On the infinity-norm and the asymptotic stability of a class of second order time-varying systems

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Abstract

In this paper we present a sufficient condition for global asymptotic stability of certain classes of second order time-varying systems. While the proof is based upon geometrical and non-quadratic Lyapunov arguments, the stability conditions are expressed as simple restrictions on a matrix pencil. These restrictions possess a simple geometrical interpretation and generalize to higher order systems. The conditions are invariant under similarity transformations, and are readily verifiable via root-locus checking. Further, the conditions establish a direct connection between a matrix pencil and the generalized infinity norm, and can therefore be viewed as extending the link between known matrix pencil conditions and quadratic stability criteria.

1 Introduction

We consider a class of second order linear switching systems $\Sigma_{\boldsymbol{A}(t)}$ of the form

$$\Sigma_{\boldsymbol{A}(t)}: \qquad \dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}, \qquad \boldsymbol{A}(t) \in \{\boldsymbol{A}_1, \boldsymbol{A}_2\} \quad (1)$$

where $\boldsymbol{A}_1, \boldsymbol{A}_2$ are constant Hurwitz matrices in $\mathbb{R}^{2\times 2}$. Switching occurs according to an arbitrary switching sequence $\mathcal{S}$, defined by

$$\mathcal{S}: \quad (\boldsymbol{A}_1, t_0), (\boldsymbol{A}_2, t_1), (\boldsymbol{A}_1, t_2), (\boldsymbol{A}_2, t_3), \ldots \qquad (2)$$

where t_k denotes the switching instant at which the dynamics change to those specified by the respective matrix $\boldsymbol{A}_i$. While systems of this form have been widely studied [1], many fundamental questions pertaining to their stability remain unanswered. Of these, questions pertaining to non-quadratic stability criteria have been almost entirely developed from a numerical viewpoint in recent literature. In this paper we present non-quadratic stability criteria for second order systems. These conditions can be simply verified by checking that the locus of eigenvalues of the matrix pencil $\alpha\boldsymbol{B}_1 + (1-\alpha)\boldsymbol{B}_2$ remains exclusively in a sector of the complex plane, where the matrices $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are related to $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ in Equation (1). This region is depicted in Figure 1, and is referred to as the 45^o *region*. In this paper we illustrate that this region arises naturally when discussing the stability of two distinct sub-classes of switching systems; the first considers systems where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have real distinct eigenvalues; the second where these matrices are in companion form.

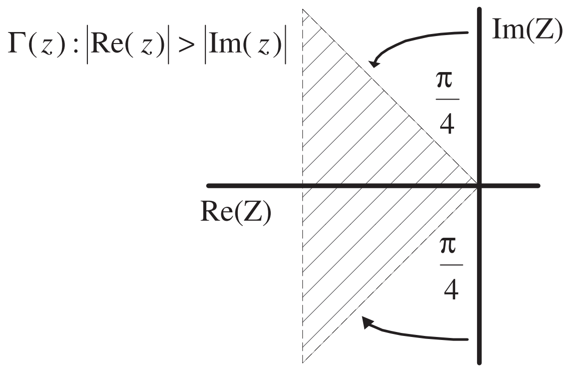

Figure 1: The 45^o region.

This paper is structured as follows. We present some mathematical preliminaries in Section 2. In Section 3 we present the main results of the paper. The use of the infinity norm in the derivation of the main results is discussed in Section 4. The relationship between the results presented in this paper and those discussed in [2, 3] are discussed briefly in the Section 5.

2 Mathematical Preliminaries

In this section we present general results and definitions that are useful in stating the main results of this paper. For ease of exposition the results are simply quoted. Full proofs can be found in [4].

Definition 2.1 (Matrix Pencil) Matrix pencils form the basis of much of the discussion in this paper. We use two forms of the matrix pencil interchangeably. These are defined as follows:

$$\boldsymbol{P}_{\alpha}[\boldsymbol{A}_1, \boldsymbol{A}_2] = \alpha \boldsymbol{A}_1 + (1-\alpha)\boldsymbol{A}_2, \quad \alpha \in [0,1], \tag{3}$$

and

$$\boldsymbol{P}_{\gamma}[\boldsymbol{A}_1, \boldsymbol{A}_2] = \boldsymbol{A}_1 + \gamma \boldsymbol{A}_2, \quad \gamma \in [0,\infty) \tag{4}$$

<u>Comment:</u> Note that $\boldsymbol{P}_{\alpha}$ can be written as $(\frac{1-\alpha}{\alpha})\boldsymbol{P}_{\gamma}$ for $\alpha \neq 0$. Hence, for all $\alpha \neq 0$, the eigenvalues of $\boldsymbol{P}_{\gamma}$ and $\boldsymbol{P}_{\alpha}$ are identical up to a positive scaling factor.

