Global Reconstruction Of Nonlinear Systems From Families of Linear Systems

D.J.Leith, W.E.Leithead
Department of Electronic & Electrical Engineering, University of Strathclyde,
50 George St., Glasgow G1 1QE, U.K.
Tel. +44 141 548 2407, Fax. +44 141 548 4203, Email. doug@icu.strath.ac.uk

Abstract
This note concerns a fundamental issue in the modelling and realisation of nonlinear systems; namely, whether it is possible to uniquely reconstruct a nonlinear system from a suitable collection of transfer functions and, if so, under what conditions. It is established that a family of frozen-parameter linearisations may be associated with a class of nonlinear systems to provide an alternative realisation of such systems. Nevertheless, knowledge of only the input-output dynamics (transfer functions) of the frozen-parameter linearisations is insufficient to permit unique reconstruction of a nonlinear system. The difficulty with the transfer function family arises from the degree of freedom available in the choice of state-space realisation of each linearisation. Under mild structural conditions, it is shown that knowledge of a family of augmented transfer functions is sufficient to permit a large class of nonlinear systems to be uniquely reconstructed. Essentially, the augmented family embodies the information necessary to select state-space realisations for the linearisations which are compatible with one another and with the underlying nonlinear system. The results are constructive, with a state-space realisation of the nonlinear system associated with a transfer function family being obtained as the solution to a number of linear equations.

1. Introduction

This note concerns a fundamental issue in the modelling and realisation of nonlinear systems; namely, whether it is possible to uniquely reconstruct a nonlinear system from a suitable collection of transfer functions and, if so, under what conditions. Families of linear systems play an important role in many areas of nonlinear systems theory and practice. The construction of nonlinear systems related to a family of linear systems is, for example, the subject of the pseudo-linearisation (*e.g.* Reboulet & Champetier 1984) and extended linearisation (*e.g.* Rugh 1986) approaches and plays a central role in the choice of realisation of gain-scheduled controllers (*e.g.* Lawrence & Rugh 1995, Leith & Leithead 1996,1998a). Families of linear systems also play an important role in system identification practice (*e.g.* Skeppstedt *et al.* 1992, McLoone & Irwin 2000).

A key issue in many application domains is that the linear systems are specified only to within a linear state transformation; that is, the choice of state realisation is available as a degree of freedom. This is usually the situation, for example, in divide and conquer identification (because only input-output data is measurable) and many forms of gain-scheduling design (because the linear methods used to carry out point designs are generally insensitive to the choice of state-space realisation). The objective of this note is to investigate the conditions, if any, under which unique, global reconstruction of a nonlinear system is possible. In order to focus on structural factors and to improve the clarity of the development, attention is restricted here to situations where the linearisation family is well-posed and known exactly; that is, stochastic issues are considered outwith the scope of the present note.

The note is organised as follows. In section 2, families of frozen-parameter linearisations are introduced and discussed. The non-uniqueness associated with standard transfer function information of these linearisations is introduced in section 3 and in section 4 sufficient conditions permitting global, unique reconstruction of a nonlinear system from an appropriate transfer function family are derived. A number of areas of application of these results are indicated in section 5 and the conclusions are summarised in section 6.

2. Preliminaries

It is well known that the family of classical perturbation linearisations of a nonlinear system need not fully characterise the dynamics of a nonlinear system. It is not possible to distinguish between systems having the same equilibrium dynamics but different dynamics away from equilibrium. For example, consider a family of equilibrium linearisations for which the member associated with the equilibrium operating point, (r_o, x_o, y_o), is

$$\delta\dot{x} = -10.1\delta x + 1.01\delta r, \quad \delta y = \delta x \tag{1}$$

$$\delta r = r - r_o, \delta x = x - x_o, y = \delta y + y_o$$

The linearised dynamics are the *same* at every equilibrium point and so might, for example, trivially be associated with the linear system

$$\dot{x} = -10.1x + 1.01r, \quad y = x \qquad (2)$$

However, it is straightforward to confirm that the linearised dynamics might equally be associated with any member of the family of nonlinear systems

$$\dot{x} = G(r - 10x), \qquad y = x \qquad (3)$$

for which G($\bullet$) is any differentiable function such that $\nabla G(0)=1.01$. To enable the nonlinear system to be reconstructed, it is necessary to adopt a different linearisation approach which provides additional information about the dynamics of the system.

Borrowing notation from the LPV/quasi-LPV literature, consider systems of the form

$$\begin{aligned} \dot{\mathbf{z}} &= \big(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{M}\big)\mathbf{z} + \big(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{N}\big)\mathbf{u} \\ \mathbf{v} &= \big(\mathbf{C} + \boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{M}\big)\mathbf{z} + \big(\mathbf{D} + \boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{N}\big)\mathbf{u} \end{aligned} \qquad (4)$$

where $\mathbf{u} \in \Re^m$, $\mathbf{v} \in \Re^p$, $\boldsymbol{\theta} \in \Re^q$, $\mathbf{z} \in \Re^n$, $\boldsymbol{\phi}$, $\boldsymbol{\varphi}$ are nonlinear matrix functions and **A**, **B**, **C**, **D**, **M**, **N** are appropriately dimensioned constant matrices. The defining characteristic of the systems in (4) is that the parameter variation enters via the nonlinear functions $\boldsymbol{\phi}$, $\boldsymbol{\varphi}$ which are, in turn, linearly coupled into the system equations through **M**, **N**. It is assumed that the "parameter" $\boldsymbol{\theta}$ is either measured directly or estimated from measurable signals but no restriction is otherwise placed on $\boldsymbol{\theta}$. In particular, $\boldsymbol{\theta}$ need not be an exogenous variable but may depend via a static or dynamic mapping on the state, **z**, of the system. Confining attention to the class of systems (4) is not overly restrictive as it is easy to verify that any LPV/quasi-LPV system can be formulated as in (4) by, if necessary, appropriately augmenting the parameter vector (trivially by including all the states and all the inputs when required).

Instead of the classical equilibrium linearisations, consider the family of linear systems with members

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \big(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{M}\big)\hat{\mathbf{z}} + \big(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{N}\big)\mathbf{u} \\ \hat{\mathbf{v}} &= \big(\mathbf{C} + \boldsymbol{\varphi}(\boldsymbol{\theta}_1)\mathbf{M}\big)\hat{\mathbf{z}} + \big(\mathbf{D} + \boldsymbol{\varphi}(\boldsymbol{\theta}_1)\mathbf{N}\big)\mathbf{u} \end{aligned} \qquad (5)$$

obtained by "freezing" the parameter, $\boldsymbol{\theta}$, of the system (4). It is important to note that the frozen-parameter linearisation family includes information regarding not only the dynamics relating the input and output (characterised by the transfer function) but also the state-space realisation of each linearisation. As will become clearer in the sequel, the latter plays a key role in the reconstruction of the nonlinear dynamics from a family of frozen-parameter linearisations. Evidently, and quite unlike the situation with classical equilibrium linearisations, knowledge of the state-space frozen-parameter linearisation family, (5), *does* completely define the nonlinear system (4) since it can be recovered by simply allowing $\boldsymbol{\theta}$ to vary in (5); that is, the family of frozen-parameter linearisations is an alternative representation of the nonlinear system (4). Observe that, when $\boldsymbol{\theta}$ depends on the state, **z**, of the system, there is a frozen-parameter linearisation associated with every value of $\boldsymbol{\theta}$ even though in general some may only occur for off-equilibrium operating points. The restriction to near equilibrium operation inherent in the use of classical equilibrium linearisations is thereby avoided. Moreover, expanding, with respect to *time*, the solution **z**(t) of the system (4) relative to an initial time, t_1,

$$\mathbf{z}(t) = \mathbf{z}(t_1) + \dot{\mathbf{z}}(t_1)\delta t + \varepsilon_z \qquad (6)$$

with $\varepsilon_z = \mathbf{z}(t) - \{\mathbf{z}(t_1) + \dot{\mathbf{z}}(t_1)\,\delta t\}$, $\dot{\mathbf{z}}(t_1) = \big(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{M}\big)\mathbf{z}(t_1) + \big(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{N}\big)\mathbf{u}(t_1)$, $\delta t = t - t_1$ and $\boldsymbol{\theta}_1 = \boldsymbol{\theta}(\mathbf{z}(t_1), \mathbf{Nu}(t_1))$. Similarly, expanding the solution of the corresponding frozen-parameter linearisation, $\hat{\mathbf{z}}$, relative to time t_1 then

