Conditional Probability

EE304 - Probability and Statistics

October 7, 2010

Conditional Probability

We are often interested in the likelihood of an event occurring given that another has occurred. Examples of this type include:

- Probability that a train arrives on time given that it left on time.
- Probability that a PC crashes given the operating system installed.
- Probability that a bit is transmitted over a channel is received as a 1 given that the bit transmitted was a 1.
- Probability that a website is visited given its number of inlinks.

Questions of this type are handled using conditional probability.

Recall the frequency interpretation of probability.

- The probability of an event E is the proportion of times E occurs - $\frac{k_E}{N}$.
- If we are given that F has occurred and want the probability of E occurring, this restricts our sample space to the outcomes in F.
- If we write $k_{E\cap F}$ for the number of times that both F and E occur, a reasonable definition of the probability of E given F would be

$$\frac{k_{E\cap F}}{k_F}$$

This represents the fraction of the times that F occurs where both E and F occur.

This motivates the following definition of the probability of E given F.

Definition

The probability of E given F, written $P(E|F)$ is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

Conditional Probability

Example

A sample of 150 plastic panels was selected and tested for shock resistance and scratch resistance. The results are summarised in the table below.

		shock	**resistance**
		high	low
scratch	high	125	12
resistance	low	7	6

A panel is selected at random. E is the event that it has high shock resistance. F denotes the event that it has high scratch resistance. Calculate:

- $P(E|F)$ Answer: $\frac{125}{137}$.
- $P(F|E)$ Answer: $\frac{125}{132}$.
- $P(F^c|E)$ Answer: $\frac{7}{132}$.

Example

A family has two children (not twins). What is the probability that the younger child is a girl given that at least one of the children is a girl? (assume that boys and girls are equally likely to be born)

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- Let E be the event that the second child is a girl - $E = \{GG, BG\}$
- Let F be the event that at least one child is a girl - $F = \{GG, BG, GB\}$
- $$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2}{3}.$$

Conditional Probability and the Multiplication Rule

It follows from the formula for conditional probability that for any events E and F,

$$P(E \cap F) = P(F|E)P(E) = P(E|F)P(F).$$

Example

Two cards are chosen at random without replacement from a well-shuffled pack. What is the probability that the second card drawn is also a king given that the first one drawn was a king?

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Two cards are chosen at random without replacement from a well shuffled pack. What is the probability that they are both aces?

$$\frac{4}{52}\frac{3}{51} = \frac{1}{221}.$$

Example

An urn contains 3 red and 5 green balls. Two balls are selected from the urn without replacement. Calculate the probability that:

(i) both balls are green;

(ii) the second ball is green.

Let G_1, G_2 be the events that the first and second ball is green respectively.
(i) Here we want $P(G_2 \cap G_1) = P(G_2|G_1)P(G_1)$. But $P(G_1) = 5/8$ and $P(G_2|G_1) = 4/7$.
So

$$P(G_2 \cap G_1) = (4/7)(5/8) = 20/56 = 5/14.$$

(ii) There are two possibilities here. $G_2 \cap G_1$ or $G_2 \cap G_1^c$. As these are mutually exclusive events,

$$\begin{aligned} P(G_2) &= P(G_2 \cap G_1) + P(G_2 \cap G_1^c) \\ &= 5/14 + (5/7)(3/8) = 35/56. \end{aligned}$$

Bayes Theorem

A collection of events $F_1, \ldots, F_k$ is *exhaustive* if

$$F_1 \cup F_2 \cup \cdots \cup F_k = S$$

where S is the sample space.
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The collection is exhaustive if at least one of the events must occur.
If $F_1, \ldots, F_k$ are exhaustive and mutually exclusive events then for any event E,

$$P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_k)P(F_k).$$

This is known as the rule of total probability.