Definition 2.2 (The Matrix $\bar{\boldsymbol{A}}$) The arbitrary matrix $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ with distinct eigenvalues d_1, d_2 can be decomposed as follows:

$$\boldsymbol{A} = \boldsymbol{M}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\boldsymbol{M}^{-1}$$

where $\boldsymbol{M}$ is the modal matrix associated with $\boldsymbol{A}$. Then,

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{M}\left[\begin{pmatrix} d_1 & 0 \\ 0 & d_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d_2 - d_1 \end{pmatrix}\right]\boldsymbol{M}^{-1}, \\ &= \boldsymbol{M} d_1 \boldsymbol{I}\boldsymbol{M}^{-1} + \boldsymbol{M}\begin{pmatrix} 0 & 0 \\ 0 & d_2 - d_1 \end{pmatrix}\boldsymbol{M}^{-1} \\ &= d_1 \boldsymbol{I} + \bar{\boldsymbol{A}} \end{aligned} \tag{5}$$

where $\mid d_2 \mid > \mid d_1 \mid$.

Lemma 2.1 [4] Consider the switching system $\Sigma_{\boldsymbol{A}(t)}$ in Equation (1) with real negative eigenvalues. If the solution to the corresponding switching system

$$\Sigma_{\bar{\boldsymbol{A}}(t)}: \qquad \dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}(t)\bar{\boldsymbol{x}}, \qquad \bar{\boldsymbol{A}}(t) \in \{\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2\} \tag{6}$$

is bounded, then the system $\Sigma_{\boldsymbol{A}(t)}$ is globally asymptotically stable.

Lemma 2.2 [4] Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ and $\boldsymbol{v}_3$ be vectors in $\mathbb{R}^2$ where every two vectors $\{\boldsymbol{v}_i, \boldsymbol{v}_j\}$ are linearly independent. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be linearly independent vectors in $\mathbb{R}^2$ with $\boldsymbol{w}_3 = \boldsymbol{w}_1 + \boldsymbol{w}_2$. Then, there exists an invertible matrix $\boldsymbol{T}$ such that $\boldsymbol{T}\boldsymbol{v}_1 = \boldsymbol{w}_1$, $\boldsymbol{T}\boldsymbol{v}_2 = \boldsymbol{w}_2$ and $\boldsymbol{T}\boldsymbol{v}_3 = \boldsymbol{w}_3$.

Corollary 2.1 Let $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathbb{R}^{2\times 2}$, be invertible matrices with real eigenvalues and real eigenvectors. Let the eigenvectors of $\boldsymbol{A}_1$ be denoted $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, and those of $\boldsymbol{A}_2$ by $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$. Let every pair of vectors $\{\boldsymbol{v}_i, \boldsymbol{v}_j\} \in \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$, $i \neq j$ be linearly independent. Then, there exists an invertible matrix $\boldsymbol{T}$ such that three of the eigenvectors of the transformed matrices $\boldsymbol{T}\boldsymbol{A}_1\boldsymbol{T}^{-1}$ and $\boldsymbol{T}\boldsymbol{A}_2\boldsymbol{T}^{-1}$ are given by $\boldsymbol{w}_1 = [1,\ 1]^{\mathsf{T}}, \boldsymbol{w}_2 = [1, -1]^{\mathsf{T}}$ and $\boldsymbol{w}_3 = [0,\ 2]^{\mathsf{T}}$, respectively.

3 Main results

We now present the main results of this paper. We begin with the case where $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ have real distinct eigenvalues.

Theorem 3.1 Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be two Hurwitz matrices with real distinct eigenvalues. Let $\bar{\boldsymbol{A}}_i$, $i \in \{1,2\}$ be defined as in Section 2. A necessary and sufficient condition for the dynamic systems $\Sigma_{\bar{\boldsymbol{A}}_1}$ and $\Sigma_{\bar{\boldsymbol{A}}_2}$ to have a common Lyapunov function, is that the eigenvalues of the matrix pencil $\boldsymbol{P}_{\alpha}[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ satisfy $\mathrm{Re}\{\lambda(\boldsymbol{P}_{\alpha})\} \leq -|\mathrm{Im}\{\lambda(\boldsymbol{P}_{\alpha})\}|$. This is a sufficient condition for the global asymptotic stability of the system (1).