$$\hat{\mathbf{z}}(t) = \hat{\mathbf{z}}(t_1) + \dot{\hat{\mathbf{z}}}(t_1)\delta t + \varepsilon_{\hat{z}} \qquad (7)$$

with $\dot{\hat{\mathbf{z}}} = \big(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{M}\big)\hat{\mathbf{z}}(t_1) + \big(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{N}\big)\mathbf{u}(t_1)$. For initial condition $\hat{\mathbf{z}}(t_1) = \mathbf{z}(t_1)$, it can therefore be seen that the solution to (5) approximates the solution to (4) with error $O(\delta t^2)$; that is, to first-order in time. By combining the solutions to the members of the frozen-parameter linearisation family in an appropriate manner, a *global* approximation to **z**(t) can be obtained. Over any time interval, $[t_1, t_2]$, an approximation is obtained by partitioning the interval into a number of short sub-intervals. Over each sub-interval, the approximate solution is the solution to (5) with $\boldsymbol{\theta}_1$ equal to the value of $\boldsymbol{\theta}$ at the operating point reached at the initial time for the sub-interval (with the initial conditions chosen to ensure continuity of the approximate solution). The approximation error over each sub-interval is proportional to the duration of the sub-interval squared. Hence, as the number of sub-intervals increases the number of local solutions pieced together increases, the approximation error associated with each decreases more quickly and the overall approximation error reduces. Indeed, since this construction is just Euler integration, it is straightforward to confirm that the overall approximation error tends to zero as the number of sub-intervals becomes unbounded.

3. Conventional Transfer Function[1] Knowledge Alone Is Insufficient

The family of frozen-parameter linearisations, (5), completely defines the system (4) since it can be recovered by simply allowing $\boldsymbol{\theta}$ to vary in (5); that is, the family of frozen-parameter linearisations is an alternative representation of the nonlinear system (4). Nevertheless, this equivalence is dependent on knowledge of the appropriate state co-ordinates for the frozen-parameter linearisations. For example, consider a system in the quasi-LPV form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(\boldsymbol{\theta})\mathbf{x} + \mathbf{B}(\boldsymbol{\theta})\mathbf{r} \\ \mathbf{y} &= \mathbf{C}(\boldsymbol{\theta})\mathbf{x} + \mathbf{D}(\boldsymbol{\theta})\mathbf{r}\end{aligned} \tag{8}$$

with $\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{x},\mathbf{r})$. It can be seen immediately that any quasi-LPV system

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathrm{T}(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})\mathrm{T}^{-1}(\boldsymbol{\theta})\tilde{\mathbf{x}} + \mathrm{T}(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta})\mathbf{r} \\ \tilde{\mathbf{y}} &= \mathbf{C}(\boldsymbol{\theta})\mathrm{T}^{-1}(\boldsymbol{\theta})\tilde{\mathbf{x}} + \mathbf{D}(\boldsymbol{\theta})\mathbf{r}\end{aligned} \tag{9}$$

with $\mathbf{T}(\boldsymbol{\theta})$ non-singular, has frozen-parameter linearisations with transfer functions [1] identical to those of (8). A system (9) may, of course, have quite different dynamics from those of (8): applying the state transformation $\overline{\mathbf{x}} = \mathbf{T}^{-1}(\boldsymbol{\theta})\tilde{\mathbf{x}}$ yields

$$\begin{aligned}\dot{\overline{\mathbf{x}}} &= \mathbf{A}(\boldsymbol{\theta})\overline{\mathbf{x}} + \mathbf{B}(\boldsymbol{\theta})\mathbf{r} + \dot{\mathbf{T}}^{-1}(\boldsymbol{\theta})\mathbf{T}(\boldsymbol{\theta})\overline{\mathbf{x}} \\ \tilde{\mathbf{y}} &= \mathbf{C}(\boldsymbol{\theta})\overline{\mathbf{x}} + \mathbf{D}(\boldsymbol{\theta})\mathbf{r}\end{aligned} \tag{10}$$

Evidently, the dynamics, (10) (equivalent to (9)) differ from (8).

The impact of variations in T(•) may also be seen in the context of constructing the solution to the nonlinear system from the piecewise combination of the solutions to the frozen-parameter linearisations (see §2). For example, consider the piecewise-linear system

$$\dot{\mathbf{z}} = \mathbf{A}(t)\mathbf{z}, \quad \mathbf{A}(t) \in \{\mathbf{A}_1, \mathbf{A}_2, \ldots\} \tag{11}$$

where $\mathbf{A}(t)=\mathbf{A}_i$ on the interval $(t_i,t_{i-1}]$ with $t_1 \le t_2 \le t_3 \ldots$ and $\mathbf{A}_i=\mathbf{T}_i\mathbf{A}\mathbf{T}_i^{-1}$. The solution may be written explicitly as

$$\begin{aligned}\mathbf{z}(t_k) &= \mathbf{e}^{\mathbf{A}_k(t_k-t_{k-1})}\mathbf{e}^{\mathbf{A}_{k-1}(t_{k-1}-t_{k-2})}\cdots\mathbf{e}^{\mathbf{A}_1(t_1-t_o)}\mathbf{z}(t_o) \\ &= \mathbf{e}^{\mathbf{T}_k\mathbf{A}\mathbf{T}_k^{-1}(t_k-t_{k-1})}\mathbf{e}^{\mathbf{T}_{k-1}\mathbf{A}\mathbf{T}_{k-1}^{-1}(t_{k-1}-t_{k-2})}\cdots\mathbf{e}^{\mathbf{T}_o\mathbf{A}\mathbf{T}_o^{-1}(t_1-t_o)}\mathbf{z}(t_o)\end{aligned} \tag{12}$$

The solution is strongly dependent on the properties of the $\mathbf{T}_i$. For example, when the $\mathbf{T}_i$ are identical, the system is precisely linear and thus stable for $\mathbf{A}$ Hurwitz, whereas when the $\mathbf{T}_i$ differ the system behaviour may be highly nonlinear and, in particular, unstable even when $\mathbf{A}$ is Hurwitz (*e.g.* with $\mathbf{A}(t) \in \left\{\begin{bmatrix} 3.5 & -4.5 \\ 13.5 & -14.5\end{bmatrix}, \begin{bmatrix} 3.5 & 4.5 \\ -13.5 & -14.5\end{bmatrix}\right\}$ the $\mathbf{A}_i$ are Hurwitz and similar yet it is straightforward to confirm, using for example the results of Shorten & Narendra (1998), that there exist switching sequences such that (11) is unstable).

The objective of the present paper is to study the situation where a nonlinear system (4) is to be reconstructed from the members of its frozen-parameter linearisations when the latter are specified only to within a linear state transformation (that is, only the transfer functions[1] are specified and the choice of state realisation is an available as a degree of freedom). This is the situation, for example, in divide and conquer identification (because only input-output data is measurable, see for example McLoone & Irwin 2000) and many forms of gain-scheduling design (because the linear methods used to carry out point designs are generally insensitive to the choice of state-space realisation, see for example Leith & Leithead 2000). It is clear that, for each linear system, it is necessary to determine the appropriate choice of state which *cannot* be uniquely inferred from conventional transfer function information alone.

4. Conditions for Reconstructing a Nonlinear System

It is evident from the foregoing discussion that additional information is required in order to permit a nonlinear system to be reconstructed in a unique manner from an associated family of linear transfer functions. Neither the

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[1] Throughout this paper the term 'transfer function' is used as shorthand to denote a linear model based only on measurable input-output data since this is the situation generally encountered in, for example, system identification and gain-scheduling contexts. It includes, in addition to actual transfer function models, linear state-space models where the choice of state co-ordinates is only defined to within a linear transformation. No restriction to frequency-domain methods is implied or necessary.

conventional family of input/output transfer functions associated with the classical equilibrium linearisations nor the family of input/output transfer functions associated with the frozen-parameter linearisations satisfy this requirement. The requirement is thus to determine a suitable family of linear state-space systems which both uniquely defines (to within a non-singular state transformation) a nonlinear system and which is, in turn, uniquely defined by its associated family of transfer functions.

