal{V} = \{v_1, \dots, v_{n+1}\}$ be a set of real vectors, where each $v_i \in \mathrm{I\!R}^n$. Suppose any n vectors in $\mathcal{V}$ are linearly independent. For each $i \in \{1, 2, \dots, n+1\}$, construct an $n \times n$ matrix M_i as follows: $M_1 = [v_1, v_2, \dots, v_{i-1}, v_i, \dots, v_{n-1}, v_n]$ and for $2 \le i \le n+1$ we define $M_i =$

$[v_1, v_2, \ldots v_{n+1}, v_i, \ldots v_{n-1}, v_n]$, i.e. M_i is obtained by replacing the $(i-1)^{th}$ column in M_1 with the vector v_{n+1}. Thus, the columns of each M_j come from n different vectors in $\mathcal{V}$, and so each M_j is invertible. Suppose we also have p different diagonal matrices $D_1, D_2, \ldots, D_p$ with all diagonal entries negative.

Define $A_{i,h} = M_i D_h M_i^{-1}$, for $1 \leq i \leq n+1$ and $1 \leq h \leq p$. This gives us m different diagonalisable matrices $A_{i,h}$, where $m = p(n+1)$, because the eigenvectors of $A_{i,h}$ are the columns of M_i and the eigenvalues of $A_{i,h}$ are the diagonal entries of D_h. Let $\mathcal{A}$ be a subset of $\{A_{i,h} : 1 \leq i \leq n+1,\ 1 \leq h \leq p\}$. $\mathcal{A}$ is a set of at most m different diagonalisable matrices such that any two matrices in $\mathcal{A}$ have at least $n-1$ linearly independent common eigenvectors. Therefore, each pair of matrices in $\mathcal{A}$ can be simultaneously upper (lower) triangularised; see [7].

Statement of Theorem:

If we consider the switching system (1) with the set $\mathcal{A}$ defined as in the above preamble, then the origin is globally attractive.

Proof :

5 Example

As an example of the application of the results presented in the previous sections we consider the design of automobile speed control system. A simple model for the longtitudinal dynamics of an automobile is given by the following second order linear parameter varying (LPV) structure,

$$T_{g,v,u} : \frac{d^2 v}{dt^2} + a1_g \frac{dv}{dt} + a0_g v = L_g u, \qquad (5)$$

where $v \in \mathbb{R}$ is the vehicle velocity, $u \in \mathbb{R}$ the throttle angle, $g \in \{1, 2, \cdots, m\}$ is the engaged gear, and where the parameters $a1_g, a0_g, L_g$ vary depending on the engaged gear [7]. The task of speed regulation requires the design of a control system that not only maintains a constant velocity in the presence of gear changes and road disturbances (rolling hills, inclines, wind resistance, etc.), but that also guarantees stability of the nominal closed loop system. In this section we domonstrate that the results presented in this paper may be used as the basis for a controller design that accomodates all of the these design considerations.

The control strategy advocated here consists of a

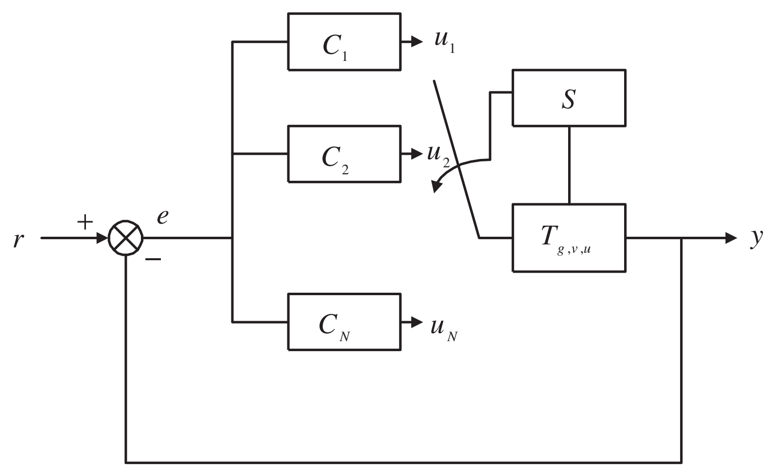

Figure 2: The closed loop switching system. S denotes a switching element

bank of linear controllers (one for each gear) and a switching mechamism that are connected in feedback as depicted in Figure 2. The individual controllers are of the form,