<u>Outline of Proof:</u> Consider the switching system $\Sigma_{\bar{\boldsymbol{A}}}$. Without loss of generality we assume that the eigenvector of $\bar{\boldsymbol{A}}_1$ whose eigenvalue is zero (referred to as the *slow* eigenvector of $\bar{\boldsymbol{A}}_1$) is $\boldsymbol{v}_{11} = [1, -1]^{\mathsf{T}}$. Similarly, the eigenvector of $\bar{\boldsymbol{A}}_1$ associated with the non-zero eigenvalue (the *fast* eigenvalue) can be assumed to be $\boldsymbol{v}_{12} = [0, 1]^{\mathsf{T}}$, and the slow eigenvector of $\bar{\boldsymbol{A}}_2$ to be $\boldsymbol{v}_{21} = [1, 1]^{\mathsf{T}}$ (see Lemma 2.2 and Corollary 2.1).

The system matrices are therefore given by:

$$\begin{aligned} \bar{\boldsymbol{A}}_1 &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}, \\ \bar{\boldsymbol{A}}_2 &= \begin{pmatrix} 1 & y \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -y \\ -1 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} y & -y \\ 1 & -1 \end{pmatrix}, \end{aligned}$$

where $y \in \mathbb{R}$ denotes the unknown component of the non-fixed eigenvector and where the constant scaling factor $\frac{1}{1-y}$, $y \neq 1$ is neglected.

We show (bounded) stability of the origin using a Lyapunov argument, namely we choose a Lyapunov function $V(\boldsymbol{x})$ such that $\dot{V}(\boldsymbol{x}) \leq 0$ along the trajectories of $\Sigma_{\bar{\boldsymbol{A}}_1}$ and $\Sigma_{\bar{\boldsymbol{A}}_2}$. This will guarantee asymptotic stability of (1) for arbitrary switching between the vector fields defined by $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$.

As Lyapunov function candidate we choose the L_{∞}-norm:

$$V(\boldsymbol{x}) = L_{\infty}(\boldsymbol{x}) = \max_i\{|x_i|\}. \tag{7}$$

This is depicted in Figure 2 along with the eigenvectors of $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$.

This function is positive definite. To show that this function is a common Lyapunov function (CLF) for both

Figure 2 and the start of the page.

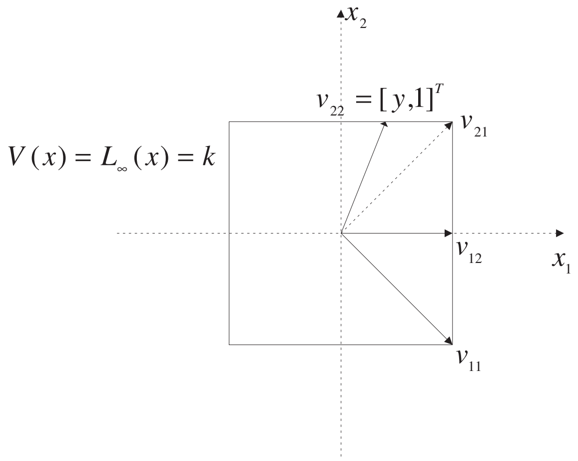

Figure 2: Proof of Theorem 3.1

$\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ we need to show that the time-derivative of the function along these trajectories is non-increasing.

We consider each side of the Lyapunov 'square' separately. Straightforward manipulation reveals that $\dot{V}(\boldsymbol{x})$ is 0 or negative along the trajectories of $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ provided that y is non-positive. When y is positive, $\dot{V}(\boldsymbol{x})$ is positive along some of the trajectories of $\bar{\boldsymbol{A}}_2$, and instability of the system $\Sigma_{\bar{\boldsymbol{A}}(t)}$ (Equation (6)) is possible by switching between the flows of $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$. Hence, $V(\boldsymbol{x})$ is a CLF provided that y is non-positive. A CLF cannot exist when $\dot{V}(\boldsymbol{x})$ is positive.