4.1 Conditions for Uniqueness

Consider two nonlinear systems

$$\begin{aligned}\dot{\mathbf{z}} &= \left(\mathbf{A}+\boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{M}\right)\mathbf{z}+\left(\mathbf{B}+\boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{N}\right)\mathbf{u}\\ \mathbf{v} &= \left(\mathbf{C}+\boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{M}\right)\mathbf{z}+\left(\mathbf{D}+\boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{N}\right)\mathbf{u}\end{aligned} \tag{13}$$

and

$$\begin{aligned}\dot{\tilde{\mathbf{z}}} &= \left(\tilde{\mathbf{A}}+\tilde{\boldsymbol{\phi}}(\tilde{\boldsymbol{\theta}})\tilde{\mathbf{M}}\right)\tilde{\mathbf{z}}+\left(\tilde{\mathbf{B}}+\tilde{\boldsymbol{\phi}}(\tilde{\boldsymbol{\theta}})\tilde{\mathbf{N}}\right)\mathbf{u}\\ \tilde{\mathbf{v}} &= \left(\tilde{\mathbf{C}}+\tilde{\boldsymbol{\varphi}}(\tilde{\boldsymbol{\theta}})\tilde{\mathbf{M}}\right)\tilde{\mathbf{z}}+\left(\tilde{\mathbf{D}}+\tilde{\boldsymbol{\varphi}}(\tilde{\boldsymbol{\theta}})\tilde{\mathbf{N}}\right)\mathbf{u}\end{aligned} \tag{14}$$

The system (13) may be reformulated as

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{Az}+\mathbf{Bu}+\boldsymbol{\phi}(\boldsymbol{\theta})\boldsymbol{\vartheta}\\ \mathbf{v}_{\mathbf{aug}} &= \begin{bmatrix}\mathbf{v}\\ \boldsymbol{\vartheta}\end{bmatrix}=\begin{bmatrix}\mathbf{C}\\ \mathbf{M}\end{bmatrix}\mathbf{z}+\begin{bmatrix}\mathbf{D}\\ \mathbf{N}\end{bmatrix}\mathbf{u}+\begin{bmatrix}\boldsymbol{\varphi}(\boldsymbol{\theta})\boldsymbol{\vartheta}\\ 0\end{bmatrix}\end{aligned} \tag{15}$$

and similarly for (14). Assume that the following conditions are satisfied

(i) the members of the frozen-parameter linearisations families corresponding to (15) are controllable and observable and $[\mathbf{M} \quad \mathbf{N}]$ is full rank

(ii) $\boldsymbol{\phi}(\theta_o)$, $\tilde{\boldsymbol{\phi}}(\theta_o)$, $\boldsymbol{\varphi}(\theta_o)$, $\tilde{\boldsymbol{\varphi}}(\theta_o)$ are equal to zero, for some value of $\boldsymbol{\theta}$ equal to $\boldsymbol{\theta}_o$

(iii) there exist no non-zero solutions $\boldsymbol{\Delta}$, $\mathbf{X}$ and $\mathbf{Y}$, satisfying

$$\begin{bmatrix}\boldsymbol{\Delta}\mathbf{A}-\mathbf{A}\boldsymbol{\Delta} & \boldsymbol{\Delta}\mathbf{B}\\ \mathbf{C}\boldsymbol{\Delta} & \mathbf{0}\end{bmatrix}=\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}[\mathbf{M} \quad \mathbf{N}],\ \mathbf{M}\boldsymbol{\Delta}=\mathbf{0} \tag{16}$$

(iv) corresponding members of the frozen-parameter linearisation families (i.e. for which $\boldsymbol{\theta}=\boldsymbol{\theta}_1$, $\tilde{\boldsymbol{\theta}}=\boldsymbol{\theta}_1$) have, respectively, the same transfer function from $\mathbf{u}$ to $\mathbf{v}_{aug}$ and from $\mathbf{u}$ to $\tilde{\mathbf{v}}_{aug}$.

Condition (i) is a standard minimality condition from linear theory whilst condition (ii) removes the possible ambiguity regarding the linear component, if any, of $\boldsymbol{\phi}$, $\tilde{\boldsymbol{\phi}}$, $\boldsymbol{\varphi}$, $\tilde{\boldsymbol{\varphi}}$. Note, condition (iii) needs to be tested for only one member of the linearisation family since it is then automatically satisfied by the entire family. Condition (iv) requires that the transfer function relating the input, $\mathbf{u}$, to $\boldsymbol{\vartheta}$ is known in addition to the transfer function relating $\mathbf{u}$ to $\mathbf{v}$. More information than was available in section 3 is thus available

Proposition (Uniqueness) Assume that conditions (i)-(iv) are satisfied. Then the nonlinear systems (13) and (14) are identical (to within a constant linear state transformation); that is, under structural conditions (i)-(iii) the transfer function information specified in condition (iv) uniquely defines a nonlinear system.

Proof It follows immediately from standard linear theory that when condition (iv) is satisfied

$$\begin{aligned}&\tilde{\mathbf{A}}+\tilde{\boldsymbol{\phi}}(\boldsymbol{\theta}_1)\tilde{\mathbf{M}}=\mathbf{T}(\boldsymbol{\theta}_1)\left(\mathbf{A}+\boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{M}\right)\mathbf{T}^{-1}(\boldsymbol{\theta}_1),\quad \tilde{\mathbf{B}}+\tilde{\boldsymbol{\phi}}(\boldsymbol{\theta}_1)\tilde{\mathbf{N}}=\mathbf{T}(\boldsymbol{\theta}_1)\left(\mathbf{B}+\boldsymbol{\phi}(\boldsymbol{\theta}_1)\mathbf{N}\right)\\ &\tilde{\mathbf{C}}+\tilde{\boldsymbol{\varphi}}(\boldsymbol{\theta}_1)\tilde{\mathbf{M}}=\left(\mathbf{C}+\boldsymbol{\varphi}(\boldsymbol{\theta}_1)\mathbf{M}\right)\mathbf{T}^{-1}(\boldsymbol{\theta}_1),\quad \tilde{\mathbf{D}}+\tilde{\boldsymbol{\varphi}}(\boldsymbol{\theta}_1)\tilde{\mathbf{N}}=\mathbf{D}+\boldsymbol{\varphi}(\boldsymbol{\theta}_1)\mathbf{N} \qquad \forall\boldsymbol{\theta}_1\in\Re^q\\ &\tilde{\mathbf{M}}=\mathbf{MT}^{-1}(\boldsymbol{\theta}_1),\quad \tilde{\mathbf{N}}=\mathbf{N}\end{aligned} \tag{17}$$

where $\mathbf{T}(\boldsymbol{\theta}_1)$ is a non-singular linear state transformation (which may be different for each member of a linear family). Let $\mathbf{T}(\boldsymbol{\theta}_o)$ be the identity matrix; this involves no loss of generality since, by (i), it can always be achieved by applying an appropriate constant linear state transformation. Then, owing to the minimality conditions (ii), it follows that (17) reduces at θ_o to $\tilde{\mathbf{A}}=\mathbf{A}$, $\tilde{\mathbf{B}}=\mathbf{B}$, $\tilde{\mathbf{C}}=\mathbf{C}$, $\tilde{\mathbf{D}}=\mathbf{D}$, $\tilde{\mathbf{M}}=\mathbf{M}$, $\tilde{\mathbf{N}}=\mathbf{N}$. Hence,

$$\begin{bmatrix}\Delta(\theta_1)\mathbf{A}-\mathbf{A}\Delta(\theta_1) & \Delta(\theta_1)\mathbf{B}\\ -\mathbf{C}\Delta(\theta_1) & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\Delta(\theta_1)\tilde{\phi}(\theta_1)\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{M} & \mathbf{N}\end{bmatrix}=\begin{bmatrix}(\phi(\theta_1)-\tilde{\phi}(\theta_1))\\ (\varphi(\theta_1)-\tilde{\varphi}(\theta_1))\end{bmatrix}\begin{bmatrix}\mathbf{M} & \mathbf{N}\end{bmatrix}\quad \forall\theta_1\in\Re^q \tag{18}$$

$$\mathbf{M}\Delta(\theta_1)=\mathbf{0}$$

where $\Delta(\theta_1)=\mathbf{T}^{-1}(\theta_1)-\mathbf{I}$. Condition (iii) ensures that $\Delta(\theta_1)=0$, $\mathbf{X}=0$ and $\mathbf{Y}=0$ is the only solution to

$$\begin{bmatrix}\Delta(\theta_1)\mathbf{A}-\mathbf{A}\Delta(\theta_1) & \Delta(\theta_1)\mathbf{B}\\ -\mathbf{C}\Delta(\theta_1) & \mathbf{0}\end{bmatrix}=\begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\end{bmatrix}\begin{bmatrix}\mathbf{M} & \mathbf{N}\end{bmatrix},\quad \mathbf{M}\Delta(\rho_1)=\mathbf{0} \tag{19}$$

and so by (18) and (iii)

$$\tilde{\phi}(\theta_1)=\phi(\theta_1),\ \tilde{\varphi}(\theta_1)=\varphi(\theta_1)\qquad \forall\theta_1\in\Re^q \tag{20}$$

as required. Consequently, under the foregoing conditions the nonlinear systems (13) and (14) must be identical to within a constant linear state transformation.

