Some Simple Counting Rules

EE304 - Probability and Statistics

Semester 1

Counting

If all outcomes are equally likely, the probability of an event E is given by

$$\frac{|E|}{|S|}$$

where $|E|$ $(|S|)$ denotes the number of elements in E (S).

To apply this rule, we need to be able to count the number of elements in events. We shall look at:

- Multiplication Rules;
- Permutations of distinct objects;
- Permutations where some objects are identical;
- Combinations .

Some Simple Counting Rules

Multiplication Rule Basic idea

If one operation can be done in n_1 ways and a second operation can be done in n_2 ways then the number of different ways of doing both is $n_1 n_2$.

Example

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Example

- If we roll a fair die and toss a coin, the total number of possible outcomes is $6 \times 2 = 12$.
- If we roll a fair 4-sided die 3 times, the total number of possible outcomes is $4 \times 4 \times 4 = 64$.

Example

A simple survey consists of three multiple choice questions. The first question has 3 possible answers, the second has 4 possible answers and the third has 3 possible answers. What is the total number of different ways in which this survey could be completed?

$3 \times 4 \times 3 = 36$.

Some Simple Counting Rules

Example

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Example

A circuit board contains 4 relays each of which can be set to any of three positions. What is the total number of distinct configurations for the 4 relays?

- Each relay can be set in 3 ways and we have 4 relays.
- So the total number of configurations is $(3)(3)(3)(3) = 81$.

Permutations

How many different arrangements/permutations of n distinct objects are possible?

- The first object can be chosen in n ways;
- The second object can then be chosen in $n-1$ ways and so on;
- The number of ways of permuting (arranging in order) n distinguishable objects is

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Or compactly, $n!$ (n factorial).

Permutations

- The total number of different ways in which the letters of the word "count" can be arranged is $5! = (5)(4)(3)(2)(1) = 120$. It is important here that the letters are all different.
- 6 horses run a race. The total number of possible results of this race (assuming no ties) is $6! = (6)(5)(4)(3)(2)(1) = 720$.
- A search engine ranks 11 websites related to a particular query in order of relevance. How many different rankings are possible? Answer $11!$.

Permutations

What if not all the objects are distinct?

- What is the total number of different arrangements of the letters in the word "stat"?

- Suppose the two "t"s can be distinguished. st_1at_2.
- Then we would have 4! arrangements.
- Each arrangement of the original word "stat" would generate 2! arrangements of st_1at_2.
- So the number of arrangements of the word stat is $\frac{4!}{2!} = 12$.

Permutations with some Objects Identical

In general if we have n items k of which are identical, the total number of distinct permutations is $\frac{n!}{k!}$.

Example

How many different ways can we arrange the letters BBBACDE?
Answer $\frac{7!}{3!} = (7)(6)(5)(4) = 840$.

Example

How many different numbers can be formed by rearranging 2212562?
Answer $\frac{7!}{4!} = 210$.

Permutations with some Objects Identical

Example

How many different ways can we rearrange the letters of MISSISSIPPI?

M	I	S	P
	I	S	P
	I	S	
	I	S	

We have 11 letters in total, of which 4 are 'I', 4 are 'S' and '2' are 'P'. In this situation, the total number of different rearrangements is

$$\frac{11!}{4!4!2!}.$$

Some Simple Counting Rules

r-Permutations

How many permutations of n distinct objects, taken r at a time are possible?

- Again, we have n ways of choosing the first object.
- We then have $n-1$ ways of choosing the second object and so on.
- When choosing the r^{th} object, we have already chosen $r-1$ objects, so there are still $n-(r-1)=n-r+1$ possible choices.

Permutations

- The total number of r permutations of a set of n distinguishable objects is

$$n \cdot (n-1) \cdots (n-r+1).$$

There are r terms in the product.

The number of r-permutations of a set of n distinguishable objects is written ${}^nP_r$.

$${}^nP_r = \frac{n!}{(n-r)!}.$$

Examples

Example

- In a race with eight competitors, how many different possibilities are there for who finishes finish first, second and third?
- Three major new roads are to be constructed and eight companies have tendered for the three projects. If at most one construction project is to be given to any one company, what is the total number of ways in which the three contracts can be awarded?

Example

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Both of these have the same answer - ${}^{8}P_3$.

Combinations

How many different ways can we select a set of size r from a larger set of n distinguishable objects? The order of selection does not matter.

We are asking for the number of combinations of n objects taken r at a time. This number is written as $\binom{n}{r}$.

- Each combination/set of r-objects can be permuted in exactly $r!$ distinct ways;
- This means that

$$\binom{n}{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!(n-r)!}$$

.

Note that $\binom{n}{r} = \binom{n}{n-r}$.

Example

- How many ways can a company select 3 candidates to interview from a short list of 15?

  Answer

  $$\binom{15}{3} = \frac{(15)(14)(13)}{(3)(2)(1)} = 455$$

- In how many ways can a subcommittee of 5 be chosen from a panel of 20?

  Answer

  $$\binom{20}{5} = \frac{(20)(19)(18)(17)(16)}{(5)(4)(3)(2)(1)} = 15504.$$

Combinations - Examples

Example

A drum contains 3 black balls, 5 red balls and 6 green balls. If 4 balls are selected at random what is the probability that the 4 selected contain

(i) No red ball?

(ii) Exactly 1 black ball?

(iii) Exactly 1 red ball and exactly 2 green balls?

Examples

Example

(i) Total number of ways of choosing 4 balls from 14 is $\binom{14}{4}$.
Total number of ways of choosing 4 balls, none of which is red is $\binom{9}{4}$.
So the probability that the 4 balls contain no red ball is

$$\frac{\binom{9}{4}}{\binom{14}{4}}.$$

(ii) 1 black ball can be chosen in $\binom{3}{1}$ ways, while the other three balls can be chosen in $\binom{11}{3}$ ways. So the probability of choosing exactly 1 black ball is

$$\frac{\binom{3}{1}\binom{11}{3}}{\binom{14}{4}}.$$

Example

Probability of choosing 1 red ball and two green balls is

$$\frac{\binom{5}{1}\binom{6}{2}\binom{3}{1}}{\binom{14}{4}}.$$

Example

A hand of 5 cards is dealt from a well-shuffled deck. What is the probability that the hand contains:

(i) No aces?

(ii) 5 clubs?

(iii) At least one ace?

(iv) 3 clubs and 2 hearts?

Example

Example

(i) Answer: $\frac{\binom{48}{5}}{\binom{52}{5}}$.

(ii) Answer: $\frac{\binom{13}{5}}{\binom{52}{5}}$.

(iii) Answer: $1 - \frac{\binom{48}{5}}{\binom{52}{5}}$.

(iv) Answer: $\frac{\binom{13}{3}\binom{13}{2}}{\binom{52}{5}}$.