Solutions 4

1. Notice first that there is exactly one event in $[0, t]$ if and only if $N_t = 1$ (but it is wrong to say that there is exactly one event on $[0, t]$ if and only if $T_1 \leq t$). So the probaility we are looking for is:

$$\mathbb{P}(N_s = 1 | N_t = 1) = \frac{\mathbb{P}(N_s = 1,\, N_t = 1)}{\mathbb{P}(N_t = 1)} = \frac{\mathbb{P}(N_s = 1,\, N_t - N_s = 0)}{\mathbb{P}(N_t = 1)}.$$

By independence, we further obtain

$$\mathbb{P}(N_s = 1 | N_t = 1) = \frac{\mathbb{P}(N_s = 1)\, \mathbb{P}(N_t - N_s = 0)}{\mathbb{P}(N_t = 1)} = \frac{\lambda s\, e^{-\lambda s}\, e^{-\lambda(t-s)}}{\lambda t\, e^{-\lambda t}} = \frac{s}{t}.$$

So given that exactly one event takes place during the time interval $[0, t]$, its distribution is uniform over $[0, t]$.

2. Notice first that is T if a non-negative random variable, then $\mathbb{E}(T) = \int_0^\infty dt\, \mathbb{P}(T \geq t)$. Indeed,

$$\mathbb{E}(T) = \mathbb{E}\left(\int_0^T dt\right) = \mathbb{E}\left(\int_0^\infty 1_{\{T \geq t\}}\, dt\right) = \int_0^\infty dt\, \mathbb{E}(1_{\{T \geq t\}}) = \int_0^\infty dt\, \mathbb{P}(T \geq t)$$

(provided that we allow ourselves to shamelessly commute the expectation and the integral). The following equality can be shown in a similar manner:

$$\mathbb{E}(T | T \geq a) = \int_0^\infty dt\, \mathbb{P}(T \geq t | T \geq a).$$

Using this together with the fact that

$$\mathbb{P}(T \geq t | T \geq a) = \frac{\mathbb{P}(T \geq \max\{t, a\})}{\mathbb{P}(T \geq a)} = \begin{cases} 1, & \text{if } t \leq a, \\ \dfrac{\mathbb{P}(T \geq t)}{\mathbb{P}(T \geq a)}, & \text{if } t > a, \end{cases}$$

we obtain for a)

$$\mathbb{E}(T | T \geq a) = a + \int_a^\infty dt\, \exp(-(t - a)) = a + 1 = 1 + a.$$

While for b), we have

$$\mathbb{E}(T | T \geq a) = a + \int_a^2 dt\, \frac{2 - t}{2 - a} = a + 1 - a/2 = 1 + a/2.$$

c) So the expected time in a) is clearly the largest. This can be interpreted as follows. In a), T is exponentially distributed and can therefore be considered as the first arrival time of a Poisson process. As the exponential distribution is memoryless, knowing that $T \geq a$ just adds the value a to the expectation of T.

In the second case b), T is uniform between 0 and 2, which means that T can also be interpreted as the first arrival time of a Poisson process, *given that there is exactly one arrival between* 0 *and* 2 (see Exercise 1). Conditioning now on $T \geq a$ reduces therefore the expectation with respect to the former case, as in this case, we know "in advance" that the time T will occur before $t = 2$.

3. We have

$$\begin{aligned}\mathbb{P}(T \le t) &= \mathbb{P}(\min\{T_1, T_2\} \le t) = 1 - \mathbb{P}(\min\{T_1, T_2\} > t) = 1 - \mathbb{P}(\{T_1 > t\} \cap \{T_2 > t\}) \\ &= 1 - \mathbb{P}(T_1 > t)\, \mathbb{P}(T_2 > t) = 1 - \exp(-(\lambda_1 + \lambda_2)t).\end{aligned}$$

So the pdf of T is given by

$$p_T(t) = \begin{cases} (\lambda_1 + \lambda_2)\exp(-(\lambda_1 + \lambda_2)t), & \text{if } t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

which is an exponential random variable with parameter $\lambda_1 + \lambda_2$. This is in concordance with the fact that the superposition of two Poisson processes with intensity λ_1 and λ_2 respectively is again a Poisson process, with intensity $\lambda_1 + \lambda_2$ (the first arrival time of this new process is the minimum of the first arrival times of the two original processes).