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Remark: *Genericity of condition (iii)*. It is evident that violation of condition (iii) requires the simultaneous satisfaction of many linear constraints. Specific systems violating condition (iii) do, of course, exist; for example, in the case of a system for which

$$\mathbf{A}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{bmatrix},\mathbf{B}=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ 0 & 0\end{bmatrix},\mathbf{C}=\begin{bmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\end{bmatrix},\mathbf{D}=\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix},\mathbf{M}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\mathbf{N}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \tag{21}$$

it is straightforward to confirm that

$$\Delta(\rho_1)=\begin{bmatrix}0 & 0 & -\delta(\rho_1)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\mathbf{X}(\rho_1)=\begin{bmatrix}-a_{32} & (a_{11}-a_{33})\\ 0 & a_{21}\\ 0 & 0\end{bmatrix}\delta(\rho_1),\mathbf{Y}(\rho_1)=\begin{bmatrix}0 & -c_{11}\\ 0 & -c_{21}\end{bmatrix}\delta(\rho_1) \tag{22}$$

are solutions to (16). Nevertheless, the class of linearisation families for which there exist non-zero solutions to (16) is *non-generic*. This can be seen as follows. The number of unknowns in (16) is n^2+nq+pn while the number of linear equations is n^2+nq+pn+nm, where n, m, p, q are the dimensions, respectively, of the state, input, output and parameter vectors. From standard linear theory, for any singular matrix there exists an arbitrarily small perturbation which makes it non-singular; that is, non-singular matrices are generic. When m is zero, the number of linear equations is the same as the number of unknowns and it follows immediately from the genericity of non-singular matrices that condition (iii) is also generically satisfied. When m is non-zero, violation of condition (iii) requires singularity of an n^2+nq+pn subsets of equations subject to nm equality constraints and again genericity follows immediately. Hence, for most practical purposes condition (iii) may be assumed to always be satisfied (that is, except in singular circumstances where there exist specific application-related constraints such that consideration of the non-generic solutions to (16) is essential).

4.2 θ and ϑ Linearly Related

Condition (iv) in Section 4.1 requires knowledge of the transfer functions relating ϑ to the input **u**. When ϑ is linearly related to θ and known *a priori*, the transfer function of one may be inferred from that of the other. In these circumstances, condition (iv) can be modified to a requirement for knowledge of the frozen-parameter transfer functions relating **u** to $\begin{bmatrix}\mathbf{v}\\ \theta\end{bmatrix}$. The Uniqueness Proposition may therefore be readily specialised as follows.

Corollary (ϑ Linearly Related to θ) When ϑ is linearly related to θ, conditions (i)-(iii) of section 4.1 together with knowledge of the frozen-parameter transfer functions relating **u** to $\begin{bmatrix}\mathbf{v}\\ \theta\end{bmatrix}$ and the relationship between ϑ and θ uniquely defines a nonlinear system (4). (A similar situation pertains when, for example, the elements of θ are a subset the elements of $\boldsymbol{\vartheta}$ or when the mapping from $\boldsymbol{\vartheta}$ to $\boldsymbol{\theta}$ is defined indirectly via some third quantity, $\boldsymbol{\xi}$ say; note that $\boldsymbol{\xi}$ may be measurable when $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$ are not).

The proof follows directly from the observation in the Uniqueness Proposition that when the relationship between $\boldsymbol{\vartheta}$ and $\boldsymbol{\theta}$ is linear and known *a priori*, the transfer function of one may be inferred from that of the other.

Remark: In this context Condition (iv) of the Uniqueness Condition is a very natural requirement. Information concerning the local evolution of the state is provided by the transfer function relating **u** to **v**. However, the frozen linearisation evolves as the state evolves. Hence, to construct non-local solutions, the information is required to also update the member of the frozen linearisation family being used to define the evolution of the state. This additional information is provided by the transfer function relating **u** to $\boldsymbol{\theta}$.

Examples in the literature to which this corollary is directly relevant include:

(1) <u>State-dependent systems</u> (Priestley 1988, Young 2000)

One particularly interesting special case (studied by, for example, Priestley 1988, Young 2000) is nonlinear systems of the form

$$\dot{\mathbf{z}} = \mathbf{Az} + \mathbf{Bu} + \boldsymbol{\phi}(\boldsymbol{\rho})\boldsymbol{\rho}$$
$$\begin{bmatrix}\mathbf{v}\\ \boldsymbol{\rho}\end{bmatrix} = \begin{bmatrix}\mathbf{C}\\ \mathbf{M}\end{bmatrix}\mathbf{z} + \begin{bmatrix}\mathbf{D}\\ \mathbf{N}\end{bmatrix}\mathbf{u} + \begin{bmatrix}\boldsymbol{\varphi}(\boldsymbol{\rho})\boldsymbol{\rho}\\ 0\end{bmatrix} \tag{23}$$

with $\boldsymbol{\theta}$ and $\boldsymbol{\vartheta}$ equal to $\boldsymbol{\rho}$. The notation, $\boldsymbol{\rho}$, is used here rather than $\boldsymbol{\theta}$ or $\boldsymbol{\vartheta}$, in order to emphasise that for such systems the parameter $\boldsymbol{\rho}$ embodies the nonlinear dependence of the dynamics. Consequently, for example, the series expansion of the right-hand side is solely in terms of $\boldsymbol{\rho}$.

(2) <u>Velocity-based systems</u> (Leith & Leithead 1998b,c)

Following Leith and Leithead (1998b,c), any nonlinear dynamics

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{24}$$

where $\mathbf{r}\in\Re^m$, $\mathbf{y}\in\Re^p$, $\mathbf{x}\in\Re^n$, $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable nonlinear functions may be reformulated as

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Br} + \mathbf{f}(\boldsymbol{\rho}), \quad \mathbf{y} = \mathbf{Cx} + \mathbf{Dr} + \mathbf{g}(\boldsymbol{\rho}) \tag{25}$$

where **A**, **B**, **C** , **D** are appropriately dimensioned constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ are differentiable nonlinear functions and $\boldsymbol{\rho}(\mathbf{x},\mathbf{r})\in\Re^q$, $q\leq m+n$, embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_{\mathbf{x}}\boldsymbol{\rho}$, $\nabla_{\mathbf{r}}\boldsymbol{\rho}$ constant. Trivially, this reformulation can always be achieved by letting $\boldsymbol{\rho} = [\,\mathbf{x}^T \;\; \mathbf{r}^T]^T$, in which case q=m+n. However, the nonlinearity of the system is frequently dependent on only a subset of the states and inputs, in which case the dimension, q, of $\boldsymbol{\rho}$ is less than m+n. Under an appropriate state and input transformation, (25) may be reformulated as a system of the form (4). For example, the archetypal transformation is to differentiate (25), yielding the alternative representation of the nonlinear dynamics

$$\dot{\mathbf{w}} = (\mathbf{A} + \nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{x}}\boldsymbol{\rho})\mathbf{w} + (\mathbf{B} + \nabla\mathbf{f}(\boldsymbol{\rho})\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}}$$
$$\dot{\mathbf{y}} = (\mathbf{C} + \nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{x}}\boldsymbol{\rho})\mathbf{w} + (\mathbf{D} + \nabla\mathbf{g}(\boldsymbol{\rho})\nabla_{\mathbf{r}}\boldsymbol{\rho})\dot{\mathbf{r}} \tag{26}$$

with

$$\dot{\boldsymbol{\rho}} = \nabla_{\mathbf{x}}\boldsymbol{\rho}\,\mathbf{w} + \nabla_{\mathbf{r}}\boldsymbol{\rho}\,\dot{\mathbf{r}} \tag{27}$$

The velocity-based (VB) formulation, (26), is dynamically equivalent to (25) in the sense that, for appropriate initial conditions, they have the same solution. Identifying, for example, **u** with $\dot{\mathbf{r}}$, **z** with **w**, **v** with $\dot{\mathbf{y}}$ and $\boldsymbol{\rho}$ with $\boldsymbol{\theta}$ it is evident that (26) is precisely of the form (4). In this case, it can be seen that $\boldsymbol{\vartheta}$ is associated with $\dot{\boldsymbol{\rho}}$ and so related to $\boldsymbol{\theta}$ by a linear differentiation operator.

