EE304 - Tutorial 3 Solutions

Question 1
A communications system sends the bits 1 and 0 over a noisy channel. 52% of all bits transmitted are 1. The probability of a transmitted 1 being received as a 0 is 0.06 while the probability of a transmitted 0 being received as a 1 is 0.04.

(i) What proportion of bits received are 0?
Solution:
Using the same notation as in Question 3 on Tutorial Sheet 2, we want $P(0R)$. This is

$$\begin{aligned} P(0R) &= P(0R|0S)P(0S) + P(0R|1S)P(1S) \\ &= (0.96)(0.48) + (0.06)(0.52) = 0.492. \end{aligned}$$

(ii) A sender wishes to send the message 1011 over this channel. If each bit is transmitted independently, what is the probability that the message is received correctly?
Solution:
The probability of correctly receiving this message will be

$$P(1R|1S)P(0R|0S)P(1R|1S)P(1R|1S) = (0.94)(0.96)(0.94)(0.94) = 0.797.$$

(iii) The message 0011 is received. What is the probability that 0011 is the message that was transmitted?
Solution:
We need to compute the probabilities $P(0S|0R)$ and $P(1S|1R)$. The first one is

$$P(0S|0R) = \frac{(0.96)(0.48)}{0.492} = 0.937.$$

Using Bayes Law again, we find

$$P(1S|1R) = \frac{(0.94)(0.52)}{(0.94)(0.52) + (0.04)(0.48)} = 0.962.$$

The desired probability is now

$$(0.937)^2(0.962)^2 = 0.813.$$

Question 2
A fair coin is tossed three times. The random variable X is given by the number of heads minus the number of tails that come up on the three tosses.

(i) What is the range of X?
Solution:
The range of X is

$$\{-3, -1, 1, 3\}.$$

(ii) Write down the probability mass function $f(x)$ of X.
Solution:
For each x in the range of X, we need to compute $f(x) = P(X = x)$. First, to compute $P(X = -3)$, note that $X = -3$ corresponds to the event of getting 3 tails. Thus $f(-3) = (1/2)^3 = 1/8$. Similarly, $X = -1$ corresponds to getting 2 tails. This can happen in 3 different ways to $f(-1) = 3/8$. Similar calculations show that $f(1) = 3/8$ and $f(3) = 1/8$. This gives us the pmf

x	-3	-1	1	3
$f(x)$	1/8	3/8	3/8	1/8

(iii) Write down the cumulative distribution function $F(x)$ of X.
Solution:
Remember that $F(x) = \sum_{y \leq x} f(x)$. Thus

$$F(x) = \begin{cases} 0 & x < -3 \\ 1/8 & -3 \leq x < -1 \\ 1/2 & -1 \leq x < 1 \\ 7/8 & 1 \leq x < 3 \\ 1 & 3 \leq x. \end{cases}$$

Question 3
A continuous random variable X has probability density function

$$f(x) = \begin{cases} 0 & x < 5.5 \\ 10e^{-10(x-5.5)} & x \geq 5.5. \end{cases}$$

Determine:

(i) $P(X < 5.7)$;
Solution:

$$\begin{aligned} P(X < 5.7) &= \int_{5.5}^{5.7} 10e^{-10(x-5.5)} dx \\ &= -e^{-10(x-5.5)}\Big|_{5.5}^{5.7} \\ &= 1 - e^{-2} = 0.865. \end{aligned}$$

(ii) $P(X > 6)$;
Solution:

$$\begin{aligned} P(X > 6) &= \int_6^\infty 10e^{-10(x-5.5)}dx \\ &= -e^{-10(x-5.5)}\Big|_6^\infty \\ &= e^{-5} = 0.007. \end{aligned}$$

(iii) $P(5.6 < X \leq 6)$;
Solution:

$$\begin{aligned} P(5.6 < X \leq 6) &= \int_{5.6}^6 10e^{-10(x-5.5)}dx \\ &= -e^{-10(x-5.5)}\Big|_{5.6}^6 \\ &= e^{-1} - e^{-5} = 0.361. \end{aligned}$$

(iv) The cumulative distribution function F of X.
Solution:
For $x < 5.5, F(x) = 0$. For $x \geq 5.5$,

$$\begin{aligned} F(x) &= \int_{5.5}^x 10e^{-10(t-5.5)}dt \\ &= -e^{-10(t-5.5)}\Big|_{5.5}^x \\ &= 1 - e^{-10(x-5.5)}. \end{aligned}$$

Question 4
A continuous random variable has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & x < 1 \\ \sqrt{x} - 1 & 1 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

Determine

(i) The probability density function of X;
Solution:
To calculate the pdf of X, we differentiate.

$$f(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2\sqrt{x}} & 1 \leq x < 4 \\ 0 & x \geq 4. \end{cases}$$

(ii) $P(X < 3)$;
Solution:
This is most easily calculated from the cumulative distribution function.

$$P(X < 3) = F(3) = \sqrt{3} - 1 = 0.732.$$

(iii) $P(X > 2.5)$;
Solution:
Again from the cumulative distribution:

$$P(X > 2.5) = 1 - F(2.5) = 2 - \sqrt{2.5} = 0.4189.$$

(iv) $P(1.5 < X < 2.5)$
Solution:

$$\begin{aligned} P(1.5 < X < 2.5) &= P(X < 2.5) - P(X < 1.5) \\ &= F(2.5) - F(1.5) \\ &= \sqrt{2.5} - \sqrt{1.5} = 0.356. \end{aligned}$$