The Binomial Distribution

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2. Checking items from a production line: success = not defective, failure = defective.
3. Phoning a call centre: success = operator free; failure = no operator free.

Bernoulli Random Variables

A Bernoulli random variable X takes the values 0 and 1 and

$$\begin{aligned} P(X = 1) &= p \\ P(X = 0) &= 1 - p. \end{aligned}$$

It can be easily checked that the mean and variance of a Bernoulli random variable are

$$\begin{aligned} E(X) &= p \\ V(X) &= p(1 - p). \end{aligned}$$

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Binomial Experiments

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1. The experiment consists of n repeated Bernoulli trials - each trial has only two possible outcomes labelled as success and failure;
2. The trials are independent - the outcome of any trial has no effect on the probability of the others;
3. The probability of success in each trial is constant which we denote by p.

The Binomial Distribution

Definition

The random variable X that counts the number of successes, k, in the n trials is said to have a binomial distribution with parameters n and p, written $bin(k; n, p)$.

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$\binom{n}{k}$ counts the number of outcomes that include exactly k successes and $n - k$ failures.

Binomial Distribution - Examples

Example

A biased coin is tossed 6 times. The probability of heads on any toss is 0.3. Let X denote the number of heads that come up. Calculate:

(i) $P(X = 2)$

(ii) $P(X = 3)$

(iii) $P(1 < X \leq 5)$.

Binomial Distribution - Examples

Example

(i) If we call heads a success then this X has a binomial distribution with parameters $n = 6$ and $p = 0.3$.

$$P(X = 2) = \binom{6}{2}(0.3)^2(0.7)^4 = 0.324135$$

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$$\begin{aligned} & P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) \\ = \; & 0.324 + 0.185 + 0.059 + 0.01 \\ = \; & 0.578 \end{aligned}$$

Binomial Distribution - Example

Example

A quality control engineer is in charge of testing whether or not 90% of the DVD players produced by his company conform to specifications. To do this, the engineer randomly selects a batch of 12 DVD players from each day's production. The day's production is acceptable provided no more than 1 DVD player fails to meet specifications. Otherwise, the entire day's production has to be tested.

(i) What is the probability that the engineer incorrectly passes a day's production as acceptable if only 80% of the day's DVD players actually conform to specification?

(ii) What is the probability that the engineer unnecessarily requires the entire day's production to be tested if in fact 90% of the DVD players conform to specifications?

Example

Example

(i) Let X denote the number of DVD players in the sample that fail to meet specifications. In part (i) we want $P(X \leq 1)$ with binomial parameters $n = 12$, $p = 0.2$.

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{12}{0}(0.2)^0(0.8)^{12} + \binom{12}{1}(0.2)^1(0.8)^{11} \\ &= 0.069 + 0.206 = 0.275 \end{aligned}$$

Example

Example

(ii) We now want $P(X > 1)$ with parameters $n = 12$, $p = 0.1$.

$$\begin{aligned} P(X \leq 1) &= P(X = 0) + P(X = 1) \\ &= \binom{12}{0}(0.1)^0(0.9)^{12} + \binom{12}{1}(0.1)^1(0.9)^{11} \\ &= 0.659 \end{aligned}$$

So $P(X > 1) = 0.341$.

Binomial Distribution - Mean and Variance

1. Any random variable with a binomial distribution X with parameters n and p is a sum of n independent *Bernoulli* random variables in which the probability of success is p.

$$X = X_1 + X_2 + \cdots + X_n.$$

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3. Hence the mean and variance of X are given by (remember the X_i are independent)

$$E(X) = np, V(X) = np(1 - p).$$

Binomial Distribution - Examples

Example

Bits are sent over a communications channel in packets of 12. If the probability of a bit being corrupted over this channel is 0.1 and such errors are independent, what is the probability that no more than 2 bits in a packet are corrupted?

If 6 packets are sent over the channel, what is the probability that at least one packet will contain 3 or more corrupted bits?

Let X denote the number of packets containing 3 or more corrupted bits. What is the probability that X will exceed its mean by more than 2 standard deviations?

Binomial Distribution - Examples

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Let C denote the number of corrupted bits in a packet. Then in the first question, we want
$P(C \leq 2) = P(C = 0) + P(C = 1) + P(C = 2)$.

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\begin{aligned}
P(C = 0) &= \binom{12}{0}(0.1)^0(0.9)^{12} = 0.282 \\
P(C = 1) &= \binom{12}{1}(0.1)^1(0.9)^{11} = 0.377 \\
P(C = 2) &= \binom{12}{2}(0.1)^2(0.9)^{10} = 0.23
\end{aligned}
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So $P(C \leq 2) = 0.282 + 0.377 + 0.23 = 0.889$.

Example

The probability of a packet containing 3 or more corrupted bits is $1 - 0.889 = 0.111$.

Let X be the number of packets containing 3 or more corrupted bits. X can be modelled with a binomial distribution with parameters $n = 6$, $p = 0.111$. The probability that at least one packet will contain 3 or more corrupted bits is:

Binomial Distribution - Examples

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Let X be the number of packets containing 3 or more corrupted bits. X can be modelled with a binomial distribution with parameters $n = 6$, $p = 0.111$. The probability that at least one packet will contain 3 or more corrupted bits is:

$$1 - P(X = 0) = 1 - \binom{6}{0}(0.111)^0(0.889)^6 = 0.494.$$

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$$1 - P(X = 0) = 1 - \binom{6}{0}(0.111)^0(0.889)^6 = 0.494.$$

The mean of X is $\mu = 6(0.111) = 0.666$ and its standard deviation is $\sigma = \sqrt{6(0.111)(0.889)} = 0.77$.

Binomial Distribution - Examples

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As X is discrete, this is equal to

$$P(X \geq 3)$$

$$\begin{aligned} P(X = 1) &= \binom{6}{1}(0.111)^1(0.889)^5 = 0.37 \\ P(X = 2) &= \binom{6}{2}(0.111)^2(0.889)^4 = 0.115. \end{aligned}$$

So $P(X \geq 3) = 1 - (.506 + .37 + .115) = 0.009$.