4.3 A Reconstruction Methodology

In state-space terms, under conditions (i)-(iv) of section 4.1 the linear family with members

$$\begin{aligned}\dot{\hat{\mathbf{z}}} &= \mathbf{T}(\boldsymbol{\theta}_1)\hat{\mathbf{A}}(\boldsymbol{\theta}_1)\mathbf{T}^{-1}(\boldsymbol{\theta}_1)\hat{\mathbf{z}} + \mathbf{T}(\boldsymbol{\theta}_1)\hat{\mathbf{B}}(\boldsymbol{\theta}_1)\mathbf{u} \\ \begin{bmatrix}\hat{\mathbf{v}} \\ \hat{\boldsymbol{\vartheta}}\end{bmatrix} &= \begin{bmatrix}\hat{\mathbf{C}}(\boldsymbol{\theta}_1) \\ \hat{\mathbf{M}}(\boldsymbol{\theta}_1)\end{bmatrix}\mathbf{T}^{-1}(\boldsymbol{\theta}_1)\hat{\mathbf{z}} + \begin{bmatrix}\hat{\mathbf{D}}(\boldsymbol{\theta}_1) \\ \hat{\mathbf{N}}\end{bmatrix}\mathbf{u}\end{aligned} \tag{28}$$

is known but the state transformation, $\mathbf{T}(\bullet)$, relating the co-ordinates of one member to another is unknown. Note that ^ notation is used to emphasise the distinction between the frozen-parameter linearisations and the associated nonlinear system. Assume, without loss of generality, that $\mathbf{T}(\boldsymbol{\theta}_o)=\mathbf{I}$ (recalling that the system is defined to within a constant linear state transformation, this assumption corresponds to one choice of linear transformation). Assume, also without loss of generality, that the constant matrices associated with the dynamics are

$$\mathbf{A} = \hat{\mathbf{A}}(\boldsymbol{\theta}_o), \mathbf{B} = \hat{\mathbf{B}}(\boldsymbol{\theta}_o), \mathbf{C} = \hat{\mathbf{C}}(\boldsymbol{\theta}_o), \mathbf{D} = \hat{\mathbf{D}}(\boldsymbol{\theta}_o), \mathbf{M} = \hat{\mathbf{M}}(\boldsymbol{\theta}_o) \tag{29}$$

(this simply serves to fix any linear component of the system nonlinearity). The coefficients of the nonlinear system associated with (28) can be obtained as the solution, $\{\mathbf{T}(\bullet), \boldsymbol{\phi}(\bullet), \boldsymbol{\varphi}(\bullet)\}$, to the following linear equalities.

$$\begin{aligned}&\begin{bmatrix}\mathbf{T}(\boldsymbol{\theta}_1)\hat{\mathbf{A}}(\boldsymbol{\theta}_1) - \mathbf{A}\mathbf{T}(\boldsymbol{\theta}_1) & \mathbf{T}(\boldsymbol{\theta}_1)\hat{\mathbf{B}} - \mathbf{B}(\boldsymbol{\theta}_1) \\ \hat{\mathbf{C}}(\boldsymbol{\theta}_1) - \mathbf{C}\mathbf{T}(\boldsymbol{\theta}_1) & \hat{\mathbf{D}}(\boldsymbol{\theta}_1) - \mathbf{D}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\phi}(\boldsymbol{\theta}_1) \\ \boldsymbol{\varphi}(\boldsymbol{\theta}_1)\end{bmatrix}\begin{bmatrix}\hat{\mathbf{M}}(\boldsymbol{\theta}_1) & \mathbf{N}\end{bmatrix} \\ &\hat{\mathbf{M}}(\boldsymbol{\theta}_1) - \mathbf{M}\mathbf{T}(\boldsymbol{\theta}_1) = 0, \quad \hat{\mathbf{N}} = \mathbf{N}\end{aligned} \tag{30}$$

A solution to (30) is guaranteed to be unique by the conditions in the foregoing proposition and corollaries; the nonlinear system thus reconstructed is described by

$$\begin{aligned}\dot{\mathbf{z}} &= \left(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{M}\right)\mathbf{z} + \left(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{N}\right)\mathbf{u} \\ \mathbf{v} &= \left(\mathbf{C} + \boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{M}\right)\mathbf{z} + \left(\mathbf{D} + \boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{N}\right)\mathbf{u}\end{aligned} \tag{31}$$

Example Missile lateral dynamics

Consider a skid-to-turn missile with lateral dynamics described (Leith *et al.* 2000) by the family of frozen-parameter transfer functions

$$\begin{aligned}\dot{\hat{\mathbf{z}}} &= \mathbf{T}(\theta_1)\begin{bmatrix}a_1 + a_2|\theta_1| & a_3 \\ b_3 + b_2|\theta_1| & b_5 + b_4|\theta_1|\end{bmatrix}\mathbf{T}^{-1}(\theta_1)\hat{\mathbf{z}} + \begin{bmatrix}a_5 + a_4|\theta_1| \\ b_7 + b_6|\theta_1|\end{bmatrix}u \\ \mathbf{y} = \hat{\boldsymbol{\vartheta}} &= \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}\mathbf{T}^{-1}(\theta_1)\hat{\mathbf{z}} + \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}u\end{aligned} \tag{32}$$

where $\hat{\mathbf{z}} = [v\ r]^T$, with r the yaw rate (rad/s), v the lateral velocity (m/s), u is the fin angle (rad) and θ_1 ranges over some appropriate set. $\mathbf{T}(\bullet)$ is an unknown state transformation as before and the ^ notation is used distinguish between the frozen-parameter linearisations and the associated nonlinear system. In this example θ is lateral velocity and $\boldsymbol{\vartheta}$ consists of the state and input, with $\hat{\theta} = \begin{bmatrix}1 & 0\end{bmatrix}\hat{\boldsymbol{\vartheta}}$. Note that the availability of measurements of the state and input is not uncommon in an aerospace context. It straightforward to confirm that the transfer functions (32) relating the input $\mathbf{u}$ to $\mathbf{v}_{\mathbf{aug}}$ are controllable, observable and condition (iii). Assume, without loss of generality, that $\mathbf{T}(0)=\mathbf{I}$ and, consequently, the constant matrices associated with the nonlinear dynamics are

$$\mathbf{A} = \begin{bmatrix}a_1 & a_3 \\ b_3 & b_5\end{bmatrix}, \mathbf{B} = \begin{bmatrix}a_5 \\ b_7\end{bmatrix}, \mathbf{C} = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}, D = 0, \mathbf{M} = \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}, \mathbf{N} = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} \tag{33}$$