$$C_i : \frac{du_i}{dt} = -b_i u_i + K1_i e + K2_i \frac{de}{dt}, \qquad (6)$$

where $b_i \in \mathbb{R}, K1_i \in \mathbb{R}, K2_i \in \mathbb{R}$, $r \in \mathbb{R}$ is a constant reference velocity, and where $e = r - v$. A controller of this form is a standard lead-lag controller that is described in elementary text-books [8]. We make the additional assumption that the switching logic selects the appropriate individual controller as the instant of gear change. To keep the discussion as simple as possible we assume that $m = 3$, yielding an expression for the closed loop dynamics given by,

$$\dot{x} = A_g x + Br,\ A_g \in \{A_1, A_2, A_3\}, \qquad (7)$$

where

$$x^T = [v,\ \frac{dv}{dt},\ u_1,\ u_2, u_3], \qquad (8)$$

$$A_g = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -a0_g & -a1_g & L_1 d_1(g) & L_2 d_2(g) & L_3 d_3(g) \\ -K2_1 & -K1_1 & -b_1 & 0 & 0 \\ -K2_2 & -K1_2 & 0 & -b_2 & 0 \\ -K2_3 & -K1_3 & 0 & 0 & -b_m \end{bmatrix} \qquad (9)$$

$$B^T = [0,\ 0,\ K1_1,\ K1_2,\ K1_3], \qquad (10)$$

where $d_g(g) = 1$ if the g'th controller is engaged and zero otherwise. We emphasise that the choice of $m = 3$ is motivated by a desire to aid exposition; the arguments and results developed in the sequel generalise to m arbitrary and finite.

Lemma 6.1 : Let $\mathcal{A} = \{A_1, A_2, A_3\}$ with A_g defined by Equation (7). Let the characteristic polynomials of A_1, A_2, A_3 be denoted $\Delta_1(\lambda), \Delta_2(\lambda)$, and $\Delta_3(\lambda)$ respectively, with,

$$\Delta_1(\lambda) = (\lambda + b_3)(\lambda + b_2)H_1(\lambda),$$
$$\Delta_2(\lambda) = (\lambda + b_3)(\lambda + b_1)H_2(\lambda),$$
$$\Delta_3(\lambda) = (\lambda + b_1)(\lambda + b_2)H_3(\lambda),$$

where

$$H_g(\lambda) = \lambda^3 + (a1_g + b_g)\lambda^2 + (a0_g + a1_g b_g + L_g K1_g)\lambda + L_g K2_g + a0_g b_g. \quad (11)$$

Let $H_1(\lambda) = H_2(\lambda) = H_3(\lambda) = H(\lambda)$, and let the roots of the polynomial $P(\lambda) = (\lambda + b_3)(\lambda + b_2)(\lambda + b_1)H(\lambda)$ be distinct. Then the matrices A_1, A_2 and A_3 have exactly $n - 1$ common eigenvectors.

Proof : We show the result for A_1 and A_2 (identical arguments can be developed for the matrix pairs (A_1, A_3) and (A_2, A_3)):

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -a0_1 & -a1_1 & L_1 & 0 & 0 \\ -K2_1 & -K1_1 & -b_1 & 0 & 0 \\ -K2_2 & -K1_2 & 0 & -b_2 & 0 \\ -K2_3 & -K1_3 & 0 & 0 & -b_3 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -a0_2 & -a1_2 & L_2 & 0 & 0 \\ -K2_1 & -K1_1 & -b_1 & 0 & 0 \\ -K2_2 & -K1_2 & 0 & -b_2 & 0 \\ -K2_3 & -K1_3 & 0 & 0 & -b_3 \end{bmatrix}.$$

We need to show that for each common eigenvalue, the matrices A_1 and A_2 have a common eigenvector, and that for the eigenvalue that is not common, the matrices have no common eigenvector. We also note that, by definition, A_1 and A_2 have $n - 1$ distinct common eigenvalues. These eigenvalues correspond to the roots of $H(\lambda)$ and $\lambda = -b_3$. The eigenvalues $\lambda = -b_1$ (corresponding to A_1) and $\lambda = -b_2$ (corresponding to A_2) are not common to both matrices.