4. T is the random variable describing the time elapsed between the arrival of the first student and the departure of the second, and let T_1 and T_2 be the random variables describing the respective durations of the first and second meetings, which are two exponential random variables with the common parameter $\lambda = 1/30 \text{ min}^{-1}$. We have

$$\begin{aligned}\mathbb{E}(T) &= \mathbb{E}(T|T_1 \le 5)\, \mathbb{P}(T_1 \le 5) + \mathbb{E}(T|T_1 > 5)\, \mathbb{P}(T_1 > 5) \\ &= \mathbb{E}(5 + T_2|T_1 \le 5)\, \mathbb{P}(T_1 \le 5) + \mathbb{E}(T_1 + T_2|T_1 > 5)\, \mathbb{P}(T_1 > 5) \\ &= (5 + \mathbb{E}(T_2))\, \mathbb{P}(T_1 \le 5) + (5 + \mathbb{E}(T_1) + \mathbb{E}(T_2))\, \mathbb{P}(T_1 > 5) \\ &= 5 + \mathbb{E}(T_2) + \mathbb{E}(T_1)\, \mathbb{P}(T_1 > 5),\end{aligned}$$

where we have used the independence of T_1 and T_2, as well as the fact that the exponential random variable T_1 is memoryless. This gives finally:

$$\mathbb{E}(T) = 5 + \frac{1}{\lambda} + \frac{1}{\lambda}\exp(-5\lambda) = 35 + 30\exp(-1/6) \simeq 60.4 \text{ minutes}$$

(remember that $1/\lambda = 30$ minutes).

5. a) The embedded discrete-time Markov chain is actually a deteministic process moving from k to $k + 1$ with probability 1 at each time step.

b) The Kolmogorov equations read:

$$\frac{d\pi_0}{dt}(t) = -\lambda\pi_0(t), \quad \frac{d\pi_1}{dt}(t) = \lambda\pi_0(t) - \lambda\pi_1(t), \quad \frac{d\pi_2}{dt}(t) = \lambda\pi_1(t) - \lambda\pi_2(t), \quad \ldots$$

with the initial conditions

$$\pi_0(0) = 1, \quad \pi_1(0) = 0, \quad \pi_2(0) = 0, \quad \ldots$$

The solution is found by induction:

$$\pi_0(t) = e^{-\lambda t}, \quad \pi_1(t) = \lambda t\, e^{-\lambda t}, \quad \pi_2(t) = \frac{(\lambda t)^2}{2}\, e^{-\lambda t}, \quad \ldots, \quad \pi_n(t) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}.$$

c) The Poisson process does not admit a stationary distribution, because it is transient. Alternatively, it can be argued that if we set $\frac{d\pi_n}{dt}(t) = 0$ in the above equations, then all π_n^* become equal to zero, which is impossible for a distribution.

6. a) This is the two-state Markov chain example covered in class. We consider r to be the "under repair" state, while w is the "working" state. μ is the rate at which a machine breaks down and λ is the rate at which it is repaired (measured in number of events per day). For each machine, we have

$$\pi_w(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t},$$
$$\pi_r(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

For the Windows machine, $\mu = 1$ and $\lambda = 24$, so $\pi_w(1) \approx 0.960$.
For the Linux machine, $\mu = \frac{1}{7}$ and $\lambda = 4$, so $\pi_w(1) \approx 0.966$.

b) This probabillity is given by $e^{-\mu}$. So for the Windows machine, it is approx. 0.368, while for the Linux machine, it is approx. 0.867.

7. a) From the transition graph of the Markov chain, it is clear that the student will spend more time on the information theory class than on the probability class.

b) Let us denote the states of the system as p for probability, i for information theory and w for wireless communications. From the problem set, we see that $\nu_p = 1/2$, $\nu_i = 1/4$, $\nu_w = 1/6$ (hour^{-1}), and also that $\hat{q}_{pi} = 1/2$, $\hat{q}_{pw} = 1/2$, $\hat{q}_{iw} = 1$ (so $\hat{q}_{ip} = 0$), $\hat{q}_{wp} = 2/3$ and $\hat{q}_{wi} = 1/3$. So the transition matrix Q of the continuous-time Markov chain reads

$$Q = \begin{pmatrix} -1/2 & 1/4 & 1/4 \\ 0 & -1/4 & 1/4 \\ 1/9 & 1/18 & -1/6 \end{pmatrix}.$$

Solving the stationary distribution equation $\pi^* Q = 0$ (along with the condition $\pi_p^* + \pi_i^* + \pi_w^* = 1$) gives

$$(\pi_p^*, \pi_i^*, \pi_w^*) = \left(\frac{2}{15}, \frac{4}{15}, \frac{3}{5}\right)$$

and the corresponding average numbers of hours spent in each class are given by

$$(T_p, T_i, T_w) = (3.2, 6.4, 14.4).$$