The reconstructive linear equalities, (30), for this example therefore are

$$\begin{bmatrix} T_{11}(\theta_1) & T_{12}(\theta_1) \\ T_{21}(\theta_1) & T_{22}(\theta_1) \end{bmatrix}\begin{bmatrix} a_1 + a_2|\theta_1| & a_3 \\ b_3 + b_2|\theta_1| & b_5 + b_4|\theta_1| \end{bmatrix} - \begin{bmatrix} a_1 & a_3 \\ b_3 & b_5 \end{bmatrix}\begin{bmatrix} T_{11}(\theta_1) & T_{12}(\theta_1) \\ T_{21}(\theta_1) & T_{22}(\theta_1) \end{bmatrix} = \begin{bmatrix} \phi_{11}(\theta_1) & \phi_{12}(\theta_1) & \phi_{13}(\theta_1) \\ \phi_{21}(\theta_1) & \phi_{22}(\theta_1) & \phi_{23}(\theta_1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_5 + a_4|\theta_1| \\ b_7 + b_6|\theta_1| \end{bmatrix} - \begin{bmatrix} a_5 \\ b_7 \end{bmatrix}\begin{bmatrix} T_{11}(\theta_1) & T_{12}(\theta_1) \\ T_{21}(\theta_1) & T_{22}(\theta_1) \end{bmatrix} = \begin{bmatrix} \phi_{11}(\theta_1) & \phi_{12}(\theta_1) & \phi_{13}(\theta_1) \\ \phi_{21}(\theta_1) & \phi_{22}(\theta_1) & \phi_{23}(\theta_1) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{11}(\theta_1) & T_{12}(\theta_1) \\ T_{21}(\theta_1) & T_{22}(\theta_1) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(\theta_1) & \varphi_{12}(\theta_1) & \varphi_{13}(\theta_1) \\ \varphi_{21}(\theta_1) & \varphi_{22}(\theta_1) & \varphi_{23}(\theta_1) \\ \varphi_{31}(\theta_1) & \varphi_{32}(\theta_1) & \varphi_{33}(\theta_1) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \varphi_{11}(\theta_1) & \varphi_{12}(\theta_1) & \varphi_{13}(\theta_1) \\ \varphi_{21}(\theta_1) & \varphi_{22}(\theta_1) & \varphi_{23}(\theta_1) \\ \varphi_{31}(\theta_1) & \varphi_{32}(\theta_1) & \varphi_{33}(\theta_1) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0 \tag{34}$$

The unique solution to (34) defines the nonlinear missile dynamics

$$\dot{\mathbf{z}} = \begin{bmatrix} a_1 & a_3 \\ b_3 & b_5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} a_5 \\ b_7 \end{bmatrix}u + \begin{bmatrix} a_2|\theta| & 0 & a_4|\theta| \\ b_2|\theta| & b_4|\theta| & b_6|\theta| \end{bmatrix}\boldsymbol{\vartheta}$$
$$\boldsymbol{\vartheta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \tag{35}$$

with $\theta = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{z}$. ■

5. Some Applications

5.1 Extended local linear equivalence systems

Let $E = \{\mathbf{z}_o, \mathbf{u}_o : \mathbf{A}\mathbf{z}_o + \mathbf{B}\mathbf{u}_o + \boldsymbol{\phi}(\boldsymbol{\theta}(\mathbf{z}_o, \mathbf{u}_o))[\mathbf{M}z_o + \mathbf{N}u_o] = 0\}$ denote the set of equilibrium points of the system (4), $R_\theta(E)$ denote the range of $\boldsymbol{\theta}$ on E (*i.e.* $R_\theta(E)=\{\boldsymbol{\theta}(\mathbf{z},\mathbf{u}): ((\mathbf{z},\mathbf{u}) \in E\})$ and $R_\theta(\Phi)$ the range of $\boldsymbol{\theta}$ on the full operating space of the system, $\Phi=\{(\mathbf{z},\mathbf{u}): \mathbf{z}\in \Re^n, \mathbf{u}\in\Re^m\}$. Systems, (4), for which

$$R_\theta(E) = R_\theta(\Phi) \tag{36}$$

are referred to here as extended local linear equivalence (ELLE) systems. The condition, (36), simply corresponds to the requirement that $\boldsymbol{\theta}$ is parameterised by the equilibrium points. It follows immediately that the equilibrium information, $\left\{\begin{bmatrix} \mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{M} & \mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta})\mathbf{N} \\ \mathbf{C} + \boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{M} & \mathbf{D} + \boldsymbol{\varphi}(\boldsymbol{\theta})\mathbf{N} \end{bmatrix} : \boldsymbol{\theta} \in R_\theta(E)\right\}$, together with knowledge of $\boldsymbol{\theta}$, completely defines an ELLE system. In view of the importance of equilibrium information in classical theory (particularly gain-scheduling theory), and the relative ease with which equilibrium dynamics may be identified from measured data, the class of ELLE systems is of considerable interest in its own right. Note that even if not exactly satisfied, it is often possible to utilise, within a useful operating envelope, an ELLE approximation to a non-ELLE system.

The results of section 4 can be immediately specialised to ELLE systems, as summarised by the following corollary.

Corollary (Uniqueness of ELLE Systems) Assume that conditions (i)-(iii) of section 4 are satisfied and that the frozen-parameter linearisations associated with the *equilibrium operating points* of (13) and (14) have the same transfer function from $\mathbf{u}$ to $\mathbf{v}_{aug}$ and from $\mathbf{u}$ to $\tilde{\mathbf{v}}_{aug}$. Assume, in addition, that (13) and (14) belong to the class of ELLE systems. Then the nonlinear systems (13) and (14) are identical (to within a fixed linear state transformation); that is, under conditions (i)-(iii) a nonlinear system is uniquely defined by appropriate equilibrium transfer function information. The proof follows trivially from the foregoing proposition and the definition of ELLE systems.

■

Example Wiener-Hammerstein system

Suppose that the frozen-parameter linearisation transfer functions relating $\mathbf{v_{aug}}$ and u are known and are given by

$$\mathbf{V_{aug}}(s) = \begin{bmatrix} \dfrac{K(\theta_1)}{(s+a)(s+b)} \\ \dfrac{1}{(s+a)} \end{bmatrix} U(s) \tag{37}$$

where $\mathbf{V_{aug}}(s)$, U(s) denote, respectively, the Laplace transforms of $\mathbf{v_{aug}}$, u. Assume also that the structure of the dynamics is such that θ equals ϑ. Equivalently, in state-space terms, we have that

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \mathbf{T}(\theta_1)\begin{bmatrix} -a & 0 \\ K(\theta_1) & -b \end{bmatrix}\mathbf{T}^{-1}(\theta_1)\hat{\mathbf{z}} + \mathbf{T}(\theta_1)\begin{bmatrix} 1 \\ 0 \end{bmatrix}u \\ \hat{\mathbf{v}}_{aug} &= \begin{bmatrix} \hat{v} \\ \hat{\vartheta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{T}^{-1}(\theta_1)\hat{\mathbf{z}} \end{aligned} \tag{38}$$

where $\mathbf{T}(\bullet)$ is an unknown state transformation and the ^ notation is used to emphasise the distinction between the frozen-parameter linearisations and the associated nonlinear system. The linearisations, (38), are controllable and observable. Assume, without loss of generality, that $\mathbf{T}(\theta_o)=\mathbf{I}$ (recalling that the system is defined to within a global linear state transformation, this assumption corresponds to one choice of global linear transformation). Assume, also without loss of generality, that the constant matrices associated with the nonlinear dynamics are

$$\mathbf{A} = \begin{bmatrix} -a & 0 \\ K(\theta_o) & -b \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0, \mathbf{M} = \begin{bmatrix} 1 & 0 \end{bmatrix}, N = 0 \tag{39}$$

(this simply serves to fix any linear component of the system nonlinearity). Uniqueness condition, (16), only requires to be evaluated for a single member of the linearisation family; taking the member corresponding to θ equal to θ_o yields

$$\begin{bmatrix} \Delta_{12}K(\theta_o) & (a-b)\Delta_{12} & \Delta_{11} \\ (b-a)\Delta_{21} + (\Delta_{22}-\Delta_{11})K(\theta_o) & -\Delta_{12}K(\theta_o) & \Delta_{21} \\ \Delta_{21} & \Delta_{22} & 0 \end{bmatrix} = \begin{bmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \\ Y_1 & 0 & 0 \end{bmatrix} \tag{40}$$

where Δ_{ij} denotes the ij[th] element of **Δ**, and similarly for X_i. and Y_1 Evidently, **Δ**=**0**, **X**=**0**, **Y**=**0** is the sole solution, as required. Conditions (i)-(iv) are satisfied and it therefore follows from the foregoing proposition that (37) uniquely defines a nonlinear system. From (30), the coefficients of the nonlinear system associated with (37) (equivalently, (38)) are obtained as the solution to the following linear equalities (note that the existence of a unique solution is guaranteed by the foregoing conditions).

