EXTREMALITY CONDITIONS IN THE TRACE ZERO NONNEGATIVE INVERSE EIGENVALUE PROBLEM

Thomas J. Laffey
(University College Dublin, Ireland)

Let $\sigma = (\lambda_1, \quad \ldots \quad , \lambda_n)$ be a list of complex numbers. The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and sufficient conditions on σ in order that it be the spectrum of an (entry-wise) nonnegative real matrix. We say that σ is **realizable** if it is the spectrum of a nonnegative real matrix. In earlier work (LAA 275/276 (1998) 349-357), we introduced the concept of an **extreme spectrum**. We say that σ is **extreme** if σ is realizable, but, for all $\epsilon > 0$,

$$\sigma - \epsilon := (\lambda_1 - \epsilon, \quad \ldots \quad , \lambda n - \epsilon)$$

is not realizable. We showed that if σ is extreme and A is a nonnegative matrix having spectrum σ, then there exists a nonzero nonnegative matrix Y with $AY = YA$ and $\mathrm{trace}(AY) = 0$. This result has proved useful in the study of the NIEP. However, if σ has trace 0, then it is extreme, but $Y = I$ satisfies those conditions, so the result is of no value in this case. This motivated the current investigation.

We say that a realizable trace 0 spectrum $\sigma = (\lambda_1, \quad \ldots \quad , \lambda_n)$ with Perron root λ_1 is **very extreme** if for all $\epsilon > 0$,

$$\sigma_\epsilon := (\lambda_1 - (n-1)\epsilon, \lambda_2 + \epsilon, \quad \ldots \quad , \lambda n + \epsilon)$$

is not realizable. Note that σ_ϵ also has trace 0. A nice result of Guo Wuwen (LAA 266 (1997) 261-270) implies that if σ_ϵ is realizable for some $\epsilon > 0$, then so is σ. His result enables one to reduce the NIEP for trace 0 spectra to the case of very extreme spectra.

We prove that if σ is very extreme, then there exists a non-diagonal matrix Y having nonnegative off-diagonal entries such that $AY = YA$ and $\mathrm{trace}(AY) = 0$. As in the case of the earlier result quoted above, the proof uses duality results from linear programming, but it is technically more complicated.

We will consider the corresponding problem (known as SNIEP) for nonnegative symmetric matrices and prove a corresponding result in this case when σ has distinct entries.. The proof uses analyticity results related to Rellich's Theorem.