Continuous Random Variables

October 8, 2010

Continuous Random Variables

Many practical random variables are continuous. For example:

1. The speed of a car;
2. The concentration of a chemical in a water sample;
3. Tensile strengths;
4. Heights of people in a population;
5. Lengths or areas of manufactured components;
6. Measurement Errors;
7. Electricity consumption in kilowatt hours.

Definition

The cumulative distribution function F of a continuous random variable X is the function

$$F(x) = P(X \leq x)$$

For all of our examples, we shall assume that there is some function f such that

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for all real numbers x. f is known as a probability density function for X.

Probability Density Functions

The probability density function f of a continuous random variable X satisfies

(i) $f(x) \geq 0$ for all x;

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f(x)dx$ for all a, b.

Probabilities correspond to areas under the curve $f(x)$.

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For any single value a, $P(X = a) = 0$.

$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$.

Probability Density Functions

Example

Let X denote the width in mm of metal pipes from an automated production line. If X has the probability density function $f(x) = 10e^{-10(x-5.5)}$ for $x \geq 5.5$, $f(x) = 0$ for $x < 5.5$.

Determine:

(i) $P(X < 5.7)$;

(ii) $P(X > 6)$;

(iii) $P(5.6 < X \leq 6)$.

Probability Density Functions

Example

(i)

$$
\begin{aligned}
P(X < 5.7) &= \int_{5.5}^{5.7} 10e^{-10(x-5.5)}dx \\
&= -e^{-10(x-5.5)}\Big|_{5.5}^{5.7} \\
&= 1 - e^{-2} = 0.865.
\end{aligned}
$$

Example

(i)

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(ii)

$$\begin{aligned} P(X > 6) &= \int_{6}^{\infty} 10e^{-10(x-5.5)} dx \\ &= -e^{-10(x-5.5)}\Big|_{6}^{\infty} \\ &= e^{-5} = 0.007. \end{aligned}$$

Example

(iii)

$$
\begin{aligned}
P(5.6 < X \leq 6) &= \int_{5.6}^{6} 10e^{-10(x-5.5)}dx \\
&= -e^{-10(x-5.5)}\Big|_{5.6}^{6} \\
&= e^{-1} - e^{-5} = 0.361.
\end{aligned}
$$

Example

The random variable X measures the width in mm of metal pipes from an automated production line X has the probability density function $f(x) = 10e^{-10(x-5.5)}$ for $x \geq 5.5$, $f(x) = 0$ for $x < 5.5$. What is the cumulative distribution function of X?

For $x < 5.5, F(x) = 0$. For $x \geq 5.5$,

$$\begin{aligned} F(x) &= \int_{5.5}^{x} 10e^{-10(t-5.5)} dt \\ &= -e^{-10(t-5.5)} \Big|_{5.5}^{x} \\ &= 1 - e^{-10(x-5.5)}. \end{aligned}$$

It follows from the fundamental theorem of calculus that if we are given the cumulative distribution function F of a random variable, we can calculate the probability density function by differentiating.

$$f(x) = \frac{dF(x)}{dx}.$$

provided the derivative exists.

Example

Let X denote the time in milliseconds for a chemical reaction to complete. The cumulative distribution function of X is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.05x} & x \geq 0. \end{cases}$$

What is the probability density function of X? What is the probability that a reaction completes within 40 milliseconds?

Distribution and Density Functions

Example

The probability density function will be given by $\frac{dF(x)}{dx}$.

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.05e^{-0.05x} & x \geq 0 \end{cases}$$

The probability that the reaction completes within 40 milliseconds is

$$P(X \leq 40) = F(40) = 1 - e^{-2} = 0.865.$$