$$\begin{aligned} &\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} -a & 0 \\ K(\theta_1) & -b \end{bmatrix} - \begin{bmatrix} -a & 0 \\ K(\theta_o) & -b \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \phi_1(\theta_1) \\ \phi_2(\theta_1) \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} \\ &\begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \varphi(\theta_1)\begin{bmatrix} 1 & 0 \end{bmatrix} \\ &\begin{bmatrix} 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = 0, \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \end{aligned} \tag{41}$$

It is straightforward to verify that the solution to (41) is

$$\mathbf{T}(\theta_1) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \phi(\theta_1) = \begin{bmatrix} 0 \\ K(\theta_1) - K(\theta_o) \end{bmatrix}, \varphi(\theta_1) = 0 \tag{42}$$

That is, the nonlinear system uniquely defined by the input-output information (37) is

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u + \begin{bmatrix} 0 \\ K(\theta)\vartheta \end{bmatrix} \\ \begin{bmatrix} y \\ \vartheta \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{z} \end{aligned} \tag{43}$$

with $\theta = \vartheta$. This system, depicted in figure 1, is of Wiener-Hammerstein form. The frozen-parameter linearisation family is parameterised by the quantity, θ, while the family of equilibrium points of (43) may be parameterised by the value of the input, u, at equilibrium. Since θ=u/a at equilibrium, the family of equilibrium points may therefore also be parameterised by θ, and *vice-versa*. Hence, (43) belongs to the class of ELLE systems and, in accordance with the definition of this class, the frozen-parameter linearisation family (and so the global nonlinear dynamics) is completely defined by the family of frozen-parameter linearisations at the equilibrium points taken together with appropriate knowledge of θ.

■

Remark *Correspondence between equilibrium linearisations and frozen-parameter linearisations*
In the particular situation where the frozen-parameter linearisations considered are, in fact, VB linearisations, a strong link can be established between the frozen-parameter linearisations and the classical equilibrium linearisations. The classical series expansion linearisation of (25) relative to the equilibrium operating point at which **x** and **r** are, respectively, equal to $\mathbf{x}_o$ and $\mathbf{r}_o$, is

$$\begin{aligned}\delta\dot{\mathbf{x}} &= (\mathbf{A} + \nabla\mathbf{f}(\boldsymbol{\rho}_o)\nabla_{\mathbf{x}}\boldsymbol{\rho})\delta\mathbf{x} + (\mathbf{B} + \nabla\mathbf{f}(\boldsymbol{\rho}_o)\nabla_{\mathbf{r}}\boldsymbol{\rho})\delta\mathbf{r} \\ \delta\mathbf{y} &= (\mathbf{C} + \nabla\mathbf{g}(\boldsymbol{\rho}_o)\nabla_{\mathbf{x}}\boldsymbol{\rho})\delta\mathbf{x} + (\mathbf{D} + \nabla\mathbf{g}(\boldsymbol{\rho}_o)\nabla_{\mathbf{r}}\boldsymbol{\rho})\delta\mathbf{r}\end{aligned} \tag{44}$$

together with the input, output and state transformations

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r}_0, \quad \mathbf{x} = \mathbf{x}_0 + \delta\mathbf{x}, \quad \mathbf{y} = \mathbf{y}_0 + \delta\mathbf{y} \tag{45}$$

where $\boldsymbol{\rho}_o=\boldsymbol{\rho}(\mathbf{x}_o,\mathbf{r}_o)$. In contrast to the classical equilibrium linearisations, the frozen-parameter linearisation family associated with the velocity-based system (26) includes linearisations of the plant at both non-equilibrium and equilibrium operating points. Nevertheless, it is clear that the members of the classical equilibrium linearisation family defined by (44) are closely related to the members of the VB frozen-parameter linearisation family even though the state, input and output are different. In particular, the VB frozen-parameter linearisation family can be determined directly, by inspection, from the classical equilibrium linearisation family provided that there exists an equilibrium operating point corresponding to every value in the range of $\boldsymbol{\rho}$. This correspondence is certainly not the case in general but rather is a feature of systems possessing the ELLE property (and systems for which a sufficiently accurate ELLE approximation exists). It follows immediately that, for ELLE systems, the nonlinear dynamics can be uniquely reconstructed from the classical equilibrium linearisation family taken together with appropriate knowledge of $\boldsymbol{\rho}$.

5.2 Finite parameterisation of linearisation family by blending local models.

The frozen-parameter linearisation family associated with a nonlinear system generally has infinitely many members. In many situations it is preferable to work with a small number of "representative" linearisations and recover the full linearisation family by blending or interpolating between these linearisations. Similar issues arise in many application domains and the literature on blended representations is extensive (see, for example, the survey by Johansen & Murray-Smith 1997, Leith & Leithead 1999, 2000), including numerous approaches related to gain-scheduling. A typical blended multiple model formulation of the nonlinear system (4) blends the linear local models

$$\begin{aligned}\dot{\mathbf{z}} &= \big(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta}_i)\mathbf{M}\big)\mathbf{z} + \big(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta}_i)\mathbf{N}\big)\mathbf{u} \\ \mathbf{v} &= \big(\mathbf{C} + \boldsymbol{\varphi}(\boldsymbol{\theta}_i)\mathbf{M}\big)\mathbf{z} + \big(\mathbf{D} + \boldsymbol{\varphi}(\boldsymbol{\theta}_i)\mathbf{N}\big)\mathbf{u}\end{aligned} \tag{46}$$

together via the weighting functions μ_i i=1,… to yield the nonlinear dynamics

$$\begin{aligned}\dot{\mathbf{z}} &= \sum_i \Big(\big(\mathbf{A} + \boldsymbol{\phi}(\boldsymbol{\theta}_i)\mathbf{M}\big)\mathbf{z} + \big(\mathbf{B} + \boldsymbol{\phi}(\boldsymbol{\theta}_i)\mathbf{N}\big)\mathbf{u}\Big)\, \mu_i(\boldsymbol{\theta}) \\ \mathbf{v} &= \sum_i \Big(\big(\mathbf{C} + \boldsymbol{\varphi}(\boldsymbol{\theta}_i)\mathbf{M}\big)\mathbf{z} + \big(\mathbf{D} + \boldsymbol{\varphi}(\boldsymbol{\theta}_i)\mathbf{N}\big)\mathbf{u}\Big)\, \mu_i(\boldsymbol{\theta})\end{aligned} \tag{47}$$

Provided $\mu_k(\boldsymbol{\theta}_j)$ is unity when j=k and zero when j ≠k, the frozen-parameter linearisations of (47) corresponding to parameter value $\boldsymbol{\theta}_i$ is just the local model (46). Consider relaxing condition (iv) of the Uniqueness Proposition to the weaker requirement that the frozen-parameter transfer functions relating **u** to $\mathbf{v}_{aug}$ are known only for parameter values { $\boldsymbol{\theta}_i$, i=1,…} (rather than for all parameter values). It follows directly from the proof of the Uniqueness Proposition that this relaxed condition (iv), together with conditions (i)-(iii) of section 4, uniquely defines the local models, (46). Compatible state-space realisations of the local models can be determined using the procedure described in the section 4.3 above. The blended nonlinear system, (47), is then defined by an appropriate choice of weighting functions μ_i (for example, the use of triangular weighting functions corresponds to linear interpolation between the local models (46)).

Remark *Choice of weighting function dependence*
It is worth emphasising that the weighting functions, μ_i , must depend on the same parameter, $\boldsymbol{\theta}$, as the local models in order to ensure consistency across the reconstructed nonlinear system. Inference of the parameter, $\boldsymbol{\theta}$**,** is of course one outcome of the reconstruction process. This observation is a trivial consequence of the present development, but nevertheless an issue of considerable practical importance (see, for example, the discussion in Johansen & Murray-Smith (1997)).