Bayes Theorem

For any events E and F with $P(E) \neq 0$, we have:

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^c)P(F^c)} \tag{1}$$

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Theorem

If we are given a collection of mutually exclusive and exhaustive events $F_1, F_2, \ldots, F_k$ and E is an event with $P(E) > 0$, then

$$P(F_1|E) = \frac{P(E|F_1)P(F_1)}{P(E|F_1)P(F_1) + P(E|F_2)P(F_2) + \cdots + P(E|F_k)P(F_k)}.$$

Example

A car manufacturer receives its air conditioning units from 3 different suppliers, A, B, C. 20% of its units come from supplier A, 30% from supplier B and 50% from supplier C. It is known that 10% of the units from supplier A are defective, 8% of units from supplier B are defective and 5% of units from supplier C are defective. If a unit is selected at random and is found to be defective, what is the probability that it came from:

(i) supplier A?

(ii) supplier C?

Let A, B, C denote the events that a unit comes from supplier A, B, C respectively. Also, let D denote the event that a unit is defective. In part (i), we want $P(A|D)$. Bayes theorem tells us that

Bayes Theorem - Examples

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Bayes Theorem - Examples

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$$\begin{aligned} P(A|D) &= \frac{P(D|A)P(A)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\ &= \frac{(0.1)(0.2)}{(0.1)(0.2) + (0.08)(0.3) + (0.05)(0.5)} \\ &= \frac{0.02}{0.069} = 0.29. \end{aligned}$$

Part (ii) is done similarly.

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&= \frac{(0.1)(0.2)}{(0.1)(0.2) + (0.08)(0.3) + (0.05)(0.5)} \\
&= \frac{0.02}{0.069} = 0.29.
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Part (ii) is done similarly.

$$
\begin{aligned}
P(C|D) &= \frac{P(D|C)P(C)}{P(D|A)P(A) + P(D|B)P(B) + P(D|C)P(C)} \\
&= \frac{(0.05)(0.5)}{(0.1)(0.2) + (0.08)(0.3) + (0.05)(0.5)} \\
&= \frac{0.025}{0.069} = 0.362.
\end{aligned}
$$

Example

Two urns contain coloured balls. Urn 1 contains 4 blue, 3 green and 5 red balls. Urn 2 contains 6 blue, 2 green and 3 red balls. A ball is selected at random from urn 1 and transferred to urn 2. A ball is then selected at random from urn 2. If the ball selected from urn 2 is red, what is the probability that the ball transferred from urn 1 was

(i) red?

(ii) green?

(iii) blue?

Example

Let R_1, G_1, B_1 denote the events that the ball transferred from Urn 1 to Urn 2 was red, green and blue respectively. Let R_2, G_2, B_2 denote the events that the ball selected from Urn 2 was red, green and blue respectively.

(i) We want $P(R_1|R_2)$. Bayes Theorem tells us that this is equal to:

Bayes Theorem - Examples

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(i) We want $P(R_1|R_2)$. Bayes Theorem tells us that this is equal to:

$$\frac{P(R_2|R_1)P(R_1)}{P(R_2|R_1)P(R_1) + P(R_2|G_1)P(G_1) + P(R_2|B_1)P(B_1)}$$

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$$\begin{aligned} & \frac{P(R_2|R_1)P(R_1)}{P(R_2|R_1)P(R_1) + P(R_2|G_1)P(G_1) + P(R_2|B_1)P(B_1)} \\ = & \frac{(1/3)(5/12)}{(1/3)(5/12) + (1/4)(1/4) + (1/4)(1/3)} \\ = & \frac{20}{41} \end{aligned}$$

Parts (ii) and (iii) are done in the same way.

Example

A test for internal corrosion in sections of pipe in a chemical plant correctly identifies corrosion in a corroded section with probability 0.8. On the other hand, the test incorrectly indicates corrosion in an uncorroded section with probability 0.2. If the probability that any section of pipe is corroded is 0.1, calculate:

(i) The probability that a section is corroded given that the test indicates that it is.

(ii) The probability that a section of pipe is corroded given that the test indicates that it is not.

Let C be the event that a section of pipe is corroded. Let T be the event that the test indicates corrosion.
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Bayes Theorem - Examples

Let C be the event that a section of pipe is corroded. Let T be the event that the test indicates corrosion.