<u>Common eigenvalues :</u>
It follows from the form of A_1 and A_2 that the common eigenvector that corresponds to $\lambda = -b_3$ is given by $v_1^T = [0,\ 0,\ 0,\ 0,\ 1]$.

Let λ be an eigenvector that is common to both matrices that is not equal to $-b_3$. The eigenvector of A_1 that corresponds to the eigenvalue λ can be obtained by determining the null space of $\lambda I - A_1$:

$$\lambda I - A_1 = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 \\ a0_2 & \lambda + a1_2 & L_1 & 0 & 0 \\ K2_1 & K1_1 & \lambda + b_1 & 0 & 0 \\ K2_2 & K1_2 & 0 & \lambda + b_2 & 0 \\ K2_3 & K1_3 & 0 & 0 & \lambda + b_m \end{bmatrix}.$$

Let $[rA1_1, rA1_2, rA1_3, rA1_4, rA1_5]$ denote the row vectors of the matrix $\lambda I - A_1$, where $rA1_i$ is the i'th row vector. From the above discussion $\lambda \neq b \in \{-b_1, -b_2, -b_3\}$. Hence, it immediately follows that the row vectors $[rA1_1,\ rA1_3,\ rA1_4,\ rA1_5]$ are linearly independent. However, the matrix $\lambda I - A_1$ is nilpotent. Hence, it must be possible to write $rA1_2$ as a linear combination of $[rA1_1,\ rA1_3,\ rA1_4,\ rA1_5]$. This implies that the eigenvector corresponding to λ, is completely specified by the vectors $[rA1_1,\ rA1_3,\ rA1_4,\ rA1_5]$.

Now consider the matrix A_2. The eigenvector of A_1 that corresponds to the eigenvalue λ can be obtained by determining the null space of $\lambda I - A_2$:

$$\lambda I - A_2 = \begin{bmatrix} \lambda & -1 & 0 & 0 & 0 \\ a0_2 & \lambda + a1_2 & L_2 & 0 & 0 \\ K2_1 & K1_1 & \lambda + b_1 & 0 & 0 \\ K2_2 & K1_2 & 0 & \lambda + b_2 & 0 \\ K2_3 & K1_3 & 0 & 0 & \lambda + b_m \end{bmatrix}.$$

As before, let $[rA2_1, rA2_2, rA2_3, rA2_4, rA2_5]$ denote the row vectors of the matrix $\lambda I - A_2$, where $rA2_i$ is the i'th row vector. Again, $\lambda \neq b \in \{-b_1, -b_2, -b_3\}$. Hence, the row vectors $[rA2_1,\ rA2_3,\ rA2_4,\ rA2_5]$ are linearly independent. However, as the matrix $\lambda I - A_2$ is nilpotent, it must again be possible to write $rA2_2$ as a linear combination of $[rA2_1,\ rA2_3,\ rA2_4,\ rA2_5]$. This implies that the eigenvector corresponding to λ, is completely specified by the vectors $[rA2_1,\ rA2_3,\ rA2_4,\ rA2_5]$.

But, the matrices A_1 and A_2 are identical except for the second row. Hence, it follows that $rA1_i = rA2_i,\ \forall i = \{1, 3, 4, 5\}$, and that the matrices A_1 and A_2 have a common eigenvector for all common eigenvalues λ, $\lambda \neq b \in \{-b_1, -b_2, -b_3\}$. **Q.E.D.**

<u>Eigenvalue that are not common to both matrices :</u>
Consider the matrix A_1. The eigenvalue of A_1

that is not common to A_2 is $-b_1$. The eigenvector of A_1 that corresponds to this eigenvalue is $v_1^T = [0,\ 0,\ 0,\ 1,\ 0]$. Now consider the matrix A_2. The eigenvalue of A_2 that is not common to A_1 is $-b_1$. The eigenvector of A_2 that corresponds to this eigenvalue is $v_2^T = [0,\ 0,\ 1,\ 0,\ 0]$. Clearly, $v_1 \neq v_2$. **Q.E.D.**

We now note the following facts concerning the matrix A_g.