<u>*Example (cont*</u>) Missile lateral dynamics
Returning to the missile example of section 4.3, suppose that the frozen-parameter linearisations are now known only for the *discrete* parameter values θ_i, i=1,2..N. As before, assume without loss that the constant matrices associated with the nonlinear dynamics are

$$\mathbf{A}=\begin{bmatrix} a_1 & a_3 \\ b_3 & b_5 \end{bmatrix}, \mathbf{B}=\begin{bmatrix} a_5 \\ b_7 \end{bmatrix}, \mathbf{C}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, D=0, \mathbf{M}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{N}=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{48}$$

It follows from the previous analysis of this example that conditions (i)-(iii) are satisfied by this collection of N linearisations. The reconstructive linear equalities for this example therefore are

$$\begin{bmatrix} T_{11}(\theta_i) & T_{12}(\theta_i) \\ T_{21}(\theta_i) & T_{22}(\theta_i) \end{bmatrix}\begin{bmatrix} a_1+a_2|\theta_i| & a_3 \\ b_3+b_2|\theta_i| & b_5+b_4|\theta_i| \end{bmatrix}-\begin{bmatrix} a_1 & a_3 \\ b_3 & b_5 \end{bmatrix}\begin{bmatrix} T_{11}(\theta_i) & T_{12}(\theta_i) \\ T_{21}(\theta_i) & T_{22}(\theta_i) \end{bmatrix}=\begin{bmatrix} \phi_{11}(\theta_i) & \phi_{12}(\theta_i) & \phi_{13}(\theta_i) \\ \phi_{21}(\theta_i) & \phi_{22}(\theta_i) & \phi_{23}(\theta_i) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_5+a_4|\theta_i| \\ b_7+b_6|\theta_i| \end{bmatrix}-\begin{bmatrix} a_5 \\ b_7 \end{bmatrix}\begin{bmatrix} T_{11}(\theta_i) & T_{12}(\theta_i) \\ T_{21}(\theta_i) & T_{22}(\theta_i) \end{bmatrix}=\begin{bmatrix} \phi_{11}(\theta_i) & \phi_{12}(\theta_i) & \phi_{12}(\theta_i) \\ \phi_{21}(\theta_i) & \phi_{22}(\theta_i) & \phi_{23}(\theta_i) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_{11}(\theta_i) & T_{12}(\theta_i) \\ T_{21}(\theta_i) & T_{22}(\theta_i) \end{bmatrix}=\begin{bmatrix} \varphi_{11}(\theta_i) & \varphi_{12}(\theta_i) & \varphi_{13}(\theta_i) \\ \varphi_{21}(\theta_i) & \varphi_{22}(\theta_i) & \varphi_{23}(\theta_i) \\ \varphi_{31}(\theta_i) & \varphi_{32}(\theta_i) & \varphi_{33}(\theta_i) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}=0$$

$$\begin{bmatrix} \varphi_{11}(\theta_i) & \varphi_{12}(\theta_i) & \varphi_{13}(\theta_i) \\ \varphi_{21}(\theta_i) & \varphi_{22}(\theta_i) & \varphi_{23}(\theta_i) \\ \varphi_{31}(\theta_i) & \varphi_{32}(\theta_i) & \varphi_{33}(\theta_i) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}=0 \tag{49}$$

with i=1,2..N. Note that there are now only a finite number, N, of equalities and the unique solution to (49) reconstructs the state-space realisations of the frozen-parameter linearisations as

$$\begin{aligned} \dot{\hat{\mathbf{z}}} &= \begin{bmatrix} a_1 & a_3 \\ b_3 & b_5 \end{bmatrix}\hat{\mathbf{z}}+\begin{bmatrix} a_5 \\ b_7 \end{bmatrix}u+\begin{bmatrix} a_2|\theta_i| & 0 & a_4|\theta_i| \\ b_2|\theta_i| & b_4|\theta_i| & b_6|\theta_i| \end{bmatrix}\hat{\boldsymbol{\vartheta}} \\ \hat{\boldsymbol{\vartheta}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\hat{\mathbf{z}}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \end{aligned}\quad, \quad i=1,2..N \tag{50}$$

Blending between these N state-space linearisations using appropriate weighting functions μ_i i=1,2..N yields a state-space frozen-parameter linearisation family for which the corresponding nonlinear system is

$$\begin{aligned} \dot{\mathbf{z}} &= \begin{bmatrix} a_1 & a_3 \\ b_3 & b_5 \end{bmatrix}\mathbf{z}+\begin{bmatrix} a_5 \\ b_7 \end{bmatrix}u+\begin{bmatrix} a_2\sum_{i=1}^{N}|\theta_i|\mu_i(\theta) & 0 & a_4\sum_{i=1}^{N}|\theta_i|\mu_i(\theta) \\ b_2\sum_{i=1}^{N}|\theta_i|\mu_i(\theta) & b_4\sum_{i=1}^{N}|\theta_i|\mu_i(\theta) & b_6\sum_{i=1}^{N}|\theta_i|\mu_i(\theta)| \end{bmatrix}\boldsymbol{\vartheta} \\ \boldsymbol{\vartheta} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{z}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \end{aligned}\quad, \quad i=1,2..N \tag{51}$$

with $\theta=\begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{z}$. In general, the requirement is to reconstruct the nonlinear dynamics from knowledge of as few linearisations as possible; that is, to minimise the number N of linearisations needed to achieve an accurate reconstruction. In the present example, it is known from (35) that the coefficients of the missile equations depend linearly on θ. Hence, using triangular weighting functions and the linearisations associated with the extremal values of θ associated with the required operating envelope, accurate reconstruction can in fact be achieved on the basis of knowledge of input-output information pertaining to only *two* linearisations.

■

Remark *Correspondence between equilibrium linearisations and frozen-parameter linearisations (cont)*
As noted in section 5.1, by adopting the velocity-based formalism a direct relationship exists between the frozen-parameter linearisations and the classical equilibrium linearisations for the class of systems possessing the ELLE property. The missile example considered here does not belong to the class of ELLE systems. However, it can be shown (Leith *et al.* 2000) that the velocity-based form of the missile dynamics may be accurately approximated by an appropriate ELLE system. The reconstruction of a blended type of representation as considered above may therefore be carried out in terms of the classical equilibrium linearisations (indeed, by blending only a small number of linearisations). This is clearly of considerable practical relevance.

6. Conclusions

This paper concerns a fundamental issue in the modelling and realisation of nonlinea r systems; namely, whether it is possible to uniquely reconstruct a nonlinear system from a suitable collection of transfer functions and, if so, under what conditions. (Here, 'transfer function' is used as shorthand to denote a linear model based only on measurable input-output data. It includes, in addition to actual transfer function models, linear state-space models where the choice of state co-ordinates is only defined to within a linear transformation. No restriction to frequency-domain methods is implied or necessary). It is established that

- A family of frozen-parameter linearisations may be associated with a nonlinear LPV/quasi-LPV type of system. While the dynamics of individual members of the family are only weakly related to the dynamics of the nonlinear system, the state-space *family* of linearisations nevertheless does provide an alternative realisation of the nonlinear system without loss of information. This is, of course, quite different from the situation with classical equilibrium linearisations.
- Knowledge of the input-output dynamics (transfer functions) of the frozen-parameter linearisations of a system is, however, *not* sufficient to permit reconstruction of the associated nonlinear system. This result is interesting since the state-space frozen-parameter linearisation family *does* provide a unique representation of a nonlinear system which embodies all of its dynamic characteristics. The difficulty with the transfer function family arises from the degree of freedom available in the choice of state-space realisation of each linearisation.
- Under mild structural conditions, knowledge of a family of augmented transfer functions *is* sufficient to permit a large class of nonlinear systems to be uniquely reconstructed. That is, the family of augmented transfer functions provides an alternative, and entirely input-output based, representation of a nonlinear system. Essentially, the augmented family embodies the information necessary to select state-space realisations for the linearisations which are compatible with one another and with the underlying nonlinear system. The results are constructive, with a state-space realisation of the nonlinear system associated with a transfer function family being obtained as the solution to a number of linear equations.

Acknowledgement
D.J. Leith gratefully acknowledges the support provided by the Royal Society for the work presented.

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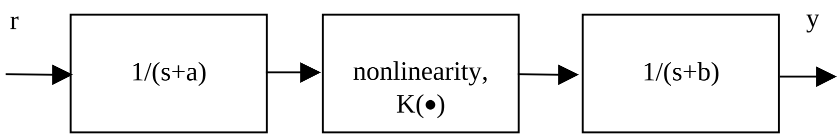

Figure 1 Structure of nonlinear system in Example 3