We now show that the existence of a CLF as discussed above can be characterized in the form of matrix pencil conditions. Consider the matrix pencil $\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ with $\bar{\boldsymbol{A}}_1$ and $\bar{\boldsymbol{A}}_2$ as defined in (6) and (7).

$$\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2] = \begin{pmatrix} y-\gamma & -y-\gamma \\ 1 & -1 \end{pmatrix}, \qquad \gamma \in [0,\infty)$$

The eigenvalues of $\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ are then

$$\lambda_{1,2} = \frac{1}{2}\Big[y-1-\gamma\Big] \pm \sqrt{\frac{1}{4}\Big[y-1-\gamma\Big]^2 - 2\gamma}$$

Straightforward manipulation reveals that the eigenvalues of $\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ lie in the 45^o region for all γ if y is non-positive. For $y > 0$ the eigenvalues of $\boldsymbol{P}_\gamma[\bar{\boldsymbol{A}}_1, \bar{\boldsymbol{A}}_2]$ lie outside the 45^o region for some γ. **QED**

Secondly we consider the following system:

$$\ddot{x} + a\dot{x} + bx + f(x,t)\,[\dot{x} + zx] = 0, \tag{8}$$

where $0 \le f(x,t)$ for all x, t and where real function f has suitable properties to ensure the existence and uniqueness of solutions for all initial conditions and where a, b, z are real constants.

Theorem 3.2 The system (8) is globally asymptotically stable for every function f of the above form provided the root locus of the polynomial $s^2+as+b+k(s+z)$ for $k \ge 0$ is restricted to the interior of the 45^o region.

<u>Outline of Proof:</u> Aizerman's well-known necessary condition for global asymptotic stability for all f requires that polynomial $s^2 + as + b + k(s+z)$ be strictly Hurwitz for all $k \ge 0$. This holds iff $a > 0$, $b > 0$ and $z \ge 0$. The celebrated circle criterion can be applied to this system and yields the following sufficient condition for global asymptotic stability for all f: $a > 0$, $b > 0$, $0 < z \le a$. We may employ a geometrical technique pioneered by Brockett to investigate stability of the system in the case where $a > 0$, $b > 0$, $z > a$.

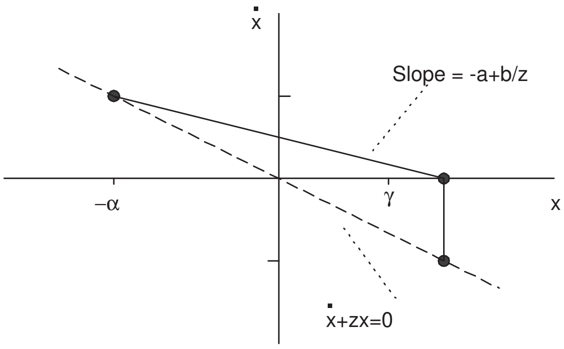

Figure 3: Brockett's construction

The method is based upon the identification of regions in the phase plane to which the system trajectories are restricted. As indicated in Figure 3 this method leads to the conclusion that the trajectory commencing at the point $(x,\dot{x}) = (-\alpha, z\alpha)$, $\alpha > 0$ on the half line $\dot{x} = 0$, $\dot{x} \ge 0$ is restricted to lie in a known triangular region of the phase plane and, consequently, reaches the half line $\dot{x} = 0$, $\dot{x} \le 0$ in finite time at some point $(x,\dot{x}) = (\gamma, -z\gamma)$, $\gamma \ge 0$ where

$$0 \;\le\; \gamma \;\le\; \alpha\frac{\left(z - a + \frac{b}{z}\right)}{\left(a - \frac{b}{z}\right)} \tag{9}$$

The system is globally asymptotically stable if the latter point is closer to the origin than the initial point, i.e. if $\gamma^2\left(1+z^2\right) < \alpha^2\left(1+z^2\right)$. Employing the upper bound on γ and undertaking an elementary algebraic manipulation one obtains the following sufficient condition for global asymptotic stability for all f: $a > 0$, $b > 0$, $z > a$, $z^2 - 2az + 2b < 0$.

Now suppose that the root locus of polynomial $s^2+as+b+k(s+z)$ for $k \ge 0$ is restricted to the 45^o region. It follows readily that $a > 0$, $b > 0$, $z > 0$, i.e. restriction of the root locus to the 45^o region implies that the

Aizerman necessary condition holds. Moreover, if $z > a$ then it follows from this restriction on the root locus that $z^2 - 2az + 2b < 0$. Hence Brockett's sufficient condition for global asymptotic stability for all f holds. On the other hand, if $z \leq a$ then the circle criterion holds. Hence the system is globally asymptotically stable for all f if the root locus lies in the 45^o region.