(i) $rank\{A_i - A_j\} = 1,\ i \neq j,\ i,j \in \{1,2,3\}$.

(ii) The characteristic polynomials $\Delta_1(\lambda), \Delta_2(\lambda)$, and $\Delta_3(\lambda)$ share $n-1$ common eigenvalues if $H_i(\lambda) = H(\lambda),\ i \in \{1,2,3\}$.

(iii) Let $H_i(\lambda) = H(\lambda),\ i \in \{1,2,3\}$. Then the matrices A_i and A_j, $i \neq j$, $i,j \in \{1,2,3\}$ satisfy Lemma 6.1 and share $n-1$ common real eigenvectors.

Therefore, sufficient conditions for the matrices A_g, $A_g \in \{A_1, A_2, A_3\}$ to satisfy the conditions of Theorem 3.1, and hence for the stability of the unforced system,

$$\dot{x} = A_g x,\ g \in \{A_1, A_2, A_3\} \tag{12}$$

are given by:

(i) the target polynomials $H_i(\lambda)$ have real negative eigenvalues for all $i \in \{1,2,3\}$;

(ii) $H_i(\lambda) = H_j(\lambda),\ i,j \in \{1,2,3\}$;

(iii) $b_i < 0,\ i \in \{1,2,3\}$;

(iv) the roots of the polynomial $P(\lambda) = (\lambda+b_3)(\lambda+b_2)(\lambda+b_1)H(\lambda)$ are distinct.

Each of these conditions can be achieved in an elementary manner with the proposed controller structure. BIBO (bounded-input bounded-output) stability of the forced system (7) follows directly from elementary arguments (see [9]).

6 Concluding remarks

In this paper we have shown global attractivity for a several classes of switching systems. These results were derived using a technique know as state-space embedding. This technique does not rely on the existence of a CQLF, and can therefore be used in situations where such a function does not exist. It is likely that derived methodology is applicable to a wide class of related switching systems.We have also shown that the condition of pairwise triangularisability is not a sufficient condition for the existence of of CQLF, or for the stability of the switching system. However, the derived results also suggest that by imposing additional minor assumptions, the condition of simultaneous triangularisability may be relaxed significantly without the loss of asymptotic stability[2].

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References

[1] Y. Mori, T. Mori, and Y. Kuroe, "Classes of Systems Having Common Quadratic Lyapunov Functions: Triangular System Matrices," in *Proceedings of Electronic Information and Systems Conference (in Japanese)*, 1996.

[2] Y. Mori, T. Mori, and Y. Kuroe, "A Solution to the Common Lyapunov Function Problem for Continuous Time Systems," in *Proceedings of 36th Conference on Decision and Control*, 1997.

[3] D. Liberzon, J. P. Hespanha, and S. Morse, "Stability of Switched Linear Systems: A Lie Algebraic Condition," tech. rep., Laboratory for Control Science and Engineering, Yale University, 1998.

[4] R. Shorten and K. Narendra, "On the Stability and Existence of Common Lyapunov Functions for Linear Stable Switching Systems," in In Proceedings *of Conference on Decision and Control, Florida Dec 15th-18th*, 1998.

[5] R. N. Shorten and O. Cairbre, "On the global attractivity of a class of switching systems," in *proceedings of the American Control Conference*, 2000.

[6] D. Angeli, "A note on the stability of arbitrarily switched homogeneous systems." Submitted to systems and control letters, 1999.

[7] R. Shorten, *A Study of Hybrid Dynamical Systems with Application to Automotive Control.* PhD thesis, Department of Electrical and Electronic Engineering, University College Dublin, Republic of Ireland, 1996.

[2] Note that the concepts of stability and attractivity are not equivalent; see [10].

[8] H. Power and R. Simpson, *An Introduction to Dynamics and Control.* McGraw-Hill, 1978.

[9] W. Rugh, *Linear system theory.* Prentice Hall, 1996.

[10] K. Narendra and A. Annaswamy, *Stable Adaptive Systems.* Prentice-Hall, 1989.