(i) We want $P(C|T)$. Bayes theorem tells us that it is given by

$$\begin{aligned} P(C|T) &= \frac{P(T|C)P(C)}{P(T|C)P(C) + P(T|C^c)P(C^c)} \\ &= \frac{(0.8)(0.1)}{(0.8)(0.1) + (0.2)(0.9)} \\ &= \frac{0.08}{0.26} = 0.31 \end{aligned}$$

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Bayes Theorem - Examples

In part (ii) we want $P(C|T^c)$. This is

$$\begin{aligned} P(C|T^c) &= \frac{P(T^c|C)P(C)}{P(T^c|C)P(C) + P(T^c|C^c)P(C^c)} \\ &= \frac{(0.2)(0.1)}{(0.2)(0.1) + (0.8)(0.9)} \\ &= \frac{0.02}{0.74} = 0.027. \end{aligned}$$

Independent events play an important role in much of what we will discuss later.

- Two events E and F are independent if knowing that F occurred changes nothing about the probability of E occurring.
- Formally, E and F are independent if $P(F) \neq 0, P(E) \neq 0$

$$P(E|F) = P(E)$$

- This is equivalent to requiring

$$P(E \cap F) = P(E)P(F).$$

Independent Events

If E and F are independent then:

- $P(F|E) = P(F)$ (so are F and E).
- E^c and F are independent.
- E^c and F^c are independent.
- E and F are NOT mutually exclusive.

Independent Events

Example

A fair coin is tossed and a fair 6 sided die is rolled. What is the probability that of getting heads on the coin and 3 on the die?

Let E be the event that the coin comes up heads - $P(E) = 1/2$
Let F be the event that 3 comes up on the die - $P(F) = 1/6$
As these events are independent,
$P(E \cap F) = P(E)P(F) = (1/2)(1/6) = (1/12)$

Independent Events

Example

A fair die is rolled twice. Let E be the event that a 1 comes up on the first roll. Let F be the event that the sum of the two scores is 5. Are E and F independent?

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$$P(E) = 1/6, P(F) = 4/36 = 1/9, P(E \cap F) = 1/36.$$

As $P(E \cap F) \neq P(E)P(F)$, these are not independent.

Independent Sets of Events

A set of events $E_1, E_2, \ldots E_k$ is independent if for any subset $E_{i_1}, \ldots, E_{i_p}$

$$P(\cap_{j=1}^{p} E_{i_j}) = \prod_{j=1}^{p} p(E_{i_j}).$$

Example

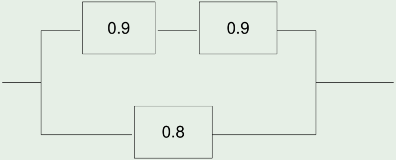
A fair 6-sided die is rolled 4 times. What is the probability of getting 4 on the first three rolls followed by a number strictly less than 3?

The results on each roll are independent and if we let E_i be the event of getting a 4 on the i^{th} roll for $i = 1, 2, 3$ and F be the event of getting a number strictly less than 3 on the 4^{th} roll, then

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3 \cap F) &= P(E_1)P(E_2)P(E_3)P(F) \\ &= (1/6)(1/6)(1/6)(1/3) = 1/648. \end{aligned}$$

Example

The circuit below operates provided there is a path of functional devices from left to right. The probability of each component functioning correctly is shown in the diagram and the components function independently. What is the probability of the circuit



functioning correctly?

Independence

The circuit functions correctly provided that either the upper or lower path is functioning. The only way this can fail to happen is if *both* the upper and lower paths fail.

$$P(\text{upper path fails}) = 1 - (0.9)(0.9) = 1 - 0.81 = 0.19$$

$$P(\text{lower path fails}) = 1 - 0.8 = 0.2.$$

So the probability that both the upper and lower paths fail is $(0.19)(0.2) = 0.038$ and the probability that the circuit operates correctly is $1 - 0.038 = 0.962$.

Example

A signal is transmitted over a noisy channel. It is known that the signal will be correctly received with probability 0.85. If three copies of the signal are independently sent over this channel, what is the probability that at least one of them is received correctly?

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- The probability that a copy is incorrectly received $=$ 1 - 0.85 $=$ 0.15.
- The probability that all 3 copies are incorrectly received $= (0.15)^3 = 0.0034$.
- Probability that at least 1 copy is correctly received $=$ 1 - 0.0034.