4 The infinity norm

In view of the central role played by the L_∞-norm in the discussion in Section 3, we comment here on its significance in the context of existing results. Matrix pencils arise naturally in the context of quadratic stability criteria. A quadratic Lyapuonov function for the system $\Sigma_{\boldsymbol{A}_1}$ is defined as a function $V(\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T}\boldsymbol{P}\boldsymbol{x}$, $\boldsymbol{P} = \boldsymbol{P}^\mathsf{T} > 0$, $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ such that $\dot{V}(\boldsymbol{x}) < 0$ along the trajectories of the system. This can be expressed mathematically by requiring that $\boldsymbol{A}^\mathsf{T}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} < 0$. When $V(\boldsymbol{x})$ is a quadratic Lyapunov function for $\Sigma_{\boldsymbol{A}_1}$ *and* $\Sigma_{\boldsymbol{A}_2}$ then $V(\boldsymbol{x})$ is said to be a common quadratic Lyapunov function (CQLF) for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$. Necessary and sufficient conditions for the existence of a CQLF for $\boldsymbol{A}_1,\ \boldsymbol{A}_2 \in \mathbb{R}^{2\times 2}$ are known and are given by the requirement that the matrix pencils $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2]$ and $\boldsymbol{P}_\alpha[\boldsymbol{A}_1, \boldsymbol{A}_2^{-1}]$ are Hurwitz [5, 6].

An alternative viewpoint on the CQLF existence problem is obtained by writing $V(\boldsymbol{x})$ in the following manner:

$$\begin{aligned} V(\boldsymbol{x}) &= \boldsymbol{x}^\mathsf{T}\boldsymbol{P}\boldsymbol{x}, \\ &= \boldsymbol{x}^\mathsf{T}\boldsymbol{H}^\mathsf{T}\boldsymbol{H}\boldsymbol{x}, \\ &= \boldsymbol{z}^\mathsf{T}\boldsymbol{z}, \end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{H}^\mathsf{T} \in \mathbb{R}^{n\times n}$ is the square root of the matrix $\boldsymbol{P}$. Hence, the existence of a CQLF for $\Sigma_{\boldsymbol{A}_1}$ and $\Sigma_{\boldsymbol{A}_2}$ is equivalent to the existence of a coordinate transformation $\boldsymbol{H}$, $\boldsymbol{z} = \boldsymbol{H}\boldsymbol{x}$ such that the Euclidean norm (L_2) is a Lyapunov function for $\Sigma_{\boldsymbol{H}^{-1}\boldsymbol{A}_1\boldsymbol{H}}$ and $\Sigma_{\boldsymbol{H}^{-1}\boldsymbol{A}_2\boldsymbol{H}}$.

The discussion in Section 2 is concerned with establishing the existence of a CLF such that $V(\boldsymbol{x}) = \max_i\{|x_i|\}$ (L_∞-norm). As in the quadratic case, conditions for the existence of such a function can be expressed in matrix pencil form. The conservatism of the quadratic function with respect to the infinity norm is evident from Figure 4.

5 Concluding remarks

In this paper we have presented sufficient conditions for global asymptotic stability of second order switching systems. These conditions are related to various results and conjectures in the stability literature [2, 3, 7]. Work under progress is concerned with investigating whether these results generalize to n-dimensional systems.

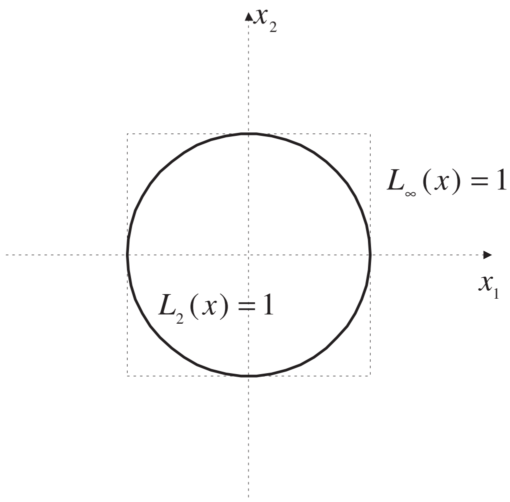

Figure 4: Conservatism of quadratic Lyapunov function

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